
Preface

Lie algebras, Lie groups, and their representation theories are parts of the mathematical language describing symmetries, and they have played a central role in modern mathematics. An early motivation of studying Lie superalgebras as a generalization of Lie algebras came from supersymmetry in mathematical physics. Ever since a Cartan-Killing type classification of finite-dimensional complex Lie superalgebras was obtained by Kac [60] in 1977, the theory of Lie superalgebras has established itself as a prominent subject in modern mathematics. An independent classification of the finite-dimensional complex simple Lie superalgebras whose even subalgebras are reductive (called simple Lie superalgebras of classical type) was given by Scheunert, Nahm, and Rittenberg in [106].

The goal of this book is a systematic account of the structure and representation theory of finite-dimensional complex Lie superalgebras of classical type. The book intends to serve as a rigorous introduction to representation theory of Lie superalgebras on one hand, and, on the other hand, it covers a new approach developed in the past few years toward understanding the Bernstein-Gelfand-Gelfand (BGG) category for classical Lie superalgebras. In spite of much interest in representations of Lie superalgebras stimulated by mathematical physics, these basic topics have not been treated in depth in book form before. The reason seems to be that the representation theory of Lie superalgebras is dramatically different from that of complex semisimple Lie algebras, and a systematic, yet accessible, approach toward the basic problem of finding irreducible characters for Lie superalgebras was not available in a great generality until very recently.

We are aware that there is an enormous literature with numerous partial results for Lie superalgebras, and it is not our intention to make this book an encyclopedia. Rather, we treat in depth the representation theory of the three most important classes of Lie superalgebras, namely, the general linear Lie superalgebras $\mathfrak{gl}(m|n)$,

the ortho-symplectic Lie superalgebras $\mathfrak{osp}(m|2n)$, and the queer Lie superalgebras $\mathfrak{q}(n)$. To a large extent, representations of $\mathfrak{sl}(m|n)$ can be understood via $\mathfrak{gl}(m|n)$. The lecture notes [32] by the authors can be considered as a prototype for this book. The presentation in this book is organized around three dualities with a unifying theme of determining irreducible characters:

Schur duality, Howe duality, and super duality.

The new book of Musson [90] treats in detail the ring theoretical aspects of the universal enveloping algebras of Lie superalgebras as well as the basic structures of simple Lie superalgebras.

There are two superalgebra generalizations of Schur duality. The first one, due to Sergeev [110] and independently Berele-Regev [7], is an interplay between the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ and the symmetric group, which incorporates the trivial and sign modules in a unified framework. The irreducible polynomial characters of $\mathfrak{gl}(m|n)$ arising this way are given by the super Schur polynomials. The second one, called Sergeev duality, is an interplay between the queer Lie superalgebra $\mathfrak{q}(n)$ and a twisted hyperoctahedral group algebra. The Schur Q -functions and related combinatorics of shifted tableaux appear naturally in the description of the irreducible polynomial characters of $\mathfrak{q}(n)$.

It has been observed that much of the classical invariant theory for the polynomial algebra has a parallel theory for the exterior algebra as well. The First Fundamental Theorem (FFT) for both polynomial invariants and skew invariants for classical groups admits natural reformulation and extension in the theory of Howe's reductive dual pairs [51, 52]. Lie superalgebras allow an elegant and uniform treatment of Howe duality on the polynomial and exterior algebras (cf. Cheng-Wang [29]). For the general linear Lie groups, Schur duality, Howe duality, and FFT are equivalent. Unlike Schur duality, Howe duality treats classical Lie groups and (super)algebras beyond type A equally well. The Howe dualities allow us to determine the character formulas for the irreducible modules appearing in the dualities.

The third duality, super duality, has a completely different flavor. It views the representation theories of Lie superalgebras and Lie algebras as two sides of the same coin, and it is an unexpected and rather powerful approach developed in the past few years by the authors and their collaborators, culminating in Cheng-Lam-Wang [24]. The super duality approach allows one to overcome in a conceptual way various major obstacles in super representation theory via an equivalence of module categories of Lie algebras and Lie superalgebras.

Schur, Howe, and super dualities provide approaches to the irreducible character problem in increasing generality and sophistication. Schur and Howe dualities only offer a solution to the irreducible character problem for modules in some semisimple subcategories. On the other hand, super duality provides a conceptual solution to the long-standing irreducible character problem in fairly general BGG

categories (including all finite-dimensional modules) over classical Lie superalgebras in terms of the usual Kazhdan-Lusztig polynomials of classical Lie algebras. Totally different and independent approaches to the irreducible character problem of *finite-dimensional* $\mathfrak{gl}(m|n)$ -modules have been developed by Serganova [107] and Brundan [11]. Also Brundan's conjecture on irreducible characters of $\mathfrak{gl}(m|n)$ in the full BGG category \mathcal{O} has recently been proved in [26]. Super duality again plays a crucial role in the proof. However, this latest approach to the full BGG category \mathcal{O} is beyond the scope of this book.

The book is largely self-contained and should be accessible to graduate students and non-experts as well. Besides assuming basic knowledge of entry-level graduate algebra (and some familiarity with basic homological algebra in the final Chapter 6), the other prerequisite is a one-semester course in the theory of finite-dimensional semisimple Lie algebras. For example, either the book by Humphreys or the first half of the book by Carter on semisimple Lie algebras is sufficient. Some familiarity with symmetric functions and representations of symmetric groups can be sometimes useful, and Appendix A provides a quick summary for our purpose. It is possible that super experts may also benefit from the book, as several "folklore" results are rigorously proved and occasionally corrected in great detail here, sometimes with new proofs. The proofs of some of these results can be at times rather difficult to trace or read in the literature (and not merely because they might be in a different language).

Here is a broad outline of the book chapter by chapter. Each chapter ends with exercises and historical notes. Though we have tried to attribute the main results accurately and fairly, we apologize beforehand for any unintended omissions and mistakes.

Chapter 1 starts by defining various classes of Lie superalgebras. For the basic Lie superalgebras, we introduce the invariant bilinear forms, root systems, fundamental systems, and Weyl groups. Positive systems and fundamental systems for basic Lie superalgebras are not conjugate under the Weyl group, and the notion of odd reflections is introduced to relate non-conjugate positive systems. The PBW theorem for the universal enveloping algebra of a Lie superalgebra is formulated, and highest weight theory for basic Lie superalgebras and $\mathfrak{q}(n)$ is developed.

In Chapter 2, we focus on Lie superalgebras of types \mathfrak{gl} , \mathfrak{osp} and \mathfrak{q} . We classify their finite-dimensional simple modules using odd reflection techniques. We then formulate and establish precisely the images of the respective Harish-Chandra homomorphisms and linkage principles. We end with a Young diagrammatic description of the extremal weights in the simple polynomial $\mathfrak{gl}(m|n)$ -modules and finite-dimensional simple $\mathfrak{osp}(m|2n)$ -modules. It takes considerably more effort to formulate and prove these results for Lie superalgebras than for semisimple Lie algebras because of the existence of non-conjugate Borel subalgebras and the limited role of Weyl groups for Lie superalgebras.

Schur duality for Lie superalgebras is developed in Chapter 3. We start with some results on the structure of associative superalgebras including the super variants of the Wedderburn theorem, Schur's lemma, and the double centralizer property. The Schur-Sergeev duality for $\mathfrak{gl}(m|n)$ is proved, and it provides a classification of irreducible polynomial $\mathfrak{gl}(m|n)$ -modules. As a consequence, the characters of the simple polynomial $\mathfrak{gl}(m|n)$ -modules are given by the super Schur polynomials. On the algebraic combinatorial level, there is a natural super generalization of the notion of semistandard tableau, which is a hybrid of the traditional version and its conjugate counterpart. The Schur-Sergeev duality for $\mathfrak{q}(n)$ requires understanding the representation theory of a twisted hyperoctahedral group algebra, which we develop from scratch. The characters of the simple polynomial $\mathfrak{q}(n)$ -modules are given by the Schur Q -polynomials up to some 2-powers.

In Chapter 4, we give a quick introduction to classical invariant theory, which serves as a preparation for Howe duality in the next chapter. We describe several versions of the FFT for the classical groups, i.e., a tensor algebra version, a polynomial algebra version, and a supersymmetric algebra version.

Howe duality is the main topic of Chapter 5. Like Schur duality, Howe duality involves commuting actions of a classical Lie group G and a classical superalgebra \mathfrak{g}' on a supersymmetric algebra. The precise relation between the classical Lie superalgebras and Weyl-Clifford algebras \mathfrak{WC} is established. According to the FFT for classical invariant theory in Chapter 4 when applied to the G -action on the associated graded algebra $\text{gr } \mathfrak{WC}$, the basic invariants generating $(\text{gr } \mathfrak{WC})^G$ turn out to form the associated graded space for a Lie superalgebra \mathfrak{g}' . From this it follows that the algebra of G -invariants \mathfrak{WC}^G is generated by \mathfrak{g}' . Multiplicity-free decompositions for various (G, \mathfrak{g}') -Howe dualities are obtained explicitly. Character formulas for the irreducible \mathfrak{g}' -modules appearing in (G, \mathfrak{g}') -Howe duality are then obtained via a comparison with Howe duality involving classical groups G and infinite-dimensional Lie algebras, which we develop in detail.

Finally in Chapter 6, we develop a super duality approach to obtain a complete and conceptual solution of the irreducible character problem in certain parabolic Bernstein-Gelfand-Gelfand categories for general linear and ortho-symplectic Lie superalgebras. This chapter is technically more sophisticated than the earlier chapters. Super duality is an equivalence of categories between parabolic categories for Lie superalgebras and their Lie algebra counterparts at an infinite-rank limit, and it matches the corresponding parabolic Verma modules, irreducible modules, Kostant u -homology groups, and Kazhdan-Lusztig-Vogan polynomials. Truncation functors are introduced to relate the BGG categories for infinite-rank and finite-rank Lie superalgebras. In this way, we obtain a solution à la Kazhdan-Lusztig of the irreducible character problem in the corresponding parabolic BGG categories for finite-dimensional basic Lie superalgebras.

There is an appendix in the book. In Appendix A, we have included a fairly self-contained treatment of some elementary aspects of symmetric function theory, including Schur functions, supersymmetric functions and Schur Q -functions. The celebrated boson-fermion correspondence serves as a prominent example relating superalgebras to mathematical physics and algebraic combinatorics. The Fock space therein is used in setting up the Howe duality for infinite-dimensional Lie algebras in Chapter 5.

For a one-semester introductory course on Lie superalgebras, we recommend two plausible ways of using this book. A first approach uses Chapters 1, 2, 3, with possible supplements from Chapter 5 and Appendix A. A second approach uses Chapters 1, 3, 5 with possible supplements from Chapter 4 and Appendix A. It is also possible to use this book for a course on the interaction between representations of Lie superalgebras and algebraic combinatorics. The more advanced Chapter 6 can be used in a research seminar.

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