

---

# Index

## A

absolute value, derivative of, 154  
additivity, 5  
algebra of sets, 9  
almost everywhere, 7  
  with respect to  $\mu$ , 7  
anticonformal, 111  
antisymmetric functions, 313  
approximation of  
   $L^p$ -functions by  $C^\infty$ -functions, 64  
   $L^p$ -functions by  $C_c^\infty$ -functions, 69  
  distributions by  $C^\infty$ -functions, 149  
area of a sphere, 6

## B

Banach–Alaoglu theorem, 68  
bathtub principle, 28  
Bessel inequality, 73  
Birman–Schwinger kernel, 308  
bootstrap process, 261  
Borel measure, 5  
  sets, 4  
boundary, 176  
  area, 322  
bounded linear functional, 54  
  sequences have weak limits, 68  
bounded variation, 27

## C

$C_c^\infty(\mathbb{R}^n)$  is dense in  $H_A^1(\mathbb{R}^n)$ , 196  
Cantor diagonal argument, 68

capacitor problem, 291  
  countable subadditivity, 299  
  solution of, 295  
  minimal, 298  
Carathéodory criterion, 29  
Cauchy sequence, 52  
chain rule, 152  
characteristic functions, 3  
closed interval, 3  
  sets, 3  
closure, 3  
CLR bounds, 312  
coherent states, 301, 318–321  
coherent state transform, 318  
compact sets, 3  
comparison function, 303  
competing symmetries, 117  
complement, 4  
completeness of  $H^1(\Omega)$ , 174  
  of  $L^p$ , 52  
completion, 6  
concave, 44  
cone property, 215  
conformal group, 111  
  transformations, 110  
connected sets (arcwise, topologically), 3,  
  39  
connections, 193  
continuous linear functional, 54  
contraction semigroup, 227  
convergence  
  in  $\mathcal{D}$  and  $\mathcal{D}'$ , 138  
  uniform, 31  
convex function, 44  
  set, 44

convex sets, projection on, 53  
 convexity inequality for gradients, 179  
   for the relativistic kinetic energy, 187  
 convexity of the norm, 42  
 convolution, 64  
   and distributions, 144  
   and Fourier transform, 132  
   of a distribution with a  $C_c^\infty$ -function, 149  
   of distributions by  $C^\infty$ -functions, 148  
   of functions and continuity, 70  
 Coulomb energy, 239  
   positivity properties of, 252  
 Coulomb potential, 239  
 countable additivity, 5  
 counting measure, 39  
 covariant derivative, 194

## D

$D^1(\mathbb{R}^n)$ , definition of, 203  
 $D^{1/2}(\mathbb{R}^n)$ , definition of, 203  
 density of  $C^\infty(\Omega)$  in  $H^1(\Omega)$ , 176  
   of  $C^\infty(\Omega)$  in  $W_{\text{loc}}^{1,p}(\Omega)$ , 151  
   of  $C_c^\infty(\mathbb{R}^n)$  in  $H^{1/2}(\mathbb{R}^n)$ , 188  
 density matrix, 315  
 derivative of the absolute value, 154  
   of distributions, 141  
   of distributions and classical derivative,  
     146  
   left and right, 44  
 diamagnetic inequality, 195  
 diameter of a set, 237  
 dimension of a Hilbert-space, 74  
 Dirac delta-measure, 5  
 Dirac ‘delta-function’, 140  
 direct method in the calculus of variations,  
   269  
 directional derivative, 51  
 Dirichlet eigenvalues, 305  
   problem, 305  
 distribution, definition of, 138  
 distributional derivative, 141  
   gradient, 141  
   Laplacian, 158  
 distributions and convolution, 144  
 distributions and derivative, 141  
   and the fundamental theorem of calculus,  
     145  
   approximation by  $C^\infty$ -functions, 149  
   convergence of, 138  
   determined by functions, 140  
   linear dependence of, 150  
   positivity and measures, 161  
 domain of the generator, 228  
 dominated convergence theorem, 19

dual of  $L^p$ , 55, 61  
   of  $W^{1,p}(\Omega)$ , 168  
   index, 43  
   space, 138

## E

Earnshaw theorem, 246  
 Egoroff’s theorem, 31, 40  
 eigenfunction, 271  
   higher, 279  
   second, 279  
 eigenvalues, 271  
   Dirichlet, 305  
   higher, 279  
   Neumann, 305, 329, 330  
   second, 279  
   sums, 306, 308, 322, 325  
 elliptic regularity theory, 260  
 ellipticity constants, 232  
   uniform, 232  
 equimeasurability, 81  
 essential spectrum, 302  
   support, 13  
   supremum, 42  
 Euclidean distance, 2  
   group, 111  
   space,  $n$ -dimensional, 2

## F

Fatou lemma, 18  
   missing term in, 21  
 finite cone, 215  
 first excited state, 279  
 Fourier characterization of  $H^1(\mathbb{R}^n)$ , 181  
 Fourier series, 74  
   coefficients, 74  
 Fourier transform and convolutions, 132  
   definition of, 125  
   in  $L^2$ , 129  
   in  $L^p$ , 130  
   of  $|x|^{\alpha-n}$ , 132  
   of a Gaussian function, 127  
   inversion formula, 130  
 Fubini theorem, 16, 25  
 function as a distribution, 140  
   vanishing at infinity, 80  
 fundamental theorem of calculus for  
   distributions, 145

## G

Gateaux derivative, 51  
 gauge invariance, 194

Gaussian function, 98, 121  
 and Fourier transform, 129  
 Gauss measure, 224  
 general rearrangement inequality, 93  
 generator, 228  
 gradients, convexity inequality for, 179  
 vanishing on the inverse of small sets, 157  
 Gram matrix, 198  
 Gram–Schmidt procedure, 73  
 Green’s function of the Laplacian, 158  
 and Poisson equation in  $\mathbb{R}^n$ , 158  
 ground state, 271  
 energy, 271

## H

$H^1(\Omega)$ , completeness of, 174  
 definition of, 173  
 density of  $C^\infty(\Omega)$  in, 176  
 multiplication by functions in  $C^\infty(\Omega)$ ,  
 175  
 $H^1(\mathbb{R}^n)$ , Fourier characterization of, 181  
 $H^1$  and  $W^{1,2}$ , 176  
 $H_0^1(\Omega)$ , 176  
 $H_A^1(\mathbb{R}^n)$ , density of  $C_c^\infty(\mathbb{R}^n)$  in, 196  
 $H_A^1$ , 193  
 definition of, 194  
 $H^{1/2}(\mathbb{R}^n)$ , density of  $C_c^\infty(\mathbb{R}^n)$  in, 188  
 $H^{1/2}$ , definition of, 183  
 Hahn–Banach theorem, 56  
 half open rectangles, 33  
 Hanner inequality for  $L^p(\Omega)$ , 49  
 for  $W^{m,p}(\Omega)$ , 171  
 Hardy–Littlewood–Sobolev inequality, 106  
 conformal invariance of, 114  
 sharp version of, 106  
 harmonic functions, 240  
 Harnack inequality, 247  
 Hausdorff–Young inequality, 131  
 heat equation, 182, 231  
 kernel, 182, 234  
 Helly’s selection principle, 89, 118  
 Hessian matrix, 242  
 Hilbert-space, 71, 173  
 separable, 73  
 Hölder inequality, 45  
 local continuity, 260  
 hydrogen atom, 284  
 hypoelliptic, 261

## I

inequality, Bessel, 73  
 convexity for gradients, 179  
 convexity for relativistic kinetic energy,  
 187

diamagnetic, 195  
 fully generalized Young, 100  
 Hanner inequality for  $L^p(\Omega)$ , 49  
 for  $W^{m,p}(\Omega)$ , 171  
 Hardy–Littlewood–Sobolev, 106  
 Harnack, 247  
 Hausdorff–Young, 131  
 Hölder, 45  
 Jensen, 44  
 Kato, 198  
 Lieb–Thirring, 307  
 logarithmic Sobolev, 224  
 mean value inequality for  $\Delta - \mu^2$ , 254  
 mean value inequality for Laplacian, 241  
 Minkowski, 47  
 nonexpansivity of rearrangement, 83  
 Nash, 222  
 Poincaré, 198, 220  
 Poincaré–Sobolev, 221  
 rearrangement, general, 93  
 simplest, 82  
 strict, 93  
 Riesz rearrangement inequality, 87  
 Riesz rearrangement inequality in  
 one-dimension, 84  
 Schwarz, 46  
 Sobolev for  $W^{m,p}(\Omega)$ , 215  
 Sobolev for  $|p|$ , 206  
 Sobolev for gradients, 204  
 Sobolev in 1 and 2 dimensions, 207  
 triangle, 2, 42, 47  
 weak Young, 107  
 Young, 98  
 infinitesimal generator of the heat kernel,  
 183  
 inner product, 71, 175  
 space, 71  
 inner regularity, 7  
 integrable functions, 14  
 locally, 139  
 integrals, 12  
 interior regularity, 260  
 inversion formula for the Fourier transform,  
 130  
 inversion on the unit sphere, 111  
 isometry, 113, 128

## J

Jensen inequality, 44

## K

Kato inequality, 198  
 kernel, 150  
 Birman–Schwinger, 308

heat, 182, 234  
 Poisson, 185  
 kinetic energy, 174, 195, 271  
 and coherent states, 320, 321  
 relativistic, 184  
 with magnetic field, 195

## L

$L^2$  Fourier transform, 129  
 $L^p$ -spaces and convolution, 70  
 completeness of, 52  
 definition of, 41  
 dual of, 61  
 local, 139  
 separability of, 67  
 $L^p$  and Fourier transform, 132  
 Laplacian, 158  
 Green's function of, 158  
 infinitesimal generator of the heat kernel,  
 183  
 layer cake representation, 26  
 Lebesgue measure, 6  
 level set, 12  
 linear dependence of distributions, 150  
 linear functionals, 54  
 separation property, 56  
 Liouville's theorem, 316  
 Lipschitz continuity, 170, 260  
 locally Hölder continuous, 260  
 summable functions, 139  
 $p^{\text{th}}$ -power summable functions, 139  
 lower semicontinuity of norms, 57  
 lower semicontinuous, 12, 241  
 Lusin's theorem, 40

## M

magnetic fields, 193  
 maximum of  $W^{1,p}$ -functions, 155  
 maximizers, 98  
 mean value inequality for  $\Delta - \mu^2$ , 254  
 for Laplacian, 241  
 measure, 5  
 Borel, 5  
 counting, 39  
 Gauss, 224  
 Lebesgue, 6  
 outer, 29  
 positive, 5  
 restriction, 5  
 space, 5  
 theory, 4  
 and distributions, 161  
 measurable, 4  
 functions, 12

sets, 5, 29  
 minimum of  $W^{1,p}$ -functions, 155  
 minimizers, 98  
 existence of, 277  
 Minkowski content, 322  
 inequality, 47  
 min-max principles, 302  
 generalized, 304  
 momentum, 316  
 monotone class, 9  
 theorem, 9  
 monotone convergence theorem, 17

## N

Nash inequality, 222  
 negative part, 15  
 Neumann eigenvalues, 305, 329, 330  
 problem, 222  
 Newton's theorem, 251  
 nonexpansivity of rearrangement, 83  
 norm, 42  
 differentiability of, 51  
 norm closed set, 53  
 normal vector, 72  
 normalization condition, 271  
 null-space, 150

## O

one parameter group, 227  
 open balls, 4  
 interval, 3  
 sets, 3  
 optimizer, 98  
 order preserving, 81  
 orthogonal, 71  
 complement, 72  
 group, 110  
 sum, 72  
 orthonormal basis, 74  
 set, 72  
 outer measure, 29, 163  
 outer regularity, 7

## P

$p, q, r$  theorem, 77  
 parallelogram identity, 49, 72  
 partial integration for functions in  $H^1(\mathbb{R}^n)$ ,  
 177  
 phase space, 316  
 Plancherel theorem, 128  
 Poincaré inequality, 198, 220  
 Poincaré–Sobolev inequality, 221  
 points of a set, 4

Poisson equation, continuity of solutions, 262  
 first differentiability of solutions, 262  
 higher differentiability of solutions, 264  
 solution of, 159  
 Poisson kernel, 185  
 polarization, 129  
 Pólya conjecture, 307  
 positive distributions, 161  
 positive measure, 5  
 positive part, 15  
 positivity preserving, 234  
 potential, 271  
 potential energy, 271  
 domination of by the kinetic energy, 272  
 weak continuity of, 276  
 product measure, 7, 23  
 associativity of, 24  
 commutativity of, 24  
 product sigma-algebras, 7  
 product space, 7  
 projection on convex sets, 53  
 Pythagoras theorem, 71

## R

really simple function, 33, 34  
 rearrangement inequality, general, 93  
 nonexpansivity, 83  
 Riesz, 87  
 Riesz in one dimension, 84  
 simplest, 82  
 strict, 93  
 rearrangement, decreasing of kinetic energy, 190  
 nonexpansivity of, 83  
 of functions, 80  
 of sets, 80  
 rectangles, 7  
 half open, 33  
 relativistic kinetic energy, 184  
 convexity inequality for, 187  
 Rellich-Kondrashov theorem, 216  
 restriction of a measure, 5  
 Riemann integrable, 14  
 Riemann integral, 14  
 Riemann–Lebesgue lemma, 126  
 Riesz–Markov representation theorem, 162  
 Riesz rearrangement inequality, 87  
 in one-dimension, 84  
 Riesz representation theorem, 61

## S

scalar product, 175  
 scaling symmetry, 111

Schrödinger equation, 271  
 existence of minimizer, 277  
 lower bound on the wave function, 256  
 regularity of solutions, 281  
 time independent, 271  
 uniqueness of minimizers, 282  
 uniqueness of positive solutions, 283  
 Schwarz symmetrization, 88  
 inequality, 46  
 second eigenfunction, 279  
 eigenvalue, 279  
 section property, 8  
 semiclassical approximation, 301, 316  
 semicontinuous, 12  
 semigroup, 227  
 contraction, 227  
 separability of  $L^p$ , 67  
 sets, 3  
 algebra of, 9  
 Borel, 4  
 closed, 3  
 compact, 3  
 connected, 3, 39  
 convex, 44  
 level, 12  
 measurable, 5, 29  
 norm closed, 53  
 open, 3  
 orthonormal, 72  
 points of, 4  
 sigma-algebra, 4  
 smallest, 4  
 sigma-finiteness, 7  
 signed measure, 27  
 signum function, 199  
 simple function, 32  
 smoothing estimate, 229  
 Sobolev inequalities for  $W^{m,p}(\Omega)$ , 215  
 inequalities in 1 and 2 dimensions, 207  
 inequality for  $|p|$ , 206  
 inequality for gradients, 204  
 logarithmic, 224  
 spaces, 143  
 spherical charge distributions, and point charges, 251  
 standard metric on  $\mathbb{S}^n$ , 113  
 volume element on  $\mathbb{S}^n$ , 113  
 star-shaped set, 237  
 Steiner symmetrization, 87  
 stereographic coordinates, 112  
 projection, 112  
 strict rearrangement inequality, 93  
 strictly convex, 44  
 strictly positive measurable function, 13  
 strictly symmetric-decreasing, 81  
 strong convergence, 52, 139, 143  
 strong maximum principle, 246

strongly convergent convex combinations,  
60  
 subharmonic functions, 240  
   and potentials, 248  
 subspace of a Hilbert-space, 72  
   closed, 72  
 summable function, 14  
   locally, 139  
 superharmonic functions, 240  
 support of a continuous function, 3  
 support plane, 44  
 symmetric, 229  
   difference, 35  
 symmetric-decreasing rearrangement, 80  
   of a function, 80  
   of a set, 80  
 symmetrization, Schwarz, 88  
   Steiner, 87

## T

tangent plane, 44  
 test functions (the space  $\mathcal{D}(\Omega)$ ), 138  
 Thomas–Fermi energy, 285  
   TF equation, 287  
   TF minimizer, 286, 289  
   TF potential, 289  
   TF problem, 285  
 tiling domains, 307  
 time independent Schrödinger equation,  
271  
 total energy, 271  
 translation invariance, 110  
 trial function, 303  
 triangle inequality, 2, 42, 47

## U

uncertainty principles, 202, 273  
 uniform boundedness principle, 58  
 uniform convergence, 31  
 uniform convexity, 49  
 unitary transformation, 129  
 upper semicontinuous, 12, 241  
 Urysohn lemma, 4, 38

## V

vanish at infinity, 80  
 variational function, 303  
 vector potential, 194

## W

$W^{1,2}$  and  $H^1$ , 176  
 $W^{1,p}(\Omega)$ , definition of, 142  
 $W_0^{1,p}(\Omega)$ , definition of, 214  
 $W_{\text{loc}}^{1,p}(\Omega)$ , definition of, 142  
   density of  $C^\infty(\Omega)$  in, 151  
 weak  $L^q$ -space, 106  
 weak continuity of the potential energy, 276  
 weak convergence, 54, 139, 143  
   implying a.e. convergence, 214  
   implying strong convergence, 210  
   nonzero after translations, 217  
 weak derivative, 141  
 weak limits, 193  
   bounded sequences and, 68  
 weak Young inequality, 107  
 weakly lower semicontinuous, 57, 270  
 Weyl's law, 316  
 Weyl's lemma, 258

## Y

Young inequality, 98  
   fully generalized, 100  
   weak, 107  
 Yukawa potential, 166, 257  
   uniqueness, 257