Preface

The first version of these lecture notes was drafted in 2010 for a course at the Pennsylvania State University. The book is addressed to graduate students in mathematics or other disciplines, who wish to understand the essential concepts of functional analysis and their application to partial differential equations. Most of its content can be covered in a one-semester course at the first-year graduate level.

In writing this textbook, I followed a number of guidelines:

- Keep it short, presenting all the fundamental concepts and results, but not more than that.
- Explain clearly the connections between theorems in functional analysis and familiar results of finite-dimensional linear algebra.
- Cover enough of the theory of Sobolev spaces and semigroups of linear operators as needed to develop significant applications to elliptic, parabolic, and hyperbolic PDEs.
- Include a large number of homework problems and illustrate the main ideas with figures, whenever possible.

In functional analysis one finds a wealth of beautiful results that could be included in a monograph. However, for a textbook of this nature one should resist such a temptation.

After the Introduction, Chapters 2 to 6 cover classical topics in linear functional analysis: Banach spaces, Hilbert spaces, and linear operators. Chapter 4 is devoted to spaces of continuous functions, including the Stone-Weierstrass approximation theorem and Ascoli's compactness theorem. In view of applications to linear PDEs, in Chapter 6 we prove some basic results on Fredholm operators and the Hilbert-Schmidt theorem on compact symmetric operators in a Hilbert space.

Chapter 7 provides an introduction to the theory of semigroups, extending the definition of the exponential function e^{tA} to a suitable class of (possibly unbounded) linear operators. We stress the connection with finite-dimensional ODEs and the close relation between the resolvent operators and backward Euler approximations.

After an introduction explaining the concepts of distribution and weak derivative, Chapter 8 develops the theory of Sobolev spaces. These spaces provide the most convenient abstract framework where techniques of functional analysis can be applied toward the solution of ordinary and partial differential equations.

The first three sections in Chapter 9 describe applications of the previous theory to elliptic, parabolic, and hyperbolic PDEs. Since differential operators are unbounded, it is often convenient to recast a linear PDE in a "weak form", involving only bounded operators on a Hilbert-Sobolev space. This new equation can then be studied using techniques of abstract functional analysis, such as the Lax-Milgram theorem, Fredholm's theory, or the representation of the solution in terms of a series of eigenfunctions.

The last chapter consists of an Appendix, collecting background material. This includes: definition and properties of metric spaces, the contraction mapping theorem, the Baire category theorem, a review of Lebesgue measure theory, mollification techniques and partitions of unity, integrals of functions taking values in a Banach space, a collection of inequalities, and a version of Gronwall's lemma.

These notes are illustrated by 41 figures. Nearly 180 homework problems are collected at the end of the various chapters. A complete set of solutions to the exercises is available to instructors. To obtain a PDF file of the solutions, please contact the author, including a link to your department's web page listing you as an instructor or professor.

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