Chapter 7

Poincaré–Bendixson Theory

This chapter is an introduction to the Poincaré–Bendixson theory. After introducing the notion of invariant set, we consider the α-limit and ω-limit sets and we establish some of their basic properties. In particular, we show that bounded semiorbits give rise to connected compact α-limit and ω-limit sets. We then establish one of the important results of the qualitative theory of differential equations in the plane, the Poincaré–Bendixson theorem, which characterizes the α-limit and ω-limit sets of bounded semiorbits. In particular, it allows one to establish a criterion for the existence of periodic orbits. For additional topics we refer the reader to [**9, 13, 15, 17**].

7.1. Limit sets

Let the function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be continuous and locally Lipschitz (see Definition 3.15). Then the equation

$$x' = f(x) \tag{7.1}$$

has unique solutions. We denote by $\varphi_t(x_0)$ the solution with $x(0) = x_0$, for $t \in I_{x_0}$, where I_{x_0} is the corresponding maximal interval.

7.1.1. Basic notions.

We first introduce the notion of invariant set.

Definition 7.1. A set $A \subset \mathbb{R}^n$ is said to be *invariant* (with respect to equation (7.1)) if $\varphi_t(x) \in A$ for every $x \in A$ and $t \in I_x$.

Example 7.2. Consider the equation

$$\begin{cases} x' = y, \\ y' = -x. \end{cases} \tag{7.2}$$

Its phase portrait is the one shown in Figure 1.4. The origin and each circle centered at the origin are invariant sets. More generally, any union of circles and any union of circles together with the origin are invariant sets.

We denote the *orbit* of a point $x \in \mathbb{R}^n$ (see Definition 1.50) by

$$\gamma(x) = \gamma_f(x) = \{\varphi_t(x) : t \in I_x\}.$$

It is also convenient to introduce the following notions.

Definition 7.3. Given $x \in \mathbb{R}^n$, the set

$$\gamma^+(x) = \gamma_f^+(x) = \{\varphi_t(x) : t \in I_x \cap \mathbb{R}^+\}$$

is called the *positive semiorbit* of x, and the set

$$\gamma^-(x) = \gamma_f^-(x) = \{\varphi_t(x) : t \in I_x \cap \mathbb{R}^-\}$$

is called the *negative semiorbit* of x.

One can easily verify that a set is invariant if and only if it is a union of orbits. In other words, a set $A \subset \mathbb{R}^n$ is invariant if and only if

$$A = \bigcup_{x \in A} \gamma(x).$$

Now we introduce the notions of α-limit and ω-limit sets.

Definition 7.4. Given $x \in \mathbb{R}^n$, the *α-limit* and *ω-limit sets* of x (with respect to equation (7.1)) are defined respectively by

$$\alpha(x) = \alpha_f(x) = \bigcap_{y \in \gamma(x)} \overline{\gamma^-(y)}$$

and

$$\omega(x) = \omega_f(x) = \bigcap_{y \in \gamma(x)} \overline{\gamma^+(y)}.$$

Example 7.5 (continuation of Example 7.2)**.** For equation (7.2), we have

$$\alpha(x) = \omega(x) = \gamma^+(x) = \gamma^-(x) = \gamma(x)$$

for any $x \in \mathbb{R}^2$.

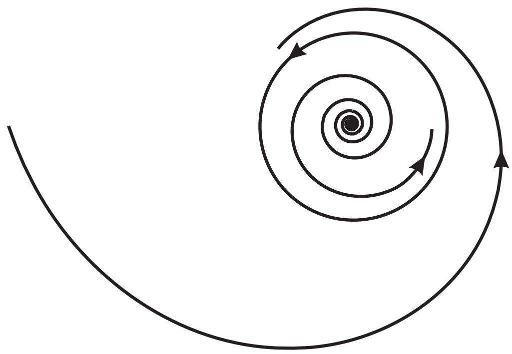

Figure 7.1. Phase portrait of equation (7.3).

Example 7.6. Consider the equation in polar coordinates

$$\begin{cases} r' = r(1-r), \\ \theta' = 1. \end{cases} \tag{7.3}$$

Its phase portrait is the one shown in Figure 7.1. Let

$$S = \{x \in \mathbb{R}^2 : \|x\| = 1\}.$$

We have

$$\alpha(x) = \omega(x) = \gamma^+(x) = \gamma^-(x) = \gamma(x)$$

for $x \in \{(0,0)\} \cup S$,

$$\alpha(x) = \{(0,0)\} \quad \text{and} \quad \omega(x) = S$$

for $x \in \mathbb{R}^2$ with $0 < \|x\| < 1$, and finally,

$$\alpha(x) = \varnothing \quad \text{and} \quad \omega(x) = S$$

for $x \in \mathbb{R}^2$ with $\|x\| > 1$.

Example 7.7. For the phase portrait in Figure 7.2 we have:

$$\alpha(x_1) = \omega(x_1) = \varnothing;$$

$$\alpha(x_2) = \varnothing, \quad \omega(x_2) = \{q\};$$

$$\alpha(x_3) = \omega(x_3) = \{p\};$$

$$\alpha(x_4) = \omega(x_4) = \gamma(x_4).$$

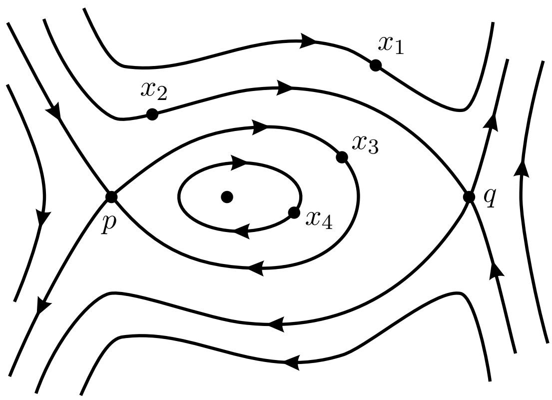

Figure 7.2. Phase portrait for Example 7.7.

7.1.2. Additional properties. In this section we establish some properties of the α-limit and ω-limit sets for equation (7.1).

Proposition 7.8. *If the positive semiorbit $\gamma^+(x)$ of a point $x \in \mathbb{R}^n$ is bounded, then:*

a) $\omega(x)$ *is compact, connected and nonempty;*

b) $y \in \omega(x)$ *if and only if there exists a sequence* $t_k \nearrow +\infty$ *such that* $\varphi_{t_k}(x) \to y$ *when* $k \to \infty$*;*

c) $\varphi_t(y) \in \omega(x)$ *for every* $y \in \omega(x)$ *and* $t > 0$*;*

d) $\inf\{\|\varphi_t(x) - y\| : y \in \omega(x)\} \to 0$ *when* $t \to +\infty$.

Proof. Let $K = \overline{\gamma^+(x)}$. It follows readily from the definition of ω-limit set that $\omega(x)$ is closed. On the other hand, $\omega(x) \subset K$ and thus $\omega(x)$ is also bounded. Hence, the ω-limit set is compact.

Moreover, since the semiorbit $\gamma^+(x)$ is bounded, we have $\mathbb{R}^+ \subset I_x$ (by Theorem 1.46), and thus,

$$\omega(x) = \bigcap_{t>0} A_t, \tag{7.4}$$

where

$$A_t = \overline{\{\varphi_s(x) : s > t\}}.$$

Identity (7.4) yields the second property in the proposition. Indeed, if $y \in \omega(x)$, then there exists a sequence $t_k \nearrow +\infty$ such that $y \in A_{t_k}$ for $k \in \mathbb{N}$. Thus, there is also a sequence $s_k \nearrow +\infty$ with $s_k \geq t_k$ for $k \in \mathbb{N}$ such that $\varphi_{s_k}(x) \to y$ when $k \to \infty$. On the other hand, if there exists a sequence $t_k \nearrow +\infty$ as in the second property in the proposition, then $y \in A_{t_k}$ for

$k \in \mathbb{N}$, and hence,

$$y \in \bigcap_{k=1}^{\infty} A_{t_k} = \bigcap_{t>0} A_t,$$

because $A_t \subset A_{t'}$ for $t > t'$.

Now we consider a sequence $(\varphi_k(x))_k$ contained in the compact set K. By compactness, there exists a subsequence $(\varphi_{t_k}(x))_k$, with $t_k \nearrow +\infty$, converging to a point of K. This shows that $\omega(x)$ is nonempty.

Now we show that $\omega(x)$ is connected. Otherwise, by Definition 6.10, we would have $\omega(x) = A \cup B$ for some nonempty sets A and B such that $\overline{A} \cap B = A \cap \overline{B} = \varnothing$. Since $\omega(x)$ is closed, we have

$$\begin{aligned}\overline{A} &= \overline{A} \cap \omega(x) = \overline{A} \cap (A \cup B) \\ &= (\overline{A} \cap A) \cup (\overline{A} \cap B) = A\end{aligned}$$

and analogously $\overline{B} = B$. This shows that the sets A and B are closed, and hence, they are at a positive distance, that is,

$$\delta := \inf \big\{ \|a - b\| : a \in A, b \in B \big\} > 0.$$

Now we consider the set

$$C = \left\{ z \in \mathbb{R}^2 : \inf_{y \in \omega(x)} \|z - y\| \geq \frac{\delta}{4} \right\}.$$

One can easily verify that $C \cap K$ is compact and nonempty. Hence, it follows from the second property in the proposition that $C \cap K \cap \omega(x) \neq \varnothing$. But by the definition of the set C we know that $C \cap K$ does not intersect $\omega(x)$. This contradiction shows that $\omega(x)$ is connected.

In order to verify the third property in the proposition, we recall that, by the second property, if $y \in \omega(x)$, then there exists a sequence $t_k \nearrow +\infty$ such that $\varphi_{t_k}(x) \to y$ when $k \to \infty$. By Theorem 1.40, the function $y \mapsto \varphi_t(y)$ is continuous for each fixed t. Thus, given $t > 0$, we have

$$\varphi_{t_k+t}(y) = \varphi_t(\varphi_{t_k}(y)) \to \varphi_t(y)$$

when $k \to \infty$. Since $t_k + t \nearrow +\infty$ when $k \to \infty$, it follows from the second property that $\varphi_t(y) \in \omega(x)$.

Finally, we establish the last property in the proposition. Otherwise, there would exist a sequence $t_k \nearrow +\infty$ and a constant $\delta > 0$ such that

$$\inf_{y \in \omega(x)} \|\varphi_{t_k}(x) - y\| \geq \delta \tag{7.5}$$

for $k \in \mathbb{N}$. Since the set K is compact, there exists a convergent subsequence $(\varphi_{t'_k}(x))_k$ of $(\varphi_{t_k}(x))_k \subset K$, which by the second property in the proposition has a limit $p \in \omega(x)$. On the other hand, it follows from (7.5) that

$$\|\varphi_{t'_k}(x) - y\| \geq \delta$$

for every $y \in \omega(x)$ and $k \in \mathbb{N}$. Thus, $\|p-y\| \geq \delta$ for $y \in \omega(x)$, which implies that $p \notin \omega(x)$. This contradiction yields the desired result. □

We have an analogous result for the α-limit set.

Proposition 7.9. *If the negative semiorbit $\gamma^-(x)$ of a point $x \in \mathbb{R}^n$ is bounded, then:*

a) *$\alpha(x)$ is compact, connected and nonempty;*

b) *$y \in \alpha(x)$ if and only if there exists a sequence $t_k \searrow -\infty$ such that $\varphi_{t_k}(x) \to y$ when $k \to \infty$;*

c) *$\varphi_t(y) \in \alpha(x)$ for every $y \in \alpha(x)$ and $t < 0$;*

d) *$\inf\{\|\varphi_t(x) - y\| : y \in \alpha(x)\} \to 0$ when $t \to -\infty$.*

Proof. As in the proof of Theorem 5.4, let $g\colon \mathbb{R}^n \to \mathbb{R}^n$ be the function $g(x) = -f(x)$. We recall that if $\varphi_t(x_0)$ is the solution of the equation $x' = f(x)$ with $x(0) = x_0$, for t in the maximal interval $I_{x_0} = (a, b)$, then $\psi_t(x_0) = \varphi_{-t}(x_0)$ is the solution of the equation $x' = g(x)$ with $x(0) = x_0$, for $t \in (-b, -a)$. This implies that

$$\gamma_g(x) = \gamma_f(x) \quad \text{and} \quad \gamma_g^+(x) = \gamma_f^-(x) \tag{7.6}$$

for every $x \in \mathbb{R}^n$, and thus,

$$\alpha_f(x) = \bigcap_{y \in \gamma_f(x)} \overline{\gamma_f^-(y)} = \bigcap_{y \in \gamma_g(x)} \overline{\gamma_g^+(x)} = \omega_g(x). \tag{7.7}$$

Now we assume that the negative semiorbit $\gamma_f^-(x)$ is bounded. Since $\gamma_g^+(x) = \gamma_f^-(x)$, it follows from Proposition 7.8 that:

a) $\omega_g(x)$ is compact, connected and nonempty;

b) $y \in \omega_g(x)$ if and only if there exists a sequence $t_k \nearrow +\infty$ such that $\psi_{t_k}(x) \to y$ when $k \to \infty$;

c) $\psi_t(y) \in \omega_g(x)$ for every $y \in \omega_g(x)$ and $t > 0$;

d) $\inf\left\{\|\psi_t(x) - y\| : y \in \omega_g(x)\right\} \to 0$ when $t \to +\infty$.

Since $\psi_t = \varphi_{-t}$, the proposition now follows readily from (7.7) and these four properties. □

7.2. The Poincaré–Bendixson theorem

Now we turn to $\mathbb{R}^2$ and we establish one of the important results of the qualitative theory of differential equations: the Poincaré–Bendixson theorem.

7.2.1. Intersections with transversals. We first establish an auxiliary result. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function of class C^1. Also, let L be a line segment *transverse* to f. This means that for each $x \in L$ the directions of L and $f(x)$ generate $\mathbb{R}^2$ (see Figure 7.3). We then say that L is a *transversal* to f.

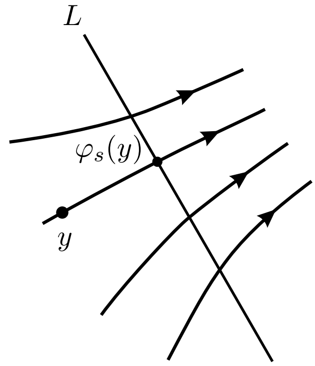

Figure 7.3. Orbits in the neighborhood of a transversal.

Proposition 7.10. *Given $x \in \mathbb{R}^2$, the intersection $\omega(x) \cap L$ contains at most one point.*

Proof. Assume that $\omega(x) \cap L$ is nonempty and take $q \in \omega(x) \cap L$. By Proposition 7.8, there exists a sequence $t_k \nearrow +\infty$ such that $\varphi_{t_k}(x) \to q$ when $k \to \infty$. On the other hand, since L is a transversal to f, it follows from the Flow box theorem (Theorem 1.54) that for each $y \in \mathbb{R}^2$ sufficiently close to L there exists a unique time s such that $\varphi_s(y) \in L$ and $\varphi_t(y) \notin L$ for $t \in (0, s)$ when $s > 0$, or for $t \in (s, 0)$ when $s < 0$ (see Figure 7.3); in particular, for each $k \in \mathbb{N}$ there exists $s = s_k$ as above such that $x_k = \varphi_{t_k+s_k}(x) \in L$.

Now we consider two cases: either $(x_k)_k$ is a constant sequence, in which case the orbit of x is periodic, or $(x_k)_k$ is not a constant sequence. In the first case, since the orbit of x is periodic, the ω-limit set $\omega(x) = \gamma(x)$ only intersects L at the constant value of the sequence $(x_k)_k$, and thus $\omega(x) \cap L = \{q\}$. In the second case, let us consider two successive points of intersection x_k and x_{k+1}, that can be disposed in L in the two forms in Figure 7.4. We note that along L the vector field f always points to the same side (in other words, the projection of f on the perpendicular to L always has the same direction). Otherwise, since f is continuous, there would exist at least one point $z \in L$ with $f(z)$ equal to zero or with the direction of L, but then L would not be a transversal. We also note that the segment of orbit between x_k and x_{k+1} together with the line segment between these two points form a continuous curve C whose complement

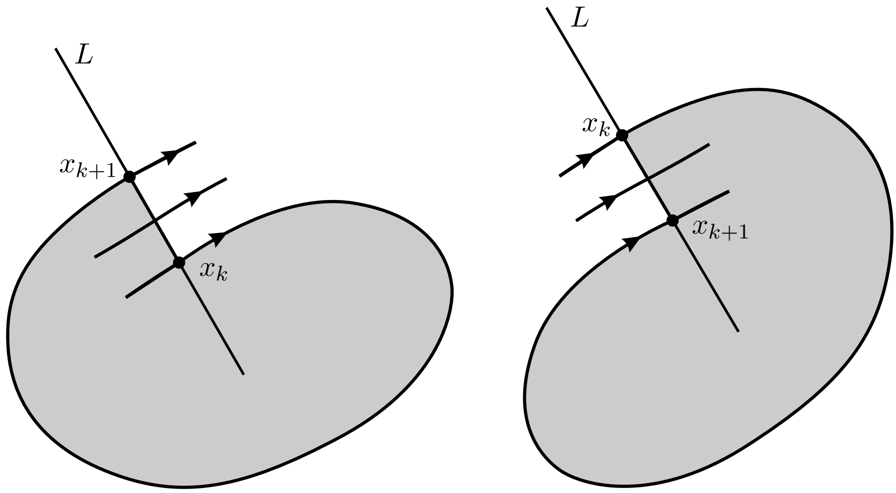

Figure 7.4. Intersections with the transversal L.

$\mathbb{R}^2 \setminus C$ has two connected components, as a consequence of Jordan's curve theorem (Proposition 6.12). The bounded connected component is marked in gray in Figure 7.4. Due to the direction of f on the segment between x_k and x_{k+1} (see Figure 7.4), the positive semiorbit $\gamma^+(x_k)$ is contained in the unbounded connected component. This implies that the next intersection x_{k+2} does not belong to the line segment between x_k and x_{k+1}. Therefore, the points x_k, x_{k+1} and x_{k+2} are ordered on the transversal L as shown in Figure 7.5. Due to the monotonicity of the sequence $(x_k)_k$ along L, it has at most one accumulation point in L and hence $\omega(x) \cap L = \{q\}$. ☐

7.2.2. The Poincaré–Bendixson theorem. The following is the main result of this chapter.

Theorem 7.11 (Poincaré–Bendixson)**.** *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function of class C^1. For equation* (7.1)*, if the positive semiorbit $\gamma^+(x)$ of a point x is bounded and $\omega(x)$ contains no critical points, then $\omega(x)$ is a periodic orbit.*

Proof. Since the semiorbit $\gamma^+(x)$ is bounded, it follows from Proposition 7.8 that $\omega(x)$ is nonempty. Take a point $p \in \omega(x)$. It follows from the first and third properties in Proposition 7.8, together with the definition of ω-limit set, that $\omega(p)$ is nonempty and $\omega(p) \subset \omega(x)$. Now take a point $q \in \omega(p)$. By hypothesis, q is not a critical point, and thus there exists a line segment L containing q that is transverse to f. Since $q \in \omega(p)$, by the second property in Proposition 7.8, there exists a sequence $t_k \nearrow +\infty$ such that $\varphi_{t_k}(p) \to q$ when $k \to \infty$. Proceeding as in the proof of Proposition 7.10, one can always

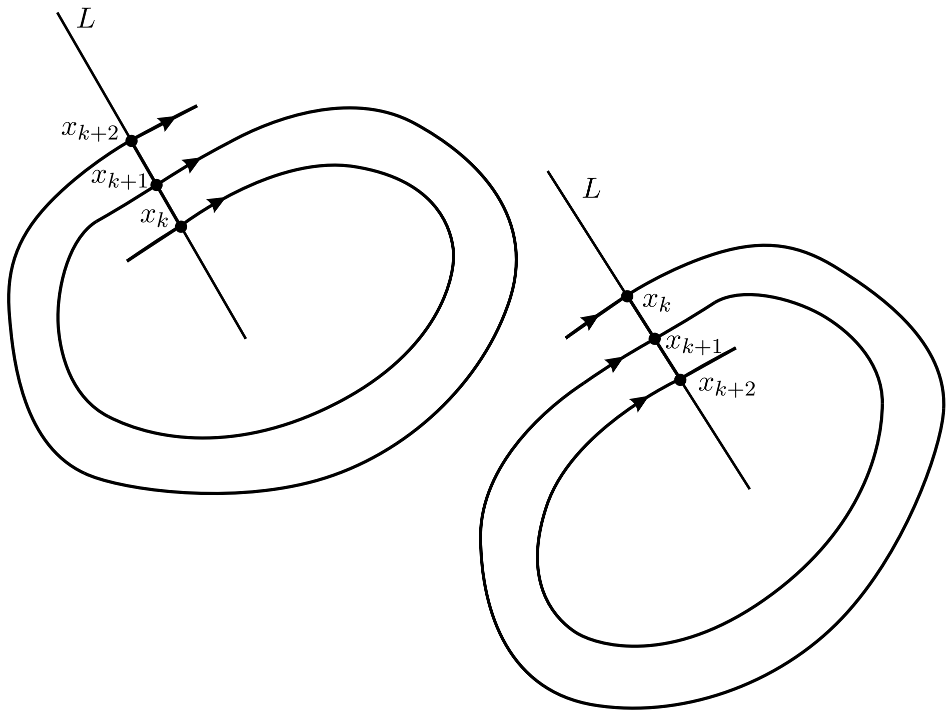

Figure 7.5. Intersections x_k, x_{k+1} and x_{k+2} with the transversal L.

assume that $\varphi_{t_k}(p) \in L$ for $k \in \mathbb{N}$. On the other hand, since $p \in \omega(x)$, it follows from the third property in Proposition 7.8 that $\varphi_{t_k}(p) \in \omega(x)$ for $k \in \mathbb{N}$. Since $\varphi_{t_k}(p) \in \omega(x) \cap L$, by Proposition 7.10 we obtain

$$\varphi_{t_k}(p) = q \quad \text{for every} \quad k \in \mathbb{N}.$$

This implies that $\gamma(p)$ is a periodic orbit. In particular, $\gamma(p) \subset \omega(x)$.

It remains to show that $\omega(x) = \gamma(p)$. If $\omega(x) \setminus \gamma(p) \neq \varnothing$, then since $\omega(x)$ is connected, in each neighborhood of $\gamma(p)$ there exist points of $\omega(x)$ that are not in $\gamma(p)$. We note that any neighborhood of $\gamma(p)$ that is sufficiently small contains no critical points. Thus, there exists a transversal L' to f containing one of these points, which is in $\omega(x)$, and a point of $\gamma(p)$. That is, $\omega(x) \cap L'$ contains at least two points, because $\gamma(p) \subset \omega(x)$; but this contradicts Proposition 7.10. Therefore, $\omega(x) = \gamma(p)$ and the ω-limit set of x is a periodic orbit. □

One can obtain an analogous result to Theorem 7.11 for bounded negative semiorbits.

Theorem 7.12. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function of class C^1. For equation* (7.1)*, if the negative semiorbit $\gamma^-(x)$ of a point x is bounded and $\alpha(x)$ contains no critical points, then $\alpha(x)$ is a periodic orbit.*

Proof. As in the proof of Proposition 7.9, consider the function $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(x) = -f(x)$ and the equation $x' = g(x)$. By (7.6) and (7.7), we have

$$\gamma_f^-(x) = \gamma_g^+(x) \quad \text{and} \quad \alpha_f(x) = \omega_g(x).$$

The result is now an immediate consequence of Theorem 7.11. □

Example 7.13 (continuation of Example 7.6)**.** We already know that equation (7.3) has a periodic orbit (see Figure 7.1), namely the circle of radius 1 centered at the origin. Now we deduce the existence of a periodic orbit as an application of the Poincaré–Bendixson theorem. Consider the ring

$$D = \left\{ x \in \mathbb{R}^2 : \frac{1}{2} < \|x\| < 2 \right\}.$$

For $r = 1/2$ we have $r' = 1/4 > 0$, and for $r = 2$ we have $r' = 2 < 0$. This implies that any orbit entering D no longer leaves D (for positive times). This corresponds to the qualitative behavior shown in Figure 7.6. In particular, any positive semiorbit $\gamma^+(x)$ of a point $x \in D$ is contained in D

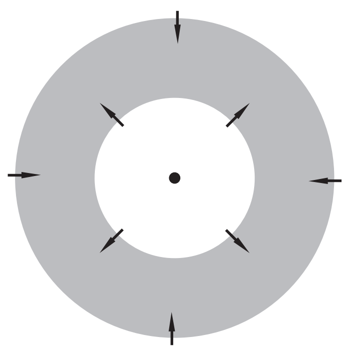

Figure 7.6. Behavior in the boundary of D.

and hence it is bounded. Moreover, it follows from (7.3) that the origin is the only critical point. By the Poincaré–Bendixson theorem (Theorem 7.11), we conclude that $\omega(x)$ is a periodic orbit for each $x \in D$.

Example 7.14. Consider the equation

$$\begin{cases} x' = x(x^2 + y^2 - 3x - 1) - y, \\ y' = y(x^2 + y^2 - 3x - 1) + x, \end{cases} \tag{7.8}$$

which in polar coordinates takes the form

$$\begin{cases} r' = r(r^2 - 3r\cos\theta - 1), \\ \theta' = 1. \end{cases}$$

For any sufficiently small r, we have

$$r^2 - 3r\cos\theta - 1 < 0,$$

and thus $r' < 0$. Moreover, for any sufficiently large r, we have

$$r^2 - 3r\cos\theta - 1 > 0,$$

and thus $r' > 0$. On the other hand, the origin is the only critical point. Now we use an analogous argument to that in Example 7.13. Namely, for $r_1 > 0$ sufficiently small and $r_2 > 0$ sufficiently large, there are no critical points in the ring

$$D' = \{x \in \mathbb{R}^2 : r_1 < \|x\| < r_2\}.$$

Moreover, any negative semiorbit $\gamma^-(x)$ of a point $x \in D'$ is contained in D', and hence it is bounded. It follows from Theorem 7.12 that $\alpha(x) \subset D'$ is a periodic orbit for each $x \in D'$. In particular, equation (7.8) has at least one periodic orbit in D'.

Now we formulate a result generalizing the Poincaré–Bendixson theorem to the case when $\omega(x)$ contains critical points.

Theorem 7.15. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function of class C^1. For equation* (7.1)*, if the positive semiorbit $\gamma^+(x)$ of a point x is contained in a compact set where there are at most finitely many critical points, then one of the following alternatives holds:*

a) *$\omega(x)$ is a critical point;*
b) *$\omega(x)$ is a periodic orbit;*
c) *$\omega(x)$ is a union of a finite number of critical points and homoclinic or heteroclinic orbits.*

Proof. Since $\omega(x) \subset \overline{\gamma^+(x)}$, the set $\omega(x)$ contains at most finitely many critical points. If it only contains critical points, then it is necessarily a single critical point, since by Proposition 7.8 the set $\omega(x)$ is connected.

Now we assume that $\omega(x)$ contains noncritical points and that it contains at least one periodic orbit $\gamma(p)$. We show that $\omega(x)$ is the periodic orbit. Otherwise, since $\omega(x)$ is connected, there would exist a sequence $(x_k)_k \subset \omega(x) \setminus \gamma(p)$ and a point $x_0 \in \gamma(p)$ such that $x_k \to x_0$ when $k \to \infty$. Now we consider a transversal L to the vector field f such that $x_0 \in L$. It follows from Proposition 7.10 that $\omega(x) \cap L = \{x_0\}$. On the other hand, proceeding as in the proof of Proposition 7.10, we conclude that $\gamma^+(x_k) \subset$

$\omega(x)$ intersects L for any sufficiently large k. Since $\omega(x) \cap L = \{x_0\}$, this shows that $x_k \in \gamma(x_0) = \gamma(p)$ for any sufficiently large k, which contradicts the choice of the sequence $(x_k)_k$. Therefore, $\omega(x)$ is a periodic orbit.

Finally, we assume that $\omega(x)$ contains noncritical points but no periodic orbits. We show that for any noncritical point $p \in \omega(x)$ the sets $\omega(p)$ and $\alpha(p)$ are critical points. We only consider $\omega(p)$, because the argument for $\alpha(p)$ is analogous. Let $p \in \omega(x)$ be a noncritical point. We note that $\omega(p) \subset \omega(x)$. If $q \in \omega(p)$ is not a critical point and L is transversal to f containing q, then, by Proposition 7.10,

$$\omega(x) \cap L = \omega(p) \cap L = \{q\};$$

in particular, the orbit $\gamma^+(p)$ intersects L at a point x_0. Since $\gamma^+(p) \subset \omega(x)$, we have $x_0 = q$, and thus $\gamma^+(p)$ and $\omega(p)$ have the point q in common. Proceeding again as in the proof of Proposition 7.10, we conclude that $\omega(p) = \gamma(p)$ is a periodic orbit. This contradiction shows that $\omega(p)$ contains only critical points and since it is connected it contains a single critical point. □

We recall that by Proposition 7.8 the set $\omega(x)$ is connected. Under the assumptions of Theorem 7.15, this forbids, for example, that $\omega(x)$ is a (finite) union of critical points.

One can also formulate a corresponding result for negative semiorbits.

7.3. Exercises

Exercise 7.1. Consider the matrices

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 3 & 0 \end{pmatrix}.$$

a) For the equation $x' = Ax$ show that $\alpha(x) = \{0\}$ for every $x \in \mathbb{R}^5$.

b) For the equation $x' = Bx$ show that a solution x is bounded if and only if $x(0) \in \{0\}^3 \times \mathbb{R}^2$.

Exercise 7.2. By sketching the phase portrait, verify that there exist equations in the plane with at least one disconnected ω-limit set.

Exercise 7.3. Consider the equation

$$\begin{cases} x' = x^2 - xy, \\ y' = y^2 - x^2 - 1. \end{cases}$$

a) Show that the straight line $x = 0$ is a union of orbits.

b) Find whether there exist other straight lines passing through the origin and having the same property.

Exercise 7.4. For each $\varepsilon \in \mathbb{R}$, consider the equation in polar coordinates

$$\begin{cases} r' = r(1-r), \\ \theta' = \sin^2\theta + \varepsilon. \end{cases}$$

a) Sketch the phase portrait for each $\varepsilon \in \mathbb{R}$.

b) Find all values of ε for which the equation is conservative.

c) Find the period of each periodic orbit when $\varepsilon = 1$.

d) Find whether the smallest invariant set containing the open ball of radius $1/2$ centered at $(1, 0)$ is an open set when $\varepsilon = 0$.

Exercise 7.5. Consider the equation

$$\begin{cases} x' = x^2 - y^2, \\ y' = x^2 + y^2. \end{cases}$$

a) Show that there is an invariant straight line containing $(0, 0)$.

b) Show that there are no periodic orbits.

c) Sketch the phase portrait.

Exercise 7.6. For the function $B(x, y) = xy(1-x-y)$, consider the equation

$$x' = \frac{\partial B(x,y)}{\partial y} \quad \text{and} \quad y' = -\frac{\partial B(x,y)}{\partial x}.$$

a) Find all critical points and verify that the straight lines $x = 0$ and $y = 0$ are invariant.

b) Show that the straight line $x + y = 1$ is invariant.

c) Find an invariant compact set with infinitely many points.

d) Sketch the phase portrait.

Exercise 7.7. Given a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ of class C^2, consider the equation $x' = \nabla f(x)$. Show that any nonempty ω-limit set is a critical point.

Exercise 7.8. Verify that there exists an autonomous equation in $\mathbb{R}^3$ with a periodic orbit but without critical points.

Solutions.

7.3 b) There are none.

7.4 b) The equation is conservative for no values of ε.
c) The only periodic orbit is the circle of radius 1 centered at the origin and its period is $\int_0^{2\pi} 1/(\sin^2\theta + 1)\, d\theta = \sqrt{2}\pi$.
d) It is open.

7.6 a) $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1/3, 1/3)$.
c) Triangle determined by $(0, 0)$, $(0, 1)$ and $(1, 0)$.