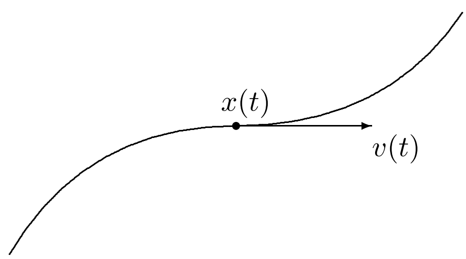
Chapter 1

Introduction

1.1. Newton's equations

Let us begin with an example from physics. In classical mechanics a particle is described by a point in space whose location is given by a function

$$x: \mathbb{R} \to \mathbb{R}^3. \tag{1.1}$$



The derivative of this function with respect to time is the velocity of the particle

$$v = \dot{x}: \mathbb{R} \to \mathbb{R}^3 \tag{1.2}$$

and the derivative of the velocity is the acceleration

$$a = \dot{v}: \mathbb{R} \to \mathbb{R}^3. \tag{1.3}$$

In such a model the particle is usually moving in an external force field

$$F: \mathbb{R}^3 \to \mathbb{R}^3 \tag{1.4}$$

which exerts a force $F(x)$ on the particle at x. Then **Newton's second law of motion** states that, at each point x in space, the force acting on the particle must be equal to the acceleration times the mass m (a positive constant) of the particle, that is,

$$m\,\ddot{x}(t) = F(x(t)), \qquad \text{for all } t \in \mathbb{R}. \tag{1.5}$$

Such a relation between a function $x(t)$ and its derivatives is called a **differential equation**. Equation (1.5) is of second order since the highest derivative is of second degree. More precisely, we have a system of differential equations since there is one for each coordinate direction.

In our case x is called the dependent and t is called the independent variable. It is also possible to increase the number of dependent variables by adding v to the dependent variables and considering $(x, v) \in \mathbb{R}^6$. The advantage is that we now have a *first-order* system

$$\begin{aligned} \dot{x}(t) &= v(t) \\ \dot{v}(t) &= \frac{1}{m} F(x(t)). \end{aligned} \tag{1.6}$$

This form is often better suited for theoretical investigations.

For given force F one wants to find solutions, that is, functions $x(t)$ that satisfy (1.5) (respectively (1.6)). To be more specific, let us look at the motion of a stone falling towards the earth. In the vicinity of the surface of the earth, the gravitational force acting on the stone is approximately constant and given by

$$F(x) = -m\,g \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.7}$$

Here g is a positive constant and the x_3 direction is assumed to be normal to the surface. Hence our system of differential equations reads

$$\begin{aligned} m\,\ddot{x}_1 &= 0, \\ m\,\ddot{x}_2 &= 0, \\ m\,\ddot{x}_3 &= -m\,g. \end{aligned} \tag{1.8}$$

The first equation can be integrated with respect to t twice, resulting in $x_1(t) = C_1 + C_2 t$, where C_1, C_2 are the integration constants. Computing the values of x_1, $\dot{x}_1$ at $t = 0$ shows $C_1 = x_1(0)$, $C_2 = v_1(0)$, respectively. Proceeding analogously with the remaining two equations we end up with

$$x(t) = x(0) + v(0)\,t - \frac{g}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t^2. \tag{1.9}$$

Hence the entire fate (past and future) of our particle is uniquely determined by specifying the initial location $x(0)$ together with the initial velocity $v(0)$.

From this example you might get the impression that solutions of differential equations can always be found by straightforward integration. However, this is not the case in general. The reason why it worked here is that the force is independent of x. If we refine our model and take the real gravitational force

$$F(x) = -\gamma\, m\, M \frac{x}{|x|^3}, \qquad \gamma, M > 0, \tag{1.10}$$

our differential equation reads

$$\begin{aligned} m\,\ddot{x}_1 &= -\frac{\gamma\, m\, M\, x_1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}, \\ m\,\ddot{x}_2 &= -\frac{\gamma\, m\, M\, x_2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}, \\ m\,\ddot{x}_3 &= -\frac{\gamma\, m\, M\, x_3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} \end{aligned} \tag{1.11}$$

and it is no longer clear how to solve it. Moreover, it is even unclear whether solutions exist at all! (We will return to this problem in Section 8.5.)

Problem 1.1. *Consider the case of a stone dropped from the height h. Denote by r the distance of the stone from the surface. The initial condition reads $r(0) = h$, $\dot{r}(0) = 0$. The equation of motion reads*

$$\ddot{r} = -\frac{\gamma M}{(R + r)^2} \qquad \textit{(exact model)}$$

respectively

$$\ddot{r} = -g \qquad \textit{(approximate model)},$$

where $g = \gamma M/R^2$ and R, M are the radius, mass of the earth, respectively.

(i) *Transform both equations into a first-order system.*

(ii) *Compute the solution to the approximate system corresponding to the given initial condition. Compute the time it takes for the stone to hit the surface ($r = 0$).*

(iii) *Assume that the exact equation also has a unique solution corresponding to the given initial condition. What can you say about the time it takes for the stone to hit the surface in comparison to the approximate model? Will it be longer or shorter? Estimate the difference between the solutions in the exact and in the approximate case. (Hints: You should not compute the solution to the exact equation! Look at the minimum, maximum of the force.)*

(iv) *Grab your physics book from high school and give numerical values for the case $h = 10m$.*

Problem 1.2. *Consider again the exact model from the previous problem and write*

$$\ddot{r} = -\frac{\gamma M \varepsilon^2}{(1+\varepsilon r)^2}, \qquad \varepsilon = \frac{1}{R}.$$

It can be shown that the solution $r(t) = r(t,\varepsilon)$ *to the above initial conditions is* C^∞ *(with respect to both* t *and* ε*). Show that*

$$r(t) = h - g(1 - 2\frac{h}{R})\frac{t^2}{2} + O(\frac{1}{R^4}), \qquad g = \frac{\gamma M}{R^2}.$$

(Hint: Insert $r(t,\varepsilon) = r_0(t) + r_1(t)\varepsilon + r_2(t)\varepsilon^2 + r_3(t)\varepsilon^3 + O(\varepsilon^4)$ *into the differential equation and collect powers of* ε*. Then solve the corresponding differential equations for* $r_0(t)$*,* $r_1(t)$*, ... and note that the initial conditions follow from* $r(0,\varepsilon) = h$ *respectively* $\dot{r}(0,\varepsilon) = 0$*. A rigorous justification for this procedure will be given in Section* 2.5*.)*

1.2. Classification of differential equations

Let $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ and $k \in \mathbb{N}_0$. Then $C^k(U,V)$ denotes the set of functions $U \to V$ having continuous derivatives up to order k. In addition, we will abbreviate $C(U,V) = C^0(U,V)$, $C^\infty(U,V) = \bigcap_{k\in\mathbb{N}} C^k(U,V)$, and $C^k(U) = C^k(U,\mathbb{R})$.

A classical **ordinary differential equation** (ODE) is a functional relation of the form

$$F(t, x, x^{(1)}, \dots, x^{(k)}) = 0 \tag{1.12}$$

for the unknown function $x \in C^k(J)$, $J \subseteq \mathbb{R}$, and its derivatives

$$x^{(j)}(t) = \frac{d^j x(t)}{dt^j}, \qquad j \in \mathbb{N}_0. \tag{1.13}$$

Here $F \in C(U)$ with U an open subset of $\mathbb{R}^{k+2}$. One frequently calls t the **independent** and x the **dependent variable**. The highest derivative appearing in F is called the **order** of the differential equation. A **solution** of the ODE (1.12) is a function $\phi \in C^k(I)$, where $I \subseteq J$ is an interval, such that

$$F(t, \phi(t), \phi^{(1)}(t), \dots, \phi^{(k)}(t)) = 0, \qquad \text{for all } t \in I. \tag{1.14}$$

This implicitly implies $(t, \phi(t), \phi^{(1)}(t), \dots, \phi^{(k)}(t)) \in U$ for all $t \in I$.

Unfortunately there is not too much one can say about general differential equations in the above form (1.12). Hence we will assume that one can solve F for the highest derivative, resulting in a differential equation of the form

$$x^{(k)} = f(t, x, x^{(1)}, \dots, x^{(k-1)}). \tag{1.15}$$

By the implicit function theorem this can be done at least locally near some point $(t,y) \in U$ if the partial derivative with respect to the highest derivative does not vanish at that point, $\frac{\partial F}{\partial y_k}(t,y) \neq 0$. This is the type of differential equations we will consider from now on.

We have seen in the previous section that the case of real-valued functions is not enough and we should admit the case $x : \mathbb{R} \to \mathbb{R}^n$. This leads us to **systems of ordinary differential equations**

$$\begin{aligned} x_1^{(k)} &= f_1(t, x, x^{(1)}, \dots, x^{(k-1)}), \\ &\vdots \\ x_n^{(k)} &= f_n(t, x, x^{(1)}, \dots, x^{(k-1)}). \end{aligned} \tag{1.16}$$

Such a system is said to be **linear** if it is of the form

$$x_i^{(k)} = g_i(t) + \sum_{l=1}^{n} \sum_{j=0}^{k-1} f_{i,j,l}(t) x_l^{(j)}. \tag{1.17}$$

It is called **homogeneous** if $g_i(t) \equiv 0$.

Moreover, any system can always be reduced to a first-order system by changing to the new set of dependent variables $y = (x, x^{(1)}, \dots, x^{(k-1)})$. This yields the new **first-order system**

$$\begin{aligned} \dot{y}_1 &= y_2, \\ &\vdots \\ \dot{y}_{k-1} &= y_k, \\ \dot{y}_k &= f(t, y). \end{aligned} \tag{1.18}$$

We can even add t to the dependent variables $z = (t, y)$, making the right-hand side independent of t

$$\begin{aligned} \dot{z}_1 &= 1, \\ \dot{z}_2 &= z_3, \\ &\vdots \\ \dot{z}_k &= z_{k+1}, \\ \dot{z}_{k+1} &= f(z). \end{aligned} \tag{1.19}$$

Such a system, where f does not depend on t, is called **autonomous**. In particular, it suffices to consider the case of autonomous first-order systems, which we will frequently do.

Of course, we could also look at the case $t \in \mathbb{R}^m$ implying that we have to deal with partial derivatives. We then enter the realm of **partial differential equations** (PDE). However, we will not pursue this case here.

Finally, note that we could admit complex values for the dependent variables. It will make no difference in the sequel whether we use real or complex dependent variables. However, we will state most results only for the real case and leave the obvious changes to the reader. On the other hand, the case where the independent variable t is complex requires more than obvious modifications and will be considered in Chapter 4.

Problem 1.3. *Classify the following differential equations. Is the equation linear, autonomous? What is its order?*

(i) $y'(x) + y(x) = 0$.

(ii) $\frac{d^2}{dt^2}u(t) = t\sin(u(t))$.

(iii) $y(t)^2 + 2y(t) = 0$.

(iv) $\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) = 0$.

(v) $\dot{x} = -y$, $\dot{y} = x$.

Problem 1.4. *Which of the following differential equations for $y(x)$ are linear?*

(i) $y' = \sin(x)y + \cos(y)$.

(ii) $y' = \sin(y)x + \cos(x)$.

(iii) $y' = \sin(x)y + \cos(x)$.

Problem 1.5. *Find the most general form of a second-order linear equation.*

Problem 1.6. *Transform the following differential equations into first-order systems.*

(i) $\ddot{x} + t\sin(\dot{x}) = x$.

(ii) $\ddot{x} = -y$, $\ddot{y} = x$.

The last system is linear. Is the corresponding first-order system also linear? Is this always the case?

Problem 1.7. *Transform the following differential equations into autonomous first-order systems.*

(i) $\ddot{x} + t\sin(\dot{x}) = x$.

(ii) $\ddot{x} = -\cos(t)x$.

The last equation is linear. Is the corresponding autonomous system also linear?

Problem 1.8. *Let $x^{(k)} = f(x, x^{(1)}, \dots, x^{(k-1)})$ be an autonomous equation (or system). Show that if $\phi(t)$ is a solution, then so is $\phi(t - t_0)$.*

1.3. First-order autonomous equations

Let us look at the simplest (nontrivial) case of a first-order autonomous equation and let us try to find the solution starting at a certain point x_0 at time $t = 0$:

$$\dot{x} = f(x), \quad x(0) = x_0, \qquad f \in C(\mathbb{R}). \tag{1.20}$$

We could of course also ask for the solution starting at x_0 at time t_0. However, once we have a solution $\phi(t)$ with $\phi(0) = x_0$, the solution $\psi(t)$ with $\psi(t_0) = x_0$ is given by a simple shift $\psi(t) = \phi(t - t_0)$. (This holds in fact for any autonomous equation — compare Problem 1.8.)

This equation can be solved using a small ruse. If $f(x_0) \neq 0$, we can divide both sides by $f(x)$ and integrate both sides with respect to t:

$$\int_0^t \frac{\dot{x}(s)ds}{f(x(s))} = t. \tag{1.21}$$

Abbreviating $F(x) = \int_{x_0}^x \frac{dy}{f(y)}$ we see that every solution $x(t)$ of (1.20) must satisfy $F(x(t)) = t$. Since $F(x)$ is strictly monotone near x_0, it can be inverted and we obtain a unique solution

$$\phi(t) = F^{-1}(t), \qquad \phi(0) = F^{-1}(0) = x_0, \tag{1.22}$$

of our initial value problem. Here $F^{-1}(t)$ is the inverse map of $F(t)$.

Now let us look at the maximal interval where ϕ is defined by this procedure. If $f(x_0) > 0$ (the case $f(x_0) < 0$ follows analogously), then f remains positive in some interval (x_1, x_2) around x_0 by continuity. Define

$$T_+ = \lim_{x \uparrow x_2} F(x) \in (0, \infty], \quad \text{respectively} \quad T_- = \lim_{x \downarrow x_1} F(x) \in [-\infty, 0). \tag{1.23}$$

Then $\phi \in C^1((T_-, T_+))$ and

$$\lim_{t \uparrow T_+} \phi(t) = x_2, \quad \text{respectively} \quad \lim_{t \downarrow T_-} \phi(t) = x_1. \tag{1.24}$$

In particular, ϕ is defined for all $t > 0$ if and only if

$$T_+ = \int_{x_0}^{x_2} \frac{dy}{f(y)} = +\infty, \tag{1.25}$$

that is, if $1/f(x)$ is *not* integrable near x_2. Similarly, ϕ is defined for all $t < 0$ if and only if $1/f(x)$ is *not* integrable near x_1.

If $T_+ < \infty$ there are two possible cases: Either $x_2 = \infty$ or $x_2 < \infty$. In the first case the solution ϕ diverges to $+\infty$ and there is no way to extend it beyond T_+ in a continuous way. In the second case the solution ϕ reaches the point x_2 at the finite time T_+ and we could extend it as follows: If $f(x_2) > 0$ then x_2 was not chosen maximal and we can increase it, which provides the required extension. Otherwise, if $f(x_2) = 0$, we can extend ϕ by setting $\phi(t) = x_2$ for $t \geq T_+$. However, in the latter case this might not be the only possible extension, as we will see in the examples below. Clearly, similar arguments apply for $t < 0$.

Now let us look at some examples.

Example. If $f(x) = x$, $x_0 > 0$, we have $(x_1, x_2) = (0, \infty)$ and

$$F(x) = \log(\frac{x}{x_0}). \tag{1.26}$$

Hence $T_\pm = \pm\infty$ and

$$\phi(t) = x_0 \mathrm{e}^t. \tag{1.27}$$

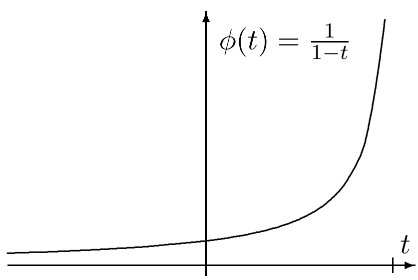
Thus the solution is globally defined for all $t \in \mathbb{R}$. Note that this is in fact a solution for all $x_0 \in \mathbb{R}$. ⋄

Example. Let $f(x) = x^2$, $x_0 > 0$. We have $(x_1, x_2) = (0, \infty)$ and

$$F(x) = \frac{1}{x_0} - \frac{1}{x}. \tag{1.28}$$

Hence $T_+ = 1/x_0$, $T_- = -\infty$ and

$$\phi(t) = \frac{x_0}{1 - x_0 t}. \tag{1.29}$$



In particular, the solution is no longer defined for all $t \in \mathbb{R}$. Moreover, since $\lim_{t \uparrow 1/x_0} \phi(t) = \infty$, there is no way we can possibly extend this solution for $t \geq T_+$. ⋄

Now what is so special about the zeros of $f(x)$? Clearly, if $f(x_0) = 0$, there is a trivial solution

$$\phi(t) = x_0 \tag{1.30}$$

to the initial condition $x(0) = x_0$. But is this the only one? If we have

$$\left| \int_{x_0}^{x_0+\varepsilon} \frac{dy}{f(y)} \right| < \infty, \tag{1.31}$$

then there is another solution

$$\varphi(t) = F^{-1}(t), \qquad F(x) = \int_{x_0}^{x} \frac{dy}{f(y)}, \tag{1.32}$$

with $\varphi(0) = x_0$, which is different from $\phi(t)$!

Example. Consider $f(x) = \sqrt{|x|}$, $x_0 > 0$. Then $(x_1, x_2) = (0, \infty)$,

$$F(x) = 2(\sqrt{x} - \sqrt{x_0}), \tag{1.33}$$

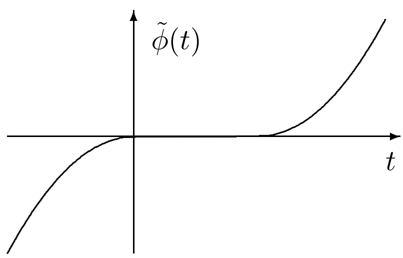
and

$$\varphi(t) = (\sqrt{x_0} + \frac{t}{2})^2, \quad -2\sqrt{x_0} < t < \infty. \tag{1.34}$$

So for $x_0 = 0$ there are several solutions which can be obtained by patching the trivial solution $\phi(t) = 0$ with the above solution as follows

$$\tilde{\phi}(t) = \begin{cases} -\frac{(t-t_0)^2}{4}, & t \le t_0, \\ 0, & t_0 \le t \le t_1, \\ \frac{(t-t_1)^2}{4}, & t_1 \le t. \end{cases} \tag{1.35}$$

The solution $\tilde{\phi}$ for $t_0 = 0$ and $t_1 = 1$ is depicted below:



◇

As a conclusion of the previous examples we have:

- Solutions might exist only locally in t, even for perfectly nice f.
- Solutions might not be unique. Note, however, that $f(x) = \sqrt{|x|}$ is not differentiable at the point $x_0 = 0$ which causes the problems.

Note that the same ruse can be used to solve so-called **separable** equations

$$\dot{x} = f(x)g(t) \tag{1.36}$$

(see Problem 1.11).

Problem 1.9. *Solve the following differential equations:*

(i) $\dot{x} = x^3$.

(ii) $\dot{x} = x(1-x)$.

(iii) $\dot{x} = x(1-x) - c$.

Problem 1.10. *Show that the solution of* (1.20) *is unique if* $f \in C^1(\mathbb{R})$.

Problem 1.11 (Separable equations)**.** *Show that the equation* ($f, g \in C^1$)

$$\dot{x} = f(x)g(t), \qquad x(t_0) = x_0,$$

locally has a unique solution if $f(x_0) \neq 0$. *Give an implicit formula for the solution.*

Problem 1.12. *Solve the following differential equations:*

(i) $\dot{x} = \sin(t)x$.

(ii) $\dot{x} = g(t)\tan(x)$.

(iii) $\dot{x} = \sin(t)\mathrm{e}^x$.

Sketch the solutions. For which initial conditions (if any) are the solutions bounded?

Problem 1.13. *Investigate uniqueness of the differential equation*

$$\dot{x} = \begin{cases} -t\sqrt{|x|}, & x \geq 0, \\ t\sqrt{|x|}, & x \leq 0. \end{cases}$$

Show that the initial value problem $x(0) = x_0$ *has a unique global solution for every* $x_0 \in \mathbb{R}$. *However, show that the global solutions still intersect! (Hint: Note that if* $x(t)$ *is a solution so is* $-x(t)$ *and* $x(-t)$*, so it suffices to consider* $x_0 \geq 0$ *and* $t \geq 0$.*)*

Problem 1.14. *Charging a capacitor is described by the differential equation*

$$R\dot{Q}(t) + \frac{1}{C}Q(t) = V_0,$$

where $Q(t)$ *is the charge at the capacitor,* C *is its capacitance,* V_0 *is the voltage of the battery, and* R *is the resistance of the wire.*

Compute $Q(t)$ *assuming the capacitor is uncharged at* $t = 0$. *What charge do you get as* $t \to \infty$*?*

Problem 1.15 (Growth of bacteria)**.** *A certain species of bacteria grows according to*

$$\dot{N}(t) = \kappa N(t), \qquad N(0) = N_0,$$

where $N(t)$ *is the amount of bacteria at time* t, $\kappa > 0$ *is the growth rate, and* N_0 *is the initial amount. If there is space for only* $N_{\max}$ *bacteria, this has to be modified according to*

$$\dot{N}(t) = \kappa(1 - \frac{N(t)}{N_{\max}})N(t), \qquad N(0) = N_0.$$

Solve both equations assuming $0 < N_0 < N_{\max}$ *and discuss the solutions. What is the behavior of* $N(t)$ *as* $t \to \infty$*?*

Problem 1.16 (Optimal harvest)**.** *Take the same setting as in the previous problem. Now suppose that you harvest bacteria at a certain rate* $H > 0$. *Then the situation is modeled by*

$$\dot{N}(t) = \kappa(1 - \frac{N(t)}{N_{\max}})N(t) - H, \qquad N(0) = N_0.$$

Rescale by

$$x(\tau) = \frac{N(t)}{N_{\max}}, \qquad \tau = \kappa t$$

and show that the equation transforms into

$$\dot{x}(\tau) = (1 - x(\tau))x(\tau) - h, \qquad h = \frac{H}{\kappa N_{\max}}.$$

Visualize the region where $f(x,h) = (1-x)x - h$, $(x,h) \in U = (0,1) \times (0,\infty)$, is positive respectively negative. For given $(x_0, h) \in U$, what is the behavior of the solution as $t \to \infty$? How is it connected to the regions plotted above? What is the maximal harvest rate you would suggest?

Problem 1.17 (Parachutist)**.** *Consider the free fall with air resistance modeled by*

$$\ddot{x} = \eta \dot{x}^2 - g, \qquad \eta > 0.$$

Solve this equation. (Hint: Introduce the velocity $v = \dot{x}$ as new independent variable.) Is there a limit to the speed the object can attain? If yes, find it. Consider the case of a parachutist. Suppose the chute is opened at a certain time $t_0 > 0$. Model this situation by assuming $\eta = \eta_1$ for $0 < t < t_0$ and $\eta = \eta_2 > \eta_1$ for $t > t_0$ and match the solutions at t_0. What does the solution look like?

1.4. Finding explicit solutions

We have seen in the previous section that some differential equations can be solved explicitly. Unfortunately, there is no general recipe for solving a given differential equation. Moreover, finding explicit solutions is in general impossible unless the equation is of a particular form. In this section I will show you some classes of first-order equations which are explicitly solvable.

The general idea is to find a suitable change of variables which transforms the given equation into a solvable form. In many cases the solvable equation will be the

Linear equation:

The solution of the linear homogeneous equation

$$\dot{x} = a(t)x \tag{1.37}$$

is given by

$$\phi(t) = x_0 A(t, t_0), \qquad A(t,s) = \mathrm{e}^{\int_s^t a(s)ds}, \tag{1.38}$$

and the solution of the corresponding inhomogeneous equation

$$\dot{x} = a(t)x + g(t) \tag{1.39}$$

is given by

$$\phi(t) = x_0 A(t, t_0) + \int_{t_0}^{t} A(t, s) g(s) ds. \tag{1.40}$$

This can be verified by a straightforward computation.

Next we turn to the problem of transforming differential equations. Given the point with coordinates (t, x), we may change to new coordinates (s, y) given by

$$s = \sigma(t, x), \qquad y = \eta(t, x). \tag{1.41}$$

Since we do not want to lose information, we require this transformation to be a diffeomorphism (i.e., invertible with differentiable inverse).

A given function $\phi(t)$ will be transformed into a function $\psi(s)$ which has to be obtained by eliminating t from

$$s = \sigma(t, \phi(t)), \qquad \psi = \eta(t, \phi(t)). \tag{1.42}$$

Unfortunately this will not always be possible (e.g., if we rotate the graph of a function in $\mathbb{R}^2$, the result might not be the graph of a function). To avoid this problem we restrict our attention to the special case of **fiber preserving transformations**

$$s = \sigma(t), \qquad y = \eta(t, x) \tag{1.43}$$

(which map the fibers $t = const$ to the fibers $s = const$). Denoting the inverse transform by

$$t = \tau(s), \qquad x = \xi(s, y), \tag{1.44}$$

a straightforward application of the chain rule shows that $\phi(t)$ satisfies

$$\dot{x} = f(t, x) \tag{1.45}$$

if and only if $\psi(s) = \eta(\tau(s), \phi(\tau(s)))$ satisfies

$$\dot{y} = \dot{\tau} \left(\frac{\partial \eta}{\partial t}(\tau, \xi) + \frac{\partial \eta}{\partial x}(\tau, \xi)\, f(\tau, \xi) \right), \tag{1.46}$$

where $\tau = \tau(s)$ and $\xi = \xi(s, y)$. Similarly, we could work out formulas for higher order equations. However, these formulas are usually of little help for practical computations and it is better to use the simpler (but ambiguous) notation

$$\frac{dy}{ds} = \frac{dy(t(s), x(t(s)))}{ds} = \frac{\partial y}{\partial t}\frac{dt}{ds} + \frac{\partial y}{\partial x}\frac{dx}{dt}\frac{dt}{ds}. \tag{1.47}$$

But now let us see how transformations can be used to solve differential equations.

Homogeneous equation:

A (nonlinear) differential equation is called **homogeneous** if it is of the form

$$\dot{x} = f(\frac{x}{t}). \tag{1.48}$$

This special form suggests the change of variables $y = \frac{x}{t}$ $(t \neq 0)$, which (by (1.47)) transforms our equation into

$$\dot{y} = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial x}\dot{x} = -\frac{x}{t^2} + \frac{1}{t}\dot{x} = \frac{f(y) - y}{t}. \tag{1.49}$$

This equation is separable.

More generally, consider the differential equation

$$\dot{x} = f(\frac{ax + bt + c}{\alpha x + \beta t + \gamma}). \tag{1.50}$$

Two cases can occur. If $a\beta - \alpha b = 0$, our differential equation is of the form

$$\dot{x} = \tilde{f}(ax + bt), \tag{1.51}$$

which transforms into

$$\dot{y} = a\tilde{f}(y) + b \tag{1.52}$$

if we set $y = ax + bt$. If $a\beta - \alpha b \neq 0$, we can use $y = x - x_0$ and $s = t - t_0$ which transforms (1.50) to the homogeneous equation

$$\dot{y} = \hat{f}(\frac{ay + bs}{\alpha y + \beta s}) \tag{1.53}$$

if (x_0, t_0) is the unique solution of the linear system $ax + bt + c = 0$, $\alpha x + \beta t + \gamma = 0$.

Bernoulli equation:

A differential equation is of **Bernoulli** type if it is of the form

$$\dot{x} = f(t)x + g(t)x^n, \qquad n \neq 0, 1. \tag{1.54}$$

The transformation

$$y = x^{1-n} \tag{1.55}$$

gives the linear equation

$$\dot{y} = (1 - n)f(t)y + (1 - n)g(t). \tag{1.56}$$

(Note: If $n = 0$ or $n = 1$ the equation is already linear and there is nothing to do.)

Riccati equation:

A differential equation is of **Riccati** type if it is of the form

$$\dot{x} = f(t)x + g(t)x^2 + h(t). \tag{1.57}$$

Solving this equation is possible only if a particular solution $x_p(t)$ is known. Then the transformation

$$y = \frac{1}{x - x_p(t)} \tag{1.58}$$

yields the linear equation

$$\dot{y} = -(f(t) + 2x_p(t)g(t))y - g(t). \tag{1.59}$$

These are only a few of the most important equations which can be explicitly solved using some clever transformation. In fact, there are reference books such as the ones by Kamke [**24**] or Zwillinger [**50**], where you can look up a given equation and find out if it is known to be solvable explicitly. As a rule of thumb, for a first-order equation there is a realistic chance that it is explicitly solvable. But already for second-order equations, explicitly solvable ones are rare.

Alternatively, we can also ask a symbolic computer program like *Mathematica* to solve differential equations for us. For example, to solve

$$\dot{x} = \sin(t)x \tag{1.60}$$

you would use the command

```
In[1]:= DSolve[x'[t] == x[t]Sin[t], x[t], t]
Out[1]= {{x[t] → e^(-Cos[t]) C[1]}}
```

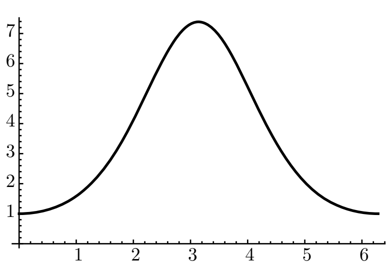
Here the constant C[1] introduced by *Mathematica* can be chosen arbitrarily (e.g. to satisfy an initial condition). We can also solve the corresponding initial value problem using

```
In[2]:= DSolve[{x'[t] == Sin[t]x[t], x[0] == 1}, x[t], t]
Out[2]= {{x[t] → e^(1-Cos[t])}}
```

and plot it using

```
In[3]:= Plot[x[t] /. %, {t, 0, 2π}]
```

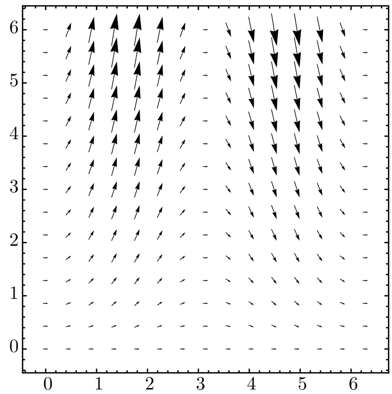
Out[3]=



In some situations it is also useful to visualize the corresponding **directional field**. That is, to every point (t, x) we attach the vector $(1, f(t, x))$. Then the solution curves will be tangent to this vector field in every point:

```
In[4]:= VectorPlot[{1, Sin[t] x}, {t, 0, 2π}, {x, 0, 6}]
```

Out[4]=



So it almost looks as if *Mathematica* can do everything for us and all we have to do is type in the equation, press enter, and wait for the solution. However, as always, life is not that easy. Since, as mentioned earlier, only very few differential equations can be solved explicitly, the `DSolve` command can help us only in very few cases. The other cases, that is, those which cannot be explicitly solved, will be the subject of the remainder of this book!

Let me close this section with a warning. Solving one of our previous examples using *Mathematica* produces

```
In[5]:= DSolve[{x'[t] == √x[t], x[0] == 0}, x[t], t]
```

Out[5]= $\{\{\mathtt{x[t]} \to \frac{\mathtt{t}^2}{4}\}\}$

However, our investigations of the previous section show that this is not the only solution to the posed problem! *Mathematica* expects you to know that there are other solutions and how to get them.

Moreover, if you try to solve the general initial value problem it gets even worse:

```
In[6]:= DSolve[{x'[t] == √x[t], x[0] == x0}, x[t], t] // Simplify
```

Out[6]= $\{\{\mathtt{x[t]} \to \frac{1}{4}(\mathtt{t} - 2\sqrt{\mathtt{x}_0})^2\}, \{\mathtt{x[t]} \to \frac{1}{4}(\mathtt{t} + 2\sqrt{\mathtt{x}_0})^2\}\}$

The first "solution" is no solution of our initial value problem at all! It satisfies $\dot{x} = -\sqrt{x}$.

Problem 1.18. *Try to find solutions of the following differential equations:*

(i) $\dot{x} = \frac{3x-2t}{t}$.

(ii) $\dot{x} = \frac{x-t+2}{2x+t+1} + 5$.

(iii) $y' = y^2 - \frac{y}{x} - \frac{1}{x^2}$.

(iv) $y' = \frac{y}{x} - \tan(\frac{y}{x})$.

Problem 1.19 (Euler equation)**.** *Transform the differential equation*

$$t^2\ddot{x} + 3t\dot{x} + x = \frac{2}{t}$$

to the new coordinates $y = x$, $s = \log(t)$. (Hint: You are not *asked to solve it.)*

Problem 1.20. *Pick some differential equations from the previous problems and solve them using your favorite computer algebra system. Plot the solutions.*

Problem 1.21 (Exact equations)**.** *Consider the equation*

$$F(x, y) = 0,$$

where $F \in C^2(\mathbb{R}^2, \mathbb{R})$. Suppose $y(x)$ solves this equation. Show that $y(x)$ satisfies

$$p(x, y)y' + q(x, y) = 0,$$

where

$$p(x, y) = \frac{\partial F(x, y)}{\partial y} \quad \text{and} \quad q(x, y) = \frac{\partial F(x, y)}{\partial x}.$$

Show that we have

$$\frac{\partial p(x, y)}{\partial x} = \frac{\partial q(x, y)}{\partial y}.$$

Conversely, a first-order differential equation as above (with arbitrary coefficients $p(x, y)$ and $q(x, y)$) satisfying this last condition is called **exact***. Show that if the equation is exact, then there is a corresponding function F as above. Find an explicit formula for F in terms of p and q. Is F uniquely determined by p and q?*

Show that

$$(4bxy + 3x + 5)y' + 3x^2 + 8ax + 2by^2 + 3y = 0$$

is exact. Find F and find the solution.

Problem 1.22 (Integrating factor)**.** *Consider*

$$p(x, y)y' + q(x, y) = 0.$$

A function $\mu(x, y)$ is called an **integrating factor** *if*

$$\mu(x, y)p(x, y)y' + \mu(x, y)q(x, y) = 0$$

is exact.

Finding an integrating factor is in general as hard as solving the original equation. However, in some cases making an ansatz for the form of μ works.

Consider

$$xy' + 3x - 2y = 0$$

and look for an integrating factor $\mu(x)$ depending only on x. Solve the equation.

Problem 1.23. *Show that*

$$\dot{x} = t^{n-1} f\left(\frac{x}{t^n}\right)$$

can be solved using the new variable $y = \frac{x}{t^n}$.

Problem 1.24 (Focusing of waves)**.** *Suppose you have an incoming electromagnetic wave along the y-axis which should be focused on a receiver sitting at the origin $(0,0)$. What is the optimal shape for the mirror?*

(Hint: An incoming ray, hitting the mirror at (x, y), is given by

$$R_{\text{in}}(t) = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} t, \quad t \in (-\infty, 0].$$

At (x, y) it is reflected and moves along

$$R_{\text{rfl}}(t) = \begin{pmatrix} x \\ y \end{pmatrix} (1 - t), \quad t \in [0, 1].$$

The laws of physics require that the angle between the normal vector of the mirror and the incoming respectively reflected ray must be equal. Considering the scalar products of the vectors with the normal vector this yields

$$\frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} -x \\ -y \end{pmatrix} \begin{pmatrix} -y' \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} -y' \\ 1 \end{pmatrix},$$

which is the differential equation for $y = y(x)$ you have to solve. I recommend the substitution $u = \frac{y}{x}$.)

Problem 1.25 (Catenary)**.** *Solve the differential equation describing the shape $y(x)$ of a hanging chain suspended at two points:*

$$y'' = a\sqrt{1 + (y')^2}, \qquad a > 0.$$

Problem 1.26 (Nonlinear boundary value problem)**.** *Show that the nonlinear boundary value problem*

$$y''(x) + y(x)^2 = 0, \qquad y(0) = y(1) = 0$$

has a unique nontrivial solution. Assume that the initial value problem $y(x_0) = y_0$, $y'(x_0) = y_1$ has a unique solution.

- *Show that a nontrivial solution of the boundary value problem must satisfy $y'(0) = p_0 > 0$.*
- *If a solution satisfies $y'(x_0) = 0$, then the solution is symmetric with respect to this point: $y(x) = y(x_0 - x)$. (Hint: Uniqueness.)*

- *Solve the initial value problem $y(0) = 0$, $y'(0) = p_0 > 0$ as follows: Set $y' = p(y)$ and derive a first-order equation for $p(y)$. Solve this equation for $p(y)$ and then solve the equation $y' = p(y)$. (Note that this works for any equation of the type $y'' = f(y)$.)*
- *Does the solution found in the previous item attain $y'(x_0) = 0$ at some x_0? What value should x_0 have for $y(x)$ to solve our boundary value problem?*
- *Can you find a value for p_0 in terms of special functions?*

1.5. Qualitative analysis of first-order equations

As already noted in the previous section, only very few ordinary differential equations are explicitly solvable. Fortunately, in many situations a solution is not needed and only some qualitative aspects of the solutions are of interest, e.g., does it stay within a certain region, what does it look like for large t, etc.

Moreover, even in situations where an exact solution can be obtained, a qualitative analysis can give a better overview of the behavior than the formula for the solution. To get more specific, let us look at the first-order autonomous initial value problem

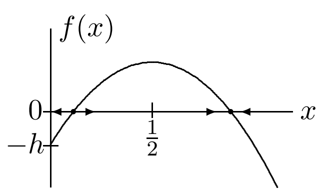
$$\dot{x} = f(x), \qquad x(0) = x_0, \tag{1.61}$$

where $f \in C(\mathbb{R})$ is such that solutions are unique (e.g. $f \in C^1(\mathbb{R})$). We already saw how to solve this equation in Section 1.3. However, for a given f we might well shipwreck when computing the integral $F(x) = \int_{x_0}^{x} \frac{dy}{f(y)}$ or when trying to solve $F(x(t)) = t$ for $x(t)$. On the other hand, to get a qualitative understanding of the solution an explicit formula turns out to be unessential.

Example. Consider the logistic growth model (Problem 1.16)

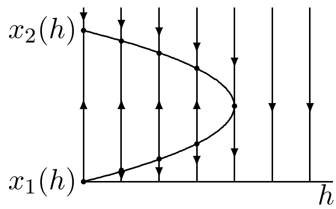
$$\dot{x}(t) = (1 - x(t))x(t) - h, \tag{1.62}$$

which can be solved by separation of variables. To get an overview we plot the corresponding right-hand side $f(x) = (1 - x)x - h$:



Since the sign of $f(x)$ tells us in what direction the solution will move, all we have to do is to discuss the sign of $f(x)$!

For $0 < h < \frac{1}{4}$ there are two zeros $x_{1,2} = \frac{1}{2}(1 \pm \sqrt{1-4h})$. If we start at one of these zeros, the solution will stay there for all t. If we start below x_1 the solution will decrease and converge to $-\infty$. If we start above x_1 the solution will increase and converge to x_2. If we start above x_2 the solution will decrease and again converge to x_2.



At $h = \frac{1}{4}$ a bifurcation occurs: The two zeros coincide $x_1 = x_2$ but otherwise the analysis from above still applies. For $h > \frac{1}{4}$ there are no zeros and all solutions decrease and converge to $-\infty$. ⋄

So we get a complete picture just by discussing the sign of $f(x)$! More generally, we have the following result (Problem 1.28).

Lemma 1.1. *Consider the first-order autonomous initial value problem* (1.61)*, where $f \in C(\mathbb{R})$ is such that solutions are unique.*

(i) *If $f(x_0) = 0$, then $x(t) = x_0$ for all t.*

(ii) *If $f(x_0) \neq 0$, then $x(t)$ converges to the first zero left ($f(x_0) < 0$) respectively right ($f(x_0) > 0$) of x_0. If there is no such zero the solution converges to $-\infty$, respectively ∞.*

If our differential equation is not autonomous, the situation becomes a bit more involved. As a prototypical example let us investigate the differential equation

$$\dot{x} = x^2 - t^2. \tag{1.63}$$

It is of Riccati type and according to the previous section, it cannot be solved unless a particular solution can be found. But there does not seem to be a solution which can be easily guessed. (We will show later, in Problem 4.13, that it is explicitly solvable in terms of special functions.)

So let us try to analyze this equation without knowing the solution. First we should make sure that solutions exist at all! Since we will attack this in full generality in the next chapter, let me just state that if $f(t,x) \in C^1(\mathbb{R}^2, \mathbb{R})$, then for every $(t_0, x_0) \in \mathbb{R}^2$ there exists a unique solution of the initial value problem

$$\dot{x} = f(t,x), \qquad x(t_0) = x_0 \tag{1.64}$$

defined in a neighborhood of t_0 (Theorem 2.2). As we already know from Section 1.3, solutions might not exist for all t even though the differential equation is defined for all $(t, x) \in \mathbb{R}^2$. However, we will show that a solution must converge to $\pm\infty$ if it does not exist for all t (Corollary 2.16).

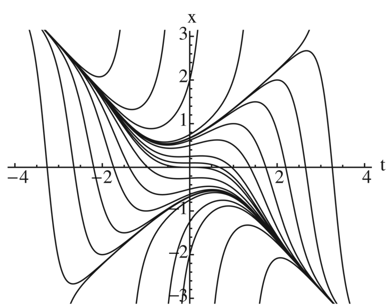
To get some feeling of what we should expect, a good starting point is a numerical investigation. Using the command

```
In[7]:= NDSolve[{x'[t] == x[t]^2 - t^2, x[0] == 1}, x[t], {t, -2, 2}]

    NDSolve::ndsz: At t == 1.0374678967709798`, step size is
      effectively zero; singularity suspected.

Out[7]= {{x[t] → InterpolatingFunction[{{-2., 1.03747}}, <>][t]}}
```

we can compute a numerical solution on the interval $(-2, 2)$. Numerically solving an ordinary differential equation means computing a sequence of points (t_j, x_j) which are, we hope, close to the graph of the real solution. (We will briefly discuss numerical methods in Section 2.7.) Instead of this list of points, *Mathematica* returns an interpolation function which — as you might have already guessed from the name — interpolates between these points and hence can be used as any other function.

Note that, in our particular example, *Mathematica* complained about the step size (i.e., the difference $t_j - t_{j-1}$) getting too small and stopped at $t = 1.037\ldots$. Hence the result is only defined on the interval $(-2, 1.03747)$, even though we have requested the solution on $(-2, 2)$. This indicates that the solution exists only for finite time.

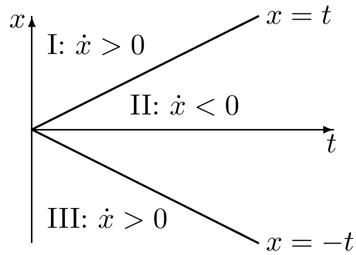
Combining the solutions for different initial conditions into one plot we get the following picture:



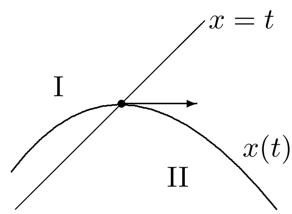
First we note the symmetry with respect to the transformation $(t, x) \to (-t, -x)$. Hence it suffices to consider $t \geq 0$. Moreover, observe that different solutions never cross, which is a consequence of uniqueness.

According to our picture, there seem to be two cases. Either the solution escapes to $+\infty$ in finite time or it converges to the line $x = -t$. But is this really the correct behavior? There could be some numerical errors accumulating. Maybe there are also solutions which converge to the line $x = t$ (we could have missed the corresponding initial conditions in our picture)? Moreover, we could have missed some important things by restricting ourselves to the interval $t \in (-2, 2)$! So let us try to prove that our picture is indeed correct and that we have not missed anything.

We begin by splitting the plane into regions according to the sign of $f(t, x) = x^2 - t^2$. Since it suffices to consider $t \geq 0$ there are only three regions: I: $x > t$, II: $-t < x < t$, and III: $x < -t$. In regions I and III the solution is increasing; in region II it is decreasing.



Furthermore, on the line $x = t$ each solution has a horizontal tangent and hence solutions can only get from region I to II but not the other way around.



Similarly, solutions can only get from III to II but not from II to III.

This already has important consequences for the solutions:

- For solutions starting in region I there are two cases; either the solution stays in I for all time and hence must converge to $+\infty$ (maybe in finite time) or it enters region II.
- A solution starting in region II (or entering region II) will stay there for all time and hence must converge to $-\infty$. (Why can't it remain bounded?) Since it must stay above $x = -t$ this cannot happen in finite time.
- A solution starting in III will eventually hit $x = -t$ and enter region II.

Hence there are two remaining questions: Do the solutions in region I which converge to $+\infty$ reach $+\infty$ in finite time, or are there also solutions which converge to $+\infty$, e.g., along the line $x = t$? Do the other solutions all converge to the line $x = -t$ as our numerical solutions indicate?

To answer these questions we need to generalize the idea from above that a solution can cross the line $x = t$ only from above and the line $x = -t$ only from below.

A differentiable function $x_+(t)$ satisfying

$$\dot{x}_+(t) > f(t, x_+(t)), \qquad t \in [t_0, T), \tag{1.65}$$

is called a **super solution** (or **upper solution**) of our equation. Similarly, a differentiable function $x_-(t)$ satisfying

$$\dot{x}_-(t) < f(t, x_-(t)), \qquad t \in [t_0, T), \tag{1.66}$$

is called a **sub solution** (or **lower solution**).

Example. The function $x_+(t) = t$ is a super solution and $x_-(t) = -t$ is a sub solution of our equation for $t \geq 0$. ◇

Lemma 1.2. *Let $x_+(t)$, $x_-(t)$ be super, sub solutions of the differential equation $\dot{x} = f(t, x)$ on $[t_0, T)$, respectively. Then for every solution $x(t)$ on $[t_0, T)$ we have*

$$x(t) < x_+(t), \quad t \in (t_0, T), \quad \textit{whenever} \quad x(t_0) \leq x_+(t_0), \tag{1.67}$$

respectively

$$x_-(t) < x(t), \quad t \in (t_0, T), \quad \textit{whenever} \quad x(t_0) \geq x_-(t_0). \tag{1.68}$$

Proof. In fact, consider $\Delta(t) = x_+(t) - x(t)$. Then we have $\Delta(t_0) \geq 0$ and $\dot{\Delta}(t) > 0$ whenever $\Delta(t) = 0$. Hence $\Delta(t)$ can cross 0 only from below. Since we start with $\Delta(t_0) \geq 0$, we have $\Delta(t) > 0$ for $t > t_0$ sufficiently close to t_0. In fact, if $\Delta(t_0) > 0$ this follows from continuity and otherwise, if $\Delta(t_0) = 0$, this follows from $\dot{\Delta}(t_0) > 0$. Now let $t_1 > t_0$ be the first value with $\Delta(t_1) = 0$. Then $\Delta(t) > 0$ for $t \in (t_0, t_1)$, which contradicts $\dot{\Delta}(t_1) > 0$. □

Similar results hold for $t < t_0$. The details are left to the reader (Problem 1.29).

Now we are able to answer our remaining questions. Since we were already successful by considering the curves given by $f(t, x) = 0$, let us look at the **isoclines** $f(t, x) = const.$

Considering $x^2 - t^2 = -2$ the corresponding curve is

$$y_+(t) = -\sqrt{t^2 - 2}, \qquad t > \sqrt{2},$$

which is easily seen to be a super solution

$$\dot{y}_+(t) = -\frac{t}{\sqrt{t^2 - 2}} > -2 = f(t, y_+(t))$$

for $t > 2\sqrt{2/3}$. Thus, as soon as a solution $x(t)$ enters the region between $y_+(t)$ and $x_-(t)$ it must stay there and hence converge to the line $x = -t$ since $y_+(t)$ does.

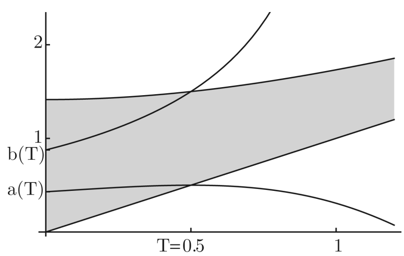
But will every solution in region II eventually end up between $y_+(t)$ and $x_-(t)$? The answer is yes: Since $x(t)$ is decreasing in region II, every solution will eventually be below $-y_+(t)$. Furthermore, every solution $x(t)$ starting at a point (t_0, x_0) below $-y_+(t)$ and above $y_+(t)$ satisfies $\dot{x}(t) < -2$ as long as it remains between $-y_+(t)$ and $y_+(t)$. Hence, by integrating this inequality, $x(t) - x_0 < -2(t - t_0)$, we see that $x(t)$ stays below the line $x_0 - 2(t - t_0)$ as long as it remains between $-y_+(t)$ and $y_+(t)$. Hence every solution which is in region II at some time will converge to the line $x = -t$.

Finally, note that there is nothing special about -2; any value smaller than -1 would have worked as well.

Now let us turn to the other question. This time we take an isocline $x^2 - t^2 = 2$ to obtain a corresponding sub solution

$$y_-(t) = \sqrt{2 + t^2}, \qquad t > 0.$$

At first this does not seem to help much because the sub solution $y_-(t)$ lies *above* the super solution $x_+(t)$. Hence solutions are able to leave the region between $y_-(t)$ and $x_+(t)$ but cannot come back. However, let us look at the solutions which stay inside at least for some finite time $t \in [0, T]$. By following the solutions with initial conditions $(T, x_+(T))$ and $(T, y_-(T))$ we see that they hit the line $t = 0$ at some points $a(T)$ and $b(T)$, respectively. See the picture below which shows two solutions entering the shaded region between $x_+(t)$ and $y_-(t)$ at $T = 0.5$:



Since different solutions can never cross, the solutions which stay inside for (at least) $t \in [0, T]$ are precisely those starting at $t = 0$ in the interval $[a(T), b(T)]$! Moreover, this also implies that $a(T)$ is strictly increasing and $b(T)$ is strictly decreasing. Taking $T \to \infty$ we see that all solutions starting in the interval $[a(\infty), b(\infty)]$ (which might be just one point) at $t = 0$, stay inside for all $t > 0$. Furthermore, since $x \mapsto f(t, x) = x^2 - t^2$ is increasing in region I, we see that the distance between two solutions

$$x_1(t) - x_0(t) = x_1(t_0) - x_0(t_0) + \int_{t_0}^{t} \Big(f(s, x_1(s)) - f(s, x_0(s)) \Big) ds$$

must increase as well. If there were two such solutions, their distance would consequently increase. But this is impossible, since the distance of $x_+(t)$

and $y_-(t)$ tends to zero. Thus there can be at most one solution $x_0(t)$ which stays between $x_+(t)$ and $y_-(t)$ for all $t > 0$ (i.e., $a(\infty) = b(\infty)$). All solutions below $x_0(t)$ will eventually enter region II and converge to $-\infty$ along $x = -t$. All solutions above $x_0(t)$ will eventually be above $y_-(t)$ and converge to $+\infty$. It remains to show that this happens in finite time.

This is not surprising, since the $x(t)^2$ term should dominate over the $-t^2$ term and we already know that the solutions of $\dot{x} = x^2$ diverge. So let us try to make this precise: First of all

$$\dot{x}(t) = x(t)^2 - t^2 > 2$$

for every solution above $y_-(t)$ implies $x(t) > x_0 + 2(t - t_0)$. Thus there is an $\varepsilon > 0$ such that

$$x(t) > \frac{t}{\sqrt{1-\varepsilon}}.$$

This implies

$$\dot{x}(t) = x(t)^2 - t^2 > x(t)^2 - (1-\varepsilon)x(t)^2 = \varepsilon x(t)^2$$

and every solution $x(t)$ is a super solution to a corresponding solution of

$$\dot{x}(t) = \varepsilon x(t)^2.$$

But we already know that the solutions of the last equation escape to $+\infty$ in finite time and so the same must be true for our equation.

In summary, we have shown the following

- There is a unique solution $x_0(t)$ which converges to the line $x = t$.
- All solutions above $x_0(t)$ will eventually converge to $+\infty$ in finite time.
- All solutions below $x_0(t)$ converge to the line $x = -t$.

It is clear that similar considerations can be applied to any first-order equation $\dot{x} = f(t, x)$ and one can usually obtain a quite complete picture of the solutions. However, it is important to point out that the reason for our success was the fact that our equation lives in two dimensions $(t, x) \in \mathbb{R}^2$. If we consider higher order equations or systems of equations, we need more dimensions. At first this seems to imply only that we can no longer plot everything, but there is another more severe difference: In $\mathbb{R}^2$ a curve splits our space into two regions: one above and one below the curve. The only way to get from one region to the other is by crossing the curve. In more than two dimensions this is no longer true and this allows for much more complicated behavior of solutions. In fact, equations in three (or more) dimensions will often exhibit *chaotic* behavior which makes a simple description of solutions impossible!

We end this section with a generalization of Lemma 1.2 which is often useful. Indeed, you might wonder what happens if we allow equality in the definition of a super solution (1.65). At first you might expect that this should not do much harm and the conclusion of Lemma 1.2 should still hold if we allow for equality there as well. However, if you apply this conclusion to two solutions of the same equation it will automatically give you uniqueness of solutions. Hence this generalization cannot be true without further assumptions on f. One assumption which will do the trick (and which will hence also guarantee uniqueness of solutions) is the following condition: We will say that f is locally **Lipschitz continuous** in the second argument, uniformly with respect to the first argument, if

$$L = \sup_{(t,x)\neq(t,y)\in V} \frac{|f(t,x) - f(t,y)|}{|x-y|} \tag{1.69}$$

is finite for every compact set V contained in the domain of f. We will meet this condition again in Section 2.2 where we will also further discuss it. For now notice that it will hold if f has a continuous partial derivative with respect to x by the mean value theorem.

Theorem 1.3. *Suppose f is locally Lipschitz continuous with respect to x uniformly in t. Let $x(t)$ and $y(t)$ be two differentiable functions such that*

$$x(t_0) \le y(t_0), \qquad \dot{x}(t) - f(t,x(t)) \le \dot{y}(t) - f(t,y(t)),\ t\in[t_0,T). \tag{1.70}$$

Then we have $x(t) \le y(t)$ for every $t \in [t_0, T)$. Moreover, if $x(t) < y(t)$ for some t this remains true for all later times.

Proof. We argue by contradiction. Suppose the first claim were not true. Then we could find some time t_1 such that $x(t_1) = y(t_1)$ and $x(t) > y(t)$ for $t \in (t_1, t_1 + \varepsilon)$. Introduce $\Delta(t) = x(t) - y(t)$ and observe

$$\dot{\Delta}(t) = \dot{x}(t) - \dot{y}(t) \le f(t,x(t)) - f(t,y(t)) \le L\Delta(t), \qquad t \in [t_1, t_1+\varepsilon),$$

where the first inequality follows from assumption and the second from (1.69). But this implies that the function $\tilde{\Delta}(t) = \Delta(t)\mathrm{e}^{-Lt}$ satisfies $\dot{\tilde{\Delta}}(t) \le 0$ and thus $\tilde{\Delta}(t) \le \tilde{\Delta}(t_1) = 0$, that is, $x(t) \le y(t)$ for $t \in [t_0, T)$, contradicting our assumption.

So the first part is true. To show the second part set $\Delta(t) = y(t) - x(t)$, which is now nonnegative by the first part. Then, as in the previous case one shows $\dot{\tilde{\Delta}}(t) \ge 0$ where $\tilde{\Delta}(t) = \Delta(t)\mathrm{e}^{Lt}$ and the claim follows. □

A few consequences are worthwhile noting:

First of all, if $x(t)$ and $y(t)$ are two solutions with $x(t_0) \le y(t_0)$, then $x(t) \le y(t)$ for all $t \ge t_0$ (for which both solutions are defined). In particular, in the case $x(t_0) = y(t_0)$ this shows uniqueness of solutions: $x(t) = y(t)$.

Second, we can extend the notion of a super solution by requiring only $x_+(t) \geq f(t, x_+(t))$. Then $x_+(t_0) \geq x(t_0)$ implies $x_+(t) \geq x(t)$ for all $t \geq t_0$ and if strict inequality becomes true at some time it remains true for all later times.

Problem 1.27. *Let x be a solution of (1.61) which satisfies $\lim_{t\to\infty} x(t) = x_1$. Show that $\lim_{t\to\infty} \dot{x}(t) = 0$ and $f(x_1) = 0$. (Hint: If you prove $\lim_{t\to\infty} \dot{x}(t) = 0$ without using (1.61), your proof is wrong! Can you give a counterexample?)*

Problem 1.28. *Prove Lemma 1.1. (Hint: This can be done either by using the analysis from Section 1.3 or by using the previous problem.)*

Problem 1.29. *Generalize the concept of sub and super solutions to the interval (T, t_0), where $T < t_0$.*

Problem 1.30. *Discuss the equation $\dot{x} = x^2 - \frac{t^2}{1+t^2}$.*

- *Make a numerical analysis.*
- *Show that there is a unique solution which asymptotically approaches the line $x = 1$.*
- *Show that all solutions below this solution approach the line $x = -1$.*
- *Show that all solutions above go to ∞ in finite time.*

Problem 1.31. *Discuss the equation $\dot{x} = x^2 - t$.*

Problem 1.32. *Generalize Theorem 1.3 to the interval (T, t_0), where $T < t_0$.*

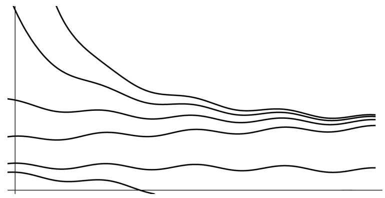
1.6. Qualitative analysis of first-order periodic equations

Some of the most interesting examples are periodic ones, where $f(t+1, x) = f(t, x)$ (without loss we have assumed the period to be one). So let us consider the logistic growth model with a time dependent harvesting term

$$\dot{x}(t) = (1 - x(t))x(t) - h \cdot (1 - \sin(2\pi t)), \tag{1.71}$$

where $h \geq 0$ is some positive constant. In fact, we could replace $1 - \sin(2\pi t)$ by any nonnegative periodic function $g(t)$ and the analysis below will still hold.

The solutions corresponding to some initial conditions for $h = 0.2$ are depicted below.



It looks as if all solutions starting above some value x_1 converge to a periodic solution starting at some other value $x_2 > x_1$, while solutions starting below x_1 diverge to $-\infty$.

The key idea is to look at the fate of an arbitrary initial value x after one period. More precisely, let us denote the solution which starts at the point x at time $t = 0$ by $\phi(t, x)$. Then we can introduce the **Poincaré map** via

$$P(x) = \phi(1, x). \tag{1.72}$$

By construction, an initial condition x_0 will correspond to a periodic solution if and only if x_0 is a fixed point of the Poincaré map, $P(x_0) = x_0$. In fact, this follows from uniqueness of solutions of the initial value problem, since $\phi(t+1, x)$ again satisfies $\dot{x} = f(t, x)$ if $f(t+1, x) = f(t, x)$. So $\phi(t+1, x_0) = \phi(t, x_0)$ if and only if equality holds at the initial time $t = 0$, that is, $\phi(1, x_0) = \phi(0, x_0) = x_0$.

We begin by trying to compute the derivative of $P(x)$ as follows. Set

$$\theta(t, x) = \frac{\partial}{\partial x}\phi(t, x) \tag{1.73}$$

and differentiate

$$\dot{\phi}(t, x) = \big(1 - \phi(t, x)\big)\phi(t, x) - h \cdot \big(1 - \sin(2\pi t)\big), \tag{1.74}$$

with respect to x. (We will justify this step in Theorem 2.10.) Then we obtain

$$\dot{\theta}(t, x) = \big(1 - 2\phi(t, x)\big)\theta(t, x) \tag{1.75}$$

and assuming $\phi(t, x)$ is known we can use (1.38) to write down the solution

$$\theta(t, x) = \exp\left(\int_0^t \big(1 - 2\phi(s, x)\big) ds\right). \tag{1.76}$$

Setting $t = 1$ we obtain

$$P'(x) = \theta(1, x) = \exp\left(1 - 2\int_0^1 \phi(s, x) ds\right). \tag{1.77}$$

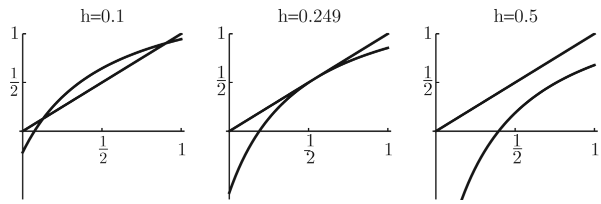
While it might look as if this formula is of little help since we do not know $\phi(t, x)$, it at least tells us that that $P'(x) > 0$, that is, $P(x)$ is strictly increasing. Note that this latter fact also follows since different solutions cannot cross in the (t, x) plane by uniqueness (show this!).

Moreover, differentiating this last expression once more we obtain

$$P''(x) = -2\left(\int_0^1 \theta(s,x)ds\right) P'(x) < 0 \tag{1.78}$$

since $\theta(t,x) > 0$ by (1.76). Thus $P(x)$ is concave and there are at most two intersections with the line x (why?). In other words, there are at most two periodic solutions. Note that so far we did not need any information on the harvesting term.

To see that all cases can occur, we will now consider the dependence with respect to the parameter h. A numerically computed picture of the Poincaré map for different values of h is shown below.



It seems to indicate that $P(x)$ is decreasing as a function of h. To prove this we proceed as before. Set

$$\psi(t,x) = \frac{\partial}{\partial h}\phi(t,x) \tag{1.79}$$

and differentiate the differential equation with respect to h (again this step will be justified by Theorem 2.10) to obtain

$$\dot{\psi}(t,x) = \big(1 - 2\phi(t,x)\big)\psi(t,x) + \big(1 - \sin(2\pi t)\big). \tag{1.80}$$

Hence, since $\psi(0,x) = \frac{\partial}{\partial h}\phi(0,x) = \frac{\partial}{\partial h}x = 0$, equation (1.40) implies

$$\psi(t,x) = -\int_0^t \exp\left(\int_s^t \big(1 - 2\phi(r,x)\big)dr\right)\big(1 - \sin(2\pi s)\big)ds < 0 \tag{1.81}$$

and setting $t = 1$ we infer

$$\frac{\partial}{\partial h}P_h(x) < 0, \tag{1.82}$$

where we have added h as a subscript to emphasize the dependence on the parameter h. Moreover, for $h = 0$ we have

$$P_0(x) = \frac{\mathrm{e}\,x}{1 + (\mathrm{e} - 1)x} \tag{1.83}$$

and there are two fixed points $x_1 = 0$ and $x_2 = 1$. As h increases these points will approach each other and collide at some critical value h_c. Above this value there are no periodic solutions and all orbits converge to $-\infty$ since $P(x) < x$ for all $x \in \mathbb{R}$ (show this).

To complete our analysis suppose $h < h_c$ and denote by $x_1 < x_2$ the two fixed points of $P(x)$. Define the iterates of $P(x)$ by $P^0(x) = x$ and $P^n(x) = P(P^{n-1}(x))$. We claim

$$\lim_{n\to\infty} P^n(x) = \begin{cases} x_2, & x > x_1, \\ x_1, & x = x_1, \\ -\infty, & x < x_1. \end{cases} \tag{1.84}$$

For example, let $x \in (x_1, x_2)$. Then, since $P(x)$ is strictly increasing we have $x_1 = P(x_1) < P(x) < P(x_2) = x_2$. Moreover, since $P(x)$ is concave we have $x < P(x)$, which shows that $P^n(x)$ is a strictly increasing sequence. Let $x_0 \in (x, x_2]$ be its limit. Then $P(x_0) = P(\lim_{n\to\infty} P^n(x)) = \lim_{n\to\infty} P^{n+1}(x) = x_0$ shows that x_0 is a fixed point, that is, $x_0 = x_2$. The other cases can be shown similarly (Problem 1.33).

So for $x < x_1$ the solution diverges to $-\infty$ and for $x > x_1$ we have

$$\lim_{n\to\infty} |\phi(n, x) - x_2| = 0, \tag{1.85}$$

which implies (show this)

$$\lim_{t\to\infty} |\phi(t, x) - \phi(t, x_2)| = 0. \tag{1.86}$$

Similar considerations can be made for the case $h = h_c$ and $h > h_c$.

Problem 1.33. *Suppose $P(x)$ is a continuous, monotone, and concave function with two fixed points $x_1 < x_2$. Show the remaining cases in* (1.84).

Problem 1.34. *Find $\lim_{n\to\infty} P^n(x)$ in the case $h = h_c$ and $h > h_c$.*

Problem 1.35. *Suppose $f \in C^2(\mathbb{R})$ and $g \in C(\mathbb{R})$ is a nonnegative periodic function $g(t+1) = g(t)$. Show that the above discussion still holds for the equation*

$$\dot{x} = f(x) + h \cdot g(t)$$

if $f''(x) < 0$ and $g(t) \geq 0$.

Problem 1.36. *Suppose $a \in \mathbb{R}$ and $g \in C(\mathbb{R})$ is a nonnegative periodic function $g(t+1) = g(t)$. Find conditions on a, g such that the linear inhomogeneous equation*

$$\dot{x} = ax + g(t)$$

has a periodic solution. When is this solution unique? (Hint: (1.40).*)*

Chapter 2

Initial value problems

Our main task in this section will be to prove the basic existence and uniqueness result for ordinary differential equations. The key ingredient will be the contraction principle (Banach fixed point theorem), which we will derive first.

2.1. Fixed point theorems

Let X be a real **vector space**. A **norm** on X is a map $\|.\| : X \to [0, \infty)$ satisfying the following requirements:

(i) $\|0\| = 0$, $\|x\| > 0$ for $x \in X \backslash \{0\}$.

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for $\alpha \in \mathbb{R}$ and $x \in X$.

(iii) $\|x + y\| \le \|x\| + \|y\|$ for $x, y \in X$ (**triangle inequality**).

From the triangle inequality we also get the **inverse triangle inequality** (Problem 2.1)

$$|\|f\| - \|g\|| \le \|f - g\|. \tag{2.1}$$

The pair $(X, \|.\|)$ is called a **normed vector space**. Given a normed vector space X, we say that a sequence of vectors f_n **converges** to a vector f if $\lim_{n\to\infty} \|f_n - f\| = 0$. We will write $f_n \to f$ or $\lim_{n\to\infty} f_n = f$, as usual, in this case. Moreover, a mapping $F : X \to Y$ between two normed spaces is called **continuous** if $f_n \to f$ implies $F(f_n) \to F(f)$. In fact, it is not hard to see that the norm, vector addition, and multiplication by scalars are continuous (Problem 2.2).

In addition to the concept of convergence we also have the concept of a **Cauchy sequence** and hence the concept of completeness: A normed space is called **complete** if every Cauchy sequence has a limit. A complete normed space is called a **Banach space**.

Example. Clearly $\mathbb{R}^n$ (or $\mathbb{C}^n$) is a Banach space with the usual Euclidean norm

$$|x| = \sqrt{\sum_{j=1}^{n} |x_j|^2}. \tag{2.2}$$

◇

We will be mainly interested in the following example: Let I be a compact interval and consider the continuous functions $C(I)$ on this interval. They form a vector space if all operations are defined pointwise. Moreover, $C(I)$ becomes a normed space if we define

$$\|x\| = \sup_{t\in I} |x(t)|. \tag{2.3}$$

I leave it as an exercise to check the three requirements from above. Now what about convergence in this space? A sequence of functions $x_n(t)$ converges to $x(t)$ if and only if

$$\lim_{n\to\infty} \|x_n - x\| = \lim_{n\to\infty} \sup_{t\in I} |x_n(t) - x(t)| = 0. \tag{2.4}$$

That is, in the language of real analysis, x_n converges uniformly to x. Now let us look at the case where x_n is only a Cauchy sequence. Then $x_n(t)$ is clearly a Cauchy sequence of real numbers for any fixed $t \in I$. In particular, by completeness of $\mathbb{R}$, there is a limit $x(t)$ for each t. Thus we get a limiting function $x(t)$. Moreover, letting $m \to \infty$ in

$$|x_n(t) - x_m(t)| \le \varepsilon \qquad \forall n, m > N_\varepsilon,\ t \in I \tag{2.5}$$

we see

$$|x_n(t) - x(t)| \le \varepsilon \qquad \forall n > N_\varepsilon,\ t \in I, \tag{2.6}$$

that is, $x_n(t)$ converges uniformly to $x(t)$. However, up to this point we do not know whether it is in our vector space $C(I)$ or not, that is, whether it is continuous or not. Fortunately, there is a well-known result from real analysis which tells us that the uniform limit of continuous functions is again continuous: Fix $t \in I$ and $\varepsilon > 0$. To show that x is continuous we need to find a δ such that $|t - s| < \delta$ implies $|x(t) - x(s)| < \varepsilon$. Pick n so that $\|x_n - x\| < \varepsilon/3$ and δ so that $|t - s| < \delta$ implies $|x_n(t) - x_n(s)| < \varepsilon/3$. Then $|t - s| < \delta$ implies

$$|x(t)-x(s)| \le |x(t)-x_n(t)|+|x_n(t)-x_n(s)|+|x_n(s)-x(s)| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon$$

as required. Hence $x(t) \in C(I)$ and thus every Cauchy sequence in $C(I)$ converges, or, in other words, $C(I)$ is a Banach space.

You will certainly ask how all these considerations should help us with our investigation of differential equations? Well, you will see in the next section that it will allow us to give an easy and transparent proof of our basic existence and uniqueness theorem based on the following result.

A **fixed point** of a mapping $K : C \subseteq X \to C$ is an element $x \in C$ such that $K(x) = x$. Moreover, K is called a **contraction** if there is a contraction constant $\theta \in [0, 1)$ such that

$$\|K(x) - K(y)\| \le \theta\|x - y\|, \qquad x, y \in C. \tag{2.7}$$

We also recall the notation $K^n(x) = K(K^{n-1}(x))$, $K^0(x) = x$.

Theorem 2.1 (Contraction principle)**.** *Let C be a (nonempty) closed subset of a Banach space X and let $K : C \to C$ be a contraction. Then K has a unique fixed point $\overline{x} \in C$ such that*

$$\|K^n(x) - \overline{x}\| \le \frac{\theta^n}{1-\theta}\|K(x) - x\|, \qquad x \in C. \tag{2.8}$$

Proof. If $\overline{x} = K(\overline{x})$ and $\tilde{x} = K(\tilde{x})$, then $\|\overline{x}-\tilde{x}\| = \|K(\overline{x})-K(\tilde{x})\| \le \theta\|\overline{x}-\tilde{x}\|$ shows that there can be at most one fixed point.

Concerning existence, fix $x_0 \in C$ and consider the sequence $x_n = K^n(x_0)$. We have

$$\|x_{n+1} - x_n\| \le \theta\|x_n - x_{n-1}\| \le \cdots \le \theta^n\|x_1 - x_0\|$$

and hence by the triangle inequality (for $n > m$)

$$\begin{aligned} \|x_n - x_m\| &\le \sum_{j=m+1}^{n} \|x_j - x_{j-1}\| \le \theta^m \sum_{j=0}^{n-m-1} \theta^j\|x_1 - x_0\| \\ &= \theta^m \frac{1-\theta^{n-m}}{1-\theta}\|x_1 - x_0\| \le \frac{\theta^m}{1-\theta}\|x_1 - x_0\|. \end{aligned} \tag{2.9}$$

Thus x_n is Cauchy and tends to a limit $\overline{x}$. Moreover,

$$\|K(\overline{x}) - \overline{x}\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$$

shows that $\overline{x}$ is a fixed point and the estimate (2.8) follows after taking the limit $n \to \infty$ in (2.9). □

Question: Why is closedness of C important?

Problem 2.1. *Show that $|\|f\| - \|g\|| \le \|f - g\|$.*

Problem 2.2. *Let X be a Banach space. Show that the norm, vector addition, and multiplication by scalars are continuous. That is, if $f_n \to f$, $g_n \to g$, and $\alpha_n \to \alpha$, then $\|f_n\| \to \|f\|$, $f_n + g_n \to f + g$, and $\alpha_n f_n \to \alpha f$.*

Problem 2.3. *Show that the space* $C(I,\mathbb{R}^n)$ *together with the sup norm* (2.3) *is a Banach space if* I *is a compact interval. Show that the same is true for* $I=[0,\infty)$ *and* $I=\mathbb{R}$ *if one considers the vector space of bounded continuous functions* $C_b(I,\mathbb{R}^n)$.

Problem 2.4. *Derive Newton's method for finding the zeros of a twice continuously differentiable function* $f(x)$,

$$x_{n+1}=K(x_n), \qquad K(x)=x-\frac{f(x)}{f'(x)},$$

from the contraction principle by showing that if $\overline{x}$ *is a zero with* $f'(\overline{x})\neq 0$, *then there is a corresponding closed interval* C *around* $\overline{x}$ *such that the assumptions of Theorem* 2.1 *are satisfied.*

2.2. The basic existence and uniqueness result

Now we will use the preparations from the previous section to show existence and uniqueness of solutions for the following **initial value problem** (IVP)

$$\dot{x}=f(t,x), \quad x(t_0)=x_0. \tag{2.10}$$

We suppose $f\in C(U,\mathbb{R}^n)$, where U is an open subset of $\mathbb{R}^{n+1}$ and $(t_0,x_0)\in U$.

First note that integrating both sides with respect to t shows that (2.10) is equivalent to the following **integral equation**

$$x(t)=x_0+\int_{t_0}^{t} f(s,x(s))\,ds. \tag{2.11}$$

At first this does not seem to help much. However, note that $x_0(t)=x_0$ is an approximating solution at least for small t. Plugging $x_0(t)$ into our integral equation we get another approximating solution

$$x_1(t)=x_0+\int_{t_0}^{t} f(s,x_0(s))\,ds. \tag{2.12}$$

Iterating this procedure we get a sequence of approximating solutions

$$x_m(t)=K^m(x_0)(t), \qquad K(x)(t)=x_0+\int_{t_0}^{t} f(s,x(s))\,ds. \tag{2.13}$$

Now this observation begs us to apply the contraction principle from the previous section to the fixed point equation $x=K(x)$, which is precisely our integral equation (2.11).

We will set $t_0=0$ for notational simplicity and consider only the case $t\geq 0$ to avoid excessive numbers of absolute values in the following estimates.

First we will need a Banach space. The obvious choice is $X=C([0,T],\mathbb{R}^n)$ for some suitable $T>0$. Furthermore, we need a closed subset $C\subseteq X$ such that $K : C \to C$. We will try a closed ball of radius δ around the constant function x_0.

Since U is open and $(0, x_0) \in U$ we can choose $V = [0, T] \times \overline{B_\delta(x_0)} \subset U$, where $B_\delta(x_0) = \{x \in \mathbb{R}^n | \, |x - x_0| < \delta\}$, and abbreviate

$$M = \max_{(t,x)\in V} |f(t, x)|, \tag{2.14}$$

where the maximum exists by continuity of f and compactness of V. Then

$$|K(x)(t) - x_0| \le \int_0^t |f(s, x(s))|ds \le t\, M \tag{2.15}$$

whenever the graph of $x(t)$ lies within V, that is, $\{(t, x(t))|t \in [0, T]\} \subset V$. Hence for $t \le T_0$, where

$$T_0 = \min\{T, \frac{\delta}{M}\}, \tag{2.16}$$

we have $T_0\, M \le \delta$ and the graph of $K(x)$ restricted to $[0, T_0]$ is again in V. In the special case $M = 0$ one has to understand this as $\frac{\delta}{M} = \infty$ such that $T_0 = \min\{T, \infty\} = T$. Moreover, note that since $[0, T_0] \subseteq [0, T]$ the same constant M will also bound $|f|$ on $V_0 = [0, T_0] \times \overline{B_\delta(x_0)} \subseteq V$.

So if we choose $X = C([0, T_0], \mathbb{R}^n)$ as our Banach space, with norm $\|x\| = \max_{0\le t\le T_0} |x(t)|$, and $C = \{x \in X \,|\, \|x - x_0\| \le \delta\}$ as our closed subset, then $K : C \to C$ and it remains to show that K is a contraction.

To show this, we need to estimate

$$|K(x)(t) - K(y)(t)| \le \int_0^t |f(s, x(s)) - f(s, y(s))|ds. \tag{2.17}$$

Clearly, since f is continuous, we know that $|f(s, x(s)) - f(s, y(s))|$ is small if $|x(s) - y(s)|$ is. However, this is not good enough to estimate the integral above. For this we need the following stronger condition: Suppose f is locally **Lipschitz continuous** in the second argument, uniformly with respect to the first argument, that is, for every compact set $V_0 \subset U$ the following number

$$L = \sup_{(t,x)\ne(t,y)\in V_0} \frac{|f(t, x) - f(t, y)|}{|x - y|} \tag{2.18}$$

(which depends on V_0) is finite. Then,

$$\begin{aligned} \int_0^t |f(s, x(s)) - f(s, y(s))|ds &\le L \int_0^t |x(s) - y(s)|ds \\ &\le L\, t \sup_{0\le s\le t} |x(s) - y(s)| \end{aligned} \tag{2.19}$$

provided the graphs of both $x(t)$ and $y(t)$ lie in V_0. In other words,

$$\|K(x) - K(y)\| \le L\, T_0 \|x - y\|, \quad x, y \in C. \tag{2.20}$$

Moreover, choosing $T_0 < L^{-1}$ we see that K is a contraction and existence of a unique solution follows from the contraction principle:

Theorem 2.2 (Picard–Lindelöf). *Suppose $f \in C(U, \mathbb{R}^n)$, where U is an open subset of $\mathbb{R}^{n+1}$, and $(t_0, x_0) \in U$. If f is locally Lipschitz continuous in the second argument, uniformly with respect to the first, then there exists a unique local solution $\overline{x}(t) \in C^1(I)$ of the IVP (2.10), where I is some interval around t_0.*

More specifically, if $V = [t_0, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$ and M denotes the maximum of $|f|$ on V. Then the solution exists at least for $t \in [t_0, t_0 + T_0]$ and remains in $\overline{B_\delta(x_0)}$, where $T_0 = \min\{T, \frac{\delta}{M}\}$. The analogous result holds for the interval $[t_0 - T, t_0]$.

Proof. We have already shown everything except for the fact that our proof requires $T_0 < L^{-1}$, in addition to $T_0 \le T$ and $T_0 \le \frac{\delta}{M}$. That this condition is indeed superfluous will be shown in the next section. □

The procedure to find the solution is called the **Picard iteration**. Unfortunately, it is not suitable for actually finding the solution since computing the integrals in each iteration step will not be possible in general. Even for numerical computations evaluating the integrals is often too time consuming. However, if $f(t,x)$ is analytic, the m'th Picard iterate $x_m(t)$ matches the Taylor expansion of the solution $\overline{x}(t)$ around t_0 up to order m and this can be used for numerical computations (cf. Problem 4.4). In any event, the important fact for us is that there exists a unique solution to the initial value problem.

In many cases, f will be even differentiable. In particular, recall that $f \in C^1(U, \mathbb{R}^n)$ implies that f is locally Lipschitz continuous in the second argument, uniformly with respect to the first, as required in Theorem 2.2 (see Problem 2.5 below).

Lemma 2.3. *Suppose $f \in C^k(U, \mathbb{R}^n)$, $k \ge 1$, where U is an open subset of $\mathbb{R}^{n+1}$, and $(t_0, x_0) \in U$. Then the local solution $\overline{x}$ of the IVP (2.10) is $C^{k+1}(I)$.*

Proof. Let $k = 1$. Then $\overline{x}(t) \in C^1$ by the above theorem. Moreover, using $\dot{\overline{x}}(t) = f(t, \overline{x}(t)) \in C^1$ we infer $\overline{x}(t) \in C^2$. The rest follows from induction. □

Problem 2.5. *Show that $f \in C^1(\mathbb{R}^m, \mathbb{R}^n)$ is locally Lipschitz continuous. In fact, show that*

$$|f(y) - f(x)| \le \sup_{\varepsilon \in [0,1]} \left\| \frac{\partial f(x + \varepsilon(y-x))}{\partial x} \right\| |x - y|,$$

where $\frac{\partial f(x_0)}{\partial x}$ *denotes the Jacobian matrix at* x_0 *and* $\|.\|$ *denotes the matrix norm (cf.* (3.8)*). Conclude that a function* $f \in C^1(U, \mathbb{R}^n)$*,* $U \subseteq \mathbb{R}^{n+1}$*, is locally Lipschitz continuous in the second argument, uniformly with respect to the first, and thus satisfies the hypothesis of Theorem* 2.2*. (Hint: Start with the case* $m = n = 1$*.)*

Problem 2.6. *Are the following functions Lipschitz continuous near* 0*? If yes, find a Lipschitz constant for some interval containing* 0*.*

(i) $f(x) = \frac{1}{1-x^2}$.

(ii) $f(x) = |x|^{1/2}$.

(iii) $f(x) = x^2 \sin(\frac{1}{x})$.

Problem 2.7. *Apply the Picard iteration to the first-order linear equation*

$$\dot{x} = x, \qquad x(0) = 1.$$

Problem 2.8. *Apply the Picard iteration to the first-order equation*

$$\dot{x} = 2t - 2\sqrt{\max(0, x)}, \qquad x(0) = 0.$$

Does it converge?

2.3. Some extensions

In this section we derive some further extensions of the Picard–Lindelöf theorem. They are of a more technical nature and can be skipped on first reading.

As a preparation we need a slight generalization of the contraction principle. In fact, looking at its proof, observe that we can replace θ^n by any other summable sequence θ_n (Problem 2.10).

Theorem 2.4 (Weissinger). *Let* C *be a (nonempty) closed subset of a Banach space* X*. Suppose* $K : C \to C$ *satisfies*

$$\|K^n(x) - K^n(y)\| \le \theta_n \|x - y\|, \qquad x, y \in C, \tag{2.21}$$

with $\sum_{n=1}^{\infty} \theta_n < \infty$*. Then* K *has a unique fixed point* $\overline{x}$ *such that*

$$\|K^n(x) - \overline{x}\| \le \left(\sum_{j=n}^{\infty} \theta_j \right) \|K(x) - x\|, \qquad x \in C. \tag{2.22}$$

Our first objective is to give some concrete values for the existence time T_0. Using Weissinger's theorem instead of the contraction principle, we can avoid the restriction $T_0 < L^{-1}$:

Theorem 2.5 (improved Picard–Lindelöf). *Suppose $f \in C(U, \mathbb{R}^n)$, where U is an open subset of $\mathbb{R}^{n+1}$, and f is locally Lipschitz continuous in the second argument. Choose $(t_0, x_0) \in U$ and $\delta, T > 0$ such that $[t_0, t_0+T] \times \overline{B_\delta(x_0)} \subset U$. Set*

$$M(t) = \int_{t_0}^{t} \sup_{x \in B_\delta(x_0)} |f(s,x)| ds, \tag{2.23}$$

$$L(t) = \sup_{x \neq y \in B_\delta(x_0)} \frac{|f(t,x) - f(t,y)|}{|x-y|}. \tag{2.24}$$

Note that $M(t)$ is nondecreasing and define T_0 via

$$T_0 = \sup\{0 < t \le T |\, M(t_0+t) \le \delta\}. \tag{2.25}$$

Suppose

$$L_1(T_0) = \int_{t_0}^{t_0+T_0} L(t) dt < \infty. \tag{2.26}$$

Then the unique local solution $\overline{x}(t)$ of the IVP (2.10) *is given by*

$$\overline{x} = \lim_{m \to \infty} K^m(x_0) \in C^1([t_0, t_0+T_0], \overline{B_\delta(x_0)}), \tag{2.27}$$

where $K^m(x_0)$ is defined in (2.13)*, and satisfies the estimate*

$$\sup_{t_0 \le t \le T_0} |\overline{x}(t) - K^m(x_0)(t)| \le \frac{L_1(T_0)^m}{m!} \mathrm{e}^{L_1(T_0)} \int_{t_0}^{t_0+T_0} |f(s,x_0)| ds. \tag{2.28}$$

An analogous result holds for $t < t_0$.

Proof. Again we choose $t_0 = 0$ for notational simplicity. Our aim is to verify the assumptions of Theorem 2.4, choosing $X = C([0,T_0], \mathbb{R}^n)$ with norm $\|x\| = \max_{0 \le t \le T_0} |x(t)|$ and $C = \{x \in X \,|\, \|x - x_0\| \le \delta\}$.

First, if $x \in C$ we have

$$|K(x)(t) - x_0| \le \int_0^t |f(s, x(s))| ds \le M(t) \le \delta, \qquad t \in [0, T_0],$$

that is, $K(x) \in C$ as well. In particular, this explains our choice for T_0.

Next we claim

$$|K^m(x)(t) - K^m(y)(t)| \le \frac{L_1(t)^m}{m!} \sup_{0 \le s \le t} |x(s) - y(s)|, \tag{2.29}$$

where $L_1(t) = \int_0^t L(s)ds$. This follows by induction:

$$\begin{aligned}
|K^{m+1}(x)(t) - K^{m+1}(y)(t)| &\le \int_0^t |f(s, K^m(x)(s)) - f(s, K^m(y)(s))|ds \\
&\le \int_0^t L(s)|K^m(x)(s) - K^m(y)(s)|ds \\
&\le \int_0^t L(s)\frac{L_1(s)^m}{m!} \sup_{r\le s} |x(r) - y(r)|ds \\
&\le \sup_{r\le t} |x(r) - y(r)| \int_0^t L_1'(s)\frac{L_1(s)^m}{m!} ds \\
&= \frac{L_1(t)^{m+1}}{(m+1)!} \sup_{r\le t} |x(r) - y(r)|.
\end{aligned}$$

Hence K satisfies the assumptions of Theorem 2.4, which finally yields

$$\sup_{0\le t\le T_0} |\overline{x}(t) - K^m(x_0)(t)| \le \sum_{j=m}^{\infty} \left(\frac{L_1(T_0)^j}{j!}\right) \int_0^{T_0} |f(s, x_0)|ds.$$

□

Note that if we set

$$M = \sup_{(t,x)\in[t_0,T]\times B_\delta(x_0)} |f(t, x)| \tag{2.30}$$

then we can chose

$$T_0 = \min(T, \frac{M}{\delta}). \tag{2.31}$$

If $f(t, x)$ is defined for all $x \in \mathbb{R}^n$ and we can find a *global* Lipschitz constant, then we can say more about the interval where the solution exists:

Corollary 2.6. *Suppose* $[t_0, T] \times \mathbb{R}^n \subset U$ *and*

$$\int_{t_0}^{T} L(t)dt < \infty, \qquad L(t) = \sup_{x\ne y\in\mathbb{R}^n} \frac{|f(t, x) - f(t, y)|}{|x - y|}, \tag{2.32}$$

then $\overline{x}$ *is defined for all* $t \in [t_0, T]$.

In particular, if $U = \mathbb{R}^{n+1}$ *and* $\int_{-T}^{T} L(t)dt < \infty$ *for all* $T > 0$, *then* $\overline{x}$ *is defined for all* $t \in \mathbb{R}$.

Proof. In this case we can simply choose our closed set C to be the entire Banach space $X = C([0, T], \mathbb{R}^n)$ (i.e., $\delta = \infty$) and proceed as in the proof of the previous theorem with $T_0 = T$. □

Note that this corollary applies, for example, if the differential equation is linear, that is, $f(t, x) = A(t)x + b(t)$, where $A(t)$ is a matrix and $b(t)$ is a vector which both have continuous entries.

Finally, let me remark that the requirement that f is continuous in Theorem 2.2 is already more than we actually needed in its proof. In fact, all one needs to require is that f is measurable with $M(t)$ finite and $L(t)$ locally integrable (i.e., $\int_I L(t)dt < \infty$ for any compact interval I).

However, then the solution of the integral equation is only absolutely continuous and might fail to be continuously differentiable. In particular, when going back from the integral to the differential equation, the differentiation has to be understood in a generalized sense. I do not want to go into further details here, but rather give you an example. Consider

$$\dot{x} = \mathrm{sgn}(t)x, \qquad x(0) = 1. \tag{2.33}$$

Then $x(t) = \exp(|t|)$ might be considered a solution even though it is not differentiable at $t = 0$. This generalization is known as **differential equations in the sense of Carathéodory**.

Problem 2.9. *Consider the initial value problem $\dot{x} = x^2$, $x(0) = x_0 > 0$. What is the maximal value for T_0 (as a function of x_0) according to Theorem 2.2, respectively Theorem 2.5? What maximal value do you get from the explicit solution? (Hint: Compute T_0 as a function of δ and find the optimal δ.)*

Problem 2.10. *Prove Theorem 2.4. Moreover, suppose $K : C \to C$ and that K^n is a contraction. Show that the fixed point of K^n is also one of K. (Hint: Use uniqueness.) Hence Theorem 2.4 (except for the estimate) can also be considered as a special case of Theorem 2.1 since the assumption implies that K^n is a contraction for n sufficiently large.*

2.4. Dependence on the initial condition

Usually, in applications several data are only known approximately. If the problem is **well-posed**, one expects that small changes in the data will result in small changes of the solution. This will be shown in our next theorem. As a preparation we need **Gronwall's inequality**.

Lemma 2.7 (Generalized Gronwall's inequality). *Suppose $\psi(t)$ satisfies*

$$\psi(t) \le \alpha(t) + \int_0^t \beta(s)\psi(s)ds, \qquad t \in [0, T], \tag{2.34}$$

with $\alpha(t) \in \mathbb{R}$ and $\beta(t) \ge 0$. Then

$$\psi(t) \le \alpha(t) + \int_0^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(r)dr\right) ds, \qquad t \in [0, T]. \tag{2.35}$$

Moreover, if in addition $\alpha(s) \le \alpha(t)$ for $s \le t$, then

$$\psi(t) \le \alpha(t) \exp\left(\int_0^t \beta(s)ds\right), \qquad t \in [0, T]. \tag{2.36}$$

Proof. Abbreviate $\phi(t) = \exp\big(-\int_0^t \beta(s)ds\big)$. Then one computes

$$\frac{d}{dt}\phi(t)\int_0^t \beta(s)\psi(s)ds = \beta(t)\phi(t)\left(\psi(t) - \int_0^t \beta(s)\psi(s)ds\right) \le \alpha(t)\beta(t)\phi(t)$$

by our assumption (2.34). Integrating this inequality with respect to t and dividing the resulting equation by $\phi(t)$ shows

$$\int_0^t \beta(s)\psi(s)ds \le \int_0^t \alpha(s)\beta(s)\frac{\phi(s)}{\phi(t)}ds.$$

Adding $\alpha(t)$ on both sides and using again (2.34) finishes the proof of the first claim. The second claim is left as an exercise (Problem 2.11). □

We will also frequently use the following simple consequence (Problem 2.12). If

$$\psi(t) \le \alpha + \int_0^t (\beta\,\psi(s) + \gamma)ds, \qquad t \in [0,T], \tag{2.37}$$

for given constants $\alpha \in \mathbb{R}$, $\beta \ge 0$, and $\gamma \in \mathbb{R}$, then

$$\psi(t) \le \alpha\,\exp(\beta t) + \frac{\gamma}{\beta}(\exp(\beta t) - 1), \qquad t \in [0,T]. \tag{2.38}$$

In the case $\beta = 0$ the right-hand side has to be replaced by its limit $\psi(t) \le \alpha + \gamma t$. Of course this last inequality does not provide any new insights.

Now we can show that our IVP is well-posed.

Theorem 2.8. *Suppose $f, g \in C(U, \mathbb{R}^n)$ and let f be locally Lipschitz continuous in the second argument, uniformly with respect to the first. If $x(t)$ and $y(t)$ are respective solutions of the IVPs*

$$\begin{array}{c} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{array} \quad \text{and} \quad \begin{array}{c} \dot{y} = g(t,y) \\ y(t_0) = y_0 \end{array}, \tag{2.39}$$

then

$$|x(t) - y(t)| \le |x_0 - y_0|\,\mathrm{e}^{L|t-t_0|} + \frac{M}{L}(\mathrm{e}^{L|t-t_0|} - 1), \tag{2.40}$$

where

$$L = \sup_{(t,x)\ne(t,y)\in V} \frac{|f(t,x) - f(t,y)|}{|x - y|}, \quad M = \sup_{(t,x)\in V} |f(t,x) - g(t,x)|, \tag{2.41}$$

with $V \subset U$ some set containing the graphs of $x(t)$ and $y(t)$.

Proof. Without restriction we set $t_0 = 0$. Then we have

$$|x(t) - y(t)| \le |x_0 - y_0| + \int_0^t |f(s, x(s)) - g(s, y(s))|ds.$$

Estimating the integrand shows

$$\begin{aligned}|f(s, x(s)) - g(s, y(s))| \\ &\leq |f(s, x(s)) - f(s, y(s))| + |f(s, y(s)) - g(s, y(s))| \\ &\leq L|x(s) - y(s)| + M.\end{aligned}$$

Hence the claim follows from (2.38). □

In particular, denote the solution of the IVP (2.10) by

$$\phi(t, t_0, x_0) \tag{2.42}$$

to emphasize the dependence on the initial condition. Then our theorem, in the special case $f = g$,

$$|\phi(t, t_0, x_0) - \phi(t, t_0, y_0)| \leq |x_0 - y_0| \, \mathrm{e}^{L|t-t_0|}, \tag{2.43}$$

shows that ϕ depends continuously on the initial value. Of course this bound blows up exponentially as t increases, but the linear equation $\dot{x} = x$ in one dimension shows that we cannot do better in general.

Moreover, we even have

Theorem 2.9. *Suppose $f \in C(U, \mathbb{R}^n)$ is locally Lipschitz continuous in the second argument, uniformly with respect to the first. Around each point $(t_0, x_0) \in U$ we can find a compact set $I \times B \subset U$ such that $\phi(t, s, x) \in C(I \times I \times B, \mathbb{R}^n)$. Moreover, $\phi(t, t_0, x_0)$ is Lipschitz continuous,*

$$|\phi(t, t_0, x_0) - \phi(s, s_0, y_0)| \leq |x_0 - y_0| \, \mathrm{e}^{L|t-t_0|} + (|t - s| + |t_0 - s_0| \mathrm{e}^{L|t-s_0|})M, \tag{2.44}$$

where

$$L = \sup_{(t,x) \neq (t,y) \in V} \frac{|f(t, x) - f(t, y)|}{|x - y|}, \qquad M = \max_{(t,x) \in V} |f(t, x)|, \tag{2.45}$$

with $V \subset U$ some compact set containing $I \times \phi(I \times I \times B)$.

Proof. Using the same notation as in the proof of Theorem 2.2 we can find a compact set $V = [t_0 - \varepsilon, t_0 + \varepsilon] \times \overline{B_\delta(x_0)}$ such that $\phi(t, t_0, x_0)$ exists for $|t - t_0| \leq \varepsilon$. But then it is straightforward to check that $V_1 = [t_1 - \varepsilon/2, t_1 + \varepsilon/2] \times \overline{B_{\delta/2}(x_1)}$ works to show that $\phi(t, t_1, x_1)$ exists for $|t - t_1| \leq \varepsilon/2$ whenever $|t_1 - t_0| \leq \varepsilon/2$ and $|x_1 - x_0| \leq \delta/2$. Hence we can choose $I = [t_0 - \varepsilon/2, t_0 + \varepsilon/2]$ and $B = \overline{B_{\delta/2}(x_0)}$.

To obtain the estimate observe

$$
\begin{aligned}
|\phi(t,t_0,x_0)-\phi(s,s_0,y_0)| &\le |\phi(t,t_0,x_0)-\phi(t,t_0,y_0)| \\
&\quad + |\phi(t,t_0,y_0)-\phi(t,s_0,y_0)| \\
&\quad + |\phi(t,s_0,y_0)-\phi(s,s_0,y_0)| \\
&\le |x_0-y_0|\,\mathrm{e}^{L|t-t_0|} \\
&\quad + |\int_{t_0}^t f(r,\phi(r,t_0,y_0))dr - \int_{s_0}^t f(r,\phi(r,s_0,y_0))dr| \\
&\quad + |\int_s^t f(r,\phi(r,s_0,y_0))dr|,
\end{aligned}
$$

where we have used (2.43) for the first term. Moreover, the third term can clearly be estimated by $M|t-s|$. To estimate the second term, abbreviate $\Delta(t)=\phi(t,t_0,y_0)-\phi(t,s_0,y_0)$ and use (assume $t_0\le s_0\le t$ without loss of generality)

$$
\begin{aligned}
\Delta(t) &\le \left|\int_{t_0}^{s_0} f(r,\phi(r,t_0,y_0))dr\right| + \int_{s_0}^t |f(r,\phi(r,t_0,y_0))-f(r,\phi(r,s_0,y_0))|dr \\
&\le |t_0-s_0|M + L\int_{s_0}^t \Delta(r)dr.
\end{aligned}
$$

Hence an application of Gronwall's inequality finishes the proof. □

Note that in the case of an autonomous system we have $\phi(t,t_0,x_0)=\phi(t-t_0,0,x_0)$ by Problem 1.8 and it suffices to consider $\phi(t,x_0)=\phi(t,0,x_0)$ in such a situation.

However, in many cases the previous result is not good enough and we need to be able to differentiate with respect to the initial condition. Hence we will assume $f\in C^k(U,\mathbb{R}^n)$ for some $k\ge 1$.

We first suppose that $\phi(t,t_0,x)$ is differentiable with respect to x. Then, by (2.10) combined with the chain rule, the same is true for $\dot\phi(t,t_0,x)$ and differentiating (2.10) yields

$$
\frac{\partial^2\phi}{\partial x\partial t}(t,t_0,x)=\frac{\partial f}{\partial x}(t,\phi(t,t_0,x))\frac{\partial\phi}{\partial x}(t,t_0,x). \tag{2.46}
$$

Hence if we further assume that we can interchange the partial derivatives on the left-hand side,

$$
\frac{\partial^2\phi}{\partial x\partial t}(t,t_0,x)=\frac{\partial^2\phi}{\partial t\partial x}(t,t_0,x), \tag{2.47}
$$

we see that

$$
\frac{\partial\phi}{\partial x}(t,t_0,x) \tag{2.48}
$$

necessarily satisfies the **first variational equation**

$$\dot{y} = A(t,x)y, \qquad A(t,x) = \frac{\partial f}{\partial x}(t, \phi(t, t_0, x)). \tag{2.49}$$

Note that this equation is linear and the corresponding integral equation reads

$$y(t) = \mathbb{I} + \int_{t_0}^{t} A(s,x)y(s)ds, \tag{2.50}$$

where we have used $\phi(t_0, t_0, x) = x$ and hence $\frac{\partial \phi}{\partial x}(t_0, t_0, x) = \mathbb{I}$. Applying similar fixed point techniques as before, one can show that the first variational equation has a solution which is indeed the derivative of $\phi(t, t_0, x)$ with respect to x.

Theorem 2.10. *Suppose $f \in C^k(U, \mathbb{R}^n)$, $k \ge 1$. Around each point $(t_0, x_0) \in U$ we can find an open set $I \times B \subseteq U$ such that $\phi(t,s,x) \in C^k(I \times I \times B, \mathbb{R}^n)$. Moreover, $\frac{\partial}{\partial t}\phi(t,s,x) \in C^k(I \times I \times B, \mathbb{R}^n)$ and if D_k is a partial derivative of order k, then $D_k\phi$ satisfies the higher order variational equation obtained from*

$$\frac{\partial}{\partial t} D_k \phi(t,s,x) = D_k \frac{\partial}{\partial t}\phi(t,s,x) = D_k f(t, \phi(t,s,x)) \tag{2.51}$$

by applying the chain rule. In particular, this equation is linear in $D_k\phi$ and it also follows that the corresponding higher order derivatives commute.

Proof. By adding t to the dependent variables it is no restriction to assume that our equation is autonomous and consider $\phi(t,x) = \phi(t,0,x)$. Existence of a set $I \times B \subseteq U$ such that $\phi(t, x_0)$ is continuous has been established in the previous theorem and it remains to investigate differentiability.

We start by showing the case $k = 1$. We have to prove that $\phi(t,x)$ is differentiable at every given point $x_1 \in B$. Without loss of generality we will assume $x_1 = 0$ for notational convenience. We will take $I = (-T, T)$ and B some open ball around x_0 such that the closure of $I \times B$ still lies in U.

Abbreviate $\phi(t) = \phi(t, x_1)$, $A(t) = A(t, x_1)$ and denote by $\psi(t)$ the solution of the first variational equation $\dot{\psi}(t) = A(t)\psi(t)$ corresponding to the initial condition $\psi(t_0) = \mathbb{I}$. Set

$$\theta(t,x) = \frac{\phi(t,x) - \phi(t) - \psi(t)x}{|x|},$$

then $\frac{\partial \phi}{\partial x}$ at $x_1 = 0$ will exist (and be equal to ψ) if we can show $\lim_{x \to 0} \theta(t,x) = 0$.

Our assumption $f \in C^1$ implies

$$f(y) = f(x) + \frac{\partial f}{\partial x}(x)\,(y-x) + \left(\int_0^1 \Big(\frac{\partial f}{\partial x}(x + t(y-x)) - \frac{\partial f}{\partial x}(x)\Big) dt\right)(y-x),$$

and thus

$$f(y) - f(x) = \frac{\partial f}{\partial x}(x)\,(y - x) + |y - x| R(y, x), \tag{2.52}$$

where

$$|R(y, x)| \le \max_{t \in [0,1]} \Big\| \frac{\partial f}{\partial x}(x + t(y - x)) - \frac{\partial f}{\partial x}(x) \Big\|.$$

Here $\|.\|$ denotes the matrix norm (cf. Section 3.1). By uniform continuity of the partial derivatives $\frac{\partial f}{\partial x}$ in a neighborhood of $x_1 = 0$ we infer $\lim_{y \to x} |R(y, x)| = 0$ again uniformly in x in some neighborhood of 0.

Using (2.52) we see

$$\begin{aligned} \dot\theta(t, x) &= \frac{1}{|x|}(f(\phi(t, x)) - f(\phi(t)) - A(t)\psi(t)x) \\ &= A(t)\theta(t, x) + \frac{|\phi(t, x) - \phi(t)|}{|x|} R(\phi(t, x), \phi(t)). \end{aligned}$$

Now integrate and take absolute values (note $\theta(0, x) = 0$ and recall (2.43)) to obtain

$$|\theta(t, x)| \le \tilde R(x) + \int_0^t \|A(s)\| |\theta(s, x)| ds,$$

where

$$\tilde R(x) = \mathrm{e}^{LT} \int_0^T |R(\phi(s, x), \phi(s))| ds.$$

Then Gronwall's inequality implies $|\theta(t, x)| \le \tilde R(x) \exp(\int_0^T \|A(s)\| ds)$. Since $\lim_{y \to x} |R(y, x)| = 0$ uniformly in x in some neighborhood of 0, we have $\lim_{x \to 0} \tilde R(x) = 0$ and hence $\lim_{x \to 0} \theta(t, x) = 0$. Moreover, $\frac{\partial \phi}{\partial x}(t, x)$ is C^0 as the solution of the first variational equation. This settles the case $k = 1$ since all partial derivatives (including the one with respect to t) are continuous.

For the general case $k \ge 1$ we use induction: Suppose the claim holds for k and let $f \in C^{k+1}$. Then $\phi(t, x) \in C^1$ and the partial derivative $\frac{\partial \phi}{\partial x}(t, x)$ solves the first variational equation. But $A(t, x) \in C^k$ and hence $\frac{\partial \phi}{\partial x}(t, x) \in C^k$, which, together with Lemma 2.3, shows $\phi(t, x) \in C^{k+1}$. □

In fact, we can also handle the dependence on parameters. Suppose f depends on some parameters $\lambda \in \Lambda \subseteq \mathbb{R}^p$ and consider the IVP

$$\dot x(t) = f(t, x, \lambda), \qquad x(t_0) = x_0, \tag{2.53}$$

with corresponding solution

$$\phi(t, t_0, x_0, \lambda). \tag{2.54}$$

Theorem 2.11. *Suppose $f \in C^k(U \times \Lambda, \mathbb{R}^n)$, $k \ge 1$. Around each point $(t_0, x_0, \lambda_0) \in U \times \Lambda$ we can find an open set $I \times B \times \Lambda_0 \subseteq U \times \Lambda$ such that $\phi(t, s, x, \lambda) \in C^k(I \times I \times B \times \Lambda_0, \mathbb{R}^n)$.*

Proof. This follows from the previous result by adding the parameters λ to the dependent variables and requiring $\dot{\lambda} = 0$. Details are left to the reader. □

Problem 2.11. *Show* (2.36).

Problem 2.12. *Show* (2.38). *(Hint: Introduce* $\tilde{\psi}(t) = \psi(t) + \frac{\gamma}{\beta}$*.)*

Problem 2.13. *Find different functions* $f(t,x) = f(x)$ *and* $g(t,x) = g(x)$ *such that the inequality in* (2.40) *becomes an equality.*

Problem 2.14. *Suppose* $f \in C(U, \mathbb{R}^n)$ *satisfies* $|f(t,x) - f(t,y)| \le L(t)|x - y|$. *Show that the solution* $\phi(t, x_0)$ *of* (2.10) *satisfies*

$$|\phi(t,x_0) - \phi(t,y_0)| \le |x_0 - y_0| \, \mathrm{e}^{|\int_{t_0}^t L(s)ds|}.$$

Problem 2.15. *Show that in the one-dimensional case, we have*

$$\frac{\partial \phi}{\partial x}(t,x) = \exp\left(\int_{t_0}^t \frac{\partial f}{\partial x}(s, \phi(s,x)) ds\right).$$

2.5. Regular perturbation theory

Using Theorem 2.11 we can now also justify the perturbation method proposed in Problem 1.2 for initial value problems depending on a small parameter ε. In general, such a problem is of the form

$$\dot{x} = f(t,x,\varepsilon), \qquad x(t_0) = x_0, \tag{2.55}$$

and known as a **regular perturbation problem**.

If we suppose $f \in C^1$ then Theorem 2.11 ensures that the same is true for the solution $\phi(t,\varepsilon)$, where we do not display the dependence on the initial conditions (t_0, x_0) for notational simplicity. In particular, we have the following Taylor expansions

$$\phi(t,\varepsilon) = \phi_0(t) + \phi_1(t)\varepsilon + o(\varepsilon) \tag{2.56}$$

with respect to ε in a neighborhood of $\varepsilon = 0$.

Clearly the unperturbed term $\phi_0(t) = \phi(t,0)$ is given as the solution of the unperturbed equation

$$\dot{\phi}_0 = f_0(t,\phi_0), \qquad \phi_0(t_0) = x_0, \tag{2.57}$$

where $f_0(t,x) = f(t,x,0)$. Moreover the derivative $\phi_1(t) = \frac{\partial}{\partial \varepsilon}\phi(t,\varepsilon)|_{\varepsilon=0}$ solves the corresponding first variational equation

$$\dot{\phi}_1 = f_{10}(t,\phi_0(t))\phi_1 + f_{11}(t,\phi_0(t)), \qquad \phi_1(t_0) = 0, \tag{2.58}$$

where $f_{10}(t,x) = \frac{\partial}{\partial x} f(t,x,0)$ and $f_{11}(t,x) = \frac{\partial}{\partial \varepsilon} f(t,x,\varepsilon)|_{\varepsilon=0}$. The initial condition $\phi_1(t_0) = 0$ follows from the fact that the initial condition x_0 does not depend on ε, implying $\phi_1(t_0) = \frac{\partial}{\partial \varepsilon}\phi(t_0,\varepsilon)|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} x_0|_{\varepsilon=0} = 0$.

Hence once we have the solution of the unperturbed problem $\phi_0(t)$, we can then compute the correction term $\phi_1(t)$ by solving another linear equation.

Note that the procedure can be equivalently described as follows: Plug the Taylor expansion for $\phi(t,\varepsilon)$ into the differential equation, expand the right-hand side with respect to ε, and compare coefficients with respect to powers of ε.

Example. Let us look at a simple example. Consider the equation

$$\dot{v} = -\varepsilon v - g, \quad v(0) = 0, \qquad \varepsilon \geq 0,$$

which models the velocity of a falling object with air resistance (cf. Problem 1.17). The solution can be easily found

$$\phi(t,\varepsilon) = g\frac{\mathrm{e}^{-\varepsilon t} - 1}{\varepsilon}$$

and there is no need for any perturbation techniques. However, we will still apply it to illustrate the method. The unperturbed problem is

$$\dot{\phi}_0 = -g, \quad \phi_0(0) = 0,$$

and the solution is given by $\phi_0(t) = -gt$. Similarly, since $f(t,v,\varepsilon) = -\varepsilon v - g$ it follows that $f_{10}(t,v) = 0$, $f_{11}(t,v) = -v$ and the equation for the first correction term is

$$\dot{\phi}_1 = -\phi_0(t), \quad \phi_1(0) = 0,$$

with solution given by $\phi_1(t) = \frac{g}{2}t^2$. Hence our approximation is

$$v(t) = -g\left(t - \varepsilon\frac{t^2}{2} + o(\varepsilon)\right)$$

which of course coincides with the Taylor expansion of the exact solution. However, note the approximation is valid only for fixed time and will in general get worse as t increases. In fact, observe that for $\varepsilon > 0$ the approximation diverges to $+\infty$ while the exact solution converges to $\frac{g}{\varepsilon}$. ⋄

Clearly we can extend this procedure to get further approximations:

Theorem 2.12. *Let Λ be some open interval. Suppose $f \in C^k(U \times \Lambda, \mathbb{R}^n)$, $k \geq 1$ and fix some values $(t_0, x_0, \varepsilon_0) \in U \times \Lambda$. Let $\phi(t,\varepsilon) \in C^k(I \times \Lambda_0, \mathbb{R}^n)$ be the solution of the initial value problem*

$$\dot{x} = f(t,x,\varepsilon), \qquad x(t_0) = x_0, \tag{2.59}$$

guaranteed to exist by Theorem 2.11*.*

Then

$$\phi(t,\varepsilon) = \sum_{j=0}^{k} \frac{\phi_j(t)}{j!}(\varepsilon - \varepsilon_0)^j + o((\varepsilon - \varepsilon_0)^k), \tag{2.60}$$

where the coefficients can be obtained by recursively solving

$$\dot{\phi}_j = f_j(t, \phi_0, \dots, \phi_j, \varepsilon_0), \qquad \phi_j(t_0) = \begin{cases} x_0, & j = 0, \\ 0, & j \geq 1, \end{cases} \tag{2.61}$$

where the function f_j is recursively defined via

$$\begin{aligned} f_{j+1}(t, x_0, \dots, x_{j+1}, \varepsilon) =& \frac{\partial f_j}{\partial \varepsilon}(t, x_0, \dots, x_j, \varepsilon) \\ &+ \sum_{k=0}^{j} \frac{\partial f_j}{\partial x_k}(t, x_0, \dots, x_j, \varepsilon) x_{k+1}, \\ f_0(t, x_0, \varepsilon) =& f(t, x_0, \varepsilon). \end{aligned} \tag{2.62}$$

If we assume $f \in C^{k+1}$ the error term will be $O((\varepsilon - \varepsilon_0)^{k+1})$ uniformly for $t \in I$.

Proof. The result follows by plugging (2.60) into the differential equation and comparing powers of ε. If $f \in C^{k+1}$ we know that $\frac{\partial^{k+1}}{\partial \varepsilon^{k+1}} \phi$ is continuous and hence bounded on $I \times \Lambda_0$, which gives the desired estimate on the remainder in the Taylor expansion. □

A few remarks are in order: Of course we could admit more than one parameter if we are willing to deal with Taylor series in more than one variable. Moreover, we could include the case where the initial condition depends on ε by simply replacing the initial conditions for $\phi_j(t_0)$ by the corresponding expansion coefficients of $x_0(\varepsilon)$.

Finally, we remark that the Taylor expansion will converge if f is analytic with respect to all variables. This will be shown in Theorem 4.2.

Problem 2.16. *Compute the next term ϕ_2 in the above example.*

Problem 2.17. *Approximate the solutions of $\ddot{x} + x + \varepsilon x^3 = 0$, $x(0) = 1$, $\dot{x}(0) = 0$ up to order one. (Hint: It is not necessary to convert this second-order equation to a first-order system. To solve the second-order equations you need to use the computer or preview Section* 3.3.*)*

2.6. Extensibility of solutions

We have already seen that solutions might not exist for all $t \in \mathbb{R}$ even though the differential equation is defined for all $t \in \mathbb{R}$. This raises the question about the maximal interval on which a solution of the IVP (2.10) can be defined.

Suppose that solutions of the IVP (2.10) exist locally and are unique (e.g., f is Lipschitz). Let ϕ_1, ϕ_2 be two solutions of the IVP (2.10) defined on the open intervals I_1, I_2, respectively. Let $I = I_1 \cap I_2 = (T_-, T_+)$ and let (t_-, t_+) be the maximal open interval on which both solutions coincide. I claim that $(t_-, t_+) = (T_-, T_+)$. In fact, if $t_+ < T_+$, both solutions would also coincide at t_+ by continuity. Next, considering the IVP with initial condition $x(t_+) = \phi_1(t_+) = \phi_2(t_+)$ shows that both solutions coincide in a neighborhood of t_+ by local uniqueness. This contradicts maximality of t_+ and hence $t_+ = T_+$. Similarly, $t_- = T_-$.

Moreover, we get a solution

$$\phi(t) = \begin{cases} \phi_1(t), & t \in I_1, \\ \phi_2(t), & t \in I_2, \end{cases} \tag{2.63}$$

defined on $I_1 \cup I_2$. In fact, this even extends to an arbitrary number of solutions and in this way we get a (unique) solution defined on some maximal interval.

Theorem 2.13. *Suppose the IVP* (2.10) *has a unique local solution (e.g. the conditions of Theorem* 2.5 *are satisfied). Then there exists a unique maximal solution defined on some maximal interval* $I_{(t_0,x_0)} = (T_-(t_0, x_0), T_+(t_0, x_0))$.

Proof. Let $\mathcal{S}$ be the set of all solutions ϕ of (2.10) which are defined on an open interval I_ϕ. Let $\mathcal{I} = \bigcup_{\phi \in \mathcal{S}} I_\phi$, which is again open. Moreover, if $t_1 > t_0 \in \mathcal{I}$, then $t_1 \in I_\phi$ for some ϕ and thus $[t_0, t_1) \subseteq I_\phi \subseteq \mathcal{I}$. Similarly for $t_1 < t_0$ and thus $\mathcal{I}$ is an open interval containing t_0. In particular, it is of the form $\mathcal{I} = (T_-, T_+)$. Now define $\phi_{max}(t)$ on $\mathcal{I}$ by $\phi_{max}(t) = \phi(t)$ for some $\phi \in \mathcal{S}$ with $t \in I_\phi$. By our above considerations any two ϕ will give the same value, and thus $\phi_{max}(t)$ is well-defined. Moreover, for every $t_1 > t_0$ there is some $\phi \in \mathcal{S}$ such that $t_1 \in I_\phi$ and $\phi_{max}(t) = \phi(t)$ for $t \in (t_0 - \varepsilon, t_1 + \varepsilon)$ which shows that ϕ_{max} is a solution. By construction there cannot be a solution defined on a larger interval. □

The solution found in the previous theorem is called the **maximal solution**. A solution defined for all $t \in \mathbb{R}$ is called a **global solution**. Clearly every global solution is maximal.

Remark: If we drop the requirement that f is Lipschitz, we still have existence of solutions (see Theorem 2.19 below), but we already know that we might lose uniqueness. Even without uniqueness, two given solutions of the IVP (2.10) can still be glued together at t_0 (if necessary) to obtain a solution defined on $I_1 \cup I_2$. Furthermore, Zorn's lemma can be used to ensure existence of maximal solutions in this case. For example, consider the differential equation $\dot{x} = \sqrt{|x|}$ where we have found global (and thus maximal) solutions which are, however, not unique.

Now let us look at how we can tell from a given solution whether an extension exists or not.

Lemma 2.14. *Let $\phi(t)$ be a solution of* (2.10) *defined on the interval* (t_-, t_+). *Then there exists an extension to the interval* $(t_-, t_+ + \varepsilon)$ *for some* $\varepsilon > 0$ *if and only if there exists a sequence* $t_m \in (t_-, t_+)$ *such that*

$$\lim_{m\to\infty} (t_m, \phi(t_m)) = (t_+, y) \in U. \tag{2.64}$$

The analogous statement holds for an extension to $(t_- - \varepsilon, t_+)$.

Proof. Clearly, if there is an extension, then (2.64) holds for any sequence $t_m \uparrow t_+$. Conversely, suppose there is a sequence satisfying (2.64). We first show that in this case

$$\lim_{t\uparrow t_+} \phi(t) = y. \tag{2.65}$$

Intuitively this follows since otherwise the solution would need to oscillate faster and faster as t approaches t_+. Consequently its derivative would need to grow, which is impossible since $f(t,x)$ is bounded near y. More precisely, since U is open there is some $\delta > 0$ such that $V = [t_+ - \delta, t_+] \times \overline{B_\delta(y)} \subset U$ and $M = \max_{(t,x)\in V} |f(t,x)| < \infty$. Moreover, after maybe passing to a subsequence, we can assume that $t_m \in (t_+ - \delta, t_+)$, $\phi(t_m) \in B_\delta(y)$, and $t_m < t_{m+1}$. If (2.65) were wrong, we could find a sequence $\tau_m \uparrow t_+$ such that $|\phi(\tau_m) - y| \geq \gamma > 0$. Without loss we can choose $\gamma < \delta$ and $\tau_m \geq t_m$. Moreover, by the intermediate value theorem we can even require $|\phi(\tau_m) - y| = \gamma$ and $|\phi(t) - y| < \delta$ for $t \in [t_m, \tau_m]$. But then

$$\begin{aligned} 0 < \gamma &= |\phi(\tau_m) - y| \leq |\phi(\tau_m) - \phi(t_m)| + |\phi(t_m) - y| \\ &\leq \int_{t_m}^{\tau_m} |f(s,\phi(s))| ds + |\phi(t_m) - y| \leq M|\tau_m - t_m| + |\phi(t_m) - y|, \end{aligned}$$

where the right-hand side converges to 0 as $m \to \infty$, which is a contradiction. Thus (2.65) holds.

Now take a solution $\tilde{\phi}(t)$ of the IVP $x(t_+) = y$ defined on the interval $(t_+ - \varepsilon, t_+ + \varepsilon)$. As before, we can glue $\phi(t)$ and $\tilde{\phi}(t)$ at t_+ to obtain a function on $(t_-, t_+ + \varepsilon)$. This function is continuous by construction and the limits of its left and right derivative are both equal to $f(t_+, y)$. Hence it is differentiable at $t = t_+$ and thus a solution defined on $(t_-, t_+ + \varepsilon)$. □

Our final goal is to show that solutions exist for all $t \in \mathbb{R}$ if $f(t,x)$ grows at most linearly with respect to x. But first we need a better criterion which does not require a complete knowledge of the solution.

Corollary 2.15. *Let $\phi(t)$ be a solution of* (2.10) *defined on the interval* (t_-, t_+). *Suppose there is a compact set* $[t_0, t_+] \times C \subset U$ *such that* $\phi(t_m) \in C$ *for some sequence* $t_m \in [t_0, t_+)$ *converging to* t_+. *Then there exists an extension to the interval* $(t_-, t_+ + \varepsilon)$ *for some* $\varepsilon > 0$.

In particular, if there is such a compact set C for every $t_+ > t_0$ (C might depend on t_+), then the solution exists for all $t > t_0$.

The analogous statement holds for an extension to $(t_- - \varepsilon, t_+)$.

Proof. Let $t_m \to t_+$. By compactness $\phi(t_m)$ has a convergent subsequence and the claim follows from the previous lemma. □

The logical negation of this result is also of interest.

Corollary 2.16. *Let $I_{(t_0,x_0)} = (T_-(t_0,x_0), T_+(t_0,x_0))$ be the maximal interval of existence of a solution starting at $x(t_0) = x_0$. If $T_+ = T_+(t_0,x_0) < \infty$, then the solution must eventually leave every compact set C with $[t_0,T_+] \times C \subset U$ as t approaches T_+. In particular, if $U = \mathbb{R} \times \mathbb{R}^n$, the solution must tend to infinity as t approaches T_+.*

Now we come to the proof of our anticipated result.

Theorem 2.17. *Suppose $U = \mathbb{R} \times \mathbb{R}^n$ and for every $T > 0$ there are constants $M(T)$, $L(T)$ such that*

$$|f(t,x)| \le M(T) + L(T)|x|, \qquad (t,x) \in [-T,T] \times \mathbb{R}^n. \tag{2.66}$$

Then all solutions of the IVP (2.10) *are defined for all $t \in \mathbb{R}$.*

Proof. Using the above estimate for f we have ($t_0 = 0$ without loss of generality)

$$|\phi(t)| \le |x_0| + \int_0^t (M + L|\phi(s)|)ds, \quad t \in [0,T] \cap I,$$

and the variant (2.38) of Gronwall's inequality shows

$$|\phi(t)| \le |x_0| \mathrm{e}^{LT} + \frac{M}{L}(\mathrm{e}^{LT} - 1).$$

Thus ϕ lies in a compact ball and the result follows by the previous lemma. □

Again, let me remark that it suffices to assume

$$|f(t,x)| \le M(t) + L(t)|x|, \qquad x \in \mathbb{R}^n, \tag{2.67}$$

where $M(t)$, $L(t)$ are locally integrable. A slight extension of the above result is outlined in Problem 2.22.

Problem 2.18. *Show that Theorem* 2.17 *is false (in general) if the estimate is replaced by*

$$|f(t,x)| \le M(T) + L(T)|x|^\alpha$$

with $\alpha > 1$.

Problem 2.19. *Consider a first-order autonomous system in $\mathbb{R}^n$ with $f(x)$ Lipschitz. Show that $x(t)$ is a solution if and only if $x(t-t_0)$ is. Use this and uniqueness to show that for two maximal solutions $x_j(t)$, $j=1,2$, the images $\gamma_j = \{x_j(t) | t \in I_j\} \subset \mathbb{R}^n$ either coincide or are disjoint.*

Problem 2.20. *Consider a first-order autonomous equation in $\mathbb{R}^1$ with $f(x)$ Lipschitz. Suppose $f(0) = f(1) = 0$. Show that solutions starting in $[0,1]$ cannot leave this interval. What is the maximal interval of definition (T_-, T_+) for solutions starting in $[0,1]$? Does such a solution have a limit as $t \to T_\pm$?*

Problem 2.21. *Consider a first-order equation in $\mathbb{R}^1$ with $f(t,x)$ defined on $\mathbb{R} \times \mathbb{R}$. Suppose $x\, f(t,x) < 0$ for $|x| > R$. Show that all solutions exist for all $t > 0$.*

Problem 2.22. *Suppose $U = \mathbb{R} \times \mathbb{R}^n$ and that*

$$|f(t,x)| \le g(|x|)$$

for some positive continuous function $g \in C([0,\infty))$ which satisfies

$$\int_0^\infty \frac{dr}{g(r)} = \infty.$$

Then all solutions of the IVP (2.10) are defined for all $t \ge 0$.

Show that the same conclusion still holds if there is such a function $g_T(r)$ for every $t \in [0,T]$.

(Hint: Look at the differential equation for $r(t)^2 = |x(t)|^2$. Estimate the right-hand side and recall the analysis from Sections 1.3 and 1.5.)

2.7. Euler's method and the Peano theorem

In this section we show that continuity of $f(t,x)$ is sufficient for existence of at least one solution of the initial value problem (2.10).

If $\phi(t)$ is a solution, then by Taylor's theorem we have

$$\phi(t_0 + h) = x_0 + \dot{\phi}(t_0)h + o(h) = x_0 + f(t_0, x_0)h + o(h). \tag{2.68}$$

This suggests that we define an approximate solution by omitting the error term and apply the procedure iteratively. That is, we set

$$x_h(t_{m+1}) = x_h(t_m) + f(t_m, x_h(t_m))h, \qquad t_m = t_0 + mh, \tag{2.69}$$

and use linear interpolation in between. This procedure is known as the **Euler method**.

We expect that $x_h(t)$ converges to a solution as $h \downarrow 0$. But how should we prove this? Well, the key observation is that, since f is continuous, it is bounded by a constant on each compact interval. Hence the derivative of $x_h(t)$ is bounded by the same constant. Since this constant is independent of h, the functions $x_h(t)$ form an equicontinuous family of functions which converges uniformly after maybe passing to a subsequence by the Arzelà–Ascoli theorem.

Theorem 2.18 (Arzelà–Ascoli). *Suppose the sequence of functions $x_m(t) \in C(I, \mathbb{R}^n)$, $m \in \mathbb{N}$, on a compact interval I is (uniformly)* **equicontinuous**, *that is, for every $\varepsilon > 0$ there is a $\delta > 0$ (independent of m) such that*

$$|x_m(t) - x_m(s)| \le \varepsilon \quad \text{if} \quad |t - s| < \delta,\ m \in \mathbb{N}. \tag{2.70}$$

If the sequence x_m is bounded, then there is a uniformly convergent subsequence.

Proof. Let $\{t_j\}_{j=1}^\infty \subset I$ be a dense subset of our interval (e.g., all rational numbers in I). Since $x_m(t_1)$ is bounded, we can choose a subsequence $x_m^{(1)}(t)$ such that $x_m^{(1)}(t_1)$ converges (Bolzano–Weierstraß). Similarly we can extract a subsequence $x_m^{(2)}(t)$ from $x_m^{(1)}(t)$ which converges at t_2 (and hence also at t_1 since it is a subsequence of $x_m^{(1)}(t)$). By induction we get a sequence $x_m^{(j)}(t)$ converging at $t_1, \dots, t_j$. The diagonal sequence $\tilde{x}_m(t) = x_m^{(m)}(t)$ will hence converge for all $t = t_j$ (why?). We will show that it converges uniformly for all t:

Fix $\varepsilon > 0$ and choose δ such that $|x_m(t) - x_m(s)| \le \frac{\varepsilon}{3}$ for $|t - s| < \delta$. The balls $B_\delta(t_j)$ cover I and by compactness even finitely many, say $1 \le j \le p$, suffice. Furthermore, choose N_ε such that $|\tilde{x}_m(t_j) - \tilde{x}_n(t_j)| \le \frac{\varepsilon}{3}$ for $n, m \ge N_\varepsilon$ and $1 \le j \le p$.

Now pick t and note that $t \in B_\delta(t_j)$ for some j. Thus

$$\begin{aligned} |\tilde{x}_m(t) - \tilde{x}_n(t)| \le & |\tilde{x}_m(t) - \tilde{x}_m(t_j)| + |\tilde{x}_m(t_j) - \tilde{x}_n(t_j)| \\ & + |\tilde{x}_n(t_j) - \tilde{x}_n(t)| \le \varepsilon \end{aligned}$$

for $n, m \ge N_\varepsilon$, which shows that $\tilde{x}_m$ is Cauchy with respect to the maximum norm. By completeness of $C(I, \mathbb{R}^n)$ it has a limit. □

More precisely, pick $\delta, T > 0$ such that $V = [t_0, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$ and let

$$M = \max_{(t,x) \in V} |f(t, x)|. \tag{2.71}$$

Then $x_h(t) \in B_\delta(x_0)$ for $t \in [t_0, t_0 + T_0]$, where $T_0 = \min\{T, \frac{\delta}{M}\}$, and

$$|x_h(t) - x_h(s)| \le M|t - s|. \tag{2.72}$$

Hence any subsequence of the family $x_h(t)$ is equicontinuous and there is a uniformly convergent subsequence $\phi_m(t) \to \phi(t)$. It remains to show that the limit $\phi(t)$ solves our initial value problem (2.10). This will be done by verifying that the corresponding integral equation (2.11) holds. Since f is uniformly continuous on V, we can find a sequence $\Delta(h) \to 0$ as $h \to 0$, such that

$$|f(s,y) - f(t,x)| \le \Delta(h) \quad \text{for} \quad |y-x| \le Mh,\ |s-t| \le h. \tag{2.73}$$

To be able to estimate the difference between left and right-hand side of (2.11) for $x_h(t)$ we choose an m with $t \le t_m$ and write

$$x_h(t) = x_0 + \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \chi(s) f(t_j, x_h(t_j)) ds, \tag{2.74}$$

where $\chi(s) = 1$ for $s \in [t_0, t]$ and $\chi(s) = 0$ else. Then

$$\begin{aligned} &\left| x_h(t) - x_0 - \int_{t_0}^{t} f(s, x_h(s)) ds \right| \\ &\quad \le \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \chi(s) |f(t_j, x_h(t_j)) - f(s, x_h(s))| ds \\ &\quad \le \Delta(h) \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \chi(s) ds = |t - t_0| \Delta(h), \end{aligned} \tag{2.75}$$

from which it follows that ϕ is indeed a solution

$$\phi(t) = \lim_{m\to\infty} \phi_m(t) = x_0 + \lim_{m\to\infty} \int_{t_0}^{t} f(s, \phi_m(s)) ds = x_0 + \int_{t_0}^{t} f(s, \phi(s)) ds \tag{2.76}$$

since we can interchange limit and integral by uniform convergence.

Hence we have proven **Peano's theorem**.

Theorem 2.19 (Peano). *Suppose f is continuous on $V = [t_0, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$ and denote the maximum of $|f|$ by M. Then there exists at least one solution of the initial value problem* (2.10) *for $t \in [t_0, t_0 + T_0]$ which remains in $\overline{B_\delta(x_0)}$, where $T_0 = \min\{T, \frac{\delta}{M}\}$. The analogous result holds for the interval $[t_0 - T_0, t_0]$.*

Of course this theorem raises the question: If there are also conditions on f, which are weaker than the Lipschitz condition but still guarantee uniqueness? One such condition is presented in Problem 2.25.

Finally, let me remark that the Euler algorithm is well suited for the numerical computation of an approximate solution since it requires only the evaluation of f at certain points. On the other hand, it is not clear how to find the converging subsequence, and so let us show that $x_h(t)$ converges uniformly if f is Lipschitz. By (2.29) with $x(t) = x_h(t)$ and $y(t) = K(x_h)(t)$

this yields

$$\|x_h - K^m(x_h)\| \le \sum_{j=0}^{m-1} \|K^j(x_h) - K^{j+1}(x_h)\|$$
$$\le \|x_h - K(x_h)\| \sum_{j=0}^{m-1} \frac{(LT_0)^j}{j!}, \tag{2.77}$$

using the same notation as in the proof of Theorem 2.2. Taking $n \to \infty$ we finally obtain

$$\|x_h - \phi\| \le T_0 \mathrm{e}^{LT_0} \Delta(h), \qquad t \in [t_0, t_0 + T_0], \tag{2.78}$$

since our above estimate (2.75) for $t = t_0 + T_0$ reads

$$\|x_h - K(x_h)\| \le T_0 \Delta(h). \tag{2.79}$$

Note that if we can find some Lipschitz constant L_0 such that $|f(t,x) - f(s,x)| \le L_0|t-s|$, then we can choose $\Delta(h) = (L_0 + LM)h$.

Thus we have a simple numerical method for computing solutions plus an error estimate. However, in practical computations one usually uses some heuristic error estimates, e.g., by performing each step using two step sizes h and $\frac{h}{2}$. If the difference between the two results becomes too big, the step size is reduced and the last step is repeated.

Of course the Euler algorithm is not the most effective one available today. Usually one takes more terms in the Taylor expansion and approximates all differentials by their difference quotients. The resulting algorithm will converge faster, but it will also involve more calculations in each step. A good compromise is usually a method, where one approximates $\phi(t_0 + h)$ up to the fourth order in h. Setting $t_m = t_0 + hm$ and $x_m = x_h(t_m)$ the resulting algorithm

$$x_{m+1} = x_m + \frac{h}{6}(k_{1,m} + 2k_{2,m} + 2k_{3,m} + k_{4,m}), \tag{2.80}$$

where

$$\begin{aligned} k_{1,m} &= f(t_m, x_m), & k_{2,m} &= f(t_m + \tfrac{h}{2}, x_m + \tfrac{h}{2}k_{1,m}),\\ k_{3,m} &= f(t_m + \tfrac{h}{2}, x_m + \tfrac{h}{2}k_{2,m}), & k_{4,m} &= f(t_{m+1}, x_m + hk_{3,m}), \end{aligned} \tag{2.81}$$

is called the **Runge–Kutta algorithm**. For even better methods see the literature on numerical methods for ordinary differential equations.

Problem 2.23. Heun's method *(or* **improved Euler***) is given by*

$$x_{m+1} = x_m + \frac{h}{2}\big(f(t_m, x_m) + f(t_{m+1}, y_m)\big), \quad y_m = x_m + f(t_m, x_m)h.$$

Show that using this method the error during one step is of $O(h^3)$ *(provided* $f \in C^2$*):*

$$\phi(t_0 + h) = x_0 + \frac{h}{2}\big(f(t_0, x_0) + f(t_1, y_0)\big) + O(h^3).$$

Note that this is not the only possible scheme with this error order since

$$\phi(t_0 + h) = x_0 + \frac{h}{2}\big(f(t_1, x_0) + f(t_0, y_0)\big) + O(h^3)$$

as well.

Problem 2.24. *Compute the solution of the initial value problem* $\dot{x} = x$, $x(0) = 1$, *using the Euler and Runge–Kutta algorithm with step size* $h = 10^{-1}$. *Compare the results with the exact solution.*

Problem 2.25 (Osgood uniqueness criterion)**.** *We call a continuous nondecreasing function* $\rho : [0, \infty) \to [0, \infty)$ *with* $\rho(0) = 0$ *a* **module of continuity**. *It is said to satisfy the* **Osgood condition** *if*

$$\int_0^1 \frac{dr}{\rho(r)} = \infty.$$

We will say that a function $f : \mathbb{R} \to \mathbb{R}$ *is* ρ*-continuous if* $|f(x) - f(y)| \le C\rho(|x - y|)$ *for some constant* C. *For example, in the case* $\rho(r) = r^\alpha$, $\alpha \in (0, 1)$, *we obtain the Hölder continuous functions and in the case* $\rho(r) = r$ *the Lipschitz continuous functions. Note that the Osgood condition holds only in the Lipschitz case. Another module satisfying the Osgood condition is* $\rho(r) = r(1 + |\log(r)|)$. *The corresponding functions are known as almost Lipschitz functions.*

Let $f(t, x)$ *be as in the Peano theorem and suppose*

$$|(x - y) \cdot (f(t, x) - f(t, y))| \le C|x - y|\rho(|x - y|),$$

$t \in [t_0, t_0 + T]$, $x, y \in B_\delta(x_0)$, *for some modulus of continuity which satisfies the Osgood condition. Here the* $\cdot$ *indicates the scalar product. Then the solution is unique.*

(Hint: Consider the difference of two solutions $R(t) = |x(t) - y(t)|^2$ *and suppose* $R(t_1) = 0$ *but* $R(t) > 0$ *for* $t \in (t_1, t_2)$. *Estimate* $\dot{R}$ *using the assumptions and proceed as with a separable equation to obtain a contradiction.)*