Chapter 1

INTRODUCTION

1.1. BASIC THEMES

A major goal of this book is understanding the relationships between dynamical systems and the behavior of solutions to various linear partial differential equations (PDE) and pseudodifferential equations containing a small positive parameter h. We outline in this opening section some of the mathematical issues and challenges.

1.1.1. PDE with small parameters. The principal realm of motivation is quantum mechanics, in which case we informally understand h as related to *Planck's constant*. With this interpretation in mind, we break down our basic task into these two subquestions:

(i) How and to what extent do *classical dynamics* determine the behavior as $h \to 0$ of solutions to *Schrödinger's equation*

$$ih\partial_t u = -h^2 \Delta u + Vu$$

and the related *Schrödinger eigenvalue equation*

$$-h^2 \Delta u + Vu = Eu?$$

The name "semiclassical" comes from this interpretation.

(ii) Conversely, given various mathematical objects associated with classical mechanics, for instance symplectic transformations, how can we profitably "quantize" them?

In fact the techniques of semiclassical analysis apply in many other settings and for many other sorts of PDE. For example we will later study the *damped wave equation*

$$\partial_t^2 u + a\partial_t u - \Delta u = 0 \tag{1.1.1}$$

for large times. A rescaling in time will introduce the requisite small parameter h.

1.1.2. Basic techniques. We will construct, mostly in Chapters 2–4, 8–9, and 14, a wide variety of mathematical tools to address these issues, among them:

- the apparatus of symplectic geometry (to record succinctly the behavior of classical dynamical systems);
- the Fourier transform (to display dependence upon both the position variables x and the momentum variables ξ);
- stationary phase (to describe asymptotics as $h \to 0$ of various expressions involving rescaled Fourier transforms); and
- pseudodifferential operators (to localize or, as is said in the trade, to *microlocalize* functional behavior in phase space).

1.1.3. Microlocal analysis. There is a close relation between asymptotic properties of PDE with a small parameter and regularity of solutions to PDE. Asymptotic properties of $\hat{u}(\xi)$ as $1/|\xi| =: h \to 0$ are related to C^∞ regularity of u. For instance, we will see in Chapter 12 how to obtain results about *propagation of singularities* for general classes of equations. Answering questions about propagation of singularities has been one of the motivations of *microlocal analysis*, and most of the techniques presented in this book, such as pseudodifferential operators, come from that subject. Roughly speaking, in standard microlocal analysis $1/|\partial_x|$ plays the role of h. These ideas are behind the study of the damped wave equation (1.1.1).

Some techniques developed for pure PDE questions, such as local solvability, have acquired a new life when translated to the semiclassical setting. An example is the study of pseudospectra of nonselfadjoint operators; see Chapter 12. Another example is the connection between tunneling and unique continuation. These were developed independently in physics and in mathematics and are unified nicely by semiclassical Carleman estimates; see Chapter 7.

1.1.4. Other directions. This book is devoted to semiclassical analysis as a branch of linear PDE theory. The ideas explored here are useful in other areas. One is the study of *quantum maps* where symplectic transformations on compact manifolds are quantized to give matrices. The semiclassical parameter is then related to the size of the matrix. These are popular models in physics partly due to the relative ease of numerical computations; see Haake [**Hak**] and references in Chapter 13 of this text. Many other *large N limit* problems enjoy semiclassical interpretation, in the sense of connecting analysis to geometry. In this book we present one example: a semiclassical proof of Quillen's Theorem (Theorem 13.18) which is related to Hilbert's 17th problem.

Semiclassical concepts also appear in the study of nonlinear PDE. One direction is provided by nonlinear equations with an asymptotic parameter which in some physically motivated problems plays a role similar to h in Section 1.1.1 above. One natural equation is the Gross-Pitaevskii nonlinear Schrödinger equation; see for instance the book by Carles [**Car**]. An example of a numerical study is given in Potter [**Po**] where a semiclassical approximation is used to describe solitons in an external field.

Another set of microlocal methods useful in nonlinear PDE is provided by the *paradifferential calculus* of Bony, Coifman, and Meyer; see for instance Métivier [**Me**], and for a brief introduction see Bényi–Maldonado–Naibo [**B-M-N**]. The semiclassical parameter appears in the Littlewood-Paley decomposition just as it does in Chapter 7, while the pseudodifferential classes are more exotic than the ones considered in Chapter 4.

1.2. CLASSICAL AND QUANTUM MECHANICS

We introduce and foreshadow a bit about quantum and classical correspondences.

1.2.1. Observables. We can think of a given function $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, $a = a(x, \xi)$, as a *classical observable* on phase space, where as above x denotes *position* and ξ denotes *momentum*. We usually call a a *symbol*.

Let $h > 0$ be given. We will associate with the observable a a corresponding *quantum observable* $a^{\mathrm{w}}(x, hD)$, an operator defined by the formula

$$a^{\mathrm{w}}(x,hD)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a\left(\tfrac{x+y}{2}, \xi\right) u(y)\, d\xi dy$$

for appropriate smooth functions u. This is *Weyl's quantization formula*, and $a^{\mathrm{w}}(x, hD)$ is a *pseudodifferential operator*.

One major task will be to understand how the analytic properties of the symbol a dictate the functional analytic properties of its quantization $a^{\mathrm{w}}(x, hD)$. We will in fact build up a *symbol calculus*, meaning systematic rules for manipulating pseudodifferential operators.

1.2.2. Dynamics. We will be concerned as well with the evolution in time of classical particles and quantum states.

Classical evolution. Our most important example will concern the symbol

$$p(x, \xi) := |\xi|^2 + V(x),$$

corresponding to the phase space flow

$$\begin{cases} \dot{x} = 2\xi \\ \dot{\xi} = -\partial V, \end{cases}$$

where $\dot{} = \partial_t$. We generalize by introducing the arbitrary Hamiltonian $p : \mathbb{R}^{2n} \to \mathbb{R}$, $p = p(x, \xi)$, and the corresponding *Hamiltonian dynamics*

$$\begin{cases} \dot{x} = \partial_\xi p(x, \xi) \\ \dot{\xi} = -\partial_x p(x, \xi). \end{cases} \tag{1.2.1}$$

It is instructive to change our viewpoint somewhat, by writing

$$\varphi_t = \exp(tH_p)$$

for the solution of (1.2.1), where

$$H_p q := \{p, q\} = \langle \partial_\xi p, \partial_x q \rangle - \langle \partial_x p, \partial_\xi q \rangle$$

is the *Poisson bracket*. Select a symbol a and define

$$a_t(x, \xi) := a(\varphi_t(x, \xi)). \tag{1.2.2}$$

Then

$$\dot{a}_t = \{p, a_t\}, \tag{1.2.3}$$

and this equation tells us how the symbol evolves in time, as dictated by the classical dynamics (1.2.1).

Quantum evolution. We can quantize the foregoing by putting

$$P = p^{\mathrm{w}}(x, hD), \qquad A = a^{\mathrm{w}}(x, hD)$$

and defining

$$A(t) := F^{-1}(t)AF(t) \tag{1.2.4}$$

for $F(t) := e^{-\frac{itP}{h}}$. The operator $A(t)$ represents, according to the so-called *Heisenberg picture* of quantum mechanics, the evolution of the quantum observable A under the flow (1.2.1). Then we have the evolution equation

$$\partial_t A(t) = \frac{i}{h}[P, A(t)], \tag{1.2.5}$$

an obvious analogue of (1.2.3). Here then is a basic principle we will later work out in some detail: *an assertion about Hamiltonian dynamics, and so the Poisson bracket* $\{\cdot,\cdot\}$*, will involve at the quantum level the commutator* $[\cdot,\cdot]$.

REMARK: h and $\hbar$. In this book h denotes a dimensionless parameter and is consequently not immediately to be identified with the dimensional physical quantity

$$\hbar = \text{Planck's constant}/2\pi = 1.05457 \times 10^{-34}\text{joule-sec.}$$

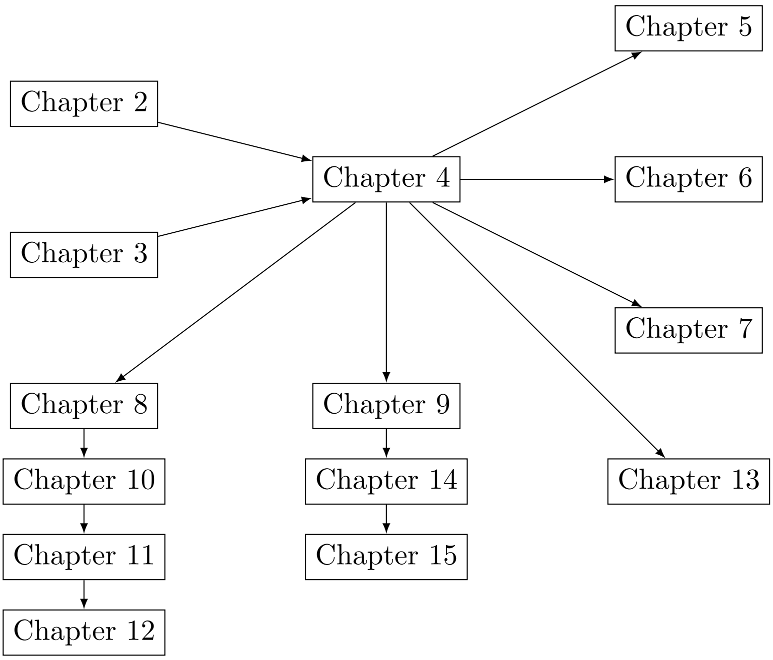
As the example of the damped wave equation (1.1.1) shows, the use of $h \to 0$ asymptotics is not restricted to problems motivated by quantum mechanics.

□

1.3. OVERVIEW

Chapters 2–4 develop the basic machinery, followed by applications to partial differential equations in Chapters 5–7. We develop more advanced theory and applications in Chapters 8–13, and in Chapters 14 and 15 we discuss semiclassical analysis on manifolds.

The following diagram indicates the dependencies of the chapters and may help in selective reading of the book:



Here is a quick overview of the book, with some of the highlights:

Chapter 2: We start with a quick introduction to symplectic analysis and geometry and their implications for classical Hamiltonian dynamical systems.

Chapter 3: This chapter provides the basics of the Fourier transform and derives also important *stationary phase* asymptotic estimates for the oscillatory integral

$$I_h := \int_{\mathbb{R}^n} e^{\frac{i\varphi}{h}} a \, dx$$

of the sort

$$I_h = (2\pi h)^{n/2} |\det \partial^2 \varphi(x_0)|^{-1/2} e^{\frac{i\pi}{4} \operatorname{sgn} \partial^2 \varphi(x_0)} e^{\frac{i\varphi(x_0)}{h}} a(x_0) + O\left(h^{\frac{n+2}{2}}\right)$$

as $h \to 0$, provided the gradient of the phase φ vanishes only at the point x_0.

Chapter 4: Next we introduce the *Weyl quantization* $a^{\mathrm{w}}(x, hD)$ of the symbol $a(x, \xi)$ and work out various properties, chief among them the composition formula

$$a^{\mathrm{w}}(x, hD) b^{\mathrm{w}}(x, hD) = c^{\mathrm{w}}(x, hD),$$

where the symbol $c := a \# b$ is computed explicitly in terms of a and b. We will prove as well the sharp *Gårding inequality*, learn when a^{w} is a bounded operator on L^2, etc.

Chapter 5: This part of the book introduces semiclassical defect measures and uses them to derive decay estimates for the damped wave equation (1.1.1), where $a \geq 0$ on the flat torus $\mathbb{T}^n$. A theorem of Rauch and Taylor provides a beautiful example of classical/quantum correspondence: the waves decay exponentially if all classical trajectories within a certain fixed time intersect the region where positive damping occurs.

Chapter 6: In Chapter 6 we begin our study of the eigenvalue problem

$$P(h)u(h) = E(h)u(h),$$

for the operator

$$P(h) := -h^2\Delta + V(x).$$

We prove *Weyl's Law* for the asymptotic distributions of eigenvalues as $h \to 0$, stating for all $a < b$ that

$$\#\{E(h) \mid a \leq E(h) \leq b\} = \frac{1}{(2\pi h)^n}(|\{a \leq |\xi|^2 + V(x) \leq b\}| + o(1))$$

as $h \to 0$. Our proof is a semiclassical analogue of the classical Dirichlet–Neumann bracketing argument of Courant.

Chapter 7: Chapter 7 deepens our study of eigenfunctions, first establishing an exponential vanishing theorem in the "classically forbidden" region. We derive as well a *Carleman-type* estimate: if $u(h)$ is an eigenfunction of a Schrödinger operator, then for any open set $U \subset\subset \mathbb{R}^n$,

$$\|u(h)\|_{L^2(U)} \geq e^{-c/h}\|u(h)\|_{L^2(\mathbb{R}^n)}.$$

This provides a quantitative estimate for quantum mechanical *tunneling*.

We also present a self-contained "semiclassical" derivation of interior Schauder estimates for the Laplacian.

Chapter 8: We return in Chapter 8 to the symbol calculus, first proving the semiclassical version of *Beals's Theorem*, characterizing pseudodifferential operators. As an application we show how quantization commutes with exponentiation at the level of order functions and then use these insights to define useful generalized Sobolev spaces. This chapter also introduces wavefront sets and the notion of microlocality.

Chapter 9: We next introduce the useful formalism of half-densities and use them to see how changing variables in a symbol affects the Weyl quantization. This motivates our introducing the new class of *Kohn–Nirenberg symbols*, which behave well under coordinate changes and are consequently useful later when we investigate the semiclassical calculus on manifolds.

Chapter 10: Chapter 10 discusses the local construction of propagators, using solutions of Hamilton–Jacobi PDE to build phase functions for *Fourier* *integral operators*. Applications include the semiclassical Strichartz estimates and L^p bounds on eigenfunction clusters.

Chapter 11: This chapter proves *Egorov's Theorem*, characterizing propagators for bounded time intervals in terms of the classical dynamics applied to symbols, up to $O(h)$ error terms. We then employ Egorov's Theorem to quantize linear and nonlinear symplectic mappings and conclude the chapter by showing that Egorov's Theorem is in fact valid until times of order $\log(h^{-1})$, the so-called *Ehrenfest time*.

Chapter 12: Chapter 12 illustrates how methods from Chapter 11 provide elegant and useful normal forms of differential and pseudodifferential operators. Among the applications, we build quasimodes for certain nonnormal operators and discuss the implications for pseudospectra.

Chapter 13: We consider the question of how close semiclassical quantization can get to multiplication. This leads to an alternative presentation of the semiclassicall calculus based on Toeplitz quantization acting on spaces of holomorphic functions. The FBI–Bargmann transform intertwines the quantization of Chapter 4 with the quantization by operators acting on holomorphic functions.

Chapter 14: Chapter 14 briefly discusses general manifolds and modifications to the symbol calculus to cover pseudodifferential operators on manifolds. Chapter 9 provides the change of variables formulas we need to work with coordinate patches.

Chapter 15: This chapter concerns the quantum implications of ergodicity for underlying dynamical systems on manifolds. A key assertion is that if the underlying dynamical system satisfies an appropriate ergodic condition, then

$$h^n \sum_{a \le E_j \le b} \left| \langle A u_j, u_j \rangle - \fint_{\{a \le p \le b\}} \sigma(A)\, dx d\xi \right|^2 \to 0$$

as $h \to 0$, for a wide class of pseudodifferential operators A. In this expression the classical observable $\sigma(A)$ is the *symbol* of A.

Appendices: Appendix A records our notation in one convenient location, and Appendix B is a very quick review of differential forms. Appendix C collects various useful functional analysis theorems (with selected proofs). Appendix D discusses Fredholm operators within the framework of Grushin problems.

1.4. NOTES

The book by Griffiths [**G**] provides a nice elementary introduction to quantum mechanics, and Hannabuss [**Ha**] is a good mathematical text. For a modern physical perspective, consult Gutzwiller [**Gut**], Haake [**Hak**], Heller–Tomsovic [**H-T**], Miller [**Mi**], or Stöckmann [**Sto**].

Chapter 6

EIGENVALUES AND EIGENFUNCTIONS

In this chapter we are given the *potential* $V : \mathbb{R}^n \to \mathbb{R}$ and investigate how the symbol

$$p(x, \xi) = |\xi|^2 + V(x) \tag{6.0.1}$$

provides interesting information about the corresponding operator

$$P(h) := P(x, hD) = -h^2\Delta + V. \tag{6.0.2}$$

We will focus mostly upon learning how p controls the asymptotic distribution of the eigenvalues of $P(h)$ in the semiclassical limit $h \to 0$.

6.1. THE HARMONIC OSCILLATOR

We investigate first the simplest case of a quadratic potential and, to simplify even more, begin in one dimension. So suppose that $n = 1$, $h = 1$, and $V(x) = x^2$. Thus we start with the *one-dimensional quantum harmonic oscillator*, meaning the operator

$$P_0 := -\partial^2 + x^2, \quad \text{where } \partial = \frac{d}{dx}\,. \tag{6.1.1}$$

6.1.1. Eigenvalues and eigenfunctions of P_0. We can as follows employ certain auxiliary first-order differential operators to compute explicitly the eigenvalues and eigenfunctions for P_0.

NOTATION. Let us write

$$A_+ := D_x + ix, \qquad A_- := D_x - ix, \tag{6.1.2}$$

where $D_x = \frac{1}{i}\partial$, and call A_+ the *creation operator* and A_- the *annihilation operator*. (This terminology is from particle physics.)

LEMMA 6.1 (Properties of $A_\pm$). *The creation and annihilation operators satisfy these identities:*

$$A_+^* = A_-, \qquad A_-^* = A_+, \tag{6.1.3}$$

$$P_0 = A_+A_- + 1 = A_-A_+ - 1. \tag{6.1.4}$$

Proof. It is easy to check that $D_x^* = D_x$ and $(ix)^* = -ix$. Furthermore,

$$\begin{aligned} A_+A_-u &= (D_x + ix)(D_x - ix)u \\ &= \left(\frac{1}{i}\partial_x + ix\right)\left(\frac{1}{i}u_x - ixu\right) \\ &= -u_{xx} - (xu)_x + xu_x + x^2u \\ &= -u_{xx} - u - xu_x + xu_x + x^2u \\ &= P_0u - u; \end{aligned}$$

and similarly,

$$\begin{aligned} A_-A_+u &= (D_x - ix)(D_x + ix)u \\ &= \left(\frac{1}{i}\partial_x - ix\right)\left(\frac{1}{i}u_x + ixu\right) \\ &= -u_{xx} + (xu)_x - xu_x + x^2u \\ &= P_0u + u. \end{aligned}$$

□

We can use $A_\pm$ to find all the eigenvalues and eigenfunctions of P_0:

THEOREM 6.2 (Eigenvalues and eigenfunctions for the harmonic oscillator).

(i) *We have*

$$\langle P_0u, u\rangle \geq \|u\|_{L^2}^2$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. That is,

$$P_0 \geq 1.$$

(ii) *The function*

$$v_0 =: e^{-\frac{x^2}{2}}$$

is an eigenfunction corresponding to the smallest eigenvalue 1.

(iii) *Set*

$$v_n := A_+^n v_0$$

for $n = 1, 2, \ldots$. *Then*

$$P_0 v_n = (2n+1) v_n. \tag{6.1.5}$$

(iv) *Define the normalized eigenfunctions*

$$u_n := \frac{v_n}{\|v_n\|_{L^2}}.$$

Then

$$u_n(x) = H_n(x) e^{-\frac{x^2}{2}} \tag{6.1.6}$$

where $H_n(x) = c_n x^n + \cdots + c_0 \quad (c_n \neq 0)$ *is a polynomial of degree* n.

(v) *We have*

$$\langle u_n, u_m \rangle = \delta_{nm} \quad (n, m \in \mathbb{N});$$

and furthermore, the collection of eigenfunctions $\{u_n\}_{n=0}^{\infty}$ *is complete in* $L^2(\mathbb{R}^n)$.

REMARKS. (i) The functions H_n mentioned in assertion (iv) are the *Hermite polynomials*.

(ii) The completeness in assertion (v) shows that we have found *all* the eigevalues of the harmonic oscillator.

Proof. 1. Note that

$$[D_x, x]u = \frac{1}{i}(xu)_x - \frac{x}{i}u_x = \frac{u}{i},$$

and consequently $i[D_x, x] = 1$. Therefore

$$\begin{aligned} \|u\|_{L^2}^2 &= \langle i[D_x, x]u, u \rangle \leq 2\|xu\|_{L^2}\|D_x u\|_{L^2} \\ &\leq \|xu\|_{L^2}^2 + \|D_x u\|_{L^2}^2 = \langle P_0 u, u \rangle. \end{aligned}$$

Next, observe that

$$A_- v_0 = \frac{1}{i}\left(e^{-\frac{x^2}{2}}\right)_x - ixe^{-\frac{x^2}{2}} = 0,$$

so that $P_0 v_0 = (A_+ A_- + 1) v_0 = v_0$.

2. We can further calculate that

$$\begin{aligned} P_0 v_n &= (A_+A_- + 1)A_+ v_{n-1} \\ &= A_+(A_-A_+ - 1)v_{n-1} + 2A_+v_{n-1} \\ &= A_+P_0v_{n-1} + 2A_+v_{n-1} \\ &= (2n-1)A_+v_{n-1} + 2A_+v_{n-1} \quad \text{(by induction)} \\ &= (2n+1)v_n. \end{aligned}$$

The form (6.1.6) of v_n, u_n follows by induction.

3. Also note that

$$\begin{aligned} [A_-, A_+] &= A_-A_+ - A_+A_- \\ &= (P_0+1) - (P_0-1) = 2. \end{aligned}$$

Hence if $m > n$,

$$\begin{aligned} \langle v_n, v_m \rangle &= \langle A_+^n v_0, A_+^m v_0 \rangle \\ &= \langle A_-^m A_+^n v_0, v_0 \rangle \quad \text{(since } A_- = A_+^*) \\ &= \langle A_-^{m-1}(A_+A_- + 2)A_+^{n-1} v_0, v_0 \rangle. \end{aligned}$$

After finitely many steps, the foregoing equals

$$\langle (\dots) A_- v_0, v_0 \rangle = 0,$$

since $A_- v_0 = 0$. Alternatively, we can simply note that $\langle P_0 v_n, v_m \rangle = \langle v_n, P_0 v_m \rangle$, $P_0 v_k = (2k+1)v_k$, $k = m, n$.

4. Finally, we demonstrate that the collection of eigenfunctions that we have found spans L^2. Suppose $\langle u_n, g \rangle = 0$ for $n = 0, 1, 2, \dots$; we must show that $g \equiv 0$.

Now since $H_n(x) = c_n x^n + \dots$, with $c_n \neq 0$, we have

$$\int_{-\infty}^{\infty} g(x) e^{-\frac{x^2}{2}} p(x)\, dx = 0$$

for each polynomial p. Hence

$$\int_{-\infty}^{\infty} g(x) e^{-\frac{x^2}{2}} e^{-ix\xi}\, dx = \int_{-\infty}^{\infty} g(x) e^{-\frac{x^2}{2}} \sum_{k=0}^{\infty} \frac{(-ix\xi)^k}{k!}\, dx;$$

and so $\mathcal{F}\left(g e^{-\frac{x^2}{2}}\right) \equiv 0$. This implies $g e^{-\frac{x^2}{2}} \equiv 0$ and consequently $g \equiv 0$. $\square$

6.1.2. Higher dimensions, rescaling. Suppose now that $n > 1$, and write

$$P_0 := -\Delta + |x|^2; \tag{6.1.7}$$

this is the *n-dimensional quantum harmonic oscillator*. We also define

$$u_\alpha(x) := \prod_{j=1}^n u_{\alpha_j}(x_j) = \prod_{j=1}^n H_{\alpha_j}(x_j) e^{-\frac{|x|^2}{2}}$$

for each multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then

$$P_0 u_\alpha = (-\Delta + |x|^2) u_\alpha = (2|\alpha| + n) u_\alpha,$$

for $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Hence u_α is an eigenfunction of P_0 corresponding to the eigenvalue $2|\alpha| + n$.

We next restore the parameter $h > 0$ by setting

$$P_0(h) := -h^2 \Delta + |x|^2, \tag{6.1.8}$$

$$u_\alpha(h)(x) := h^{-\frac{n}{4}} \prod_{j=1}^n H_{\alpha_j}\left(\frac{x_j}{\sqrt{h}}\right) e^{-\frac{|x|^2}{2h}}, \tag{6.1.9}$$

and

$$E_\alpha(h) := (2|\alpha| + n)h. \tag{6.1.10}$$

Then

$$P_0(h) u_\alpha(h) = E_\alpha(h) u_\alpha(h);$$

and upon reindexing, we can write these eigenfunction equations as

$$P_0(h) u_j(h) = E_j(h) u_j(h) \qquad (j = 1, \ldots). \tag{6.1.11}$$

6.1.3. Asymptotic distribution of eigenvalues. With these explicit formulas in hand, we can study the behavior in the semiclassical limit of the eigenvalues $E(h)$ of the harmonic oscillator:

THEOREM 6.3 (Weyl's Law for the harmonic oscillator). *Assume that $0 \leq a < b < \infty$. Then*

$$\begin{aligned} &\#\{E(h) \mid a \leq E(h) \leq b\} \\ &\qquad = \frac{1}{(2\pi h)^n}(|\{a \leq |\xi|^2 + |x|^2 \leq b\}| + o(1)) \end{aligned} \tag{6.1.12}$$

as $h \to 0$.

Proof. 1. We may assume that $a = 0$. Since $E(h) = (2|\alpha| + n)h$ for some multiindex α according to (6.1.10), we have

$$\begin{aligned} \#\{E(h) \mid 0 \leq E(h) \leq b\} &= \# \left\{ \alpha \mid 0 \leq 2|\alpha| + n \leq \frac{b}{h} \right\} \\ &= \# \{ \alpha \mid \alpha_1 + \cdots + \alpha_n \leq R \} , \end{aligned}$$

for $R := (b - nh)/2h$. Therefore

$$\begin{aligned} &\#\{E(h) \mid 0 \leq E(h) \leq b\} \\ &\quad = |\{x \mid x_i \geq 0,\ x_1 + \cdots + x_n \leq R\}| + o(R^n) \\ &\quad = \frac{1}{n!} R^n + o(R^n) \quad \text{as } R \to \infty \\ &\quad = \frac{1}{n!} \left(\frac{b}{2h} \right)^n + o(h^{-n}) \quad \text{as } h \to 0. \end{aligned}$$

Note that the volume of $\{x \mid x_i \geq 0,\ x_1 + \cdots + x_n \leq 1\}$ is $(n!)^{-1}$.

2. Next we observe that $|\{|\xi|^2 + |x|^2 \leq b\}| = \alpha(2n) b^n$, where $\alpha(k) := \pi^{\frac{k}{2}} (\Gamma(\frac{k}{2} + 1))^{-1}$ is the volume of the unit ball in $\mathbb{R}^k$. Setting $k = 2n$, we compute that $\alpha(2n) = \pi^n (n!)^{-1}$. Hence

$$\begin{aligned} \#\{E(h) \mid 0 \leq E(h) \leq b\} &= \frac{1}{n!} \left(\frac{b}{2h} \right)^n + o(h^{-n}) \\ &= \frac{1}{(2\pi h)^n} |\{|\xi|^2 + |x|^2 \leq b\}| + o(h^{-n}). \qquad \square \end{aligned}$$

6.2. SYMBOLS AND EIGENFUNCTIONS

For this section, we return to the general symbol (6.0.1) and the quantized operator (6.0.2). We assume that the potential $V : \mathbb{R}^n \to \mathbb{R}$ is smooth and satisfies the growth conditions:

$$\begin{cases} |\partial^\alpha V(x)| \leq C_\alpha \langle x \rangle^k & \text{for each multiindex } \alpha \\ V(x) \geq c \langle x \rangle^k & \text{for } |x| \geq R, \end{cases} \tag{6.2.1}$$

for appropriate constants $k, c, C_\alpha, R > 0$.

Our plan in the next section is to employ our detailed knowledge about the eigenvalues of the harmonic oscillator (6.1.8) to estimate the asymptotics of the eigenvalues of $P(h)$.

6.2.1. Concentration in phase space. First, we make the important observation that in the semiclassical limit the eigenfunctions $u(h)$ "are concentrated in phase space" on the energy surface $\{|\xi|^2 + V(x) = E\}$. (This assertion is somewhat related to the earlier Theorem 5.3.)

THEOREM 6.4 (h^∞ estimates). *Suppose that $u(h) \in L^2(\mathbb{R}^n)$ solves*

$$P(h)u(h) = E(h)u(h). \tag{6.2.2}$$

Assume as well that $a \in S$ is a symbol satisfying

$$\{|\xi|^2 + V(x) = E\} \cap \operatorname{spt}(a) = \emptyset.$$

Then if

$$|E(h) - E| < \delta$$

for some sufficiently small $\delta > 0$, we have the estimate

$$\|a^{\mathrm{w}}(x, hD)u(h)\|_{L^2} = O(h^\infty)\|u(h)\|_{L^2}. \tag{6.2.3}$$

Proof. 1. The set $K := \{|\xi|^2 + V(x) = E\} \subset \mathbb{R}^{2n}$ is compact. Hence there exists $\chi \in C_c^\infty(\mathbb{R}^{2n})$ such that

$$0 \le \chi \le 1, \quad \chi \equiv 1 \text{ on } K, \quad \chi \equiv 0 \text{ on } \operatorname{spt}(a).$$

Define the symbol

$$b := |\xi|^2 + V(x) - E(h) + i\chi = p - E(h) + i\chi$$

and the order function

$$m := \langle\xi\rangle^2 + \langle x\rangle^k.$$

Therefore if $|E(h) - E|$ is small enough,

$$|b| \ge \gamma m \quad \text{ on } \mathbb{R}^{2n}$$

for some constant $\gamma > 0$. Consequently $b \in S(m)$, with $b^{-1} \in S(m^{-1})$.

2. Thus there exist $c \in S(m^{-1})$, $r_1, r_2 \in S$ such that

$$\begin{cases} b^{\mathrm{w}}(x, hD)c^{\mathrm{w}}(x, hD) = I + r_1^{\mathrm{w}}(x, hD) \\ c^{\mathrm{w}}(x, hD)b^{\mathrm{w}}(x, hD) = I + r_2^{\mathrm{w}}(x, hD), \end{cases}$$

where $r_1^{\mathrm{w}}(x, hD), r_2^{\mathrm{w}}(x, hD)$ are $O(h^\infty)$. Then

$$a^{\mathrm{w}}(x, hD)c^{\mathrm{w}}(x, hD)b^{\mathrm{w}}(x, hD) = a^{\mathrm{w}}(x, hD) + O(h^\infty), \tag{6.2.4}$$

and

$$b^{\mathrm{w}}(x, hD) = P(h) - E(h) + i\chi^{\mathrm{w}}(x, hD). \tag{6.2.5}$$

Furthermore

$$a^{\mathrm{w}}(x, hD)c^{\mathrm{w}}(x, hD)\chi^{\mathrm{w}}(x, hD) = O(h^\infty),$$

since $\operatorname{spt}(a) \cap \operatorname{spt}(\chi) = \emptyset$. Since $P(h)u(h) = E(h)u(h)$, (6.2.4) and (6.2.5) imply that

$$\begin{aligned} a^{\mathrm{w}}(x,hD)u &= a^{\mathrm{w}}(x,hD)c^{\mathrm{w}}(x,hD)(P(h)-E(h)+i\chi^{\mathrm{w}})u + O(h^\infty) \\ &= O(h^\infty). \end{aligned} \qquad \square$$

For the next result, we temporarily return to the case of the quantum harmonic oscillator, developing some sharper estimates:

THEOREM 6.5 (Improved estimates for the harmonic oscillator). *Suppose that $u(h) \in L^2(\mathbb{R}^n)$ is an eigenfunction of the harmonic oscillator:*

$$P_0(h)u(h) = E(h)u(h). \tag{6.2.6}$$

Assume also that $a \in C^\infty_{\mathrm{c}}$.

Then there exists $E_0 > 0$, depending only on the support of a, such that for $E(h) > E_0$,

$$\|a^{\mathrm{w}}(x,hD)u(h)\|_{L^2} = O\left(\left(\tfrac{h}{E(h)}\right)^\infty\right)\|u(h)\|_{L^2}. \tag{6.2.7}$$

The precise form of the right-hand side of (6.2.7) will later let us handle eigenvalues $E(h) \to \infty$.

Proof. 1. We rescale the harmonic oscillator so that we can work near a fixed energy level E. Set

$$y := \frac{x}{E^{\frac12}}, \qquad \tilde{h} := \frac{h}{E}, \qquad E(\tilde{h}) := \frac{E(h)}{E},$$

where we choose E so that $|E(h) - E| \le E/4$. Then put

$$P_0(h) := -h^2\Delta_x + |x|^2, \qquad P_0(\tilde{h}) := -\tilde{h}^2\Delta_y + |y|^2,$$

whence

$$P_0(h) - E(h) = E(P(\tilde{h}) - \widetilde{E}(\tilde{h})).$$

We next introduce the unitary transformation

$$U\tilde{u}(y) := E^{\frac{n}{4}}\tilde{u}(E^{\frac12}y).$$

Then

$$UP_0(h)U^{-1} = EP_0(\tilde{h});$$

and more generally

$$Ub^{\mathrm{w}}(x,hD)U^{-1} = \widetilde{b}^{\mathrm{w}}(y,\tilde{h}D), \qquad \widetilde{b}(y,\eta) := b(E^{\frac12}y, E^{\frac12}\eta).$$

We will denote the symbol classes defined using $\tilde{h}$ by the symbol $\widetilde{S}_\delta$.

2. We now apply Theorem 6.4. If

$$(P_0(\tilde{h}) - E(\tilde{h}))\widetilde{u}(\tilde{h}) = 0, \qquad |E(\tilde{h}) - 1| < \delta,$$

and $\widetilde{b}(y,\eta) \in \widetilde{S}$ has its support contained in $\{|y|^2 + |\eta|^2 \le 1/2\}$, then

$$\|\widetilde{b}^{\mathrm{w}}(y,\tilde{h}D)\widetilde{u}(\tilde{h})\|_{L^2} = O(\tilde{h}^\infty)\|\widetilde{u}(\tilde{h})\|_{L^2}.$$

Translated to the original h and x as above, this assertion provides us with the bound

$$\|b^{\mathrm{w}}(x,hD)u(h)\|_{L^2} = O((h/E)^\infty)\|u(h)\|_{L^2}, \tag{6.2.8}$$

for

$$b(x,\xi) = \widetilde{b}(E^{-1/2}x, E^{-1/2}\xi) \in S.$$

Note that $\mathrm{spt}(b) \subset \{|x|^2 + |\xi|^2 \le E/2\}$.

3. We now assume that $\tilde{b} = 1$ in $\{|y|^2 + |\eta|^2 \le 1/4\}$. That corresponds to $b = 1$ in $\{|x|^2 + |\xi|^2\} \le E/4\}$. In view of (6.2.8), we only need to show that for

$$a \in C^\infty(\mathbb{R}^{2n}), \quad \mathrm{spt}(a) \subset \{|x|^2 + |\xi|^2 \le R^2\},$$

we have

$$\|(a^{\mathrm{w}}(x,hD)(1 - b^{\mathrm{w}}(x,hD))\|_{L^2\to L^2} = O((h/E)^\infty),$$

for E large enough, where b is as in (6.2.8). That is the same as showing

$$\|\widetilde{a}^{\mathrm{w}}(y,\tilde{h}D)(1 - \widetilde{b}^{\mathrm{w}}(y,\tilde{h}D))\|_{L^2\to L^2} = O(\tilde{h}^\infty), \tag{6.2.9}$$

for

$$\widetilde{a}(y,\eta) = a(E^{\frac{1}{2}}y, E^{\frac{1}{2}}\eta).$$

We first observe that $E = h/\tilde{h} < 1/\tilde{h}$ and hence

$$\widetilde{a} \in \widetilde{S}_{\frac{1}{2}}.$$

Since the support of $\tilde{a}$ is contained in $\{|y|^2 + |\eta|^2 \le R^2/E\}$, we see that for E large enough,

$$\mathrm{dist}(\mathrm{spt}(\widetilde{a}), \mathrm{spt}(1 - \widetilde{b})) \ge 1/C > 0,$$

uniformly in $\tilde{h}$. The estimate (6.2.9) is now a consequence of Theorem 4.25. □

6.2.2. Projections. We next study how projections onto the span of various eigenfunctions of the harmonic oscillator $P_0(h)$ are related to our symbol calculus.

THEOREM 6.6 (Projections and symbols). *Suppose for the symbol $a \in S$ that*

$$\mathrm{spt}(a) \subset \{|\xi|^2 + |x|^2 < R\}.$$

Let

$$\Pi := \textit{projection in } L^2 \textit{ onto}$$
$$\mathrm{span}\{u(h) \mid P_0(h)u(h) = E(h)u(h),\ E(h) \le R\}.$$

Then

$$a^{\mathrm{w}}(x,hD)(I-\Pi) = O_{L^2\to L^2}(h^\infty) \tag{6.2.10}$$

and

$$(I-\Pi)a^{\mathrm{w}}(x,hD) = O_{L^2\to L^2}(h^\infty). \tag{6.2.11}$$

Proof. First of all, observe that

$$(I-\Pi) = \sum_{E_j(h)>R} u_j(h)\otimes u_j(h),$$

meaning that

$$(I-\Pi)u = \sum_{E_j(h)>R} \langle u_j(h), u\rangle u_j(h).$$

Therefore

$$a^{\mathrm{w}}(x,hD)(I-\Pi) = \sum_{E_j(h)>R} (a^{\mathrm{w}}(x,hD)u_j(h))\otimes u_j(h);$$

and so

$$\begin{aligned}\|a^{\mathrm{w}}(x,hD)(I-\Pi)\|^2_{L^2\to L^2} &\le \sum_{E_j(h)>R} \|a^{\mathrm{w}}(x,hD)u_j(h)\|^2_{L^2}\\ &\le \left(\textstyle\sum_{R<E_j(h)\le E_1} + \sum_{E_1<E_j(h)}\right)\|a^{\mathrm{w}}(x,hD)u_j(h)\|^2_{L^2}\\ &:= A+B,\end{aligned} \tag{6.2.12}$$

where $E_1 = \max(E_0, R)$ with E_0 given in Theorem 6.5. We recall that E_0 depends only on $\mathrm{spt}(a)$.

For the term A we can use Theorems 6.3 and 6.4:

$$\begin{aligned}A &\le \#\{E_j(h) : R < E_j(h) < E_0\} \max_{R<E_j(h)<E_0} \|a^{\mathrm{w}}(x,hD)u_j(h)\|^2\\ &\le Ch^{-n}O(h^\infty) = O(h^\infty).\end{aligned}$$

(Here we noted that the assumptions of Theorem 6.4 are satisfied for any E in $R\le E\le E_0$, and hence $\|a^{\mathrm{w}}(x,hD)u_j(h)\| = O(h^\infty)$ holds for $|E_j(h)-E| < \delta_E$, with any of these E's. A finite covering of $[R, E_0]$ provides a uniform estimate.)

Next, observe that Weyl's Law for the harmonic oscillator, Theorem 6.3, implies that

$$E_j(h) \ge \gamma j^{\frac{1}{n}} h$$

for some constant $\gamma > 0$. Then according to Theorem 6.5, for each $M < N$ and for $E_j(h) > E_0$, we have

$$\begin{aligned}\|a^{\mathrm{w}}(x, hD)u_j(h)\|_{L^2}^2 &\leq C_N \left(\tfrac{h}{E_j(h)}\right)^N \\ &\leq C_N h^M \left(\tfrac{h}{E_j(h)}\right)^{N-M} \\ &\leq C h^M j^{-\frac{N-M}{n}}.\end{aligned}$$

Consequently, if we fix $N - M \geq 2n$, we obtain

$$\begin{aligned}B &= \sum_{E_j(h)>E_0} \|a^{\mathrm{w}}(x, hD)u_j(h)\|_{L^2}^2 \\ &\leq C h^M \sum_{j\geq 1} j^{-2} \leq C' h^M.\end{aligned}$$

Since M is arbitrary, we obtain $B = O(h^\infty)$. This proves (6.2.10), and the proof of (6.2.11) is similar. □

6.3. SPECTRUM AND RESOLVENTS

We next show that the spectrum of $P(h)$ consists entirely of eigenvalues.

THEOREM 6.7 (Resolvents and spectrum).

(i) *There exists a constant* $h_0 > 0$ *such that if* $0 < h \leq h_0$, *then the resolvent*

$$(P(h) - i)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is a compact operator.

(ii) *The mapping* $z \mapsto (P(h) - z)^{-1}$ *is meromorphic, with real and simple poles.*

(iii) *The spectrum of* $P(h)$ *is discrete.*

(iv) *Furthermore, there exists an orthonormal basis of* $L^2(\mathbb{R}^n)$ *comprised of eigenfunctions* $\{u_j(h)\}_{j=1}^\infty$:

$$P(h)u_j(h) = E_j(h)u_j(h) \qquad (j = 1, 2, \ldots). \tag{6.3.1}$$

Proof. 1. Let

$$m(x, \xi) := 1 + |\xi|^2 + |x|^k.$$

Then $p \in S(m)$, $C|p - i| \geq m$, and $P(h) = p^{\mathrm{w}}(x, hD)$.

For h small enough, $m^{\mathrm{w}}(x,hD)$ has a right inverse,

$$m^{\mathrm{w}}(x,hD)^{-1} := (1/m)^{\mathrm{w}}(x,hD)(I + hr^{\mathrm{w}}(x,hD))^{-1},$$

$r = (m\#(1/m) - 1)/h \in S$.

We can therefore define the Hilbert space:

$$\mathcal{H} := \{u \in \mathscr{S}' \mid (I - h^2\Delta + \langle x\rangle^k)u \in L^2\} = m^{\mathrm{w}}(x,hD)^{-1}L^2. \tag{6.3.2}$$

For small h, the inverse

$$(P(h) - i)^{-1} : L^2 \to \mathcal{H}$$

is bounded. Theorem 4.28 shows that $m(x,hD)^{-1} : L^2 \to L^2$ is compact and hence $\{u \in \mathcal{H} : \|u\|_{\mathcal{H}} \leq 1\}$ is compact in L^2. That means that the inclusion of $\mathcal{H}$ in L^2 is compact and consequently $(P(h) - i)^{-1} : L^2 \to L^2$ is a compact operator.

2. We now write

$$P(h) - z = (I - K(z,h))(P(h) - i),$$

for

$$K(z,h) := (z - i)(P(h) - i)^{-1}.$$

Since $I - K(-i,h) = I$ and $K(z,h)$ is compact, Theorem D.4 shows that $z \to (I - K(z,h))^{-1}$ is a meromorphic family of operators, with poles of finite rank. Consequently,

$$(P(h) - z)^{-1} = (P(h) - i)^{-1}(I - K(z,h))^{-1}$$

is a meromorphic family of compact operators from L^2 to L^2.

3. Since the poles of $(P(h) - z)^{-1}$ are discrete, there exists $\lambda \in \mathbb{R}$ for which $P(h) - \lambda : \mathcal{H} \to L^2$ is invertible. Hence for any $v_j \in L^2$, there exists $u_j \in \mathcal{H}$ such that $(P(h) - \lambda)u_j = v_j$ and

$$\begin{aligned}\langle (P(h) - \lambda)^{-1}v_1, v_2\rangle &= \langle (P(h) - \lambda)^{-1}(P(h) - \lambda)u_1, (P(h) - \lambda)u_2\rangle \\ &= \langle u_1, (P(h) - \lambda)u_2\rangle.\end{aligned}$$

We integrate by parts, to find

$$\langle (P(h) - \lambda)^{-1}v_1, v_2\rangle = \langle (P(h) - \lambda)u_1, u_2\rangle = \langle v_1, (P(h) - \lambda)^{-1}v_2\rangle.$$

Hence $(P(h) - \lambda)^{-1}$ is selfadjoint.

4. We now apply part (v) of Theorem C.7 to obtain an orthonormal set $\{u_j(h)\}_{j=1}^J$ and a sequence of real numbers $\{E_j(h)\}_{j=1}^J$ such that

$$(P(h) - \lambda)^{-1}v = \sum_{j=1}^{J}(E_j(h) - \lambda)^{-1}u_j(h)\langle v, u_j(h)\rangle, \tag{6.3.3}$$

for all $v \in L^2$, where either $J \in \mathbb{N}$ or else $J = \infty$.

5. Taking $v = u_j$ and applying $P(h) - \lambda$ to both sides of (6.3.3), we deduce that $P(h)u_j(h) = E_j(h)u_j$. Applying $P(h) - \lambda$ to both sides of (6.3.3) for an arbitrary $v \in L^2$, we discover that

$$v = \sum_{j=1}^{J} u_j(h)\langle v, u_j(h)\rangle.$$

Consequently the eigenfunctions $\{u_j(h)\}_{j=1}^J$ form a complete orthonormal set, and in particular $J = \infty$. □

REMARK: Eigenfunctions in $\mathscr{S}$. Using (6.3.1) and the fact that $V \in C^\infty$, we can apply Theorem 7.1 iteratively to conclude that $u \in H_h^l(\mathbb{R}^n)$ for all l and in particular that $u \in C^\infty(\mathbb{R}^n)$. Similarly we can use $V(x) \geq c\langle x\rangle^k - C$ to obtain $\langle x\rangle^N u \in H_h^l(\mathbb{R}^n)$. Putting this together, we deduce that

$$u_j(h) \in \mathscr{S}, \tag{6.3.4}$$

with seminorms depending on h. □

REMARK: An alternative proof of meromorphy. To illustrate further the semiclassical calculus, we provide a different proof of the meromorphy of $z \mapsto (P - z)^{-1}$ for h small.

1. Let $|z| \leq E$, where E is fixed; and as before let $P_0(h) = -h^2\Delta + |x|^2$ be the harmonic oscillator. As in Theorem 6.6 define

$$\Pi := \text{projection in } L^2 \text{ onto}$$
$$\text{span}\{u \mid P_0(h)u = E(h)u \text{ for } E(h) \leq R + 1\}.$$

Now suppose that $\text{spt}(a) \subset \{|x|^2 + |\xi|^2 \leq R\}$. Owing to Theorem 6.6, we have

$$a^{\mathrm{w}}(x, hD) - a^{\mathrm{w}}(x, hD)\Pi = O_{L^2 \to L^2}(h^\infty)$$

and

$$a^{\mathrm{w}}(x, hD) - \Pi a^{\mathrm{w}}(x, hD) = O_{L^2 \to L^2}(h^\infty).$$

2. Fix $R > 0$ so large that

$$\{|\xi|^2 + V(x) \leq E\} \subset \{|x|^2 + |\xi|^2 < R\}.$$

Select $\chi \in C^\infty(\mathbb{R}^{2n})$ with $\text{spt}(\chi) \subset \{|x|^2 + |\xi|^2 \leq R\}$ so that

$$|\xi|^2 + V(x) - z + \chi \geq \gamma m$$

for $m = \langle\xi\rangle^2 + \langle x\rangle^k$ and all $|z| \leq E$. Then $\chi = \Pi\chi\Pi + O(h^\infty)$. Recall that the symbolic calculus guarantees that $P(h) - z + \chi$ is invertible if h is small enough. Consequently, so is $P(h) - z + \Pi\chi\Pi$, since the two operators differ by an $O(h^\infty)$ term.

3. Now write

$$P(h) - z = P(h) - z + \Pi\chi\Pi - \Pi\chi\Pi.$$

Consequently

$$P(h) - z = (P(h) - z + \Pi\chi\Pi)(I - (P(h) - z + \Pi\chi\Pi)^{-1}\Pi\chi\Pi).$$

Note that $\Pi\chi\Pi$ is an operator of finite rank. So Theorem D.4 asserts that the family of operators

$$(I - (P(h) - z + \Pi\chi\Pi)^{-1}\Pi\chi\Pi)^{-1}$$

is meromorphic in z. It follows that $(P(h) - z)^{-1}$ is meromorphic on L^2. The poles are the eigenvalues, and the symmetry of $P(h)$ implies that these eigenvalues are real. □

6.4. WEYL'S LAW

Here is the main result of this chapter:

THEOREM 6.8 (Weyl's Law). *Suppose that V satisfies the conditions of* (6.2.1) *and that the $E(h)$ are the eigenvalues of $P(h) = -h^2\Delta + V(x)$.*

Then for each $a < b$, we have

$$\#\{E(h) \mid a \leq E(h) \leq b\} = \frac{1}{(2\pi h)^n}(|\{a \leq |\xi|^2 + V(x) \leq b\}| + o(1)) \tag{6.4.1}$$

as $h \to 0$. Here $|\{a \leq |\xi|^2 + V(x) \leq b\}|$ denotes the volume of the set of (x, ξ) such that $a \leq |\xi|^2 + V(x) \leq b$.

INTERPRETATION: Density of states. The uncertainty principle proved in Theorem 3.9 tells us the eigenfunctions cannot be arbitrarily localized in phase space. Roughly speaking, their "tightest localization" is to balls of radius $O(h^{\frac{1}{2}})$, so that $\Delta x_j \Delta\xi_j \simeq h$ for $j = 1, \dots, n$ in compliance with the uncertainty principle. Such a localized state "takes up a volume of at least order $O(h^n)$"; and since the eigenfunctions are orthogonal, they also "take up different volumes in any region of phase space." Therefore their total number is at most approximately the volume of the region times h^{-n}.

Consequently in a region of volume $|\{a \leq |\xi|^2 + V(x) \leq b\}|$ in $\mathbb{R}^{2n}$ we could plausibly expect to find at most $O(h^{-n})$ states, and Weyl's Law tells us that the eigenfunctions in fact pack the region at this maximum density. □

These heuristics in fact motivate the following proof.

Proof. 1. Let
$$N(\lambda) = \#\{E(h) \mid E(h) \le \lambda\}.$$
Select $\chi \in C_c^\infty(\mathbb{R}^{2n})$ so that
$$\chi \equiv 1 \text{ on } \{p \le \lambda + \epsilon\}, \qquad \chi \equiv 0 \text{ on } \{p \ge \lambda + 2\epsilon\}.$$
Then for M large enough
$$a := p + M\chi - \lambda \ge \gamma_\epsilon m,$$
for $m = \langle \xi \rangle^2 + \langle x \rangle^m$ and some constant $\gamma_\epsilon > 0$. Hence a is elliptic; and so for small $h > 0$, $a^{\mathrm{w}}(x, hD)$ is invertible.

2. *Claim #1*: We have
$$\langle (P(h) + M\chi^{\mathrm{w}}(x, hD) - \lambda)u, u\rangle \ge \gamma \|u\|_{L^2}^2, \quad u \in \mathcal{H}, \tag{6.4.2}$$
for some $\gamma > 0$. Here $\mathcal{H}$ is given in (6.3.2). Theorem C.14 shows that $\mathcal{H}$ is the domain of $P(h)$.

To prove (6.4.2), take $b \in S(m^{1/2})$ so that $b^2 = a$. Then $b^2 = b\#b + r_0$, where $r_0 \in hS(m)$. We also recall from the proof of Theorem 4.29 that the right inverse $b^{\mathrm{w}}(x, hD)^{-1}$ exists and
$$b^{\mathrm{w}}(x, hD)^{-1} r_0^{\mathrm{w}}(x, hD) b^{\mathrm{w}}(x, hD)^{-1} = O_{L^2 \to L^2}(h).$$
Thus
$$\begin{aligned} a^{\mathrm{w}}(x, hD) &= b^{\mathrm{w}}(x, hD) b^{\mathrm{w}}(x, hD) + r_0^{\mathrm{w}}(x, hD) \\ &= b^{\mathrm{w}}(x, hD)(1 + b^{\mathrm{w}}(x, hD)^{-1} r_0^{\mathrm{w}}(x, hD) b^{\mathrm{w}}(x, hD)^{-1}) b^{\mathrm{w}}(x, hD) \\ &= b^{\mathrm{w}}(x, hD)(1 + O_{L^2 \to L^2}(h)) b^{\mathrm{w}}(x, hD). \end{aligned}$$
Hence for sufficiently small $h > 0$,
$$\begin{aligned} \langle (P(h) + M\chi^{\mathrm{w}} - \lambda)u, u\rangle &= \langle a^{\mathrm{w}}(x, hD)u, u\rangle \\ &\ge \|b^{\mathrm{w}}(x, hD)u\|_{L^2}^2 (1 - O(h)) \\ &\ge \gamma \|u\|_{L^2}^2, \end{aligned}$$
for some $\gamma > 0$, in view of (4.7.2). This proves (6.4.2).

3. *Claim #2*: For each $\delta > 0$, there exists a bounded linear operator Q such that
$$\chi^{\mathrm{w}}(x, hD) = Q + O_{L^2 \to L^2}(h^\infty) \tag{6.4.3}$$
and
$$\operatorname{rank}(Q) \le \frac{1}{(2\pi h)^n}(|\{p \le \lambda + 2\epsilon\}| + \delta). \tag{6.4.4}$$

To prove this, cover the set $\{p \le \lambda + 2\epsilon\}$ with balls
$$B_j := B((x_j, \xi_j), r_j^2) \quad (j = 1, \ldots, N)$$

such that

$$\sum_{j=1}^{N} |B_j| \leq |\{p \leq \lambda + 2\epsilon\}| + \frac{\delta}{2}.$$

We then define the "shifted" harmonic oscillator

$$P_j(h) := |hD_x - \xi_j|^2 + |x - x_j|^2$$

and set

$$\begin{aligned} \Pi \quad := \quad & \text{orthogonal projection in } L^2 \text{ onto } V\text{, the span of} \\ & \{u \mid P_j(h)u = E_j(h)u,\ E_j(h) \leq r_j,\ j = 1, \ldots, N\}. \end{aligned}$$

We now claim that

$$(I - \Pi)\chi^{\mathrm{w}}(x, hD) = O_{L^2 \to L^2}(h^\infty). \tag{6.4.5}$$

To see this, let $\chi = \sum_{j=1}^{N} \chi_j$, where $\operatorname{spt} \chi_j \subset B((x_j, \xi_j), r_j^2)$, and put

$$\begin{aligned} \Pi_j \quad := \quad & \text{orthogonal projection in } L^2 \text{ onto the span of} \\ & \{u \mid P_j(h)u = E_j(h)u,\ E_j(h) \leq r_j\}. \end{aligned}$$

Theorem 6.6 shows that $(I - \Pi_j)\chi_j^{\mathrm{w}}(x, hD) = O(h^\infty)$. We note that $\Pi\Pi_j = \Pi_j$ and hence

$$\begin{aligned} (I - \Pi)\chi^{\mathrm{w}}(x, hD) &= \sum_{j=1}^{N} (I - \Pi)\chi_j^{\mathrm{w}}(x, hD) \\ &= \sum_{j=1}^{N} (I - \Pi)(I - \Pi_j)\chi_j^{\mathrm{w}}(x, hD) \\ &= O_{L^2 \to L^2}(h^\infty). \end{aligned}$$

This proves (6.4.5).

It now follows that

$$\chi^{\mathrm{w}}(x, hD) = \Pi\chi^{\mathrm{w}}(x, hD) + (I - \Pi)\chi^{\mathrm{w}}(x, hD) = Q + O(h^\infty)$$

for

$$Q := \Pi\chi^{\mathrm{w}}(x, hD).$$

Clearly Q has finite rank, since

$$\begin{aligned} \operatorname{rank} Q \quad &= \quad \dim(\text{image of } Q) \leq \dim(\text{image of } \Pi) \\ &\leq \quad \sum_{j=1}^{N} \#\{E_j(h) \mid E_j(h) \leq r_j\} \\ &= \quad \frac{1}{(2\pi h)^n} \left(\sum_{j=1}^{N} |B_j| + o(1) \right), \end{aligned}$$

according to Weyl's Law for the harmonic oscillator, Theorem 6.3. Consequently

$$\operatorname{rank} Q \leq \frac{1}{(2\pi h)^n}\left(|\{p \leq \lambda + 2\epsilon\}| + \frac{\delta}{2} + o(1)\right). \tag{6.4.6}$$

This proves Claim #2.

4. We next employ Claims #1 and #2 and Theorem C.15. We have

$$\begin{aligned}\langle P(h)u, u\rangle &\geq (\lambda+\gamma)\|u\|_{L^2}^2 - M\langle Qu, u\rangle - O(h^\infty)\|u\|_{L^2}^2\\ &\geq \lambda\|u\|_{L^2}^2 - M\langle Qu, u\rangle,\end{aligned}$$

where the rank of Q is bound by (6.4.6). Theorem C.15(i) implies then that

$$N(\lambda) \leq \frac{1}{(2\pi h)^n}(|\{p \leq \lambda + 2\epsilon\}| + \delta + o(1)).$$

This holds for all $\epsilon, \delta > 0$, and so

$$N(\lambda) \leq \frac{1}{(2\pi h)^n}(|\{p \leq \lambda\}| + o(1)) \tag{6.4.7}$$

as $h \to 0$.

5. We must prove the opposite inequality.

Claim #3: Suppose $B_j = B((x_j, \xi_j), r_j^2) \subset \{p < \lambda\}$ and put

$$V_j := \operatorname{span}\{u \mid P_j(h)u = E_j(h)u,\ E_j(h) \leq r_j\}.$$

We claim that for $u \in V_j$,

$$\langle P(h)u, u\rangle \leq (\lambda + \epsilon + O(h^\infty))\|u\|_{L^2}^2. \tag{6.4.8}$$

To prove this claim, select a symbol $a \in C_c^\infty(\mathbb{R}^{2n})$, with

$$a \equiv 1 \text{ on } \{p \leq \lambda\}, \qquad \operatorname{spt}(a) \subset \{p \leq \lambda + \tfrac{\epsilon}{2}\}.$$

Let $c := 1 - a$. Then $u - a^{\mathrm{w}}(x, hD)u = c^{\mathrm{w}}(x, hD)u = O(h^\infty)$ according to Theorem 6.6, since $\operatorname{spt}(1-a) \cap B_j = \emptyset$.

Define $b^{\mathrm{w}} := P(h)a^{\mathrm{w}}(x, hD)$. Now $p \in S(m)$ and $a \in S(m^{-1})$. Thus $b = pa + O(h) \in S$ and so b^{w} is bounded in L^2. Observe also that $b \leq \lambda + \frac{\epsilon}{2}$, and so

$$b^{\mathrm{w}}(x, hD) \leq \lambda + \frac{3\epsilon}{4}.$$

Therefore

$$\langle P(h)a^{\mathrm{w}}(x, hD)u, u\rangle = \langle b^{\mathrm{w}}(x, hD)u, u\rangle \leq \left(\lambda + \frac{3\epsilon}{4}\right)\|u\|_{L^2}^2.$$

Since $a^{\mathrm{w}}(x, hD)u = u + O(h^\infty)$, we deduce that

$$\langle P(h)u, u\rangle \leq (\lambda + \epsilon + O(h^\infty))\|u\|_{L^2}^2.$$

This proves Claim #3.

6. Now find disjoint balls $B_j \subset \{p < \lambda\}$ such that

$$|\{p < \lambda\}| \leq \sum_{j=1}^{N} |B_j| + \delta$$

and denote

$$V = V_1 + V_2 + \cdots + V_N.$$

The spaces V_i and V_j, $i \neq j$, are not orthogonal; but because B_i and B_j are disjoint, we see, as in Theorem 6.6, that

$$\langle u, v\rangle = O(h^\infty)\|u\|\|v\| \tag{6.4.9}$$

if $u \in V_i, v \in V_j$, and $i \neq j$. Since each V_j has an orthonormal basis of eigenvectors, (6.4.8) holds for $u \in V_j$. The approximate orthogonality (6.4.9) then gives

$$\langle Pu, u\rangle \leq (\lambda + \delta)\|u\|_{L^2}^2$$

for all $u \in V$. Also, (6.4.9) and Theorem 6.3 imply that for h small enough

$$\begin{aligned}
\dim V &= \sum_{j=1}^{N} \dim V_j \\
&= \sum_{j=1}^{N} \#\{E_j(h) \leq r_j\} \\
&= \frac{1}{(2\pi h)^n}\left(\sum_{j=1}^{N} |B_j| + o(1)\right) \\
&\geq \frac{1}{(2\pi h)^n}(|\{p < \lambda\}| - \delta + o(1)).
\end{aligned}$$

Then according to Theorem C.15(ii),

$$N(\lambda) \geq \frac{1}{(2\pi h)^n}(|\{p < \lambda\}| - \delta + o(1)).$$

□

6.5. NOTES

The proof of Weyl asymptotics is a semiclassical version of the classical Dirichlet–Neumann bracketing proof for the bounded domains.

In Chapter 12 we will present a more general form of Weyl's Law, proved using a functional calculus of pseudodifferential operators. That proof leads to many further improvements, as discussed in Dimassi–Sjöstrand [**D-S**]. The proof using min-max principle comparisons with the harmonic oscillator has the advantage of providing upper bounds for the number of eigenvalues of nonselfadjoint operators.