Chapter 1

Lie superalgebra ABC

We start by introducing the basic notions and definitions in the theory of Lie superalgebras, such as basic and queer Lie superalgebras, Cartan and Borel subalgebras, root systems, positive and fundamental systems. We formulate the main structure results for the basic Lie superalgebras and the queer Lie superalgebras. We describe in detail the structures of Lie superalgebras of type $\mathfrak{gl}$, $\mathfrak{osp}$ and $\mathfrak{q}$. A distinguishing feature for Lie superalgebras is that Borel subalgebras, positive systems, or fundamental systems of a simple finite-dimensional Lie superalgebra may not be conjugate under the action of the corresponding Weyl group; rather, they are shown to be related to each other by real and odd reflections. A highest weight theory is developed for Lie superalgebras. We describe how fundamental systems are related and how highest weights are transformed by an odd reflection.

1.1. Lie superalgebras: Definitions and examples

Throughout this book we will work over the field $\mathbb{C}$ of complex numbers. Let

$$\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$$

denote the group of two elements and let $\mathfrak{S}_n$ denote the symmetric group in n letters.

In this section, we introduce many examples of Lie superalgebras. The examples, most relevant to this book, of the general linear and ortho-symplectic Lie superalgebras are introduced first. Other series of finite-dimensional simple Lie superalgebras of classical type, namely, the queer and the periplectic Lie superalgebras, along with the three exceptional ones, are then described. The finite-dimensional Cartan type Lie superalgebras are then realized explicitly as subalgebras of the Lie superalgebra of polynomial vector fields on a purely odd dimensional superspace. The section ends with Kac's classification theorem of the finite-dimensional simple Lie superalgebras over $\mathbb{C}$, which we state without proof.

1.1.1. Basic definitions. A **vector superspace** V is a vector space endowed with a $\mathbb{Z}_2$-gradation: $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The **dimension** of the vector superspace V is the tuple $\dim V = (\dim V_{\bar{0}} | \dim V_{\bar{1}})$ or sometimes $\dim V = \dim V_{\bar{0}} + \dim V_{\bar{1}}$ (which should be clear from the context). The **superdimension** of V is defined to be $\mathrm{sdim} V := \dim V_{\bar{0}} - \dim V_{\bar{1}}$. We denote the superspace with even subspace $\mathbb{C}^m$ and odd subspace $\mathbb{C}^n$ by $\mathbb{C}^{m|n}$. It has dimension $(m|n)$. The **parity** of a homogeneous element $a \in V_i$ is denoted by $|a| = i$, $i \in \mathbb{Z}_2$. An element in $V_{\bar{0}}$ is called **even**, while an element in $V_{\bar{1}}$ is called **odd**. A **subspace** of a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a vector superspace $W = W_{\bar{0}} \oplus W_{\bar{1}} \subseteq V$ with compatible $\mathbb{Z}_2$-gradation, i.e., $W_i \subseteq V_i$, for $i \in \mathbb{Z}_2$.

Let V be a superspace. Throughout the book, when we write $|v|$ for an element $v \in V$, we will always implicitly assume that v is a homogeneous element and automatically extend the relevant formulas by linearity (whenever applicable). Also, note that if V and W are superspaces, then the space of linear transformations from V to W is naturally a vector superspace. In particular, the space of endomorphisms of V, denoted by $\mathrm{End}(V)$, is a vector superspace. When $V = \mathbb{C}^{m|n}$ we write $I = I_{m|n} = I_V$ for the **identity matrix** on V.

There is a **parity reversing functor** Π on the category of vector superspaces. For a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$, we let

$$\Pi(V) = \Pi(V)_{\bar{0}} \oplus \Pi(V)_{\bar{1}}, \quad \Pi(V)_i = V_{i+\bar{1}}, \forall i \in \mathbb{Z}_2.$$

Clearly, $\Pi^2 = \mathrm{I}$.

Definition 1.1. A **superalgebra** A, sometimes also called a $\mathbb{Z}_2$-graded algebra, is a vector superspace $A = A_{\bar{0}} \oplus A_{\bar{1}}$ equipped with a bilinear multiplication satisfying $A_i A_j \subseteq A_{i+j}$, for $i, j \in \mathbb{Z}_2$. A **module** M over a superalgebra A is always understood in the $\mathbb{Z}_2$-graded sense, that is $M = M_{\bar{0}} \oplus M_{\bar{1}}$ such that $A_i M_j \subseteq M_{i+j}$, for $i, j \in \mathbb{Z}_2$. **Subalgebras** and **ideals** of superalgebras are also understood in the $\mathbb{Z}_2$-graded sense. A superalgebra that has no nontrivial ideal is called **simple**. A **homomorphism** between A-modules M and N is a linear map $f : M \to N$ satisfying that $f(am) = af(m)$, for all $a \in A$, $m \in M$. A homomorphism $f : M \to N$ is **of degree** $|\mathbf{f}| \in \mathbb{Z}_2$ if $f(M_i) \subseteq M_{i+|f|}$ for $i \in \mathbb{Z}_2$.

A homomorphism between modules M and N of a superalgebra A is sometimes understood in the literature as a linear map $f : M \to N$ of parity $|f| \in \mathbb{Z}_2$ which satisfies $(\star)$ $f(am) = (-1)^{|a|\cdot|f|} af(m)$, for homogeneous $a \in A$, $m \in M$. Let us call such a map a $\star$-homomorphism. These two definitions can be converted to each other as follows. Given a homomorphism $f : M \to N$ of degree $|f|$ in the sense of Definition 1.1, we define $f^\dagger : M \to N$ by the formula

$$f^\dagger(x) := (-1)^{|f|\cdot|x|} f(x). \tag{1.1}$$

Then $f^\dagger$ is a $\star$-homomorphism. Conversely, (1.1) also converts a $\star$-homomorphism into a homomorphism as in Definition 1.1.

Now we come to the definition of the main object of our study.

Definition 1.2. A **Lie superalgebra** is a superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with bilinear multiplication $[\cdot,\cdot]$ satisfying the following two axioms: for homogeneous elements $a, b, c \in \mathfrak{g}$,

(1) Skew-supersymmetry: $[a,b] = -(-1)^{|a|\cdot|b|}[b,a]$.
(2) Super Jacobi identity: $[a,[b,c]] = [[a,b],c] + (-1)^{|a|\cdot|b|}[b,[a,c]]$.

A bilinear form $(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ on a Lie superalgebra $\mathfrak{g}$ is called **invariant** if $([a,b],c) = (a,[b,c])$, for all $a,b,c \in \mathfrak{g}$.

For a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, the even part $\mathfrak{g}_{\bar{0}}$ is a Lie algebra. Hence, if $\mathfrak{g}_{\bar{1}} = 0$, then $\mathfrak{g}$ is just a usual Lie algebra. A Lie superalgebra $\mathfrak{g}$ with purely odd part, i.e., $\mathfrak{g}_{\bar{0}} = 0$, has to be **abelian**, i.e., $[\mathfrak{g},\mathfrak{g}] = 0$.

Definition 1.3. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie superalgebras. A **homomorphism of Lie superalgebras** is an even linear map $f : \mathfrak{g} \to \mathfrak{g}'$ satisfying

$$f([a,b]) = [f(a), f(b)], \quad a,b \in \mathfrak{g}.$$

Example 1.4. (1) Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be an associative superalgebra. We can make A into a Lie superalgebra by letting

$$[a,b] := ab - (-1)^{|a|\cdot|b|} ba,$$

for homogeneous $a, b \in A$ and extending $[\cdot,\cdot]$ by bilinearity.

(2) Let $\mathfrak{g}$ be a Lie superalgebra. Then $\mathrm{End}(\mathfrak{g})$ is an associative superalgebra, and hence it carries a structure of a Lie superalgebra by (1). We define the **adjoint map** $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by

$$\mathrm{ad}\,(a)(b) := [a,b], \quad a,b \in \mathfrak{g}.$$

Then ad is a homomorphism of Lie superalgebras due to the super Jacobi identity. The resulting action of $\mathfrak{g}$ on itself is called the **adjoint action**.

(3) Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a superalgebra. An endomorphism $D \in \mathrm{End}(A)_s$, for $s \in \mathbb{Z}_2$, is called a **derivation** of degree s if it satisfies that

$$D(ab) = D(a)b + (-1)^{s|a|} aD(b), \quad a,b \in A.$$

Denote by $\mathrm{Der}(A)_s$ the space of derivations on A of degree s. One verifies that the superspace of derivations of A, $\mathrm{Der}(A) = \mathrm{Der}(A)_{\bar{0}} \oplus \mathrm{Der}(A)_{\bar{1}}$, is a subalgebra of the Lie superalgebra $(\mathrm{End}(A), [\cdot,\cdot])$.

In the case when $\mathfrak{g}$ is a Lie superalgebra we have $\mathrm{ad}g \in \mathrm{Der}(\mathfrak{g})$, for all $g \in \mathfrak{g}$, by the super Jacobi identity. Indeed, such derivations are called **inner derivations**. The inner derivations form an ideal in $\mathrm{Der}(\mathfrak{g})$.

(4) Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a superspace such that $\mathfrak{g}_{\bar{0}} = \mathbb{C}z$ is one-dimensional. Suppose that we have a symmetric bilinear form $B(\cdot,\cdot)$ on $\mathfrak{g}_{\bar{1}}$. We can make $\mathfrak{g}$ into a Lie superalgebra by letting z commute with $\mathfrak{g}$ and declaring

$$[v,w] := B(v,w)z, \quad v,w \in \mathfrak{g}_{\bar{1}}.$$

The special cases when $B(\cdot,\cdot)$ is zero and when $B(\cdot,\cdot)$ is non-degenerate, respectively, are of particular interest. Indeed, their corresponding universal enveloping superalgebras (see Section 1.5.1) are isomorphic to the exterior and Clifford superalgebras of Section 1.1.6 and Definition 3.33, respectively.

For a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, the restriction of the adjoint homomorphism $\mathrm{ad}|_{\mathfrak{g}_{\bar{0}}} : \mathfrak{g}_{\bar{0}} \to \mathrm{End}(\mathfrak{g}_{\bar{1}})$ is a homomorphism of Lie algebras. That is, $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module under the adjoint action.

Remark 1.5. To a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ we associate the following data:

(1) A Lie algebra $\mathfrak{g}_{\bar{0}}$.

(2) A $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ induced by the adjoint action.

(3) A $\mathfrak{g}_{\bar{0}}$-homomorphism $S^2(\mathfrak{g}_{\bar{1}}) \to \mathfrak{g}_{\bar{0}}$ induced by the Lie bracket.

(4) The condition coming from Definition 1.2(2) with $a,b,c \in \mathfrak{g}_{\bar{1}}$.

Conversely, the above data determine a Lie superalgebra structure on $\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$.

1.1.2. The general and special linear Lie superalgebras. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace so that $\mathrm{End}(V)$ is an associative superalgebra. As in Example 1.4(1), $\mathrm{End}(V)$, equipped with the supercommutator, forms a Lie superalgebra called the **general linear Lie superalgebra** and denoted by $\mathfrak{gl}(V)$. When $V = \mathbb{C}^{m|n}$ we also write $\mathfrak{gl}(m|n)$ for $\mathfrak{gl}(V)$.

Choose ordered bases for $V_{\bar{0}}$ and $V_{\bar{1}}$ that combine to a homogeneous ordered basis for V. We will make it a convention to parameterize such a basis by the set

$$I(m|n) = \{\overline{1},\ldots,\overline{m}; 1,\ldots,n\} \tag{1.2}$$

with total order

$$\overline{1} < \ldots < \overline{m} < 0 < 1 < \ldots < n. \tag{1.3}$$

Here 0 is inserted for notational convenience later on. The elementary matrices are accordingly denoted by E_{ij}, with $i,j \in I(m|n)$. With respect to such an ordered basis, $\mathrm{End}(V)$ and $\mathfrak{gl}(V)$ can be realized as $(m+n)\times(m+n)$ complex matrices of the block form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1.4}$$

where a, b, c, and d are respectively $m\times m$, $m\times n$, $n\times m$, and $n\times n$ matrices. The even subalgebra $\mathfrak{gl}(V)_{\bar 0}$ consists of matrices of the form (1.4) with $b=c=0$, while the odd subspace $\mathfrak{gl}(V)_{\bar 1}$ consists of those with $a=d=0$. In particular, $\mathfrak{gl}(V)_{\bar 0}\cong \mathfrak{gl}(m)\oplus\mathfrak{gl}(n)$, and as a $\mathfrak{gl}(V)_{\bar 0}$-module, $\mathfrak{gl}(V)_{\bar 1}$ is self-dual and is isomorphic to $(\mathbb{C}^m\otimes\mathbb{C}^{n*})\oplus(\mathbb{C}^{m*}\otimes\mathbb{C}^n)$. Here and below, $\mathbb{C}^{n*}$ denotes the dual space of $\mathbb{C}^n$.

Remark 1.6. Let Π be the parity reversing functor defined in Section 1.1.1. We have an isomorphism of Lie superalgebras from $\mathfrak{gl}(V)$ to $\mathfrak{gl}(\Pi V)$ by sending T to $\Pi T\Pi^{-1}$. When $\dim V=(m|n)$, we obtain an isomorphism of Lie superalgebras $\mathfrak{gl}(m|n)\cong\mathfrak{gl}(n|m)$.

For each element $g\in\mathfrak{gl}(m|n)$ of the form (1.4) we define the **supertrace** as

$$\mathrm{str}(g) := \mathrm{tr}(a)-\mathrm{tr}(d),$$

where $\mathrm{tr}(x)$ denotes the trace of the square matrix x. One checks that

$$\mathrm{str}([g,g'])=0, \qquad \text{for } g,g'\in\mathfrak{gl}(m|n).$$

Thus, the subspace

$$\mathfrak{sl}(m|n) := \{g\in\mathfrak{gl}(m|n) \mid \mathrm{str}(g)=0\}$$

is a subalgebra of $\mathfrak{gl}(m|n)$, and it is called the **special linear Lie superalgebra**. One verifies directly that $[\mathfrak{gl}(m|n),\mathfrak{gl}(m|n)]=\mathfrak{sl}(m|n)$. Furthermore, $\mathfrak{sl}(m|n)\cong\mathfrak{sl}(n|m)$, and when $m\neq n$ and $m+n\geq 2$, $\mathfrak{sl}(m|n)$ is simple. When $m=n$, $\mathfrak{sl}(m|m)$ contains a nontrivial center generated by the identity matrix $I_{m|m}$. For $m\geq 2$, $\mathfrak{sl}(m|m)/\mathbb{C}I_{m|m}$ is simple.

Example 1.7. Let $\mathfrak{g}=\mathfrak{gl}(1|1)$ and consider the following basis for $\mathfrak{g}$:

$$e=\begin{pmatrix}0&1\\0&0\end{pmatrix},\qquad f=\begin{pmatrix}0&0\\1&0\end{pmatrix},\qquad E_{\bar 1.\bar 1}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\qquad E_{11}=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

Set $h:=E_{\bar 1,\bar 1}+E_{11}=I_{1|1}$. Then h is central, $[e,f]=h$, and $\mathfrak{sl}(1|1)$ has a basis $\{e,h,f\}$.

Let $\mathbb{I}=\mathbb{I}_{\bar 0}\sqcup\mathbb{I}_{\bar 1}$ be a parametrization of a homogeneous basis of the superspace $V=V_{\bar 0}\oplus V_{\bar 1}$, where $\mathbb{I}_{\bar 0}$ and $\mathbb{I}_{\bar 1}$ parameterize the corresponding bases of $V_{\bar 0}$ and $V_{\bar 1}$, respectively. For an element i in $\mathbb{I}$ we define

$$|i| := \begin{cases} 0, & \text{if } i\in\mathbb{I}_{\bar 0},\\ 1, & \text{if } i\in\mathbb{I}_{\bar 1}.\end{cases}$$

For example, for the parametrization $I(m|n)$ with ordering (1.3), we have $|i|=0$ for $i<0$, and $|i|=1$ for $i>0$. Choosing a total ordering of the homogeneous basis we may identify $\mathfrak{gl}(V)$ with the space of $|\mathbb{I}| \times |\mathbb{I}|$ matrices. For such a matrix $A = \sum_{i,j\in\mathbb{I}} a_{ij}E_{ij}$, $a_{ij} \in \mathbb{C}$, we define the **supertranspose** of A to be

$$A^{\mathrm{st}} := \sum_{i,j\in\mathbb{I}} (-1)^{|j|(|i|+|j|)} a_{ij}E_{ji}. \tag{1.5}$$

We define the **Chevalley automorphism** $\tau : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ by the formula

$$\tau(A) := -A^{\mathrm{st}}. \tag{1.6}$$

It is straightforward to check that τ is an automorphism of Lie superalgebras. We note that τ restricts to an automorphism of $\mathfrak{sl}(V)$. Also, for m,n both nonzero, τ has order 4 and hence is, in general, not an involution.

1.1.3. The ortho-symplectic Lie superalgebras.

Definition 1.8. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace. A bilinear form

$$B(\cdot,\cdot) : V \times V \longrightarrow V$$

is called **even** (respectively, **odd**), if $B(V_i,V_j) = 0$ unless $i + j = \bar{0}$ (respectively, $i + j = \bar{1}$). An even bilinear form B is said to be **supersymmetric** if $B|_{V_{\bar{0}}\times V_{\bar{0}}}$ is symmetric and $B|_{V_{\bar{1}}\times V_{\bar{1}}}$ is skew-symmetric, and it is called **skew-supersymmetric** if $B|_{V_{\bar{0}}\times V_{\bar{0}}}$ is skew-symmetric and $B|_{V_{\bar{1}}\times V_{\bar{1}}}$ is symmetric.

Let B be a non-degenerate even supersymmetric bilinear form on a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$. It follows that $\dim V_{\bar{1}}$ is necessarily even. For $s \in \mathbb{Z}_2$, let

$$\begin{aligned} \mathfrak{osp}(V)_s &:= \{g \in \mathfrak{gl}(V)_s \mid B(g(x),y) = -(-1)^{s\cdot|x|}B(x,g(y)), \forall x,y \in V\}, \\ \mathfrak{osp}(V) &:= \mathfrak{osp}(V)_{\bar{0}} \oplus \mathfrak{osp}(V)_{\bar{1}}. \end{aligned}$$

One checks that $\mathfrak{osp}(V)$ is a Lie superalgebra, called the **ortho-symplectic Lie superalgebra**; that is, $\mathfrak{osp}(V)$ is the subalgebra of $\mathfrak{gl}(V)$ that preserves a non-degenerate supersymmetric bilinear form. Its even subalgebra is isomorphic to $\mathfrak{so}(V_{\bar{0}}) \oplus \mathfrak{sp}(V_{\bar{1}})$, a direct sum of the orthogonal Lie algebra on $V_{\bar{0}}$ and the symplectic Lie algebra on $V_{\bar{1}}$. When $V = \mathbb{C}^{\ell|2m}$, we write $\mathfrak{osp}(\ell|2m)$ for $\mathfrak{osp}(V)$. Note that when ℓ (respectively, m) is zero, the ortho-symplectic Lie superalgebra reduces to the classical Lie algebra $\mathfrak{sp}(2m)$ (respectively, $\mathfrak{so}(\ell)$).

Similarly, we define the Lie superalgebra $\mathfrak{spo}(V)$ as the subalgebra of $\mathfrak{gl}(V)$ that preserves a non-degenerate skew-supersymmetric bilinear form on V (here $\dim V_{\bar{0}}$ has to be even). When $V = \mathbb{C}^{2m|\ell}$, we write $\mathfrak{spo}(2m|\ell)$ for $\mathfrak{spo}(V)$.

Remark 1.9. A non-degenerate supersymmetric bilinear form B on V induces a non-degenerate skew-supersymmetric bilinear form B^{Π} on $\Pi(V)$, defined by

$$B^{\Pi}\big(\Pi(v),\Pi(v')\big) := (-1)^{|v|}B(v,v'), \quad \text{for } v,v' \in V.$$

The restriction of the isomorphism $\mathfrak{gl}(V) \cong \mathfrak{gl}(\Pi V)$ in Remark 1.6 gives rise to a Lie superalgebra isomorphism between $\mathfrak{osp}(V)$ (with respect to B) and $\mathfrak{spo}(\Pi V)$ (with respect to B^{Π}); see Exercise 1.3. It follows that $\mathfrak{osp}(\ell|2m) \cong \mathfrak{spo}(2m|\ell)$.

We now give an explicit matrix realization of the ortho-symplectic Lie superalgebra. To this end, we first observe that the supertranspose (1.5) of a matrix in the block form (1.4) is equal to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\mathrm{st}} = \begin{pmatrix} a^t & c^t \\ -b^t & d^t \end{pmatrix},$$

where x^t denotes the usual transpose of a matrix x.

Define the $(2m+2n+1)\times(2m+2n+1)$ matrix in the $(m|m|n|n|1)$-block form

$$\mathfrak{J}_{2m|2n+1} := \begin{pmatrix} 0 & I_m & 0 & 0 & 0 \\ -I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.7}$$

Let $\mathfrak{J}_{2m|2n}$ denote the $(2m+2n)\times(2m+2n)$ matrix obtained from $\mathfrak{J}_{2m|2n+1}$ by deleting the last row and column. For $\ell = 2n$ or $2n+1$, by definition $\mathfrak{spo}(2m|\ell)$ is the subalgebra of $\mathfrak{gl}(2m|\ell)$ that preserves the bilinear form on $\mathbb{C}^{2m|\ell}$ with matrix $\mathfrak{J}_{2m|\ell}$ relative to the standard basis of $\mathbb{C}^{2m|\ell}$, and hence

$$\mathfrak{spo}(2m|\ell) = \{g \in \mathfrak{gl}(2m|\ell) \mid g^{st}\mathfrak{J}_{2m|\ell} + \mathfrak{J}_{2m|\ell}\, g = 0\}.$$

By a direct computation, $\mathfrak{spo}(2m|2n+1)$ consists of the $(2m+2n+1)\times(2m+2n+1)$ matrices of the following $(m|m|n|n|1)$-block form

$$\begin{pmatrix} d & e & y_1^t & x_1^t & z_1^t \\ f & -d^t & -y^t & -x^t & -z^t \\ x & x_1 & a & b & -v^t \\ y & y_1 & c & -a^t & -u^t \\ z & z_1 & u & v & 0 \end{pmatrix}, \quad b, c \text{ skew-symmetric, } e, f \text{ symmetric.} \tag{1.8}$$

Note that $\mathfrak{spo}(2m|2n+1)_{\bar{1}} \cong \mathbb{C}^{2m} \otimes \mathbb{C}^{2n+1}$ (which is self-dual) as a module over $\mathfrak{spo}(2m|2n+1)_{\bar{0}} \cong \mathfrak{sp}(2m) \oplus \mathfrak{so}(2n+1)$.

The Lie superalgebra $\mathfrak{spo}(2m|2n)$ consists of matrices (1.8) with the last row and column removed. Note that $\mathfrak{spo}(2m|2n)_{\bar{1}} \cong \mathbb{C}^{2m} \otimes \mathbb{C}^{2n}$ (which is self-dual) as a module over $\mathfrak{spo}(2m|2n)_{\bar{0}} \cong \mathfrak{sp}(2m) \oplus \mathfrak{so}(2n)$.

Here and below, the rows and columns of the matrices $\mathfrak{J}_{2m|\ell}$ and (1.8) (or its modification) are indexed by the finite set $I(2m|\ell)$.

Proposition 1.10. *The automorphism τ in* (1.6) *restricts to an automorphism of* $\mathfrak{spo}(2m|\ell)$.

Proof. Take an element $g \in \mathfrak{spo}(2m|\ell)$. Thus, $\mathfrak{J}_{2m|\ell} g + g^{\mathrm{st}}\mathfrak{J}_{2m|\ell} = 0$, and hence

$$g^{\mathrm{st}}\mathfrak{J}_{2m|\ell}^{\mathrm{st}} + \mathfrak{J}_{2m|\ell}^{\mathrm{st}}(g^{\mathrm{st}})^{\mathrm{st}} = 0. \tag{1.9}$$

Observe that $\mathfrak{J}_{2m|\ell}^{\mathrm{st}} = (\mathfrak{J}_{2m|\ell})^{-1}$. So if we multiply (1.9) on the left and on the right by $\mathfrak{J}_{2m|\ell}$, we obtain the identity

$$\mathfrak{J}_{2m|\ell} g^{\mathrm{st}} + (g^{\mathrm{st}})^{\mathrm{st}} \mathfrak{J}_{2m|\ell} = 0.$$

This implies that $\mathfrak{J}_{2m|\ell}\tau(g) + \tau(g)^{\mathrm{st}}\mathfrak{J}_{2m|\ell} = 0$, and so $\tau(g) \in \mathfrak{spo}(2m|\ell)$. $\square$

1.1.4. The queer Lie superalgebras. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace with $\dim V_{\bar{0}} = \dim V_{\bar{1}}$. Choose $P \in \mathrm{End}(V)_{\bar{1}}$ such that $P^2 = I_{n|n}$. The subspace

$$\mathfrak{q}(V) = \{T \in \mathrm{End}(V) \mid [T,P] = 0\}$$

is a subalgebra of $\mathfrak{gl}(V)$ called the **queer Lie superalgebra**. Different choices of P give rise to isomorphic queer Lie superalgebras. If $V = \mathbb{C}^{n|n}$, then $\mathfrak{q}(V)$ is also denoted by $\mathfrak{q}(n)$.

To give an explicit matrix realization of $\mathfrak{q}(n)$, let us take P to be the $2n \times 2n$ matrix

$$P := \sqrt{-1}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{1.10}$$

Then, for $g \in \mathfrak{gl}(n|n)$ of the form (1.4), we have $g \in \mathfrak{q}(n)$ if and only if $gP - (-1)^{|g|}Pg = 0$, and in turn, if and only if

$$g = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \tag{1.11}$$

where a, b are arbitrary complex $n \times n$ matrices. Thus we have $\mathfrak{q}(n)_{\bar{0}} \cong \mathfrak{gl}(n)$ as Lie algebras, and $\mathfrak{q}(n)_{\bar{1}} \cong \mathfrak{gl}(n)$ as the adjoint $\mathfrak{q}(n)_{\bar{0}}$-module. A linear basis for $\mathfrak{q}(n)$ consists of the following elements:

$$\widetilde{E}_{ij} := E_{\bar{i}\bar{j}} + E_{ij}, \;\; \overline{E}_{ij} := E_{i\bar{j}} + E_{\bar{i}j}, \quad 1 \le i, j \le n. \tag{1.12}$$

The derived superalgebra $[\mathfrak{q}(n), \mathfrak{q}(n)]$ consists of matrices of the form (1.11), with $a \in \mathfrak{gl}(n)$ and $b \in \mathfrak{sl}(n)$, and so it contains a one-dimensional center generated by the identity matrix $I_{n|n}$. The quotient superalgebra $[\mathfrak{q}(n), \mathfrak{q}(n)]/\mathbb{C}I_{n|n}$ has even part isomorphic to $\mathfrak{sl}(n)$ and odd part isomorphic to the adjoint module, and one can show that it is simple for $n \ge 3$. For $n = 2$, the odd part of the quotient Lie superalgebra is an abelian ideal, since the adjoint module of $\mathfrak{sl}(2)$ does not appear in the symmetric square of the adjoint module.

Remark 1.11. Consider the subspace $\widetilde{\mathfrak{q}}(n)$ of $\mathfrak{gl}(n|n)$ consisting of elements that commute with P. That is,

$$\widetilde{\mathfrak{q}}(n) := \{g \in \mathfrak{gl}(n|n) | gP - Pg = 0\}.$$

In matrix form, $\widetilde{\mathfrak{q}}(n)$ consists of the following $n|n$-block matrices:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \tag{1.13}$$

where a and b are arbitrary $n \times n$ matrices. One checks that $\widetilde{\mathfrak{q}}(n)$ is closed under the Lie bracket and hence is a subalgebra of $\mathfrak{gl}(n|n)$. Indeed $\widetilde{\mathfrak{q}}(n)$ is isomorphic to $\mathfrak{q}(n)$, since the map τ in (1.6) sends $\mathfrak{q}(n)$ to $\widetilde{\mathfrak{q}}(n)$, and vice versa. Thus, (1.13) gives another realization of the queer Lie superalgebra.

1.1.5. The periplectic and exceptional Lie superalgebras. Let us describe more examples of Lie superalgebras.

The periplectic Lie superalgebras. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace with $\dim V_{\bar{0}} = \dim V_{\bar{1}}$. Let $C(\cdot,\cdot)$ be a non-degenerate odd symmetric bilinear form on V. One checks that the subspace of $\mathfrak{gl}(V)$ preserving C is closed under the Lie bracket and hence is a Lie subalgebra of $\mathfrak{gl}(V)$. This superalgebra is called the **periplectic Lie superalgebra** and will be denoted by $\mathfrak{p}(V)$. Different choices of C give rise to isomorphic periplectic Lie superalgebras. In the case $V = \mathbb{C}^{n|n}$, $\mathfrak{p}(V)$ is also denoted by $\mathfrak{p}(n)$.

To write down an explicit matrix realization of $\mathfrak{p}(n)$ as a subalgebra of $\mathfrak{gl}(n|n)$, let us take the $2n \times 2n$ matrix

$$\mathfrak{P} := \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \tag{1.14}$$

which determines an odd symmetric bilinear form C on $\mathbb{C}^{n|n}$. Then, $g \in \mathfrak{p}(n)$ if and only if $g^{st}\mathfrak{P} + \mathfrak{P}g = 0$. It follows that

$$\mathfrak{p}(n) = \left\{ \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \text{ where } b \text{ is symmetric and } c \text{ is skew-symmetric} \right\}. \tag{1.15}$$

We have

$$\mathfrak{p}(n)_{\bar{0}} \cong \mathfrak{gl}(n), \quad \mathfrak{p}(n)_{\bar{1}} \cong S^2(\mathbb{C}^n) \oplus \wedge^2(\mathbb{C}^{n*}). \tag{1.16}$$

For $n \geq 3$, the derived superalgebra $[\mathfrak{p}(n), \mathfrak{p}(n)]$ is simple, and it consists of matrices of the form $\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}$ with $\mathrm{tr}(a) = 0$, $b^t = b$, and $c^t = -c$.

Remark 1.12. One checks that the Lie subalgebra $\widetilde{\mathfrak{p}}(n)$ of $\mathfrak{gl}(n|n)$ preserving the non-degenerate odd *skew-symmetric* bilinear form corresponding to P in (1.10) consists of matrices of the form $\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}$ with $b^t = -b$ and $c^t = c$. Similar to Remark 1.11, the map τ in (1.6) restricts to an isomorphism between $\mathfrak{p}(n)$ and $\widetilde{\mathfrak{p}}(n)$.

The exceptional Lie superalgebra $D(2|1,\alpha)$.

We take three copies of the Lie algebra $\mathfrak{sl}(2)$ denoted by $\mathfrak{g}_i$ $(i = 1,2,3)$, and we associate to each $\mathfrak{g}_i$ a copy of the standard $\mathfrak{sl}(2)$-module V_i.

Clearly, as $\mathfrak{g}_i$-modules, we have an isomorphism $S^2(V_i) \cong \mathfrak{g}_i$. By Schur's Lemma we may associate a nonzero scalar $\alpha_i \in \mathbb{C}$ to any such isomorphism. Now consider $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}} \cong \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$, and $\mathfrak{g}_{\bar{1}}$ is the irreducible $\mathfrak{g}_{\bar{0}}$-module

$V_1 \otimes V_2 \otimes V_3$. We can associate three nonzero complex numbers α_i, $i = 1,2,3$, to any surjective $\mathfrak{g}_{\bar{0}}$-homomorphism from $S^2(\mathfrak{g}_{\bar{1}})$ to $\mathfrak{g}_{\bar{0}}$. Thus, our vector superspace $\mathfrak{g}$ satisfies Conditions (1)–(3) of Remark 1.5. It is easy to see that Condition (4) of Remark 1.5 is equivalent to $\sum_{i=1}^{3} \alpha_i = 0$. Thus, we obtain a Lie superalgebra $\mathfrak{g}(\alpha_1,\alpha_2,\alpha_3)$ depending on three nonzero parameters α_i, $i = 1,2,3$. For $\sigma \in \mathfrak{S}_3$, we have $\mathfrak{g}(\alpha_1,\alpha_2,\alpha_3) \cong \mathfrak{g}(\alpha_{\sigma(1)},\alpha_{\sigma(2)},\alpha_{\sigma(3)})$. Also, $\mathfrak{g}(\alpha_1,\alpha_2,\alpha_3) \cong \mathfrak{g}(\lambda\alpha_1,\lambda\alpha_2,\lambda\alpha_3)$, for any nonzero $\lambda \in \mathbb{C}$. Thus, we have a one-parameter family of Lie superalgebras $D(2|1,\alpha) := \mathfrak{g}(\alpha,1,-1-\alpha)$ that are simple for $\alpha \neq 0,-1$.

We have $\mathfrak{g}(\alpha,1,-1-\alpha) \cong \mathfrak{g}(1,\alpha,-1-\alpha) \cong \mathfrak{g}(1,\alpha^{-1},-\alpha^{-1}-1)$, and also $\mathfrak{g}(\alpha,1,-1-\alpha) \cong \mathfrak{g}(-1-\alpha,1,\alpha)$, which imply

$$D(2|1,\alpha) \cong D(2|1,\alpha^{-1}) \cong D(2|1,-1-\alpha).$$

The maps $\alpha \mapsto \alpha^{-1}$ and $\alpha \mapsto (-1-\alpha)$ generate an action of $\mathfrak{S}_3$ on $\mathbb{C} \setminus \{0,-1\}$. We have $D(2|1,\alpha) \cong D(2|1,\beta)$ if and only if $\beta \in \mathfrak{S}_3 \cdot \alpha$. This gives additional isomorphisms

$$D(2|1,\alpha) \cong D(2|1,-(1+\alpha)^{-1}\alpha) \cong D(2|1,-1-\alpha^{-1}) \cong D(2|1,-(1+\alpha)^{-1}).$$

Thus, any orbit of $(\mathbb{C} \setminus \{0,-1\})/\mathfrak{S}_3$ consists of six points, except for the orbit corresponding to the three points $\alpha = 1,-2,-\frac{1}{2}$, and the orbit corresponding to the two points $\alpha = -\frac{1}{2} \pm \sqrt{-\frac{3}{4}}$. Finally, note that $D(2|1,1) \cong \mathfrak{osp}(4|2)$.

The exceptional Lie superalgebra $F(3|1)$. There is a simple Lie superalgebra $F(3|1)$ with $F(3|1)_{\bar{0}} \cong \mathfrak{sl}(2) \oplus \mathfrak{so}(7)$. The odd part, as an $F(3|1)_{\bar{0}}$-module, is isomorphic to the tensor product of the standard $\mathfrak{sl}(2)$-module and the simple $\mathfrak{so}(7)$-spin module. Hence, $\dim F(3|1) = (24|16)$. In the literature, $F(3|1)$ is often denoted by $F(4)$, which could be confused with the simple Lie algebra of type F_4.

The exceptional Lie superalgebra $G(3)$. There is a simple Lie superalgebra $G(3)$ with $G(3)_{\bar{0}} \cong \mathfrak{sl}(2) \oplus G_2$. The odd part as a $G(3)_{\bar{0}}$-module is isomorphic to the tensor product of the standard $\mathfrak{sl}(2)$-module and the fundamental 7-dimensional G_2-module. Hence, $\dim G(3) = (17|14)$.

1.1.6. The Cartan series. In this subsection, we describe the Cartan series of finite-dimensional simple Lie superalgebras without proof. This part is included for the sake of presenting a complete classification of finite-dimensional simple Lie superalgebras in Theorem 1.13 and will not be used elsewhere in the book.

Lie superalgebra $W(n)$. Let $\wedge(n)$ be the **exterior algebra** in n indeterminates $\xi_1,\xi_2,\ldots,\xi_n$. We have $\xi_i\xi_j = -\xi_j\xi_i$, for all i,j, and in particular, $\xi_i^2 = 0$, for all i. Setting $|\xi_i| = \bar{1}$, for all i, the algebra $\wedge(n)$ becomes a superalgebra which we also refer to as an **exterior superalgebra**.

By general construction in Example 1.4, we have a finite-dimensional Lie superalgebra of derivations on $\wedge(n)$, which will be denoted by $W(n)$.

For $i = 1, \ldots, n$, the derivation $\frac{\partial}{\partial \xi_i} : \wedge(n) \to \wedge(n)$ of degree $\bar{1}$ is uniquely determined by

$$\frac{\partial}{\partial \xi_i}(\xi_j) = \delta_{ij}, \quad j = 1, \ldots, n.$$

Given an element $f = (f_1, f_2, \ldots, f_n) \in \wedge(n)^n$, with $|f_i| = |f_j|$, for all i, j, the linear map $D_f : \wedge(n) \to \wedge(n)$ of the form

$$D_f = \sum_{i=1}^{n} f_i \frac{\partial}{\partial \xi_i}$$

is a derivation of $\wedge(n)$. Furthermore, all homogeneous derivations of $\wedge(n)$ are of this form, since a derivation is determined by its values at ξ_i for all i. Therefore, sending $f \mapsto D_f$ defines a linear isomorphism from $\wedge(n)^n$ to $W(n)$, and so $W(n)$ has dimension $2^n n$.

Setting $\deg \xi_i = 1$ and $\deg \frac{\partial}{\partial \xi_i} = -1$, for all i, gives rise to a $\mathbb{Z}$-gradation on $W(n)$, called the **principal gradation**. We have

$$W(n) = \bigoplus_{j=-1}^{n-1} W(n)_j.$$

The $\mathbb{Z}$-gradation is compatible with the super structure on $W(n)$; that is, $W(n)_s = \oplus_{j \equiv s \mod 2} W(n)_j$, for $s \in \mathbb{Z}_2$. The 0th degree component $W(n)_0$ is a Lie algebra isomorphic to $\mathfrak{gl}(n)$, and each $W(n)_j$ is a $\mathfrak{gl}(n)$-module isomorphic to $\wedge^{j+1}(\mathbb{C}^n) \otimes \mathbb{C}^{n*}$. In particular, when $n = 2$, we have $W(2)_0 \cong \mathfrak{gl}(2)$, $W(2)_{-1} \cong \mathbb{C}^{2*}$ and $W(2)_1 \cong \mathbb{C}^2$ as $\mathfrak{gl}(2)$-modules. Indeed, we have isomorphisms of Lie superalgebras $W(2) \cong \mathfrak{osp}(2|2) \cong \mathfrak{sl}(2|1)$ (see Exercises 1.4 and 1.5).

The Lie superalgebra $W(n)$ is simple, for $n \geq 2$. Moreover, $W(n)$ contains the following three series of simple Lie superalgebras as subalgebras that we shall describe.

Lie superalgebra $S(n)$. The first series is the super-analogue of the Lie algebra of divergence-free vector fields given by

$$S(n) := \Big\{ \sum_{j=1}^{n} f_j \frac{\partial}{\partial \xi_j} \in W(n) \mid \sum_{j=1}^{n} \frac{\partial}{\partial \xi_j}(f_j) = 0 \Big\}.$$

The Lie superalgebra $S(n)$ is a $\mathbb{Z}$-graded subalgebra of $W(n)$ and we have $S(n) = \bigoplus_{j=-1}^{n-2} S(n)_j$. The Lie algebra $S(n)_0$ is isomorphic to $\mathfrak{sl}(n)$, and the jth degree component $S(n)_j$ is isomorphic to the top irreducible summand of the $\mathfrak{sl}(n)$-module $\wedge^{j+1}(\mathbb{C}^n) \otimes \mathbb{C}^{n*}$. The Lie superalgebra $S(n)$ is simple, for $n \geq 3$.

Lie superalgebra $\widetilde{S}(n)$. Let n be even so that $\omega = 1 + \xi_1 \xi_2 \cdots \xi_n$ is a $\mathbb{Z}_2$-homogeneous invertible element in $\wedge(n)$. Consider the subspace of $W(n)$ given by

$$\widetilde{S}(n) := \Big\{ \sum_{j=1}^{n} f_j \frac{\partial}{\partial \xi_j} \in W(n) \mid \sum_{j=1}^{n} \frac{\partial}{\partial \xi_j}(\omega f_j) = 0 \Big\}.$$

It can be shown that $\widetilde{S}(n)$ is a subalgebra of the Lie superalgebra $W(n)$. The Lie superalgebra $\widetilde{S}(n)$ is no longer $\mathbb{Z}$-graded, as the defining condition is not homogeneous. However, $\widetilde{S}(n)$ inherits a natural filtration from the filtration on $W(n)$ induced by the principal gradation. The associated graded Lie superalgebra of $\widetilde{S}(n)$ is isomorphic to $S(n)$. Explicitly, $\widetilde{S}(n)$ is the following direct sum of vector spaces inside $W(n)$:

$$\widetilde{S}(n) = \bigoplus_{j=-1}^{n-2} \widetilde{S}(n)_j, \tag{1.17}$$

where $\widetilde{S}(n)_{-1}$ is spanned by $\{(1-\xi_1\xi_2\cdots\xi_n)\frac{\partial}{\partial \xi_i} | i = 1,\ldots,n\}$, and $\widetilde{S}(n)_j = S(n)_j$, for $j = 0,\ldots,n-2$. For $n \geq 2$ and n even, $\widetilde{S}(n)$ is simple. Note that $\widetilde{S}(2) \cong \mathfrak{spo}(2|1)$.

Lie superalgebra $H(n)$. As in the classical setting, $W(n)$ contains a subalgebra $H(n)$ as defined below, which is a super-analogue of the Lie algebra of Hamiltonian vector fields. For $f, g \in \wedge(n)$, we define the Poisson bracket by

$$\{f, g\} := (-1)^{|f|} \sum_{j=1}^{n} \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial \xi_j}.$$

The Poisson bracket makes $\wedge(n)$ into a Lie superalgebra, which we will denote by $\widetilde{H}(n)$. Now putting $\deg f := k-2$, for $f \in \wedge(n)_k$, $\widetilde{H}(n)$ becomes a $\mathbb{Z}$-graded Lie superalgebra. The superalgebra $\widetilde{H}(n)$ is not simple, as it has center $\mathbb{C}1$. However, the derived superalgebra of $\widetilde{H}(n)/\mathbb{C}1$, which we denote by $H(n)$, is simple, for $n \geq 4$. Moreover, $H(n) = \bigoplus_{j=-1}^{n-3} H(n)_j$ is a graded Lie superalgebra. The 0th degree component is a Lie algebra isomorphic to $\mathfrak{so}(n)$. As an $\mathfrak{so}(n)$-module we have $H(n)_j \cong \wedge^{j+2}(\mathbb{C}^n)$, for $-1 \leq j \leq n-3$. Finally, $H(n)$ can be viewed as a subalgebra of $W(n)$, since the assignment $f \mapsto (-1)^{|f|} \sum_{j=1}^{n} \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial \xi_j}$, for $f \in \wedge(n)$, gives rise to an embedding of Lie superalgebras from $\widetilde{H}(n)/\mathbb{C}1$ into $W(n)$.

1.1.7. The classification theorem. The following theorem of Kac [**60**] gives a classification of finite-dimensional complex simple Lie superalgebras. Note the following isomorphisms of Lie superalgebras (see Exercises 1.5, 1.4, and 1.7):

$$\mathfrak{osp}(2|2) \cong \mathfrak{sl}(2|1) \cong W(2),\ \mathfrak{sl}(2|2)/\mathbb{C}I_{2|2} \cong H(4),\ [\mathfrak{p}(3), \mathfrak{p}(3)] \cong S(3).$$

Theorem 1.13. *The following is a complete list of pairwise non-isomorphic finite-dimensional simple Lie superalgebras over $\mathbb{C}$.*

(1) *A finite-dimensional simple Lie algebra in the Killing-Cartan list.*

(2) $\mathfrak{sl}(m|n)$, *for* $m > n \geq 1$ *(excluding* $(m,n) = (2,1)$*);* $\mathfrak{sl}(m|m)/\mathbb{C}I_{m|m}$, *for* $m \geq 3$*;* $\mathfrak{spo}(2m|n)$, *for* $m,n \geq 1$.

(3) $D(2|1,\alpha)$, *for* $\alpha \in \left(\mathbb{C} \setminus \{0, \pm 1, -2, -\frac{1}{2}\}\right)/\mathfrak{S}_3$*;* $F(3|1)$*; and* $G(3)$.

(4) $[\mathfrak{p}(n),\mathfrak{p}(n)]$ *and* $[\mathfrak{q}(n),\mathfrak{q}(n)]/\mathbb{C}I_{n|n}$, *for* $n \geq 3$.

(5) $W(n)$, *for* $n \geq 3$*;* $S(n)$, *for* $n \geq 4$*;* $\widetilde{S}(2n)$, *for* $n \geq 2$*; and* $H(n)$, *for* $n \geq 4$.

A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ in Theorem 1.13(1)-(4) has the property that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and the adjoint $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is semisimple. To distinguish between such a Lie superalgebra from one in the Cartan series of Theorem 1.13(5), we introduce the following terminology.

Definition 1.14. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ in Theorem 1.13(2)-(4) is called **classical**. A classical Lie superalgebra in Theorem 1.13(2)-(3) is called **basic**. The Lie superalgebra $\mathfrak{gl}(m|n)$ for $m,n \geq 1$ is also declared to be **basic**.

Remark 1.15. The basic Lie superalgebras admit non-degenerate even supersymmetric bilinear forms, and this property characterizes the simple basic Lie superalgebras among all simple Lie superalgebras in the list of Theorem 1.13.

Remark 1.16. Let us comment on the simplicity of the Lie superalgebras in Theorem 1.13. It is not difficult to check directly the simplicity of the $\mathfrak{sl}$ series, $\mathfrak{spo}(2m|2)$, and those in (4). A Lie superalgebra $\mathfrak{g}$ in the remaining cases in (2) and (3), with the exception of $\mathfrak{spo}(2m|2)$, satisfies the properties that the adjoint $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is irreducible and faithful, and $[\mathfrak{g}_{\bar{1}},\mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{0}}$. A Lie superalgebra $\mathfrak{g}$ with such properties can be easily shown to be simple (see Exercise 1.9).

For the $\mathbb{Z}$-graded Cartan type Lie superalgebras $\mathfrak{g} = \bigoplus_{j \geq -1} \mathfrak{g}_j$ in (5), we first note that they are all *transitive* (i.e., $[\mathfrak{g}_{-1},x] = 0$ implies that $x = 0$, for $x \in \mathfrak{g}_j$ and $j \geq 0$) and *irreducible* (i.e., the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible). Furthermore, we have $[\mathfrak{g}_{-1},\mathfrak{g}_1] = \mathfrak{g}_0$, $[\mathfrak{g}_0,\mathfrak{g}_1] = \mathfrak{g}_1$, and $\mathfrak{g}_j$ is *generated* by $\mathfrak{g}_1$ (i.e., $\mathfrak{g}_j = [\mathfrak{g}_{j-1},\mathfrak{g}_1]$), for $j \geq 2$. A Lie superalgebra $\mathfrak{g}$ with these properties is simple (see Exercise 1.10). Finally, the simplicity of $\widetilde{S}(n)$ can be verified with a bit of extra work using the explicit realization given in (1.17).

1.2. Structures of classical Lie superalgebras

In this section, Cartan subalgebras, root systems, Weyl groups, and invariant bilinear forms for basic Lie superalgebras are introduced and described in detail for type $\mathfrak{gl}$, $\mathfrak{osp}$ and $\mathfrak{q}$. A structure theorem is formulated for the basic and type $\mathfrak{q}$ Lie superalgebras.

1.2.1. A basic structure theorem. In this subsection, we shall assume that $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a basic Lie superalgebra (see Definition 1.14).

A **Cartan subalgebra** $\mathfrak{h}$ of $\mathfrak{g}$ is defined to be a Cartan subalgebra of the even subalgebra $\mathfrak{g}_{\bar{0}}$. Since every inner automorphism of $\mathfrak{g}_{\bar{0}}$ extends to one of Lie superalgebra $\mathfrak{g}$ and Cartan subalgebras of $\mathfrak{g}_{\bar{0}}$ are conjugate under inner automorphisms, we conclude that the Cartan subalgebras of $\mathfrak{g}$ are conjugate under inner automorphisms.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For $\alpha \in \mathfrak{h}^*$, let

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid [h,g] = \alpha(h)g, \forall h \in \mathfrak{h}\}.$$

The **root system** for $\mathfrak{g}$ is defined to be

$$\Phi = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0, \alpha \neq 0\}.$$

Define the sets of **even** and **odd roots**, respectively, to be

$$\Phi_{\bar{0}} := \{\alpha \in \Phi | \mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{0}} \neq 0\}, \quad \Phi_{\bar{1}} := \{\alpha \in \Phi | \mathfrak{g}_\alpha \cap \mathfrak{g}_{\bar{1}} \neq 0\}.$$

Definition 1.17. For a basic or queer Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, the **Weyl group** W of $\mathfrak{g}$ is defined to be the Weyl group of the reductive Lie algebra $\mathfrak{g}_{\bar{0}}$.

As we shall see, the Weyl groups play a less vital though still important role in determining central characters and the linkage principle for Lie superalgebras that is somewhat different from the theory of semisimple Lie algebras.

The following theorem shows that the structures of the basic Lie superalgebras are similar to those of semisimple Lie algebras.

Theorem 1.18. *Let $\mathfrak{g}$ be a basic Lie superalgebra with a Cartan subalgebra $\mathfrak{h}$.*

(1) *We have a root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$:*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \qquad \text{and } \mathfrak{g}_0 = \mathfrak{h}.$$

(2) $\dim \mathfrak{g}_\alpha = 1$, *for $\alpha \in \Phi$. (Now fix some nonzero $e_\alpha \in \mathfrak{g}_\alpha$.)*

(3) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, *for $\alpha, \beta, \alpha+\beta \in \Phi$.*

(4) Φ, $\Phi_{\bar{0}}$ *and* $\Phi_{\bar{1}}$ *are invariant under the action of the Weyl group W on $\mathfrak{h}^*$.*

(5) *There exists a non-degenerate even invariant supersymmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$.*

(6) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ *unless $\alpha = -\beta \in \Phi$.*

(7) *The restriction of the bilinear form $(\cdot,\cdot)$ on $\mathfrak{h} \times \mathfrak{h}$ is non-degenerate and W-invariant.*

(8) $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha$, *where h_α is the* ***coroot*** *determined by $(h_\alpha, h) = \alpha(h)$ for $h \in \mathfrak{h}$.*

(9) $\Phi = -\Phi$, $\Phi_{\bar{0}} = -\Phi_{\bar{0}}$, *and* $\Phi_{\bar{1}} = -\Phi_{\bar{1}}$.

(10) *Let $\alpha \in \Phi$. Then, $k\alpha \in \Phi$ for some integer $k \neq \pm 1$ if and only if α is an odd root such that $(\alpha,\alpha) \neq 0$; in this case, we must have $k = \pm 2$.*

Proof. For $\mathfrak{g} = \mathfrak{gl}(m|n)$ or $\mathfrak{osp}(2m|\ell)$, we will describe explicitly the root systems and trace forms below in this section, and so the theorem in these cases follows by inspection. The theorem for the other infinite series basic Lie superalgebras $\mathfrak{sl}(m|n)$ follows by some easy modification of the $\mathfrak{gl}(m|n)$ case.

Part (4) follows by the invariance, under the action of the Weyl group W, of the sets of weights for the adjoint $\mathfrak{g}_{\bar{0}}$-modules $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$, respectively.

Since the remaining three exceptional Lie superalgebras $D(2|1,\alpha)$, $G(3)$ and $F(3|1)$ will not be studied in detail in the book, we will be sketchy. Most parts of the theorem, with the exception of (5), again follow by inspection from the constructions of these superalgebras and standard arguments as for simple Lie algebras (with the help of (5)). As the exceptional Lie superalgebras are simple, any nonzero invariant supersymmetric bilinear form must be non-degenerate. For $G(3)$ and $F(3|1)$, the Killing form is such a nonzero even form. A direct and ad hoc construction of a nonzero invariant bilinear form on $D(2|1,\alpha)$ is possible. $\square$

It follows by Theorem 1.18(1) that $\mathfrak{h}$ is self-normalizing in $\mathfrak{g}$ (and $\mathfrak{h}$ is abelian), justifying the terminology of a Cartan subalgebra. Since $\mathfrak{h} \subseteq \mathfrak{g}_{\bar{0}}$ and $\dim \mathfrak{g}_\alpha = 1$ for each $\alpha \in \Phi$ by Theorem 1.18(2), there exists $i \in \mathbb{Z}_2$ such that $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_i$. Hence Φ is a disjoint union of $\Phi_{\bar{0}}$ and $\Phi_{\bar{1}}$, and we have

$$\Phi_i = \{\alpha \in \Phi \mid \mathfrak{g}_\alpha \subseteq \mathfrak{g}_i\}, \quad i \in \mathbb{Z}_2.$$

Remark 1.19. One uniform approach to establish Theorem 1.18 is as follows (see [**60**, Proposition 2.5.3]). One follows the standard construction of contragredient (i.e., Kac-Moody) (super)algebras (see [**18**, Chapter 14]) to show that any Lie superalgebra in Theorem 1.18 is a Kac-Moody superalgebra (or rather the quotient by its possibly nontrivial center) associated to some generalized Cartan matrix with $\mathbb{Z}_2$-grading. These (quotients of) Kac-Moody superalgebras carry non-degenerate even invariant supersymmetric bilinear forms by a standard Kac-Moody type argument (see [**18**, Chapter 16]).

A root $\alpha \in \Phi$ is called **isotropic** if $(\alpha, \alpha) = 0$. An isotropic root is necessarily an odd root. Denote the set of isotropic odd roots by

$$\bar{\Phi}_{\bar{1}} := \{\alpha \in \Phi_{\bar{1}} \mid (\alpha,\alpha) = 0\} = \{\alpha \in \Phi_{\bar{1}} \mid 2\alpha \notin \Phi\}. \tag{1.18}$$

The second equation above follows by Theorem 1.18(10). Moreover, we have

$$e_\alpha^2 = \frac{1}{2}[e_\alpha, e_\alpha] = 0, \qquad \text{for } \alpha \in \bar{\Phi}_{\bar{1}}.$$

We also introduce the following set of roots

$$\bar{\Phi}_{\bar{0}} = \{\alpha \in \Phi_{\bar{0}} \mid \alpha/2 \notin \Phi\}. \tag{1.19}$$

1.2.2. Invariant bilinear forms for $\mathfrak{gl}$ and $\mathfrak{osp}$. In contrast to the semisimple Lie algebras, the Killing form for a basic Lie superalgebra may be zero, and even when it is nonzero, it may not be positive definite on the real vector space spanned by Φ. In this subsection, we give a down-to-earth description of an invariant non-degenerate even supersymmetric bilinear form for Lie superalgebras of type $\mathfrak{gl}$ and $\mathfrak{osp}$.

The supertrace str on the general linear Lie superalgebra gives rise to a non-degenerate supersymmetric bilinear form

$$(\cdot,\cdot) : \mathfrak{gl}(m|n) \times \mathfrak{gl}(m|n) \to \mathbb{C}, \quad (a,b) = \mathrm{str}(ab),$$

where ab denotes the matrix multiplication. It is straightforward to check that this form is invariant. Restricting to the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices, we obtain a non-degenerate symmetric bilinear form on $\mathfrak{h}$:

$$(E_{ii}, E_{jj}) = \begin{cases} 1 & \text{if } \overline{1} \le i = j \le \overline{m}, \\ -1 & \text{if } 1 \le i = j \le n, \\ 0 & \text{if } i \neq j, \end{cases}$$

where $i, j \in I(m|n)$. We recall here that $I(m|n)$ is defined in (1.2). Denote by $\{\delta_i, \varepsilon_j\}_{i,j}$ the basis of $\mathfrak{h}^*$ dual to $\{E_{\bar{i}\bar{i}}, E_{jj}\}_{i,j}$, where $1 \le i \le m$ and $1 \le j \le n$. Using the bilinear form $(\cdot,\cdot)$ we can identify δ_i with $(E_{\bar{i},\bar{i}}, \cdot)$ and ε_j with $-(E_{jj}, \cdot)$. When it is convenient we also use the notation

$$\varepsilon_{\bar{i}} := \delta_i, \quad \text{for } 1 \le i \le m. \tag{1.20}$$

The form $(\cdot,\cdot)$ on $\mathfrak{h}$ induces a non-degenerate bilinear form on $\mathfrak{h}^*$, which will be denoted by $(\cdot,\cdot)$ as well. Then, for $i, j \in I(m|n)$, we have

$$(\varepsilon_i, \varepsilon_j) = \begin{cases} 1 & \text{if } \overline{1} \le i = j \le \overline{m}, \\ -1 & \text{if } 1 \le i = j \le n, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.21}$$

Such a bilinear form on $\mathfrak{gl}(2n|\ell)$ restricts to a non-degenerate invariant supersymmetric bilinear form on the subalgebra $\mathfrak{spo}(2n|\ell)$, which will also be denoted by $(\cdot,\cdot)$. The further restriction to a Cartan subalgebra of $\mathfrak{spo}(2n|\ell)$ remains non-degenerate. This allows us to identify a Cartan subalgebra $\mathfrak{h}$ with its dual $\mathfrak{h}^*$, and one also obtains a bilinear form on $\mathfrak{h}^*$.

1.2.3. Root system and Weyl group for $\mathfrak{gl}(m|n)$. Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\mathfrak{h}$ be the Cartan subalgebra of diagonal matrices. Its root system $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ is given by

$$\Phi_{\bar{0}} = \{\varepsilon_i - \varepsilon_j \mid i \neq j \in I(m|n),\ i, j > 0 \text{ or } i, j < 0\},$$
$$\Phi_{\bar{1}} = \{\pm(\varepsilon_i - \varepsilon_j) \mid i, j \in I(m|n),\ i < 0 < j\}.$$

Observe that E_{ij} is a root vector corresponding to the root $\varepsilon_i - \varepsilon_j$, for $i \neq j \in I(m|n)$.

The Weyl group of $\mathfrak{gl}(m|n)$, which is by definition the Weyl group of the even subalgebra $\mathfrak{g}_{\bar{0}} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, is isomorphic to $\mathfrak{S}_m \times \mathfrak{S}_n$, where we recall that $\mathfrak{S}_n$ denotes the symmetric group of n letters.

1.2.4. Root system and Weyl group for $\mathfrak{spo}(2m|2n+1)$. Now we describe the root system for the ortho-symplectic Lie superalgebra $\mathfrak{spo}(2m|2n+1)$, which is defined in matrix form (1.8) in Section 1.1.3. Recall that the rows and columns of the matrices are indexed by $I(2m|2n+1)$. The subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ of diagonal matrices has a basis given by

$$H_{\bar{i}} := E_{\bar{i},\bar{i}} - E_{\overline{m+i},\overline{m+i}}, \quad 1 \le i \le m,$$
$$H_j := E_{jj} - E_{n+j,n+j}, \quad 1 \le j \le n,$$

and it is the **standard Cartan subalgebra** for $\mathfrak{spo}(2m|2n+1)$. Let $\{\delta_i, \varepsilon_j \mid 1 \le i \le m, 1 \le j \le n\}$ be the corresponding dual basis in $\mathfrak{h}^*$. With respect to $\mathfrak{h}$, the root system $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ for $\mathfrak{spo}(2m|2n+1)$ is

$$\{\pm\delta_i \pm \delta_j, \pm 2\delta_p, \pm\varepsilon_k \pm \varepsilon_l, \pm\varepsilon_q\} \cup \{\pm\delta_p \pm \varepsilon_q, \pm\delta_p\},$$

where $1 \le i < j \le m$, $1 \le k < l \le n$, $1 \le p \le m$, $1 \le q \le n$.

A root vector is a nonzero vector in $\mathfrak{g}_\alpha$ for $\alpha \in \Phi$ and will be denoted by e_α. The root vectors for $\mathfrak{spo}(2m|2n+1)$ can be chosen explicitly as follows in (1.22)-(1.29) $(1 \le i \ne j \le m, 1 \le k \ne l \le n)$:

$$e_{\varepsilon_k} = E_{2n+1,k+n} - E_{k,2n+1}, \qquad e_{-\varepsilon_k} = E_{2n+1,k} - E_{k+n,2n+1}, \tag{1.22}$$
$$e_{2\delta_i} = E_{\bar{i},\overline{i+m}}, \qquad e_{-2\delta_i} = E_{\overline{i+m},\bar{i}}, \tag{1.23}$$
$$e_{\delta_i+\delta_j} = E_{\bar{i},\overline{j+m}} + E_{\bar{j},\overline{i+m}}, \qquad e_{-\delta_i-\delta_j} = E_{\overline{j+m},\bar{i}} + E_{\overline{i+m},\bar{j}}, \tag{1.24}$$
$$e_{\delta_i-\delta_j} = E_{\bar{i},\bar{j}} - E_{\overline{j+m},\overline{i+m}}, \qquad e_{\varepsilon_k-\varepsilon_l} = E_{kl} - E_{l+n,k+n}, \tag{1.25}$$
$$e_{\varepsilon_k+\varepsilon_l} = E_{k,l+n} - E_{l,k+n}, \qquad e_{-\varepsilon_k-\varepsilon_l} = E_{k+n,l} - E_{l+n,k}, \tag{1.26}$$

$$e_{\delta_i+\varepsilon_k} = E_{k,\overline{i+m}} + E_{\bar{i},k+n}, \qquad e_{-\delta_i-\varepsilon_k} = E_{k+n,\bar{i}} - E_{\overline{i+m},k}, \tag{1.27}$$
$$e_{\delta_i-\varepsilon_k} = E_{k+n,\overline{i+m}} + E_{\bar{i},k}, \qquad e_{-\delta_i+\varepsilon_k} = E_{k,\bar{i}} - E_{\overline{i+m},k+n}, \tag{1.28}$$
$$e_{\delta_i} = E_{2n+1,\overline{i+m}} + E_{\bar{i},2n+1}, \qquad e_{-\delta_i} = E_{2n+1,\bar{i}} - E_{\overline{i+m},2n+1}. \tag{1.29}$$

The Weyl group of $\mathfrak{spo}(2m|2n+1)$, which is by definition the Weyl group of $\mathfrak{g}_{\bar{0}} = \mathfrak{sp}(2m) \oplus \mathfrak{so}(2n+1)$, is isomorphic to $(\mathbb{Z}_2^m \rtimes \mathfrak{S}_m) \times (\mathbb{Z}_2^n \rtimes \mathfrak{S}_n)$.

1.2.5. Root system and Weyl group for $\mathfrak{spo}(2m|2n)$. Let $\mathfrak{g} = \mathfrak{spo}(2m|2n)$. The abelian Lie subalgebra $\mathfrak{h}$ spanned by $\{H_{\bar{i}}, H_j | 1 \le i \le m, 1 \le j \le n\}$ is a Cartan subalgebra for $\mathfrak{spo}(2m|2n)$. Again, when it is convenient, we will use the notation $\varepsilon_{\bar{i}} = \delta_i$, for $1 \le i \le m$. With respect to $\mathfrak{h}$, the root system $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ is given by

$$\{\pm\delta_i \pm \delta_j, \pm 2\delta_p, \pm\varepsilon_k \pm \varepsilon_l\} \cup \{\pm\delta_p \pm \varepsilon_q\},$$

where $1 \le i < j \le m$, $1 \le k < l \le n$, $1 \le p \le m$, $1 \le q \le n$.

The root vectors for $\mathfrak{spo}(2m|2n)$ are given by (1.23)–(1.28).

The Weyl group of $\mathfrak{spo}(2m|2n)$, which is by definition the Weyl group of the even subalgebra $\mathfrak{g}_{\bar{0}} = \mathfrak{sp}(2m) \oplus \mathfrak{so}(2n)$, is an index 2 subgroup of $(\mathbb{Z}_2^m \rtimes \mathfrak{S}_m) \times (\mathbb{Z}_2^n \rtimes \mathfrak{S}_n)$, with only an even number of signs in $\mathbb{Z}_2^n$ permitted.

1.2.6. Root system and odd invariant form for $\mathfrak{q}(n)$. In this subsection let $\mathfrak{g}$ be the queer Lie superalgebra $\mathfrak{q}(n)$ (see Section 1.1.4). Since the case of $\mathfrak{q}(n)$ is different from the basic Lie superalgebra case, we will describe altogether its Cartan subalgebras, root systems, positive systems, Borel subalgebras, and invariant bilinear form.

Recall that $\mathfrak{q}(n)$ can be realized as matrices in the $n|n$ block form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ indexed by $I(n|n)$. The subalgebra consisting of matrices with a, b being diagonal (which we refer to as "block diagonal matrices") will be called the **standard Cartan subalgebra**. We define a Cartan subalgebra $\mathfrak{h}$ to be any subalgebra conjugate to the standard Cartan subalgebra by the adjoint action of some element in the group $GL(n)$ associated to $\mathfrak{q}(n)_{\bar{0}}$. Note that $\mathfrak{h}$ is self-normalizing in $\mathfrak{g}$ and $\mathfrak{h}$ is nilpotent, justifying the terminology of a Cartain subalgebra. However, $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ is not abelian, since $[\mathfrak{h}_{\bar{0}}, \mathfrak{h}] = 0$ and $[\mathfrak{h}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] = \mathfrak{h}_{\bar{0}}$.

Now fix $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ to be the standard Cartan subalgebra. The vectors

$$H_i := E_{\bar{i}\bar{i}} + E_{ii}, \quad i = 1, \ldots, n \tag{1.30}$$

form a basis for $\mathfrak{h}_{\bar{0}}$, while the vectors

$$\overline{H}_i := E_{\bar{i}i} + E_{i\bar{i}}, \quad i = 1, \ldots, n \tag{1.31}$$

form a basis for $\mathfrak{h}_{\bar{1}}$. We let $\{\varepsilon_i \mid i = 1, \ldots, n\}$ denote the basis in $\mathfrak{h}_{\bar{0}}^*$ dual to $\{H_i \mid i = 1, \ldots, n\}$. With respect to $\mathfrak{h}_{\bar{0}}$, we have the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ with root system $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$, where $\Phi_{\bar{0}}$ and $\Phi_{\bar{1}}$ are understood as *distinct* isomorphic copies of the root system $\{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le n\}$ for $\mathfrak{gl}(n)$. We have $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 1$, for each $\alpha \in \Phi$, and $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_i$, for $\alpha \in \Phi_i$ and $i \in \mathbb{Z}_2$. The Weyl group of $\mathfrak{q}(n)$ is identified with the symmetric group $\mathfrak{S}_n$.

The matrices $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with a, b being upper triangular (which we refer to as "block upper triangular matrices") form a solvable subalgebra $\mathfrak{b}$, which will be called the **standard Borel subalgebra** of $\mathfrak{g}$. The positive system corresponding to the Borel subalgebra $\mathfrak{b}$ is $\Phi^+ = \Phi_{\bar{0}}^+ \cup \Phi_{\bar{1}}^+$, where $\Phi_{\bar{0}}^+ = \Phi_{\bar{1}}^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}$. Let $\Phi^- = -\Phi^+$ and so $\Phi = \Phi^+ \cup \Phi^-$. Let

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha.$$

Then, we have $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, and we have a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

To be compatible with the definition of Weyl vector ρ in (1.35) for basic Lie superalgebras later on, it makes sense to set $\rho = 0$ for $\mathfrak{q}(n)$.

Any subalgebra of $\mathfrak{g}$ that is conjugate to $\mathfrak{b}$ by $GL(n)$ will be referred to as a **Borel subalgebra** of $\mathfrak{g}$.

For $g = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ in $\mathfrak{q}(n)$, the **odd trace** is defined to be

$$\operatorname{otr}(g) := \operatorname{tr}(b). \tag{1.32}$$

Using this, we obtain an odd non-degenerate invariant symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{q}(n)$ defined by

$$(g, g') = \operatorname{otr}(gg'), \quad g, g' \in \mathfrak{q}(n). \tag{1.33}$$

Here "odd" is understood in the sense of Definition 1.8.

1.3. Non-conjugate positive systems and odd reflections

In this section, positive systems, fundamental systems, and Dynkin diagrams for basic Lie superalgebras are defined and classified, along with Borel subalgebras. In contrast to semisimple Lie algebras, the fundamental systems for a Lie superalgebra may not be conjugate under the Weyl group action.

1.3.1. Positive systems and fundamental systems. Let Φ be a root system for a basic Lie superalgebra $\mathfrak{g}$ with a given Cartan subalgebra $\mathfrak{h}$, and let E be the real vector space spanned by Φ. We have $E \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{h}^*$, for $\mathfrak{g} \neq \mathfrak{gl}(m|n)$. For $\mathfrak{g} = \mathfrak{gl}(m|n)$ the space $E \otimes_{\mathbb{R}} \mathbb{C}$ is a subspace of $\mathfrak{h}^*$ of codimension one.

A total ordering $\geq$ on E below is always assumed to be compatible with the real vector space structure; that is, $v \geq w$ and $v' \geq w'$ imply that $v + v' \geq w + w'$, $-w \geq -v$, and $cv \geq cw$ for $c \in \mathbb{R}$ and $c > 0$.

A **positive system** Φ^+ is a subset of Φ consisting precisely of all those roots $\alpha \in \Phi$ satisfying $\alpha > 0$ for some total ordering of E. Given a positive system Φ^+, we define the **fundamental system** $\Pi \subset \Phi^+$ to be the set of $\alpha \in \Phi^+$ which cannot be written as a sum of two roots in Φ^+. We refer to elements in Φ^+ as **positive roots** and elements in Π as **simple roots**. Similarly, we denote by Φ^- the corresponding set of negative roots. Set $\Phi_i^+ = \Phi^+ \cap \Phi_i$ and $\Phi_i^- = \Phi^- \cap \Phi_i$, for $i \in \mathbb{Z}_2$. By Theorem 1.18(9), we have $\Phi^- = -\Phi^+$ and $\Phi_i^- = -\Phi_i^+$, for $i \in \mathbb{Z}_2$. Then, we have

$$\Phi^+ = \Phi_{\bar{0}}^+ \cup \Phi_{\bar{1}}^+.$$

Recall $\bar{\Phi}_{\bar{1}}$ from (1.18). Associated to a positive system Φ^+, we let

$$\bar{\Phi}_{\bar{1}}^+ := \bar{\Phi}_{\bar{1}} \cap \Phi^+. \tag{1.34}$$

Proposition 1.20. *Let $\mathfrak{g}$ be a basic Lie superalgebra with a Cartan subalgebra $\mathfrak{h}$. There is a one-to-one correspondence between the set of positive systems for $(\mathfrak{g},\mathfrak{h})$ and the set of fundamental systems for $(\mathfrak{g},\mathfrak{h})$. The Weyl group of $\mathfrak{g}$ acts naturally on the set of the positive systems (respectively, fundamental systems).*

Proof. It follows by definition that the fundamental system exists and is unique for a given positive system. A positive root, if not simple, can be written as a sum of two positive roots. Continuing this way, any positive root is a $\mathbb{Z}_+$-linear combination of simple roots, and hence a positive system is determined by its fundamental system.

By Theorem 1.18, $\Phi = -\Phi$ and Φ is W-invariant. Then W acts naturally on the set of positive systems, and then on the set of fundamental systems by the above correspondence. □

We define

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \bigoplus_{\alpha\in\Phi^-} \mathfrak{g}_\alpha.$$

Then $\mathfrak{n}^\pm$ are $\mathrm{ad}\,\mathfrak{h}$-stable nilpotent subalgebras of $\mathfrak{g}$ and we obtain a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

The solvable subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is called a **Borel subalgebra** of $\mathfrak{g}$ (corresponding to Φ^+). We have $\mathfrak{b} = \mathfrak{b}_{\bar 0} \oplus \mathfrak{b}_{\bar 1}$, where $\mathfrak{b}_i = \mathfrak{b} \cap \mathfrak{g}_i$ for $i \in \mathbb{Z}_2$.

Remark 1.21. The rank one subalgebra corresponding to an isotropic simple root is isomorphic to $\mathfrak{sl}(1|1)$, which is solvable. Therefore, if we enlarge a Borel subalgebra by adding the root space corresponding to a negative isotropic simple root, then the resulting subalgebra is still solvable. Thus, a Borel subalgebra is not a maximal solvable subalgebra for Lie superalgebras in general.

Given a positive system $\Phi^+ = \Phi^+_{\bar 0} \cup \Phi^+_{\bar 1}$, the **Weyl vector** ρ is defined by

$$\rho = \rho_{\bar 0} - \rho_{\bar 1}, \tag{1.35}$$

where

$$\rho_{\bar 0} = \frac{1}{2}\sum_{\alpha\in\Phi^+_{\bar 0}} \alpha, \quad \rho_{\bar 1} = \frac{1}{2}\sum_{\beta\in\Phi^+_{\bar 1}} \beta.$$

Denote

$$\mathbf{1}_{m|n} = (\delta_1 + \ldots + \delta_m) - (\varepsilon_1 + \ldots + \varepsilon_n). \tag{1.36}$$

The following lemma is proved by a direct computation.

Lemma 1.22. *We have the following formulas for the Weyl vector ρ for the standard positive system Φ^+:*

(1) $\rho = \sum_{i=1}^m (m-i+1)\delta_i - \sum_{j=1}^n j\varepsilon_j - \frac{1}{2}(m+n+1)\mathbf{1}_{m|n}$, *for* $\mathfrak{g} = \mathfrak{gl}(m|n)$.

(2) $\rho = \sum_{i=1}^{m}(m-n-i+\frac{1}{2})\delta_i + \sum_{j=1}^{n}(n-j+\frac{1}{2})\varepsilon_j$, *for* $\mathfrak{g} = \mathfrak{spo}(2m|2n+1)$.

(3) $\rho = \sum_{i=1}^{m}(m-n-i+1)\delta_i + \sum_{j=1}^{n}(n-j)\varepsilon_j$, *for* $\mathfrak{g} = \mathfrak{spo}(2m|2n)$.

We shall next describe completely the positive and fundamental systems for Lie superalgebras of type $\mathfrak{gl}$ and $\mathfrak{osp}$ case-by-case.

1.3.2. Positive and fundamental systems for $\mathfrak{gl}(m|n)$. Recall the root system Φ for $\mathfrak{gl}(m|n)$ and the standard Cartan subalgebra $\mathfrak{h}$ described in Section 1.2.3. The subalgebra of upper triangular matrices is the **standard Borel subalgebra** of $\mathfrak{g}$ that contains $\mathfrak{h}$, and the corresponding **standard positive system** of Φ is given by $\{\varepsilon_i - \varepsilon_j \mid i,j \in I(m|n), i<j\}$. Bearing in mind $\varepsilon_{\bar{i}} = \delta_i$, the standard fundamental system for $\mathfrak{gl}(m|n)$ is

$$\{\delta_i - \delta_{i+1}, \varepsilon_j - \varepsilon_{j+1}, \delta_m - \varepsilon_1 \mid 1 \le i \le m-1, 1 \le j \le n-1\},$$

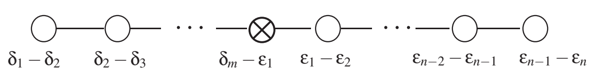
with the corresponding simple root vectors $e_i := E_{i,i+1}$, for $i \in I(m-1|n-1)$, and $e_{\overline{m}} := E_{\overline{m},1}$. The simple coroots are $h_j := E_{jj} - E_{j+1,j+1}$, for $j \in I(m-1|n-1)$, and $h_{\overline{m}} := E_{\overline{m},\overline{m}} + E_{11}$. Denote $f_i := E_{i+1,i}$, for $i \in I(m-1|n-1)$, and $f_{\overline{m}} := E_{1,\overline{m}}$. (Here we have slightly abused notation to let $i+1$ mean $\overline{\iota+1}$, for $i = \bar{\iota}$ with $1 \le \iota \le m-1$.) Then $\{e_i, h_i, f_i \mid i \in I(m|n-1)\}$ is a set of **Chevalley generators** for $\mathfrak{sl}(m|n)$.

Note that

$$\begin{aligned}
(\delta_i - \delta_{i+1}, \delta_i - \delta_{i+1}) &= 2, \quad 1 \le i \le m-1,\\
(\delta_m - \varepsilon_1, \delta_m - \varepsilon_1) &= 0,\\
(\varepsilon_j - \varepsilon_{j+1}, \varepsilon_j - \varepsilon_{j+1}) &= -2, \quad 1 \le j \le n-1.
\end{aligned}$$

Thus $\delta_m - \varepsilon_1$ is an isotropic simple root. Following the usual convention for Lie algebras, we draw the corresponding **standard Dynkin diagram** with its fundamental system attached:

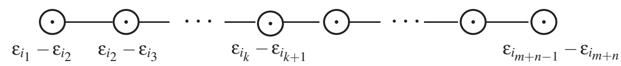
(1.37)



Here, as usual, we denote by $\bigcirc$ an even simple root α such that $\frac{1}{2}\alpha$ is not a root. Following Kac's notation, $\bigotimes$ denotes an odd isotropic simple root.

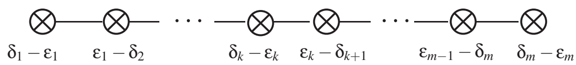
Let us classify all possible positive systems for $\mathfrak{gl}(m|n)$, keeping in mind $\varepsilon_{\bar{i}} = \delta_i$, for $1 \le i \le m$. If we ignore the parity of roots for the moment, the root system of $\mathfrak{gl}(m|n)$ is the same as the root system for $\mathfrak{gl}(m+n)$. Hence, by definition, their positive systems (respectively, fundamental systems) are exactly described in the same way, and so there are $(m+n)!$ of them in total. It follows from the well-known classification for $\mathfrak{gl}(m+n)$ that a fundamental system for $\mathfrak{gl}(m|n)$ consists of $(m+n-1)$ roots $\varepsilon_{i_1} - \varepsilon_{i_2}, \varepsilon_{i_2} - \varepsilon_{i_3}, \ldots, \varepsilon_{i_{m+n-1}} - \varepsilon_{i_{m+n}}$, where $\{i_1, i_2, \ldots, i_{m+n}\} = I(m|n)$. Then we restore the parity of the simple roots in a fundamental system for $\mathfrak{gl}(m|n)$. The corresponding Dynkin diagram is of the form

(1.38)



where $\odot$ is either $\bigcirc$ or $\bigotimes$, depending on whether the corresponding simple root is even or odd.

Example 1.23. For $n = m$, there exists a fundamental system consisting of only odd roots $\{\delta_1 - \varepsilon_1, \varepsilon_1 - \delta_2, \delta_2 - \varepsilon_2, \ldots, \delta_m - \varepsilon_m\}$, whose corresponding Dynkin diagram is



The $\varepsilon\delta$**-sequence** for a fundamental system Π as in (1.38) is obtained by switching the ordered sequence $\varepsilon_{i_1}\varepsilon_{i_2}\ldots\varepsilon_{i_{m+n}}$ for Π to the $\varepsilon\delta$-notation via the identification $\varepsilon_{\bar{i}} = \delta_i$ and then dropping the indices. Clearly, an $\varepsilon\delta$-sequence has m δ's and n ε's. In general, there exist positive systems for Φ that are not conjugate to each other under the action of the Weyl group, in contrast to the semisimple Lie algebra case.

Example 1.24. (1) The standard Borel subalgebra of $\mathfrak{gl}(m|n)$ corresponds to the sequence $\underbrace{\delta\cdots\delta}_{m}\underbrace{\varepsilon\cdots\varepsilon}_{n}$ while the Borel opposite to the standard one corresponds to $\underbrace{\varepsilon\cdots\varepsilon}_{n}\underbrace{\delta\cdots\delta}_{m}$.

(2) The three W-conjugacy classes of fundamental systems for $\mathfrak{gl}(1|2)$ correspond to the three sequences $\delta\varepsilon\varepsilon$, $\varepsilon\delta\varepsilon$, $\varepsilon\varepsilon\delta$, respectively.

1.3.3. Positive and fundamental systems for $\mathfrak{spo}(2m|2n+1)$. Now we describe the positive/fundamental systems and Dynkin diagrams for $\mathfrak{spo}(2m|2n+1)$ whose root system is described in Section 1.2.4. The standard positive system $\Phi^+ = \Phi^+_{\bar{0}} \cup \Phi^+_{\bar{1}}$ corresponding to the **standard Borel subalgebra** for $\mathfrak{spo}(2m|2n+1)$ is

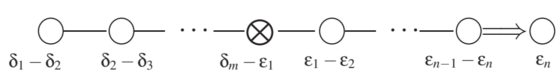
$$\{\delta_i \pm \delta_j, 2\delta_p, \varepsilon_k \pm \varepsilon_l, \varepsilon_q\} \cup \{\delta_p \pm \varepsilon_q, \delta_p\},$$

where $1 \le i < j \le m$, $1 \le k < l \le n$, $1 \le p \le m$, $1 \le q \le n$. The fundamental system Π of Φ^+ contains one odd simple root $\delta_m - \varepsilon_1$, and it is given by

$$\Pi = \{\delta_i - \delta_{i+1}, \delta_m - \varepsilon_1, \varepsilon_k - \varepsilon_{k+1}, \varepsilon_n \mid 1 \le i \le m-1, 1 \le k \le n-1\}.$$

The corresponding **standard Dynkin diagram** for $\mathfrak{spo}(2m|2n+1)$ is

(1.39)



Another often used positive system in Φ for $\mathfrak{spo}(2m|2n+1)$ is given by

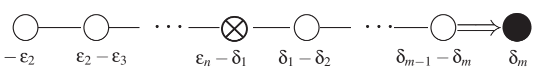
$$\{\delta_i \pm \delta_j, 2\delta_p, \varepsilon_k \pm \varepsilon_l, \varepsilon_q\} \cup \{\varepsilon_q \pm \delta_p, \delta_p\},$$

where $1 \le i < j \le m$, $1 \le k < l \le n$, $1 \le p \le m$, $1 \le q \le n$, with the following fundamental system

$$\{\varepsilon_k - \varepsilon_{k+1}, \varepsilon_n - \delta_1, \delta_i - \delta_{i+1}, \delta_m | 1 \le i \le m-1, 1 \le k \le n-1\}.$$

The corresponding Dynkin diagram is

(1.40)



where we follow Kac's convention and use ● to denote a non-isotropic odd simple root, as $(\delta_m, \delta_m) = 1$.

Now let us classify all the possible fundamental systems for $\mathfrak{spo}(2m|2n+1)$ with given Cartan subalgebra $\mathfrak{h}$, keeping in mind $\varepsilon_{\bar{i}} = \delta_i$. Note that $2\varepsilon_{\overline{p}} \in \Phi^+$ if and only if $\varepsilon_{\overline{p}} \in \Phi^+$, and that $\pm 2\varepsilon_{\overline{p}}$ are never in any fundamental system by definition. Hence, for the sake of classification of positive systems and fundamental systems in Φ, it suffices to consider the subset $\tilde{\Phi} := \Phi \setminus \{\pm 2\varepsilon_{\overline{p}} \mid 1 \le p \le m\}$. Ignoring the parity of the roots, $\tilde{\Phi}$ may be identified with the root system of the classical Lie algebra $\mathfrak{so}(2m+2n+1)$, whose fundamental systems are completely known and acted upon simply transitively by the Weyl group of $\mathfrak{so}(2m+2n+1)$ (which is $\cong \mathbb{Z}_2^{m+n} \rtimes \mathfrak{S}_{m+n}$). The number of W-conjugacy classes of fundamental systems for $\mathfrak{spo}(2m|2n+1)$ is $|W(\mathfrak{so}(2m+2n+1))|/|W| = \binom{m+n}{m}$.

The **$\varepsilon\delta$-sequence** associated to a fundamental system (or a Dynkin diagram) for $\mathfrak{spo}(2m|2n+1)$ is defined as for $\mathfrak{gl}(m|n)$, starting from the type A end of the Dynkin diagram (to fix the ambiguity). For example, the $\varepsilon\delta$-sequence associated to the standard Dynkin diagram above is m δ's followed by n ε's.

Example 1.25. Let $\mathfrak{g} = \mathfrak{osp}(1|2)$. Then its even subalgebra $\mathfrak{g}_{\bar{0}}$ is isomorphic to $\mathfrak{sl}(2) = \mathbb{C}\langle e, h, f\rangle$, and as a $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is isomorphic to the 2-dimensional natural $\mathfrak{sl}(2)$-module $\mathbb{C}E + \mathbb{C}F$. The simple root consists of a (unique) odd non-isotropic root δ_1 so that $2\delta_1$ is an even root. The Dynkin diagram is ●. The root vectors E and F associated to the odd roots δ_1 and $-\delta_1$ can be chosen such that $[E,E] = 2e, [F,F] = -2f, [E,F] = h$.

1.3.4. Positive and fundamental systems for $\mathfrak{spo}(2m|2n)$. Now we consider $\mathfrak{g} = \mathfrak{spo}(2m|2n)$, whose root system is described in Section 1.2.5. The **standard positive system $\Phi^+ = \Phi^+_{\bar{0}} \cup \Phi^+_{\bar{1}}$** in Φ corresponding to the standard Borel subalgebra is

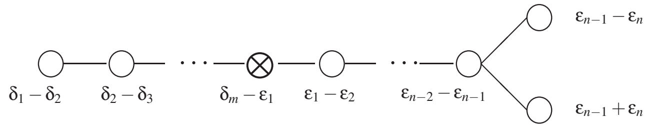
$$\{\delta_i \pm \delta_j, 2\delta_p, \varepsilon_k \pm \varepsilon_l\} \cup \{\delta_p \pm \varepsilon_q\},$$

where $1 \le i < j \le m$, $1 \le k < l \le n$, $1 \le p \le m$, $1 \le q \le n$, with its fundamental system being

$$\Pi = \{\delta_i - \delta_{i+1}, \delta_m - \varepsilon_1, \varepsilon_k - \varepsilon_{k+1}, \varepsilon_{n-1} + \varepsilon_n \mid 1 \le i \le m-1,\ 1 \le k \le n-1\}.$$

The corresponding **standard Dynkin diagram** is

(1.41)



Another often used positive system in Φ is given by

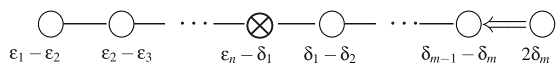
$$\{\delta_i \pm \delta_j, 2\delta_p, \varepsilon_k \pm \varepsilon_l\} \cup \{\varepsilon_q \pm \delta_p\},$$

where $1 \le i < j \le m$, $1 \le k < l \le n$, $1 \le p \le m$, $1 \le q \le n$, with its fundamental system being

$$\{\varepsilon_k - \varepsilon_{k+1}, \varepsilon_n - \delta_1, \delta_i - \delta_{i+1}, 2\delta_m \mid 1 \le i \le m-1,\ 1 \le k \le n-1\}.$$

The corresponding Dynkin diagram is

(1.42)



As we have already observed above, there are (at least) two Dynkin diagrams for $\mathfrak{spo}(2m|2n)$ of different shapes. The classification of all fundamental systems for the root system Φ of $\mathfrak{spo}(2m|2n)$ is divided into 2 cases below. As usual, we keep in mind $\varepsilon_{\bar{i}} = \delta_i$.

(1) First, we classify the fundamental systems Π in Φ that do not contain any long root (i.e., a root of the form $\pm 2\varepsilon_{\overline{p}}$). With parity ignored, the subset $\tilde{\Phi} := \Phi \setminus \{\pm 2\varepsilon_{\overline{p}} \mid 1 \le p \le m\}$ may be identified with the root system of the classical Lie algebra $\mathfrak{so}(2m+2n)$, whose fundamental systems are completely known, and they are acted upon simply transitively by the Weyl group of $\mathfrak{so}(2m+2n)$ (which is an index 2 subgroup of $\mathbb{Z}_2^{m+n} \rtimes \mathfrak{S}_{m+n}$). Observe that the positive system Φ^+ for Φ corresponding to Π is completely determined by the positive system $\tilde{\Phi} \cap \Phi^+$ for $\tilde{\Phi}$ (and vice versa). We conclude that the fundamental systems for Φ that do not contain any long root are exactly the fundamental systems for $\tilde{\Phi}$. However, not every fundamental system of $\tilde{\Phi}$ gives rise to a fundamental system of Φ. To be precise, a fundamental system of Φ cannot contain a pair of roots of the form $\{\pm\varepsilon_i \pm \varepsilon_{\overline{p}}, \pm\varepsilon_i \mp \varepsilon_{\overline{p}}\}$, $i \ne \overline{p}$ and $1 \le p \le m$. It follows that there are $\frac{n}{m+n} \cdot |W(\mathfrak{so}(2m+2n))|$ such fundamental systems of Φ, and hence the number of W-conjugacy classes for $\mathfrak{g}$ in this case is $\frac{n}{m+n} \cdot |W(\mathfrak{so}(2m+2n))|/|W| = \binom{m+n-1}{m}$.

(2) Next, we classify the fundamental systems Π that contain some long root. In this case, we consider $\widehat{\Phi} := \Phi \cup \{\pm 2\varepsilon_j, 1 \le j \le n\}$. With the parity ignored, $\widehat{\Phi}$ may be identified with the root system of $\mathfrak{sp}(2m+2n)$, whose fundamental systems

are completely described. Observe that the positive system Φ^+ for Φ that corresponds to Π is completely determined by the positive system $\widehat{\Phi}^+$ of $\widehat{\Phi}$ that contains Φ^+. We conclude that the fundamental systems for Φ that contain some long root are exactly the fundamental systems for $\widehat{\Phi}$ whose long root is of the form $\pm 2\varepsilon_{\overline{p}}$. It follows that the number of W-conjugacy classes of such fundamental systems for $\mathfrak{g}$ is $\frac{m}{m+n} \cdot |W(\mathfrak{sp}(2m+2n))|/|W| = 2\binom{m+n-1}{n}$.

The **$\varepsilon\delta$-sequence for $\mathfrak{spo}(2m|2n)$** associated to a positive system is now defined as follows. We can first obtain an ordered sequence of ε's and δ's just as for $\mathfrak{spo}(2m|2n+1)$. If this sequence has an ε as its last member, then it is the $\varepsilon\delta$-sequence. If the last member in this sequence is a δ, then the $\varepsilon\delta$-sequence is obtained from this sequence by attaching a sign to the last ε.

Example 1.26. For $\mathfrak{spo}(4|4)$, the $\varepsilon\delta$-sequences $\varepsilon\delta\delta\varepsilon$, $\varepsilon\delta\varepsilon\delta$, and $\varepsilon\delta(-\varepsilon)\delta$ are distinct.

1.3.5. Conjugacy classes of fundamental systems. We now describe the classification of the W-conjugacy classes of fundamental systems for these Lie superalgebras via the $\varepsilon\delta$-sequences.

Proposition 1.27. *Let Φ be the root system and let W be the Weyl group for a Lie superalgebra $\mathfrak{g}$ of type $\mathfrak{gl}$ or $\mathfrak{osp}$. Then the W-conjugacy classes of fundamental systems in Φ are in one-to-one correspondence with the associated $\varepsilon\delta$-sequences. In particular, there are $\binom{m+n}{m}$ W-conjugacy classes of fundamental systems for $\mathfrak{gl}(m|n)$ and $\mathfrak{spo}(2m|2n+1)$, while there are $\binom{m+n}{m} + \binom{m+n-1}{n}$ W-conjugacy classes of fundamental systems for $\mathfrak{spo}(2m|2n)$.*

Proof. By the case-by-case classification of fundamental systems in Φ and definition of $\varepsilon\delta$-sequences, we clearly have a well-defined map

$$\Theta : \{\text{fundamental systems in } \Phi\}/W \longrightarrow \{\varepsilon\delta\text{-sequences for } \mathfrak{g}\}.$$

As we can easily construct a fundamental system Π for a given $\varepsilon\delta$-sequence, Θ is surjective. To show Θ is a bijection, it remains to show that the two finite sets have the same cardinalities.

The number of W-conjugacy classes of fundamental systems equals $|W'|/|W|$, where W' denotes the Weyl group of $\mathfrak{gl}(m+n)$ when $\mathfrak{g} = \mathfrak{gl}(m|n)$ and W' denotes the Weyl group of $\mathfrak{so}(2m+2n+1)$ when $\mathfrak{g} = \mathfrak{spo}(2m|2n+1)$. In either case, $|W'|/|W| = \binom{m+n}{m}$. On the other hand, in either case, an $\varepsilon\delta$-sequence is simply an ordered arrangement of m δ's and n ε's. Thus the total number is also $\binom{m+n}{m}$.

In the case of $\mathfrak{spo}(2m|2n)$, when the last slot of an $\varepsilon\delta$-sequence is an ε, the previous $(m+n-1)$ slots can be filled with m δ's and $(n-1)$ ε's. Thus we obtain $\binom{m+n-1}{m}$ different $\varepsilon\delta$-sequences this way. When the last slot in an $\varepsilon\delta$-sequence is a δ, then the previous $(m+n-1)$ slots are filled with $(m-1)$ δ's and n ε's, with the last ε having either a positive or negative sign. This way we obtain additional

$2\binom{m+n-1}{n}$ distinct $\varepsilon\delta$-sequences. Now $\binom{m+n-1}{m}+2\binom{m+n-1}{n}$ is exactly the number of W-conjugacy classes of fundamental systems for $\mathfrak{spo}(2m|2n)$ that we have computed earlier. $\square$

The fundamental and positive systems for the exceptional Lie superalgebras $G(3), F(3|1), D(2|1,\alpha)$ are also listed completely by Kac [**60**] (with one missing for $F(3|1)$; see [**130**, 5.1]).

From the classification of the fundamental and positive systems we immediately obtain the following proposition and lemma.

Proposition 1.28. *Let $\mathfrak{g}$ be a basic Lie superalgebra, excluding $\mathfrak{sl}(n|n)/\mathbb{C}I_{n|n}$ and $\mathfrak{gl}(m|n)$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then any fundamental system for the root system of $(\mathfrak{g},\mathfrak{h})$ forms a basis in $\mathfrak{h}^*$. (For $\mathfrak{gl}(m|n)$, any fundamental system is linearly independent in $\mathfrak{h}^*$.)*

Lemma 1.29. *Let $\mathfrak{g}$ be a basic Lie superalgebra. Let Π be the fundamental system in a positive system Φ^+, and let α be an isotropic odd root in Φ^+. Then there exists $w \in W$ such that $w(\alpha) \in \Pi$.*

1.4. Odd and real reflections

In this section, beside real reflections associated to even roots as for semisimple Lie algebras, we introduce odd reflections associated to isotropic odd simple roots. Both real and odd reflections permute the fundamental systems of a root system.

1.4.1. A fundamental lemma. We have seen that the fundamental systems of a root system Φ are not always W-conjugate due to the existence of odd roots. Recall a root $\alpha \in \Phi$ is isotropic if $(\alpha,\alpha)=0$, and an isotropic root must be odd. The following lemma plays a fundamental role in the representation theory of Lie superalgebras.

Lemma 1.30. *Let $\mathfrak{g}$ be a basic Lie superalgebra and let Π be a fundamental system of a positive system Φ^+. Let α be an odd isotropic simple root. Then,*

$$\Phi^+_\alpha := \{-\alpha\} \cup \Phi^+ \backslash \{\alpha\} \tag{1.43}$$

is a new positive system whose corresponding fundamental system Π_α is given by

$$\Pi_\alpha = \{\beta \in \Pi \mid (\beta,\alpha)=0, \beta \neq \alpha\} \cup \{\beta+\alpha \mid \beta \in \Pi, (\beta,\alpha) \neq 0\} \cup \{-\alpha\}. \tag{1.44}$$

Proof. For $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{spo}(2m|2n)$, or $\mathfrak{spo}(2m|2n+1)$, which we are mostly concerned about in this book, we have already obtained complete descriptions of all fundamental systems. For most Π's, the set $\{\beta \in \Pi \mid (\beta,\alpha) \neq 0\}$ has cardinality at most 2 and consists of roots of the form $\pm\varepsilon_i \pm \varepsilon_j$ for $i \neq j \in I(m|n)$. By inspection we see that Π_α is a fundamental system. There are a few extra cases when the root α corresponds to a node in the corresponding Dynkin diagram which is either connected to a long simple root, or connected to a short non-isotropic root, or is a branching node, or is one of the (short) end nodes of a branching node, as follows:

Here the vertical dashed lines indicate an edge when it connects two $\bigotimes$'s and no edge otherwise, $\bigoplus$ means either $\bigcirc$ or $\newmoon$, and $\odot$ means either $\bigotimes$ or $\bigcirc$. It can be checked directly in these cases that Π_α is also a fundamental system.

Take $\beta \in \Phi_\alpha^+ \cap \Phi^+$. Since Π is a fundamental system for Φ^+, by Proposition 1.28 we have a unique expression $\beta = \sum_{\gamma\in\Pi\setminus\{\alpha\}} m_\gamma\gamma + m_\alpha\alpha$ for $m_\gamma \in \mathbb{Z}_+$ and $m_\alpha \in \mathbb{Z}_+$. It follows by definition of Π_α that β can be expressed as a linear combination of Π_α of the form $\beta = \sum_{\kappa\in\Pi_\alpha\setminus\{-\alpha\}} m'_\kappa\kappa + m'_\alpha(-\alpha)$ for some suitable integer m'_α. By choice of β we have $m'_\kappa > 0$ for some κ, and hence $m'_\alpha \in \mathbb{Z}_+$, since Π_α is a fundamental system. This shows that Φ_α^+ is the positive system corresponding to Π_α.

For an exceptional Lie superalgebra $\mathfrak{g} = D(2|1,\alpha)$, $G(3)$, or $F(3|1)$, which will not be studied in any detail in this book, our proof shall be rather sketchy. A conceptual approach (see [**66**]) would be to follow Remark 1.19 to regard $\mathfrak{g}$ as a Kac-Moody superalgebra with Chevalley generators e_i, f_i, h_i for $i \in I$ associated to a Cartan matrix A (and a fundamental system $\Pi = \{\alpha_i \mid i \in I\}$), where I is $\mathbb{Z}_2$-graded. For an isotropic odd root $\alpha \in \Pi$, one can construct a new set of Chevalley generators e'_i, f'_i, h'_i associated to Π_α, which gives rise to a new Cartan matrix A'. By standard machinery of the Kac-Moody theory, the Kac-Moody superalgebra associated to A and A' coincide with $\mathfrak{g}$. From this it follows that Π_α is a fundamental system for $\mathfrak{g}$. The same argument as above shows that Φ_α is the positive system associated to the fundamental system Π_α. This uniform approach is applicable to all basic Lie superalgebras. $\square$

1.4.2. Odd reflections. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^+$ corresponds to the positive system Φ^+. Then the new Borel subalgebra corresponding to Φ_α^+ in (1.44) for an isotropic odd simple root α is given by

$$\mathfrak{b}^\alpha := \mathfrak{h} \oplus \bigoplus_{\beta\in\Phi_\alpha^+} \mathfrak{g}_\beta. \tag{1.45}$$

Observe that $\mathfrak{b}_{\bar{0}} = \mathfrak{b}^\alpha_{\bar{0}}$ by Lemma 1.30. The process of obtaining Π_α (respectively, Φ_α^+ or $\mathfrak{b}^\alpha$) from Π (respectively, Φ^+ or $\mathfrak{b}$) will be referred to as **odd reflection** (with respect to α) and will be denoted by r_α. We shall write

$$r_\alpha(\Pi) = \Pi_\alpha, \quad r_\alpha(\Phi^+) = \Phi_\alpha^+, \quad r_\alpha(\mathfrak{b}) = \mathfrak{b}^\alpha. \tag{1.46}$$

Note that $r_{-\alpha}r_\alpha = 1$.

Remark 1.31. In contrast to a real reflection associated to an even root (see below), an odd reflection r_α with respect to an isotropic odd root α may not be extended to a linear transformation on $\mathfrak{h}^*$ which sends a simple root in Π to a simple root in Π_α. For example, for $\mathfrak{spo}(2m|2n)$ with a fundamental system Π corresponding to the first Dynkin diagram below, take $\alpha = \delta_m - \varepsilon_n$. The odd reflection transforms Π to Π_α corresponding to the second Dynkin diagram below (with the $\cdots$ portion unchanged). A plausible transformation would have to fix those δ_i, ε_j (for $i < m, j < n$) and interchange ε_n and δ_m, but this transformation would not send $\delta_m + \varepsilon_n$ to $2\delta_m$.

1.4.3. Real reflections. By Theorem 1.18, a basic Lie superalgebra $\mathfrak{g}$ admits an even non-degenerate supersymmetric bilinear form $(\cdot,\cdot)$, which restricts to a non-degenerate form $(\cdot,\cdot)$ on $\mathfrak{h}$ and on $\mathfrak{h}^*$. For an even root α (which is automatically non-isotropic), we define the **real reflection** r_α as a linear map on $\mathfrak{h}^*$ given by

$$r_\alpha(x) = x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha, \qquad \text{for } x \in \mathfrak{h}^*.$$

In particular, r_α preserves Φ, $\Phi_{\bar{0}}$, and $\Phi_{\bar{1}}$, respectively. The group generated by real reflections r_α, for $\alpha \in \Phi_{\bar{0}}$, is precisely the Weyl group W of $\mathfrak{g}_{\bar{0}}$ (and hence the Weyl group of $\mathfrak{g}$ by Definition 1.17).

For an even *simple* root α, we must have $\alpha/2 \notin \Phi$, that is, $\alpha \in \bar{\Phi}_{\bar{0}}$ as defined in (1.19). In this case, (1.46) can be understood with Φ_α^+ as in (1.43), $\mathfrak{b}^\alpha$ as in (1.45), and Π_α as the image of Π under r_α.

For an *odd* root α such that $2\alpha \in \Phi$, we define the reflection r_α as the real reflection $r_{2\alpha}$ associated to 2α. In this case, it is understood that $\Pi_\alpha = r_\alpha(\Pi)$, and

$$\Phi_\alpha^+ = \{-\alpha, -2\alpha\} \cup \Phi^+ \backslash \{\alpha, 2\alpha\} \tag{1.47}$$

is the new positive system associated to Π_α.

1.4.4. Reflections and fundamental systems.

Proposition 1.32. *For two fundamental systems Π and $'\Pi$ of a basic Lie superalgebra $\mathfrak{g}$, there exists a sequence consisting of real and odd reflections $r_1, r_2, \ldots, r_k$ such that $r_k \ldots r_2 r_1(\Pi) = {}'\Pi$.*

Proof. Denote by Φ^+ and $'\Phi^+$ the positive systems associated to Π and $'\Pi$ respectively. We prove the corollary by induction on $|\Phi^+ \cap {}'\Phi^-|$. If $|\Phi^+ \cap {}'\Phi^-| = 0$, then $\Pi = {}'\Pi$. Assume now $|\Phi^+ \cap {}'\Phi^-| > 0$. Then $\Pi \neq {}'\Pi$, and we pick $\alpha \in \Pi \cap {}'\Phi^- \neq \emptyset$

and apply the real or odd simple reflection r_α. Observe that Φ_α^+ is the positive system with fundamental system $\Pi_\alpha = r_\alpha(\Pi)$, regardless of the parity of the simple root α. Note that

$$|\Phi_\alpha^+ \cap {}'\Phi^-| < |\Phi^+ \cap {}'\Phi^-|.$$

By the inductive assumption, there exists a sequence of real and odd reflections $r_2, \dots, r_k$ such that $r_k \dots r_2(\Pi_\alpha) = {}'\Pi$. Hence $r_k \dots r_2 r_\alpha(\Pi) = {}'\Pi$. □

Proposition 1.33. *Let $\mathfrak{g}$ be a basic Lie superalgebra. Let Φ^+ be a positive system with Π as its fundamental system and ρ as its associated Weyl vector. Then, $(\rho, \beta) = \frac{1}{2}(\beta, \beta)$ for every simple root $\beta \in \Pi$.*

Proof. Let $\alpha \in \Pi$. Let Π_α and Φ_α^+ be respectively the fundamental and positive systems obtained by a real or odd reflection r_α defined above. Let ρ_α denote the Weyl vector for the positive system Φ_α^+. Recall Φ_α^+ is given in (1.43) for α odd or for $\alpha \in \bar{\Phi}_{\bar{0}}$, and in (1.47) otherwise. We compute that

$$\rho_\alpha = \begin{cases} \rho - \alpha, & \text{for } \alpha \in \bar{\Phi}_{\bar{0}} \text{ or non-isotropic odd}, \\ \rho + \alpha, & \text{for } \alpha \text{ isotropic odd}. \end{cases}$$

Claim. Assuming that the proposition holds for a fundamental system Π, it holds for the new fundamental system Π_α for $\alpha \in \Pi$.

Let us prove the claim. First assume $\alpha \in \Pi$ is even or non-isotropic odd. In this case, $\rho_\alpha = r_\alpha(\rho)$, $\Pi_\alpha = r_\alpha(\Pi)$, and so we have for each $\beta \in \Pi$ that

$$(\rho_\alpha, r_\alpha(\beta)) = (\rho, \beta) = \frac{1}{2}(\beta, \beta) = \frac{1}{2}(r_\alpha(\beta), r_\alpha(\beta)).$$

Now assume $\alpha \in \Pi$ is isotropic odd. We will check case-by-case, recalling the definition of Π_α from (1.44). If $(\beta, \alpha) = 0$ for $\beta \in \Pi$, then $\beta \in \Pi_\alpha$. By the assumption of the claim, we have

$$(\rho_\alpha, \beta) = (\rho + \alpha, \beta) = (\rho, \beta) = \frac{1}{2}(\beta, \beta).$$

If $(\beta, \alpha) \neq 0$ for $\beta \in \Pi$, then $\beta + \alpha \in \Pi_\alpha$. By the assumption of the claim, $(\rho, \alpha) = \frac{1}{2}(\alpha, \alpha) = 0$, and hence

$$(\rho_\alpha, \beta + \alpha) = (\rho + \alpha, \beta + \alpha) = (\rho + \alpha, \beta) = \frac{1}{2}(\beta, \beta) + (\alpha, \beta) = \frac{1}{2}(\beta + \alpha, \beta + \alpha).$$

This completes the proof of the claim.

By the claim and Proposition 1.32, it suffices to prove the proposition for just one fundamental system, e.g., the standard fundamental system for type $\mathfrak{gl}$ and $\mathfrak{osp}$. Assume that β is an even simple root or a non-isotropic odd simple root. In either case, one checks that $r_\beta(\rho) = \rho - \beta$. Since $(\cdot, \cdot)$ is W-invariant, we have

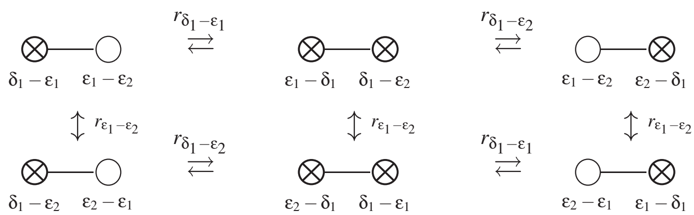
$$(\rho, \beta) = \big(r_\beta(\rho), r_\beta(\beta)\big) = (\rho - \beta, -\beta) = -(\rho, \beta) + (\beta, \beta).$$

It follows that $(\rho, \beta) = \frac{1}{2}(\beta, \beta)$ just as for semisimple Lie algebras.

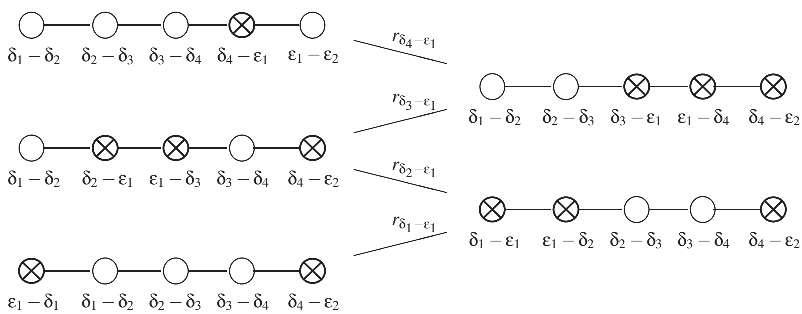
For isotropic odd simple root β, we do a case-by-case inspection. For type $\mathfrak{gl}$ or $\mathfrak{osp}$, the equation $(\rho,\beta) = \frac{1}{2}(\beta,\beta)$ follows directly from Lemma 1.22. For the three exceptional cases, it can be checked directly by using a fundamental system with exactly one isotropic simple root and the detailed description of root systems (cf. Kac [**60**]). We skip the details. □

1.4.5. Examples. Below we illustrate the notion of odd reflections by examples.

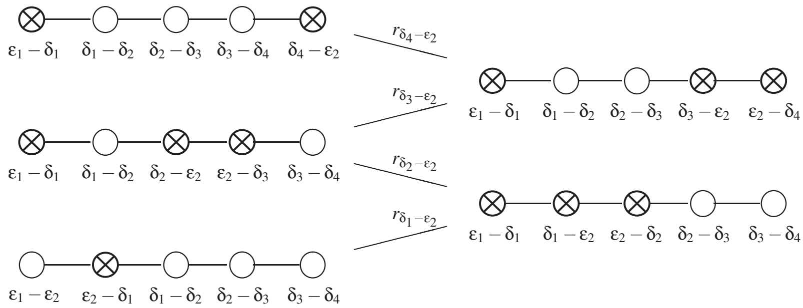
Example 1.34. Associated with $\mathfrak{gl}(1|2)$, we have $\Phi_{\bar{0}} = \{\pm(\varepsilon_1-\varepsilon_2)\}$ and $\Phi_{\bar{1}} = \{\pm(\delta_1-\varepsilon_1), \pm(\delta_1-\varepsilon_2)\}$. There are 6 fundamental systems that are related by real and odd reflections as follows. There are three conjugacy classes of Borel subalgebras corresponding to the three columns below, and each vertical pair corresponds to such a conjugacy class.



Example 1.35. Let $\mathfrak{g} = \mathfrak{gl}(4|2)$. The following sequence of fundamental systems $\Pi_0 = \Pi^{\mathrm{st}}, \Pi_1, \Pi_2, \Pi_3, \Pi_4$ (in descending order) is obtained from the standard fundamental system Π^{st} by applying consecutively the sequences of odd reflections with respect to the odd roots $\delta_4-\varepsilon_1$, $\delta_3-\varepsilon_1$, $\delta_2-\varepsilon_1$, and $\delta_1-\varepsilon_1$.



If we continue to apply to Π_4 consecutively the sequence of odd reflections with respect to the odd roots $\delta_4-\varepsilon_2$, $\delta_3-\varepsilon_2$, $\delta_2-\varepsilon_2$, and $\delta_1-\varepsilon_2$, we obtain the following fundamental systems $\Pi_4, \Pi_5, \Pi_6, \Pi_7, \Pi_8$:



1.5. Highest weight theory

In this section, we formulate the Poincaré-Birkhoff-Witt Theorem for Lie superalgebras. Finite-dimensional irreducible modules over certain solvable Lie superalgebras, including all Borel subalgebras, are classified. This is then used to develop a highest weight theory of the basic and queer Lie superalgebras.

1.5.1. The Poincaré-Birkhoff-Witt (PBW) Theorem. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. A **universal enveloping algebra** of $\mathfrak{g}$ is an associative superalgebra with unity $U(\mathfrak{g})$ together with a homomorphism of Lie superalgebras $\iota : \mathfrak{g} \to U(\mathfrak{g})$ characterized by the following universal property. Given an associative superalgebra A and a homomorphism of Lie superalgebras $\varphi : \mathfrak{g} \to A$, there exists a unique homomorphism of associative superalgebras $\psi : U(\mathfrak{g}) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & U(\mathfrak{g}) \arrow[d, dotted, "\exists\psi"] \\ & A \end{tikzcd}$$

In particular, this implies that representations of $\mathfrak{g}$ are representations of $U(\mathfrak{g})$ and vice versa. It follows by a standard tensor algebra construction that a universal enveloping algebra exists, and it is unique up to isomorphism by the defining universal property.

Theorem 1.36 (Poincaré-Birkhoff-Witt Theorem)**.** *Let $\{x_1, x_2, \ldots, x_p\}$ be a basis for $\mathfrak{g}_{\bar{0}}$ and let $\{y_1, y_2, \ldots, y_q\}$ be a basis for $\mathfrak{g}_{\bar{1}}$. Then the set*

$$\left\{x_1^{r_1} x_2^{r_2} \ldots x_p^{r_p} y_1^{s_1} y_2^{s_2} \ldots y_q^{s_q} \mid r_1, r_2, \ldots, r_p \in \mathbb{Z}_+, s_1, s_2, \ldots, s_q \in \{0,1\}\right\}$$

is a basis for $U(\mathfrak{g})$.

The proof for Theorem 1.36 is a straightforward super generalization of the Lie algebra case and will be omitted (see [**102**, Theorem 2.1] for detail).

The superalgebra $U(\mathfrak{g})$ carries a filtered algebra structure by letting

$$U_k(\mathfrak{g}) = \mathrm{span}\{x_1^{r_1} x_2^{r_2} \dots x_p^{r_p} y_1^{s_1} y_2^{s_2} \dots y_q^{s_q} \,|\, \sum_i r_i + \sum_j s_j \le k\}, \quad k \in \mathbb{Z}_+.$$

Its associated graded algebra is isomorphic to $S(\mathfrak{g}_{\bar{0}}) \otimes \wedge(\mathfrak{g}_{\bar{1}})$.

Given a Lie subalgebra $\mathfrak{l}$ of a finite-dimensional Lie superalgebra $\mathfrak{g}$ and an $\mathfrak{l}$-module V, we define the induced module as

$$\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} V = U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} V.$$

By the PBW Theorem, if V is finite-dimensional, then so is $\mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} V$.

1.5.2. Representations of solvable Lie superalgebras. Just as for Lie algebras, a finite-dimensional Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called **solvable** if $\mathfrak{g}^{(n)} = 0$ for some $n \ge 1$, where we define inductively $\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$ and $\mathfrak{g}^{(0)} = \mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional solvable Lie superalgebra such that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$. Given $\lambda \in \mathfrak{g}_{\bar{0}}^*$ with $\lambda([\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]) = 0$, we define a one-dimensional $\mathfrak{g}$-module $\mathbb{C}_\lambda = \mathbb{C} v_\lambda$ by

$$\begin{aligned} x v_\lambda &= \lambda(x) v_\lambda, \quad && \text{for } x \in \mathfrak{g}_{\bar{0}}, \\ y v_\lambda &= 0, && \text{for } y \in \mathfrak{g}_{\bar{1}}. \end{aligned}$$

There is a canonical linear isomorphism $(\mathfrak{g}_{\bar{0}}/[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}])^* \cong \{\lambda \in \mathfrak{g}_{\bar{0}}^* \mid \lambda([\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]) = 0\}$.

Lemma 1.37. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional solvable Lie superalgebra such that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$. Then every finite-dimensional irreducible $\mathfrak{g}$-module is one-dimensional. A complete list of finite-dimensional irreducible $\mathfrak{g}$-modules is given by $\mathbb{C}_\lambda$, for $\lambda \in (\mathfrak{g}_{\bar{0}}/[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}])^*$.*

Proof. We have already shown that any such $\mathbb{C}_\lambda$ is a $\mathfrak{g}$-module. Clearly every one-dimensional $\mathfrak{g}$-module is isomorphic to $\mathbb{C}_\lambda$, for some $\lambda \in (\mathfrak{g}_{\bar{0}}/[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}])^*$. So it suffices to show that every finite-dimensional irreducible $\mathfrak{g}$-module V is one dimensional.

The even subalgebra $\mathfrak{g}_{\bar{0}}$ is solvable since $\mathfrak{g}$ is solvable and $\mathfrak{g}_{\bar{0}}^{(n)} \subseteq \mathfrak{g}^{(n)}$ for every n. By applying Lie's theorem to $\mathfrak{g}_{\bar{0}}$, V contains a nonzero $\mathfrak{g}_{\bar{0}}$-invariant vector v_λ, where $\lambda \in \mathfrak{g}_{\bar{0}}^*$ with $\lambda([\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]) = 0$. Here, by a $\mathfrak{g}_{\bar{0}}$-invariant vector we mean that $x v_\lambda = \lambda(x) v_\lambda, \forall x \in \mathfrak{g}_{\bar{0}}$. Now Frobenius reciprocity says that $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} \mathbb{C} v_\lambda, V) \cong \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(\mathbb{C} v_\lambda, V)$, and thus, by the irreducibility of V, there exists a $\mathfrak{g}$-epimorphism from $\mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} \mathbb{C} v_\lambda \cong \wedge(\mathfrak{g}_{\bar{1}}) \otimes \mathbb{C} v_\lambda$ onto V. Thus, to prove that V is one dimensional, it suffices to prove that $\mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} \mathbb{C} v_\lambda$ has a composition series with one-dimensional composition factors. We shall construct such a composition series explicitly.

Set $\dim \mathfrak{g}_{\bar{1}} = n$. By Lie's theorem, the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ has an ordered basis $\{y_j | j = 1, \ldots, n\}$ such that for all $1 \leq i \leq n$ we have

$$\text{(1.48)} \qquad \operatorname{ad} \mathfrak{g}_{\bar{0}}(y_i) \subseteq \sum_{j=i}^{n} \mathbb{C} y_j, \quad \operatorname{ad} [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}](y_i) \subseteq \sum_{j=i+1}^{n} \mathbb{C} y_j.$$

Let $\mathfrak{B}_1 = \{1, y_1\}$, and define inductively $\mathfrak{B}_k = \{\mathfrak{B}_{k-1}, \mathfrak{B}_{k-1} y_k\}$, for $2 \leq k \leq n$, where the set $\mathfrak{B}_{k-1} y_k$ denotes the ordered set of elements obtained by multiplying all elements in $\mathfrak{B}_{k-1}$ on the right by y_k. Then $\mathfrak{B}_n v_\lambda$ is an ordered basis for $\operatorname{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} \mathbb{C} v_\lambda$ of cardinality 2^n. Denote this ordered basis by $\{v_1, \cdots, v_{2^n}\}$. Set $V_i := \oplus_{j=i}^{2^n} \mathbb{C} v_j$. Thanks to (1.48), we have a filtration of $\mathfrak{g}_{\bar{0}}$-modules

$$V = V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots \supseteq \cdots V_{2^n} \supseteq 0.$$

Clearly we have, for $1 \leq k, \ell \leq n$ and $i_1 < \ldots < i_\ell \leq n$,

$$\text{(1.49)} \qquad y_k y_{i_1} y_{i_2} \ldots y_{i_\ell} v_\lambda = \begin{cases} [y_k, y_{i_1}] y_{i_2} \ldots y_{i_\ell} v_\lambda - y_{i_1} y_k y_{i_2} \ldots y_{i_\ell} v_\lambda, & \text{if } k \neq i_1, \\ \frac{1}{2} [y_k, y_{i_1}] y_{i_2} \ldots y_{i_\ell} v_\lambda, & \text{if } k = i_1. \end{cases}$$

Using the assumption that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$, it follows from (1.48) and (1.49) that every y_k leaves V_i invariant, and hence each V_i is a $\mathfrak{g}$-module. Thus, the above filtration is a composition series with one-dimensional composition factors. □

Example 1.38. The Lie superalgebra $\mathfrak{g} = \mathbb{C} z \oplus \mathfrak{g}_{\bar{1}}$ in Example 1.4(4) associated to a non-degenerated symmetric bilinear form B on a nonzero space $\mathfrak{g}_{\bar{1}}$ is solvable. But an irreducible module of $\mathfrak{g}$ with the central element z acting as a nonzero scalar has dimension more than one. So the condition $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$ in Lemma 1.37 cannot be dropped.

1.5.3. Highest weight theory for basic Lie superalgebras. Let $\mathfrak{g}$ be a basic Lie superalgebra. Let $\mathfrak{h}$ be the standard Cartan subalgebra and let Φ be the root system. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ be a Borel subalgebra of $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$ and let Φ^+ be the associated positive system. The condition of Lemma 1.37 is satisfied for the solvable Lie superalgebra $\mathfrak{b}$, since we have $\mathfrak{b}_{\bar{1}} = \mathfrak{n}^+_{\bar{1}}$, and

$$[\mathfrak{b}_{\bar{1}}, \mathfrak{b}_{\bar{1}}] = [\mathfrak{n}^+_{\bar{1}}, \mathfrak{n}^+_{\bar{1}}] \subseteq \mathfrak{n}^+_{\bar{0}} = [\mathfrak{h}, \mathfrak{n}^+_{\bar{0}}] \subseteq [\mathfrak{b}_{\bar{0}}, \mathfrak{b}_{\bar{0}}].$$

Let V be a finite-dimensional irreducible representation of $\mathfrak{g}$. Then by Lemma 1.37 V contains a one-dimensional $\mathfrak{b}$-module that is of the form $\mathbb{C}_\lambda = \mathbb{C} v_\lambda$, for $\lambda \in \mathfrak{h}^* \cong (\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}])^*$. That is,

$$h v_\lambda = \lambda(h) v_\lambda \ \ (h \in \mathfrak{h}), \qquad x v_\lambda = 0 \ \ (x \in \mathfrak{n}^+).$$

By the PBW theorem and the irreducibility of V, we obtain $V = U(\mathfrak{n}^-) v_\lambda$ and thus a weight space decomposition

$$\text{(1.50)} \qquad V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu,$$

where the μ-weight space V_μ is given by

$$V_\mu := \{v \in V \mid hv = \mu(h)v, \forall h \in \mathfrak{h}\}.$$

By (1.50), $V_\mu = 0$ unless $\lambda - \mu$ is a $\mathbb{Z}_+$-linear combination of positive roots. The weight λ is called the **$\mathfrak{b}$-highest weight** (and sometime called an **extremal weight**) of V, the space $\mathbb{C}v_\lambda$ is called the **$\mathfrak{b}$-highest weight space**, and the vector v_λ is called a **$\mathfrak{b}$-highest weight vector** for V. When no confusion arises, we will simply say highest weight by dropping $\mathfrak{b}$. Hence, we have established the following.

Proposition 1.39. *Let $\mathfrak{g}$ be a basic Lie superalgebra with a Borel subalgebra $\mathfrak{b}$. Then every finite-dimensional irreducible $\mathfrak{g}$-module is a $\mathfrak{b}$-highest weight module.*

We shall denote the highest weight irreducible module of highest weight λ by $L(\lambda)$, $L(\mathfrak{g},\lambda)$, or $L(\mathfrak{g},\mathfrak{b},\lambda)$, depending on whether $\mathfrak{b}$ and $\mathfrak{g}$ are clear from the context.

Recall from Section 1.4 the notations Π_α and $\mathfrak{b}^\alpha$ associated to an isotropic odd simple root α. Denote by $\langle\cdot,\cdot\rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ the standard bilinear pairing. Denote by h_α the corresponding coroot for α, and denote by e_α and f_α the root vectors of roots α and $-\alpha$ so that $[e_\alpha, f_\alpha] = h_\alpha$.

Lemma 1.40. *Let L be a (not necessarily finite-dimensional) simple $\mathfrak{g}$-module and let v be a $\mathfrak{b}$-highest weight vector of L of $\mathfrak{b}$-highest weight λ. Let α be an isotropic odd simple root.*

(1) *If $\langle\lambda, h_\alpha\rangle = 0$, then L is a $\mathfrak{g}$-module of $\mathfrak{b}^\alpha$-highest weight λ and v is a $\mathfrak{b}^\alpha$-highest weight vector.*
(2) *If $\langle\lambda, h_\alpha\rangle \neq 0$, then L is a $\mathfrak{g}$-module of $\mathfrak{b}^\alpha$-highest weight $(\lambda-\alpha)$ and $f_\alpha v$ is a $\mathfrak{b}^\alpha$-highest weight vector.*

Proof. We first observe three simple identities:

(i) $e_\alpha f_\alpha v = [e_\alpha, f_\alpha]v = h_\alpha v = \langle\lambda, h_\alpha\rangle v$.

(ii) $e_\beta f_\alpha v = [e_\beta, f_\alpha]v = 0$ for any $\beta \in \Phi^+ \cap \Phi^+_\alpha$, since either $\beta - \alpha$ is not a root or it belongs to $\Phi^+ \cap \Phi^+_\alpha$.

(iii) $f_\alpha^2 v = 0$, since α is an isotropic odd root and so $f_\alpha^2 = 0$.

Now, we consider two cases separately.

(1) Assume that $\langle\lambda, h_\alpha\rangle = 0$. Then we must have $f_\alpha v = 0$, for otherwise $f_\alpha v$ would be a $\mathfrak{b}$-singular vector in the simple $\mathfrak{g}$-module L by (i) and (ii). Also $e_\beta v = 0$, for $\beta \in \Phi^+ \cap \Phi^+_\alpha$. Thus Lemma 1.30 implies that v is a $\mathfrak{b}^\alpha$-highest weight vector of weight λ in the $\mathfrak{g}$-module L.

(2) Assume that $\langle\lambda, h_\alpha\rangle \neq 0$. Then (i) above implies that $f_\alpha v$ is nonzero. Now it follows by (ii), (iii), and Lemma 1.30 that $f_\alpha v$ is a $\mathfrak{b}^\alpha$-highest weight vector of weight $\lambda - \alpha$ in L. □

Example 1.41. Let $\mathfrak{g} = \mathfrak{gl}(4|2)$. Denote the sequence of Borel subalgebras corresponding to the fundamental systems Π_i in Example 1.35 as $\mathfrak{b}_i$, for $0 \le i \le 8$, with $\mathfrak{b}^{\mathrm{st}} = \mathfrak{b}_0$. Consider the finite-dimensional irreducible $\mathfrak{gl}(4|2)$-module $L(\lambda)$, where $\lambda = a_1\delta_1 + a_2\delta_2 + a_3\delta_3 + a_4\delta_4 + b_1\varepsilon_1 + b_2\varepsilon_2$ with $a_1 \ge a_2 \ge a_3 \ge a_4 \ge 2$ and $b_1 \ge b_2 \ge 0$. We identify $\lambda = (a_1, a_2, a_3, a_4|b_1, b_2)$. The $\mathfrak{b}_i$-extremal weights, denoted by λ^i for $0 \le i \le 4$, are computed as follows:

$$\begin{aligned}
\lambda^0 &= \lambda = (a_1, a_2, a_3, a_4|b_1, b_2),\\
\lambda^1 &= (a_1, a_2, a_3, a_4 - 1|b_1 + 1, b_2),\\
\lambda^2 &= (a_1, a_2, a_3 - 1, a_4 - 1|b_1 + 2, b_2),\\
\lambda^3 &= (a_1, a_2 - 1, a_3 - 1, a_4 - 1|b_1 + 3, b_2),\\
\lambda^4 &= (a_1 - 1, a_2 - 1, a_3 - 1, a_4 - 1|b_1 + 4, b_2).
\end{aligned}$$

If we continue to consecutively apply to λ^4 the sequence of odd reflections with respect to the odd roots $\delta_4 - \varepsilon_2$, $\delta_3 - \varepsilon_2$, $\delta_2 - \varepsilon_2$, and $\delta_1 - \varepsilon_2$, we obtain the following $\mathfrak{b}_i$-extremal weights λ^i, for $5 \le i \le 8$:

$$\begin{aligned}
\lambda^5 &= (a_1 - 1, a_2 - 1, a_3 - 1, a_4 - 2|b_1 + 4, b_2 + 1),\\
\lambda^6 &= (a_1 - 1, a_2 - 1, a_3 - 2, a_4 - 2|b_1 + 4, b_2 + 2),\\
\lambda^7 &= (a_1 - 1, a_2 - 2, a_3 - 2, a_4 - 2|b_1 + 4, b_2 + 3),\\
\lambda^8 &= (a_1 - 2, a_2 - 2, a_3 - 2, a_4 - 2|b_1 + 4, b_2 + 4).
\end{aligned}$$

We shall see in Section 2.4 that all these weights λ^i afford very simple visualization in terms of Young diagrams.

1.5.4. Highest weight theory for $\mathfrak{q}(n)$. Let $\mathfrak{g} = \mathfrak{q}(n)$ be the queer Lie superalgebra. We recall several subalgebras of $\mathfrak{g}$ from Section 1.2.6. Let $\mathfrak{h}$ be the standard Cartan subalgebra consisting of block diagonal matrices and let $\mathfrak{b}$ be the standard Borel subalgebra of block upper triangular matrices of $\mathfrak{g}$. We have $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$. The Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_{\bar 0} \oplus \mathfrak{h}_{\bar 1}$ is a solvable but nonabelian Lie superalgebra, since $[\mathfrak{h}_{\bar 1}, \mathfrak{h}_{\bar 1}] = \mathfrak{h}_{\bar 0}$ and $[\mathfrak{h}_{\bar 0}, \mathfrak{h}_{\bar 0} + \mathfrak{h}_{\bar 1}] = 0$.

For $\lambda \in \mathfrak{h}_{\bar 0}^*$, define a symmetric bilinear form $\langle \cdot, \cdot \rangle_\lambda$ on $\mathfrak{h}_{\bar 1}$ by

$$\langle v, w \rangle_\lambda := \lambda([v, w]), \qquad \text{for } v, w \in \mathfrak{h}_{\bar 1}.$$

Denote by $\mathrm{Rad}\langle \cdot, \cdot \rangle_\lambda$ the radical of the form $\langle \cdot, \cdot \rangle_\lambda$. Then $\langle \cdot, \cdot \rangle_\lambda$ descends to a nondegenerate symmetric bilinear form on $\mathfrak{h}_{\bar 1}/\mathrm{Rad}\langle \cdot, \cdot \rangle_\lambda$, and it gives rise to a superalgebra $\mathcal{C}_\lambda$ as follows. Choosing an orthonormal basis $\{e_i\}$ of $\in \mathfrak{h}_{\bar 1}/\mathrm{Rad}\langle \cdot, \cdot \rangle_\lambda$ with respect to the form $\langle \cdot, \cdot \rangle_\lambda$, the associative superalgebra $\mathcal{C}_\lambda$ is generated by $\{e_i\}$ subject to the relations (3.23), and hence is isomorphic to the Clifford superalgebra $\mathcal{C}_k$, where $k = \dim \mathfrak{h}_{\bar 1}/\mathrm{Rad}\langle \cdot, \cdot \rangle_\lambda$ (see Definition 3.33). By definition we have an isomorphism of associative superalgebras

$$\mathcal{C}_\lambda \cong U(\mathfrak{h})/I_\lambda, \tag{1.51}$$

where I_λ denotes the ideal of $U(\mathfrak{h})$ generated by $\mathrm{Rad}\langle\cdot,\cdot\rangle_\lambda$ and $a-\lambda(a)$ for $a\in\mathfrak{h}_{\bar 0}$.

Let $\mathfrak{h}'_{\bar 1}\subseteq\mathfrak{h}_{\bar 1}$ be a maximal isotropic subspace with respect to $\langle\cdot,\cdot\rangle_\lambda$, and define the Lie subalgebra $\mathfrak{h}':=\mathfrak{h}_{\bar 0}\oplus\mathfrak{h}'_{\bar 1}$. The one-dimensional $\mathfrak{h}_{\bar 0}$-module $\mathbb{C}v_\lambda$, defined by $hv_\lambda=\lambda(h)v_\lambda$, extends to an $\mathfrak{h}'$-module by letting $\mathfrak{h}'_{\bar 1}.v_\lambda=0$. Define the induced $\mathfrak{h}$-module

$$W_\lambda:=\mathrm{Ind}_{\mathfrak{h}'}^{\mathfrak{h}}\mathbb{C}v_\lambda.$$

Recall the well-known fact that a Clifford superalgebra admits a unique irreducible ($\mathbb{Z}_2$-graded) module $\widetilde{W}_\lambda$ (see, for example, Exercise 3.11 in Chapter 3).

Lemma 1.42. *For $\lambda\in\mathfrak{h}_{\bar 0}^*$, the $\mathfrak{h}$-module W_λ is isomorphic to $\widetilde{W}_\lambda$ (viewed as an $\mathfrak{h}$-module via the pullback through* (1.51)*) and is irreducible. Furthermore, every finite-dimensional irreducible $\mathfrak{h}$-module is isomorphic to W_λ, for some $\lambda\in\mathfrak{h}_{\bar 0}^*$.*

Lemma 1.42 shows that the $\mathfrak{h}$-module W_λ is independent of a choice of a maximal isotropic subspace $\mathfrak{h}'_{\bar 1}$.

Proof. The action of $U(\mathfrak{h})$ on W_λ descends to an action of $U(\mathfrak{h})/I_\lambda$, and via (1.51) we identify W_λ with the unique irreducible module $\widetilde{W}_\lambda$ of the Clifford superalgebra $\mathcal{C}_\lambda$. Hence, the $\mathfrak{h}$-module W_λ is irreducible.

Suppose we are given an irreducible $\mathfrak{h}$-module U. Then it contains an $\mathfrak{h}_{\bar 0}$-weight vector v'_λ of weight $\lambda\in\mathfrak{h}_{\bar 0}^*$. Recall that $\mathfrak{h}'=\mathfrak{h}_{\bar 0}\oplus\mathfrak{h}'_{\bar 1}$, and consider the $\mathfrak{h}'$-submodule $\big(\pi,U(\mathfrak{h}')v'_\lambda\big)$ of U such that $\pi([\mathfrak{h}'_{\bar 1},\mathfrak{h}'_{\bar 1}])=\pi([\mathfrak{h}'_{\bar 0},\mathfrak{h}'_{\bar 0}])=0$. By applying Lemma 1.37 to Lie superalgebra $\pi(\mathfrak{h}')$, there exists a one-dimensional $\mathfrak{h}'$-submodule $\mathbb{C}v_\lambda$ of $U(\mathfrak{h}')v'_\lambda\subseteq U$. By Frobenius reciprocity we have a surjective $\mathfrak{h}$-homomorphism from W_λ onto U. Since W_λ is irreducible, we have $U\cong W_\lambda$. □

Let V be a finite-dimensional irreducible $\mathfrak{g}$-module. Pick an irreducible $\mathfrak{h}$-module W_λ in V, where $\lambda\in\mathfrak{h}_{\bar 0}^*$ can be taken to be maximal in the partial order induced by the positive system Φ^+ by the finite dimensionality of V. By definition, W_λ is $\mathfrak{h}_{\bar 0}$-semisimple of weight λ. For any $\alpha\in\Phi^+$ with associated even root vector e_α and odd root vector $\overline{e}_\alpha$ in $\mathfrak{n}^+$, the space $\mathbb{C}e_\alpha W_\lambda+\mathbb{C}\overline{e}_\alpha W_\lambda$ is an $\mathfrak{h}$-module that is $\mathfrak{h}_{\bar 0}$-semisimple of weight $\lambda+\alpha$. If $\mathbb{C}e_\alpha W_\lambda+\mathbb{C}\overline{e}_\alpha W_\lambda\neq 0$ for some $\alpha\in\Phi^+$, then it contains an isomorphic copy of $W_{\lambda+\alpha}$ as an $\mathfrak{h}$-submodule, contradicting the maximal weight assumption of λ. Hence, we have $\mathfrak{n}^+W_\lambda=0$. By irreducibility of V we must have $U(\mathfrak{n}^-)W_\lambda=V$, which gives rise to a weight space decomposition of $V=\bigoplus_{\mu\in\mathfrak{h}_{\bar 0}^*}V_\mu$. The space $W_\lambda=V_\lambda$ is the **highest weight space** of V, and it completely determines the irreducible module V. We denote V by $L(\mathfrak{g},\lambda)$, or $L(\lambda)$, if $\mathfrak{g}$ is evident from the context. Summarizing, we have proved the following.

Proposition 1.43. *Let $\mathfrak{g}=\mathfrak{q}(n)$. Any finite-dimensional irreducible $\mathfrak{g}$-module is a highest weight module.*

Let $\ell(\lambda)$ be the number of nonzero parts in a composition λ (generalizing the notation for length of a partition λ). We set

$$\delta(\lambda) = \begin{cases} 0, & \text{if } \ell(\lambda) \text{ is even}, \\ 1, & \text{if } \ell(\lambda) \text{ is odd}. \end{cases} \tag{1.52}$$

Now for $\lambda \in \mathfrak{h}^*$, recalling the notation H_i from (1.30), we identify λ with the composition $\lambda = (\lambda(H_1), \ldots, \lambda(H_n))$, and hence, $\ell(\lambda)$ is equal to the dimension of the space $\mathfrak{h}_{\bar{1}}/\mathrm{Rad}\langle\cdot|\cdot\rangle_\lambda$. We remark that the highest weight space W_λ of $L(\lambda)$ has dimension $2^{(\ell(\lambda)+\delta(\lambda))/2}$. Note that the Clifford superalgebra $\mathcal{C}_\lambda$ admits an odd automorphism if and only if $\ell(\lambda)$ is odd. Hence, the $\mathfrak{h}$-module W_λ, or equivalently the irreducible $\mathcal{C}_\lambda$-module W_λ, has an odd automorphism if and only if $\ell(\lambda)$ is an odd integer. An automorphism of the irreducible $\mathfrak{g}$-module $L(\lambda)$ clearly induces an $\mathfrak{h}$-module automorphism of its highest weight space. Conversely, any $\mathfrak{h}$-module automorphism on W_λ induces an automorphism of the $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} W_\lambda$. Since an automorphism preserves the maximal submodule, it induces an automorphism of the unique irreducible quotient $\mathfrak{g}$-module. We have proved the following.

Lemma 1.44. *Let $\mathfrak{g} = \mathfrak{q}(n)$ be the queer Lie superalgebra with a Cartan subalgebra $\mathfrak{h}$. Let $L(\lambda)$ be the irreducible $\mathfrak{g}$-module of highest weight $\lambda \in \mathfrak{h}^*_{\bar{0}}$. Then,*

$$\dim End_{\mathfrak{g}}(L(\lambda)) = 2^{\delta(\lambda)}.$$

This suggests that Schur's Lemma requires modification for superalgebras. This will be discussed in depth in Chapter 3, Lemma 3.4.

1.6. Exercises

Exercises 1.13, 1.14, 1.15, 1.16, and 1.24 below indicate that various classical theorems in the theory of Lie algebras fail for Lie superalgebras.

Exercise 1.1. Let φ be an automorphism of Lie superalgebra $\mathfrak{gl}(m|n)$, and let $J \in \mathfrak{gl}(m|n)_{\bar{0}}$. Define

$$\mathfrak{g}(\varphi, J) := \{g \in \mathfrak{gl}(m|n) \mid Jg - \varphi(g)J = 0\}.$$

Prove that $\mathfrak{g}(\varphi, J)$ is a subalgebra of $\mathfrak{gl}(m|n)$.

Exercise 1.2. Let $\mathfrak{J}_{2m|\ell}$ be as defined in 1.1.3, and let $\mathfrak{J}'_{2m|\ell}$ be the matrix obtained from $\mathfrak{J}_{2m|\ell}$ by substituting I_m with $-I_m$. Prove:

(1) There exists an automorphism φ of Lie superalgebra $\mathfrak{gl}(2m|\ell)$ given by $\varphi(g) = -g^{\mathrm{st}}$.

(2) $\mathfrak{g}(\varphi, \mathfrak{J}_{2m|\ell}) = \mathfrak{spo}(2m|\ell)$.

(3) $\mathfrak{g}(\varphi^3, \mathfrak{J}_{2m|\ell}) = \mathfrak{g}(\varphi, \mathfrak{J}'_{2m|\ell})$.

Exercise 1.3. Prove that $\Pi g^{\mathrm{st}} \Pi^{-1} = (\Pi g \Pi^{-1})^{\mathrm{st}^3}$, for $g \in \mathfrak{gl}(m|n)$; conclude that $\mathfrak{osp}(m|n) \cong \mathfrak{spo}(n|m)$, for n even (as stated in Remark 1.9).

Exercise 1.4. The space $\wedge(2)$ is naturally a $W(2)$-module with submodule $\mathbb{C}1$. Show that the action of $W(2)$ on $\wedge(2)/\mathbb{C}1$ is faithful, and thus induces, by identifying $\wedge(2)/\mathbb{C}1$ with $\mathbb{C}^{1|2}$, an isomorphism of Lie superalgebras $W(2) \cong \mathfrak{sl}(1|2)$.

Exercise 1.5. Prove that the following linear map $\psi : \mathfrak{spo}(2|2) \to \mathfrak{sl}(1|2)$ is an isomorphism of Lie superalgebras:

$$\begin{pmatrix} d & e & y_1 & x_1 \\ f & -d & -y & x \\ -x & x_1 & -a & 0 \\ y & y_1 & 0 & a \end{pmatrix} \stackrel{\psi}{\longmapsto} \begin{pmatrix} 2a & \sqrt{2}y_1 & \sqrt{2}y \\ \sqrt{2}x_1 & d+a & e \\ \sqrt{2}x & f & -d+a \end{pmatrix}.$$

Exercise 1.6. Suppose that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}$-graded Lie superalgebra such that $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$, $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$, and $[\mathfrak{g}_1, \mathfrak{g}_{-1}] = \mathfrak{g}_0$. Assume further that $\mathfrak{g}_0$ is a simple Lie algebra and $\mathfrak{g}_{\pm 1}$ are irreducible $\mathfrak{g}_0$-modules such that $\mathrm{Hom}_{\mathfrak{g}_0}(\mathfrak{g}_0, \mathfrak{g}_1 \otimes \mathfrak{g}_{-1}) \cong \mathbb{C}$. Prove that these data determine the Lie superalgebra structure on $\mathfrak{g}$ uniquely.

Exercise 1.7. Prove:

(1) The simple Lie superalgebras $S(3)$ and $[\mathfrak{p}(3), \mathfrak{p}(3)]$ are isomorphic. (Hint: use Exercise 1.6.)

(2) The Lie superalgebra $H(4)$ and $\mathfrak{sl}(2|2)/\mathbb{C}I_{2|2}$ are isomorphic.

Exercise 1.8. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional simple Lie superalgebra with $\mathfrak{g}_{\bar{1}} \neq 0$. Prove:

(1) $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{1}}$.

(2) $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{0}}$.

(3) The $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is faithful.

Exercise 1.9. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra such that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{g}_{\bar{0}}$ and the adjoint $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is faithful and irreducible. Prove that $\mathfrak{g}$ is simple.

Exercise 1.10. Let $\mathfrak{g} = \bigoplus_{j=-1}^{n} \mathfrak{g}_j$ be a $\mathbb{Z}$-graded Lie superalgebra satisfying

(1) $[\mathfrak{g}_{-1}, x] = 0$ implies that $x = 0$, for all $x \in \mathfrak{g}_j$ with $j \geq 0$;

(2) The adjoint $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible;

(3) $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$, $[\mathfrak{g}_0, \mathfrak{g}_1] = \mathfrak{g}_1$, and $\mathfrak{g}_{j+1} = [\mathfrak{g}_j, \mathfrak{g}_1]$ for $j \geq 1$.

Prove that $\mathfrak{g}$ is simple.

Exercise 1.11. Let $\mathfrak{g} = \mathfrak{sl}(2|2)/\mathbb{C}I_{2|2}$. Prove that $\dim \mathfrak{g}_\alpha = 2$, for $\alpha \in \Phi_{\bar{1}}$.

Exercise 1.12. Let $\mathfrak{g}$ be a Lie superalgebra. Prove that $\mathfrak{g} \otimes \wedge(n)$ is a Lie superalgebra with Lie bracket defined by

$$[a \otimes \lambda, b \otimes \mu] = (-1)^{|\lambda||b|}[a, b] \otimes \lambda\mu, \quad a, b \in \mathfrak{g}; \lambda, \mu \in \wedge(n).$$

Exercise 1.13. Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra. The Lie superalgebra $W(n)$ acts naturally on $\mathfrak{g} \otimes \wedge(n)$ so that the semidirect product $\mathcal{G} = (\mathfrak{g} \otimes \wedge(n)) \rtimes W(n)$ is a Lie superalgebra. Prove that $\mathcal{G}$ is **semisimple** (which by definition means that $\mathcal{G}$ has no nontrivial solvable ideal.)

Exercise 1.14. Find a filtration of subalgebras for $\mathcal{G}$ in Exercise 1.13 of the form

$$0 = \mathcal{G}_0 \subsetneq \mathcal{G}_1 \subsetneq \mathcal{G}_2 \subsetneq \cdots \subsetneq \mathcal{G}_{k-1} \subsetneq \mathcal{G}_k = \mathcal{G},$$

for some k, such that $\mathcal{G}_{i-1}$ is an ideal in $\mathcal{G}_i$ wtih $\mathcal{G}_i/\mathcal{G}_{i-1}$ simple, for all $i = 1, \ldots, k$. Prove that $\mathcal{G}$ is not a direct sum of simple Lie superalgebras.

Exercise 1.15. Let $\mathcal{G}$ be constructed as in Exercise 1.13 from a not necessarily simple Lie superalgebra $\mathfrak{g}$. Suppose that V is a faithful irreducible representation of $\mathfrak{g}$. Prove that, as a representation of $\mathcal{G}$, $V \otimes \wedge(n)$ is faithful and irreducible.

Exercise 1.16. Continuing Exercise 1.15, assume that $\mathfrak{g}$ is the one-dimensional abelian Lie algebra and V is a nontrivial irreducible representation of $\mathfrak{g}$. Prove that $\mathcal{G}$ is not reductive, i.e., $\mathcal{G}$ is not a direct sum of a semisimple Lie superalgebra and a one-dimensional even subalgebra. (It is known that a finite-dimensional Lie algebra possessing a finite-dimensional faithful representation is reductive.)

Exercise 1.17. Let $\mathfrak{g}$ be a simple Lie superalgebra whose Killing form is non-degenerate. Prove that every derivation of $\mathfrak{g}$ is inner.

Exercise 1.18. Prove that the Killing forms on the Lie superalgebras $\mathfrak{sl}(k|\ell)$ with $k \neq \ell$, $k + \ell \geq 2$, and $\mathfrak{osp}(m|2n)$ with $m - 2n \neq 2$, $m + 2n \geq 2$ are non-degenerate. Conclude that the derivations of these Lie superalgebras are all inner.

Exercise 1.19. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional simple Lie superalgebra. Suppose that $\mathfrak{g}_{\bar{0}}$ is a semisimple Lie algebra and let $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g}) \oplus W$ be a decomposition of $\mathrm{ad}(\mathfrak{g}_{\bar{0}})$-modules. Prove:

(1) W is a trivial $\mathrm{ad}(\mathfrak{g}_{\bar{0}})$-module.

(2) For $D \in W$, the map $D : \mathfrak{g} \to \mathfrak{g}$ is an $\mathrm{ad}(\mathfrak{g}_{\bar{0}})$-homomorphism vanishing on $\mathfrak{g}_{\bar{0}}$.

Exercise 1.20. Use Exercise 1.19(2) to prove the following:

(1) Every derivation of $\mathfrak{osp}(m|2n)$ is inner.

(2) The space $\mathrm{Der}(\mathfrak{g})/\mathrm{ad}(\mathfrak{g})$ for $\mathfrak{sl}(m+1|m+1)/\mathbb{C}I_{m+1|m+1}$, $[\mathfrak{p}(m), \mathfrak{p}(m)]$, and $[\mathfrak{q}(m), \mathfrak{q}(m)]/\mathbb{C}I_{m|m}$, for $m \geq 2$, are all one-dimensional.

(3) The space $\mathrm{Der}(\mathfrak{g})/\mathrm{ad}(\mathfrak{g})$ for $\mathfrak{sl}(2|2)/\mathbb{C}I_{2|2}$ is three-dimensional.

Exercise 1.21. Let $\mathfrak{g} = \bigoplus_{j \geq -1} \mathfrak{g}_j$ be the simple $\mathbb{Z}$-graded Lie superalgebra $W(m)$, $S(m)$, or $H(m+1)$, for $m \geq 3$, with principal gradation. Let $\mathfrak{b}_0$ be a Borel subalgebra of $\mathfrak{g}_0$. Set $\mathfrak{g}_{\geq 0} = \bigoplus_{j \geq 0} \mathfrak{g}_j$, $\mathfrak{g}_{>0} = \bigoplus_{j > 0} \mathfrak{g}_j$, and $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{g}_{>0}$. Prove:

(1) $\mathfrak{b}$ is a solvable Lie superalgebra satisfying $[\mathfrak{b}_{\bar{1}}, \mathfrak{b}_{\bar{1}}] \subseteq [\mathfrak{b}_{\bar{0}}, \mathfrak{b}_{\bar{0}}]$.

(2) A finite-dimensional irreducible representation of $\mathfrak{g}_{\geq 0}$ is an irreducible representation of $\mathfrak{g}_0$ on which the subalgebra $\mathfrak{g}_{>0}$ acts trivially.
(3) Any finite-dimensional irreducible module of $\mathfrak{g}$ is a quotient of $\mathrm{Ind}_{\mathfrak{g}_{\geq 0}}^{\mathfrak{g}} V$, where V is a finite-dimensional irreducible representation of $\mathfrak{g}_{\geq 0}$.

Conclude that, up to isomorphism, finite-dimensional irreducible $\mathfrak{g}_0$-modules are in one-to-one correspondence with finite-dimensional irreducible $\mathfrak{g}$-modules.

Exercise 1.22. Let $\mathfrak{g} = [\mathfrak{p}(2), \mathfrak{p}(2)]$. Prove:

(1) $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$, and hence $\mathfrak{g}$ is not simple.
(2) $\mathfrak{g}$ is semisimple.
(3) Any nontrivial finite-dimensional irreducible $\mathfrak{g}$-module is a direct sum of two copies of the same irreducible $\mathfrak{g}_{\bar{0}}$-module.

Exercise 1.23. It is known that every finite-dimensional Lie algebra has a finite-dimensional faithful representation. Prove that every finite-dimensional Lie superalgebra has a finite-dimensional faithful representation.

Exercise 1.24. Prove that the exact sequence of Lie superalgebras

$$0 \longrightarrow \mathbb{C}I_{m|m} \to \mathfrak{sl}(m|m) \longrightarrow \mathfrak{sl}(m|m)/\mathbb{C}I_{m|m} \longrightarrow 0$$

is non-split, though $\mathfrak{sl}(m|m)/\mathbb{C}I_{m|m}$ is simple, for $m \geq 2$. (Hence Levi's theorem fails for Lie superalgebras.)

Exercise 1.25. Prove that the bilinear form (1.33) induced from the odd trace form (1.32) is a symmetric non-degenerate invariant form on $\mathfrak{q}(n)$.

Notes

Section 1.1. The classification of finite-dimensional simple complex Lie superalgebras was first announced by Kac in [**59**]. The detailed proof of the classification, along with many other fundamental results on Lie superalgebras, appeared in Kac [**60**] two years later. An independent proof of the classification of the finite-dimensional complex simple Lie superalgebras whose even subalgebras are reductive was given in the two papers by Scheunert, Nahm, and Rittenberg in [**106**] around the same time.

Section 1.2. The structure theory of root systems, root space decompositions, and invariant bilinear forms of basic Lie superalgebras presented here is fairly standard. More details can be found in the standard references (see Kac [**60, 62**] and Scheunert [**105**]).

Section 1.3. The concept of odd reflections was introduced by Leites, Saveliev, and Serganova [**78**] to relate non-conjugate Borel subalgebras, fundamental and positive systems. The list of conjugacy classes of fundamental systems under the Weyl group action for the basic Lie algebras was given by Kac [**60**]. We introduce a notion of $\varepsilon\delta$-sequences (which appeared in Cheng-Wang [**32**]) to facilitate the parametrization of the conjugacy classes of fundamental systems for Lie superalgebras of type $\mathfrak{gl}$ and $\mathfrak{osp}$. For definitions of Borel subalgebras for general Lie superalgebras, see [**90, 95**].

Section 1.4. Lemma 1.30 on odd reflections appeared for more general Kac-Moody Lie superalgebras in Kac-Wakimoto [**66**, Lemma 1.2], and it is sometimes attributed to Serganova's 1988 thesis (cf., e.g., [**108**]).

Section 1.5. A detailed proof of the PBW Theorem for Lie superalgebras, Theorem 1.36, can be found in Milnor-Moore [**85**, Theorem 6.20] and Ross [**102**, Theorem 2.1]. As in the classical theory, the first step to develop a highest weight theory for the basic and queer Lie superalgebras is to study representations of Borel subalgebras. Lemma 1.37 on solvable Lie superalgebras appeared in Kac [**60**, Proposition 5.2.4]. The proof given here is different. Lemma 1.40 on the change of the extremal weights of an irreducible module under an odd reflection (which appeared in Penkov-Serganova [**94**, Lemma 1]) plays a fundamental role in representation theory of Lie superalgebras developed in the book.

Exercises 1.8, 1.9, and 1.17–1.20 are taken from [**60**] and [**105**], while Exercise 1.21 was implicit in [**60**]. Exercise 1.16 was inspired by similar examples for Lie algebras in prime characteristic, which we learned from A. Premet.