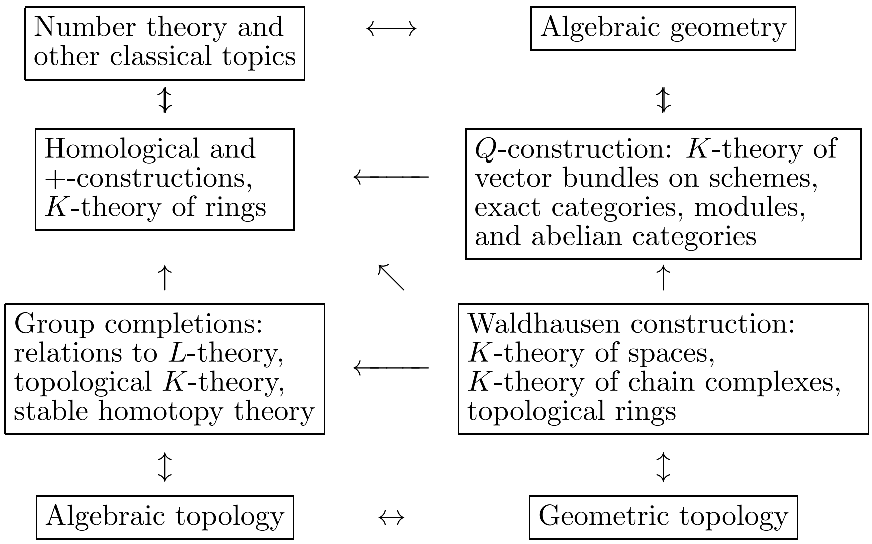
Preface

Algebraic K-theory has two components: the classical theory which centers around the Grothendieck group K_0 of a category and uses explicit algebraic presentations and higher algebraic K-theory which requires topological or homological machinery to define.

There are three basic versions of the Grothendieck group K_0. One involves the group completion construction and is used for projective modules over rings, vector bundles over compact spaces, and other symmetric monoidal categories. Another adds relations for exact sequences and is used for abelian categories as well as exact categories; this is the version first used in algebraic geometry. A third adds relations for weak equivalences and is used for categories of chain complexes and other categories with cofibrations and weak equivalences ("Waldhausen categories").

Similarly, there are four basic constructions for higher algebraic K-theory: the $+$-construction (for rings), the group completion constructions (for symmetric monoidal categories), Quillen's Q-construction (for exact categories), and Waldhausen's $wS.$ construction (for categories with cofibrations and weak equivalences). All these constructions give the same K-theory of a ring but are useful in various distinct settings. These settings fit together as in the table that follows.

All the constructions have one feature in common: some category C is concocted from the given setup, and one defines a K-theory space associated to the geometric realization BC of this category. The K-theory groups are then the homotopy groups of the K-theory space. In the first chapter, we introduce the basic cast of characters: projective modules and vector bundles (over a topological space and over a scheme). Large segments of this chapter will be familiar to many readers, but which segments are familiar will depend



upon the background and interests of the reader. The unfamiliar parts of this material may be skipped at first and referred back to when relevant. We would like to warn the complacent reader that the material on the Picard group and Chern classes for topological vector bundles is in the first chapter.

In the second chapter, we define K_0 for all the settings in the above table and give the basic definitions appropriate to these settings: group completions for symmetric monoidal categories, K_0 for rings and topological spaces, λ-operations, abelian and exact categories, Waldhausen categories. All definitions and manipulations are in terms of generators and relations. Our philosophy is that this algebraic beginning is the most gentle way to become acquainted with the basic ideas of higher K-theory. The material on K-theory of schemes is isolated in a separate section, so it may be skipped by those not interested in algebraic geometry.

In the third chapter we give a brief overview of the classical K-theory for K_1 and K_2 of a ring. Via the Fundamental Theorem, this leads to Bass's "negative K-theory," meaning groups K_{-1}, K_{-2}, etc. We cite Matsumoto's presentation for K_2 of a field from [**131**] and "Hilbert's Theorem 90 for K_2" (from [**125**]) in order to get to the main structure results. This chapter ends with a section on Milnor K-theory, including the transfer map, Izhboldin's Theorem on the lack of p-torsion, the norm residue symbol, and the relation to the Witt ring of a field.

In the fourth chapter we shall describe the four constructions for higher K-theory, starting with the original BGL^+ construction. In the case of $\mathbf{P}(R)$, finitely generated projective R-modules, we show that all the constructions give the same K-groups: the groups $K_n(R)$. The λ-operations are developed in terms of the $S^{-1}S$ construction. Nonconnective spectra and homotopy K-theory are also presented. Very few theorems are present here, in order to keep this chapter short. We do not want to get involved in the technicalities lying just under the surface of each construction, so the key topological results we need are cited from the literature when needed.

The fundamental structural theorems for higher K-theory are presented in Chapter V. This includes Additivity, Approximation, Cofinality, Resolution, Devissage, and Localization (including the Thomason-Trobaugh localization theorem for schemes). As applications, we compute the K-theory and G-theory of projective spaces and Severi-Brauer varieties (§1), construct transfer maps satisfying a projection formula (§3), and prove the Fundamental Theorem for G-theory (§6) and K-theory (§8). Several cases of Gersten's DVR (discrete valuation domain) Conjecture are established in §6 and the Gersten-Quillen Conjecture is established in §9. This is used to interpret the coniveau spectral sequence in terms of K-cohomology and establish Bloch's formula that $CH^p(X) \cong H^p(X, \mathcal{K}_p)$ for regular varieties.

In Chapter VI we describe the structure of the K-theory of fields. First we handle algebraically closed fields (§1) and the real numbers $\mathbb{R}$ (§3) following Suslin and Harris-Segal. The group $K_3(F)$ can also be handled by comparison to Bloch's group $B(F)$ using these methods (§5). In order to say more, using classical invariants such as étale cohomology, we introduce the spectral sequence from motivic cohomology to K-theory in §4 and use it in §§6–10 to describe the K-theory of local and global fields.

Text cross-references to definitions, figures, equations, and other items use the following conventions. Within Chapter IV, for example, text cross-references to Definition 1.1, Figure 4.9.1, and equation (5.3.2) of Chapter IV are referred to as Definition 1.1, Figure 4.9.1, and (5.3.2). Outside of Chapter IV, they are referred to as Definition IV.1.1, Figure IV.4.9.1, and (IV.5.3.2).

The back story

In 1985, I started hearing a persistent rumor that I was writing a book on algebraic K-theory. This was a complete surprise to me! Someone else had started the rumor, and I never knew who. After a few years, I had heard the rumor from at least a dozen people.

It actually took a decade before the rumor had become true—like the character Topsy[1], the book project was never born, it just grew. In 1988

[1] *Topsy* is a character in Harriet B. Stowe's 1852 book *Uncle Tom's Cabin*, who claimed to have never been born: "Never was born! I spect I grow'd. Don't think nobody never made me [sic]."

I wrote out a brief outline, following Quillen's paper *Higher Algebraic K-theory: I* [**153**]. It was overwhelming. I talked to Hy Bass, the author of the classic book *Algebraic K-theory* [**15**], about what would be involved in writing such a book. It was scary, because (in 1988) I didn't know how to write a book at all.

I needed a warm-up exercise, a practice book if you will. The result, *An Introduction to Homological Algebra* [**223**], took over five years to write.

By this time (1995), the K-theory landscape had changed and with it my vision of what my K-theory book should be. Was it an obsolete idea? After all, the new developments in motivic cohomology were affecting our knowledge of the K-theory of fields and varieties. In addition, there was no easily accessible source for this new material. Nevertheless, I wrote early versions of Chapters I–IV during 1994–1999. The project became known as the "K-book" at the time.

In 1999, I was asked to turn a series of lectures by Voevodsky into a book. This project took over six years, in collaboration with Carlo Mazza and Vladimir Voevodsky. The result was the book *Lecture Notes on Motivic Cohomology* [**122**], published in 2006.

In 2004–2008, Chapters IV and V were completed. At the same time, the final steps in the proof of the Norm Residue Theorem VI.4.1 were finished. (This settles not just the Bloch-Kato Conjecture, but also the Beilinson-Lichtenbaum Conjectures and Quillen-Lichtenbaum Conjectures.) The proof of this theorem is scattered over a dozen papers and preprints, and writing it spanned over a decade of work, mostly by Rost and Voevodsky. Didn't it make sense to put this house in order? It did. I am currently collaborating with Christian Haesemeyer in writing a self-contained proof of this theorem.

Acknowledgements

The author is grateful to whoever started the rumor that he was writing this book. He is also grateful to the many people who have made comments on the various versions of this manuscript over the years: R. Thomason, D. Grayson, T. Geisser, C. Haesemeyer, J.-L. Loday, M. Lorenz, J. Csirik, M. Paluch, Paul Smith, P. A. Østvær, A. Heider, J. Hornbostel, B. Calmes, G. Garkusha, P. Landweber, A. Fernandez Boix, C. Mazza, J. Davis, I. Leary, C. Crissman, P. Polo, R. Brasca, O. Braeunling, F. Calegari, K. Kedlaya, D. Grinberg, P. Boavida, R. Reis, J. Levikov, O. Schnuerer, P. Pelaez, Sujatha, J. Spakula, J. Cranch, A. Asok, ...

Charles A. Weibel

August 2012