Preface to the First Edition

A glance at the table of contents will reveal the somewhat unconventional nature of this introductory book on analysis, so perhaps we should explain our philosophy and motivation for writing a book that has elementary integration theory together with potential theory, rearrangements, regularity estimates for differential equations and the calculus of variations all sandwiched between the same covers.

Originally, we were motivated to present the essentials of modern analysis to physicists and other natural scientists, so that some modern developments in quantum mechanics, for example, would be understandable. From personal experience we realized that this task is little different from the task of explaining analysis to students of mathematics. At the present time there are many excellent texts available, but they mostly emphasize concepts in themselves rather than their useful relation to other parts of mathematics. It is a question of taste, but there are many students (and teachers) who, in the limited time available, prefer to go through a subject by doing something with the material, as it is learned, rather than wait for a full-fledged development of all basic principles.

The topics covered here are selected from those we have found useful in our own research and are among those that practicing analysts need in their kit-bag, such as basic facts about measure theory and integration, Fourier transforms, commonly used function spaces (including Sobolev spaces), distribution theory, etc. Our goal was to guide beginning students through these topics with a minimum of fuss and to lead them to the point where they can read current literature with some understanding. At the same time everything is done in a rigorous and, hopefully, pedagogical way.

Inequalities play a key role in our presentation and some of them are less standard, such as the Hardy–Littlewood–Sobolev inequality, Hanner's inequality and rearrangement inequalities. These and other unusual topics, such as $H^{1/2}$- and H^1_A-spaces, are included for a definite pedagogical reason: They introduce the student to some serious exercises in hard analysis (i.e., interesting theorems that take more than a few lines to prove), but ones that can be tackled with the elementary tools presented here. In this way we hope that relative beginners can get some of the flavor of research mathematics and the feeling that the subject is open-ended.

Throughout, our approach is 'hands on', meaning that we try to be as direct as possible and do not always strive for the most general formulation. Occasionally we have slick proofs, but we avoid unnecessary abstraction, such as the use of the Baire category theorem or the Hahn–Banach theorem, which are not needed for L^p-spaces. Our preference is to understand L^p-spaces and then have the reader go elsewhere to study Banach spaces generally (for which excellent texts abound), rather than the other way around. Another noteworthy point is that we try not to say, "there exists a constant such that ...". We usually give it, or at least an estimate of it. It is important for students of the natural sciences, *and* mathematics, to learn how to calculate. Nowadays, this is often overlooked in mathematics courses that usually emphasize pure existence theorems.

From some points of view, the topics included here are a curious mixture of the advanced-specialized together with the elementary but the reader will, we believe, see that there is a unity to it all. For example, most texts make a big distinction between 'real analysis' and 'functional analysis', but we regard this distinction as somewhat artificial. Analysis without functions doesn't go very far. On the other hand, Hilbert-space is hardly mentioned, which might seem strange in a book in which many of the examples are taken from quantum mechanics. This theory (beyond the linear algebra level) becomes truly interesting when combined with operator theory, and these topics are not treated here because they are covered in many excellent texts. Perhaps the severest rearrangement of the conventional order is in our treatment of Lebesgue integration. In Chapter 1 we introduce what is needed to understand and use integration, but we do not bother with the proof of the existence of Lebesgue measure; it suffices to *know* its existence. Finally, after the reader has acquired some sophistication, the proof is given in Exercise 6.5 as a corollary of Theorem 6.22 (positive distributions are measures).

Things the reader is expected to know: While we more or less start from 'scratch', we do expect the reader to know some elementary facts, all of which will have been learned in a good calculus course. These include: vector spaces, limits, lim inf, lim sup, open, closed and compact sets in $\mathbb{R}^n$, continuity and differentiability of functions (especially in the multivariable case), convergence and uniform convergence (indeed, the notion of 'uniform', generally), the definition and basic properties of the Riemann integral, integration by parts (of which Gauss's theorem is a special case).

How to read this book: There is a great deal of material here but the following selection hits the main points. It is possible to cover them conveniently in a year's course of 25 weeks.

CHAPTER 1. The basic facts of integration can be gleaned from 1.1, 1.2, 1.5–1.8, 1.10, 1.12 (the statement only), 1.13.

CHAPTER 2. The essential facts about L^p-spaces are in 2.1–2.4, 2.7, 2.9, 2.10, 2.14–2.19.

CHAPTER 3. 3.3, 3.4, 3.7 are enough for a first reading about rearrangements. This serves as a useful exercise in manipulating integrals.

CHAPTER 4. Read the nonsharp proofs of Young's inequality, 4.2, and the HLS inequality, 4.3.

CHAPTER 5. Fourier transforms are basic in many applications. Read 5.1–5.8.

CHAPTER 6. 6.1–6.18, 6.20, 6.21, 6.22 (statement only).

CHAPTER 7. 7.1–7.10, 7.17, 7.18. $H^{1/2}$ spaces and H^1_A spaces are specialized examples, useful in quantum mechanics, and can be ignored at first.

CHAPTER 8. All except 8.4. Sobolev inequalities are essential for partial differential equations and it is necessary to be familiar with their statements, if not their proofs.

CHAPTER 9. Potential theory is classical and basic to physics and mathematics. 9.1–9.5, 9.7, 9.8 are the most important. 9.10 is a useful extension of Harnack's inequality and is worth studying.

CHAPTER 10. It is important to know how to go from weak to strong solutions of partial differential equations. 10.1 and the statements of 10.2, 10.3, if not the proofs, should be learned.

CHAPTER 11. The calculus of variations, especially as a key to solving some differential equations, is extremely useful and important. All the examples given here, 11.1–11.17 are worth learning, not only for their intrinsic value, but because they use many of the topics presented earlier in the book.

A word about notation. The book is organized around theorems, but frequently there are some pertinent remarks before and after the statement of a theorem. The symbol ● is used to denote the introduction of a new idea or discussion, while ■ is used for the end of a proof. Equations are numbered separately in each section. The notation 1.6(2), for example, means equation number (2) in Section 1.6. Exercise 1.15, for example means exercise number 15 in Chapter 1. To avoid unnecessary enumeration, (2) means equation number (2) of the section we are presently in; similarly, Exercise 15 refers to Exercise 15 of the present chapter. Bold-face is used whenever a bit of terminology appears for the first time.

According to Walter Thirring there are three things that are easy to start but very difficult to finish. The first is a war. The second is a love affair. The third is a trill. To this may be added a fourth: a book. Many students and colleagues helped over the years to put us on the right track on several topics and helped us eliminate some of the more egregious errors and turgidities. Our thanks go to Almut Burchard, Eric Carlen, E. Brian Davies, Evans Harrell, Helge Holden, David Jerison, Richard Laugesen, Carlo Morpurgo, Bruno Nachtergaele, Barry Simon, Avraham Soffer, Bernd Thaller, Lawrence Thomas, Kenji Yajima, our students at Georgia Tech and Princeton, several anonymous referees, to Lorraine Nelson for typing most of the manuscript and to Janet Pecorelli for turning it into a book.

For the reader's convenience there is a Web page for this book where additional exercises and errata are available. The URL is http://www.math.gatech.edu/˜loss/Analysis.html

Preface to the Second Edition

Since the publication of our book four years ago we have received many helpful comments from colleagues and students. Not only were typographical errors pointed out — and duly published on our web page, whose URL is given below — but interesting suggestions were also made for improvements and clarification.

We, too, wanted to add more topics which, in the spirit of the book, are hopefully of use to students and practitioners.

This led to a second edition, which contains all the corrections and some fresh items. Chief among these is Chapter 12 in which we explain several topics concerning eigenvalues of the Laplacian and the Schrödinger operator, such as the min-max principle, coherent states, semiclassical approximation and how to use these to get bounds on eigenvalues and sums of eigenvalues. But there are other additions, too, such as more on Sobolev spaces (Chapter 8) including a compactness criterion, and Poincaré, Nash and logarithmic Sobolev inequalities. The latter two are applied to obtain smoothing properties of semigroups.

Chapter 1 (Measure and integration) has been supplemented with a discussion of the more usual approach to integration theory using simple functions, and how to make this even simpler by using 'really simple functions'. Egoroff's theorem has also been added. Several additions were made to Chapter 6 (Distributions) including one about the Yukawa potential.

There are, of course, many more Exercises as well.

In order to avoid conflict and confusion with the first edition we made the conscious decision to place the new material at the end of any given Chapter, which is not always the best place, logically, and insertions in the first edition text are kept to a minimum. (The chief exceptions are the evaluation of $\exp\{-t\sqrt{p^2+m^2}\}$ in Sect. 7.11 and a new proof of Theorem 2.16.)

We are most grateful to our numerous correspondents. Rather than inadvertently leaving someone out, we have not listed the names, but we hope our friends will be satisfied with our thanks and that they will once again let us know of any errors they find in this second edition. These will be posted on our web page.

We are especially grateful to Eric Carlen for helping us in many ways. He encouraged us to add material to Chapter 1 about the usual 'simple function' treatment of measure theory, and allowed us to use his notes freely about 'really simple functions'. He encouraged us, also, to add the material in Chapter 8 mentioned above.

Many thanks go to Donald Babbitt, the AMS publisher, who urged us to write a second edition and who made the necessary resources of the AMS available. We are extremely fortunate again in having Janet Pecorelli help us, and we are grateful to her for lending her admirable talents to this project and for patiently enduring our numerous changes. Thanks also go to Mary Letourneau for superb copy editing and Daniel Ueltschi for help with proofreading.

January, 2001

Note on the Reprinting of the Second Edition

The errata of the second edition, found on the web site noted below, have been corrected in this printing. The additional exercises on the web site, 2.24 and 4.8, have also been incorporated. We are grateful to Rupert Frank for his help in this, as well as for two new exercises, 8.5 and 8.6.

May, 2010

For the reader's convenience there is a Web page for this book where additional exercises and errata are available. The URL is http://www.math.gatech.edu/~loss/Analysis.html