Chapter 4

Elements of the Nonuniform Hyperbolicity Theory

Hyperbolicity theory is a fundamental new step in developing the Lyapunov stability theory for dynamical systems with either continuous or discrete time. It deals almost exclusively with situations where stability is conditional, i.e., appears only along some but not all directions in space. Indeed, along other directions the system is unstable. Trajectories with such conditional stability are said to be hyperbolic, either uniformly or nonuniformly according to how the rates of contraction and expansion vary from point to point. While trajectories that are completely stable (i.e., stable along all directions in space) are usually isolated, hyperbolic trajectories can occupy a "large" part of the phase space (for example, they may form a set of positive or full Lebesgue measure) and in the case of Anosov systems every trajectory is hyperbolic.

In this chapter we are mostly concerned with the concept of nonuniform hyperbolicity. In this case the rates of contraction and expansion may vary from point to point in a "wild" and, in general, uncontrollable way. It turns out, however, that for trajectories that are LP-regular one can obtain some important estimates on how the rates of contraction and expansion vary along the trajectories. Although at first glance these estimates seem to be technical, they play a crucial role in studying ergodic properties of hyperbolic systems.

In this connection one may wonder whether a given dynamical system admits LP-regular trajectories and if so how large the set of such trajectories is. A celebrated result by Oseledets known as the Multiplicative Ergodic Theorem (the name was coined by Oseledets) claims that LP-regular trajectories form a set of full measure with respect to any invariant Borel measure on M. Thus, from the measure-theoretic point of view, there is no need to impose the requirement that a given trajectory is LP-regular.

4.1. Dynamical systems with nonzero Lyapunov exponents

Consider a flow φ_t on a smooth compact Riemannian p-manifold M which is generated by the vector field $\mathfrak{X}$ on M such that

$$\mathfrak{X}(x) = \frac{d\varphi_t(x)}{dt}|_{t=0}.$$

We will always assume that the vector field $\mathfrak{X}$ depends smoothly on x and we refer to φ_t as a smooth flow on M. For every $x_0 \in M$ the trajectory $\{x(x_0, t) = \varphi_t(x_0) : t \in \mathbb{R}\}$ represents a solution of the nonlinear differential equation

$$\dot{v} = \mathfrak{X}(v) \tag{4.1}$$

on the manifold M. This solution is uniquely defined by the initial condition $x(x_0, 0) = x_0$.

With the flow φ_t one can associate a certain collection of single linear differential equations (4.1) "along" each trajectory of the flow. The stability of a given trajectory can be described by studying small perturbations of the system of variational equations. The perturbation term is of type (3.9) and since the flow is smooth, it satisfies condition (3.10). The results of the previous section apply and allow one to study the stability of trajectories via the Lyapunov exponents. Although it is still a very difficult problem to verify whether a given trajectory is Lyapunov regular, it turns out that "most" trajectories (in the sense of measure theory) have this property.

A similar but technically simpler approach can be used to establish the stability of trajectories of dynamical systems with discrete time.

Let $f\colon M \to M$ be a diffeomorphism of a compact smooth Riemannian p-dimensional manifold M. Given $x \in M$, consider the trajectory $\{f^m(x)\}_{m\in\mathbb{Z}}$. The family of maps $\{d_{f^m(x)}f\}_{m\in\mathbb{Z}}$ can be viewed as an analog of the system of variational equations in the continuous time case. Given $x \in M$ and $v \in T_xM$, the formula

$$\chi^+(x, v) = \limsup_{m\to+\infty} \frac{1}{m} \log \|d_x f^m v\| \tag{4.2}$$

defines the *Lyapunov exponent* specified by the diffeomorphism f at the point x. By the general theory of Lyapunov exponents (see Section 2.1), the function $\chi^+(x,\cdot)$ attains only finitely many values on $T_xM\setminus\{0\}$, which we denote by

$$\chi_1^+(x)<\cdots<\chi_{s^+(x)}^+(x),$$

where $s^+(x)\le p$. We also denote by $\mathcal{V}_x^+$ the filtration of T_xM associated to $\chi^+(x,\cdot)$:

$$\{0\}=V_0^+(x)\subsetneqq V_1^+(x)\subsetneqq\cdots\subsetneqq V_{s^+(x)}^+(x)=T_xM,$$

where $V_i^+(x)=\{v\in T_xM:\chi^+(x,v)\le\chi_i^+(x)\}$. The number

$$k_i^+(x)=\dim V_i^+(x)-\dim V_{i-1}^+(x)$$

is the *multiplicity* of the value $\chi_i^+(x)$. We have

$$\sum_{i=1}^{s^+(x)}k_i^+(x)=p.$$

Finally, the collection of pairs

$$\operatorname{Sp}\chi^+(x)=\{(\chi_i^+(x),k_i^+(x)):1\le i\le s^+(x)\}$$

is called the *Lyapunov spectrum* of the exponent $\chi^+(x,\cdot)$.

We note that the functions $\chi_i^+(x)$, $s^+(x)$, and $k_i^+(x)$ are *invariant* under f and are Borel *measurable* (but not necessarily continuous).

For every $x\in M$ and $v\in T_xM$ we set

$$\chi^-(x,v)=\limsup_{m\to-\infty}\frac{1}{|m|}\log\|d_xf^mv\|.$$

Again, by the general theory of Lyapunov exponents, the function $\chi^-(x,\cdot)$ takes on finitely many values on $T_xM\setminus\{0\}$:

$$\chi_1^-(x)>\cdots>\chi_{s^-(x)}^-(x),$$

where $s^-(x)\le p$. We denote by $\mathcal{V}_x^-$ the filtration of T_xM associated to $\chi^-(x,\cdot)$:

$$T_xM=V_1^-(x)\supsetneqq\cdots\supsetneqq V_{s^-(x)}^-(x)\supsetneqq V_{s^-(x)+1}^-(x)=\{0\},$$

where $V_i^-(x)=\{v\in T_xM:\chi^-(x,v)\le\chi_i^-(x)\}$. The number

$$k_i^-(x)=\dim V_i^-(x)-\dim V_{i+1}^-(x)$$

is the *multiplicity* of the value $\chi_i^-(x)$. The collection of pairs

$$\operatorname{Sp}\chi^-(x)=\{(\chi_i^-(x),k_i^-(x))\colon i=1,\dots,s^-(x)\}$$

is called the *Lyapunov spectrum* of the exponent $\chi^-(x,\cdot)$.

In order to simplify our notation, in what follows, we will often drop the superscript + from the notation of the forward Lyapunov exponents and the associated quantities if it does not cause any confusion.

We say that a point $x \in M$ is

(1) *forward regular* if the Lyapunov exponent χ^+ is forward regular;
(2) *backward regular* if the Lyapunov exponent χ^- is backward regular;
(3) *Lyapunov–Perron regular* or simply *LP-regular* if the Lyapunov exponent χ^+ is forward regular, the Lyapunov exponent χ^- is backward regular, and the filtrations $\mathcal{V}_x^-$ and $\mathcal{V}_x^-$ are coherent (see Sections 2.4 and 2.5).

Note that if x is LP-regular, then so is the point $f(x)$ and thus, one can speak of the whole trajectory $\{f^m(x)\}$ as being forward, backward, or LP-regular.

We state a result characterizing the LP-regularity, which is an analog of Theorem 2.21 for the discrete time case. Recall that the cotangent bundle T^*M consists of 1-forms on M. The diffeomorphism f acts on T^*M by its codifferential

$$d_x^* f\colon T^*_{f(x)}M \to T^*_x M$$

defined by

$$d_x^* f\varphi(v) = \varphi(d_x f v), \quad v \in T_x M, \quad \varphi \in T^*_{f(x)}M.$$

We denote the inverse map by

$$d'_x f = (d_x^* f)^{-1}\colon T^*_x M \to T^*_{f(x)}M$$

and define the Lyapunov exponent χ^* on T^*M by the formula

$$\chi^*(x, \varphi) = \limsup_{m\to+\infty} \frac{1}{m} \log \|d'_x f^m \varphi\|.$$

Given $\mathbf{v} = (v_1, \dots, v_k)$, we denote by $\Gamma_k^{\mathbf{v}}(m)$ the volume of the parallelepiped generated by the vectors $d_x f^m v_1, \dots, d_x f^m v_k$.

Theorem 4.1. *The point $x \in M$ is LP-regular if and only if there exist a decomposition*

$$T_x M = \bigoplus_{i=1}^{s(x)} E_i(x) \tag{4.3}$$

into subspaces $E_i(x)$ and numbers $\chi_1(x) < \cdots < \chi_{s(x)}(x)$ such that:

(1) *$E_i(x)$ is invariant under $d_x f$, i.e.,*

$$d_x f E_i(x) = E_i(f(x)),$$

and depends Borel measurably on x;

(2) *for $v \in E_i(x) \setminus \{0\}$,*

$$\lim_{m\to\pm\infty} \frac{1}{m} \log \|d_x f^m v\| = \chi_i(x)$$

with uniform convergence on $\{v \in E_i(x) : \|v\| = 1\}$;

(3) *if* $\mathbf{v} = (v_1, \dots, v_{k_i(x)})$ *is a basis of* $E_i(x)$*, then*

$$\lim_{m\to\pm\infty} \frac{1}{m} \log \Gamma^{\mathbf{v}}_{k_i(x)}(m) = \chi_i(x)k_i(x);$$

(4) *for any* $v, w \in T_xM \setminus \{0\}$ *with* $\angle(v, w) \neq 0$,

$$\lim_{m\to\pm\infty} \frac{1}{m} \log \angle(d_xf^m v, d_xf^m w) = 0;$$

(5) *the Lyapunov exponent* $\chi(x, \cdot)$ *is exact, that is,*

$$\liminf_{m\to\pm\infty} \frac{1}{m} \log \Gamma^{\mathbf{v}}_k(m) = \limsup_{m\to\pm\infty} \frac{1}{m} \log \Gamma^{\mathbf{v}}_k(m)$$

for any $1 \le k \le p$ *and any vectors* $v_1, \dots, v_k$*;*

(6) *there exists a decomposition of the cotangent bundle*

$$T^*_xM = \bigoplus_{i=1}^{s(x)} E^*_i(x)$$

into subspaces $E^*_i(x)$ *associated with the Lyapunov exponent* χ^**; the subspaces* $E^*_i(x)$ *are invariant under* d'_xf*, i.e.,*

$$d'_xfE^*_i(x) = E^*_i(f(x)),$$

and depend (Borel) measurably on x*; moreover, if* $\{v_i(x) : i = 1, \dots, p\}$ *is a subordinate basis with* $v_i(x) \in E_j(x)$ *for* $n_{j-1}(x) < i \le n_j(x)$ *and if* $\{v^*_i(x) : i = 1, \dots, p\}$ *is a dual basis, then* $v^*_i(x) \in E^*_j(x)$ *for* $n_{j-1}(x) < i \le n_j(x)$.

The decomposition (4.3) is called the *Oseledets decomposition* associated with the Lyapunov exponent (4.2). The following theorem of Oseledets (see [**65**]) is a key result in studying the LP-regularity of trajectories of dynamical systems. It shows that LP-regularity is "typical" from the measure-theoretic point of view.

Theorem 4.2 (Multiplicative Ergodic Theorem)**.** *If* f *is a* C^1 *diffeomorphism of a compact smooth Riemannian manifold* M*, then the set of Lyapunov–Perron regular points has full measure with respect to any* f*-invariant Borel probability measure on* M*.*

The proof of this theorem is given in Chapter 6.

We denote by $\mathcal{R} \subset M$ the set of points that are LP-regular. Although the notion of LP-regularity does not require any invariant measure to be present, the crucial consequence of Theorem 4.2 is that $\mathcal{R}$ is nonempty and indeed has full measure with respect to any f-invariant Borel probability measure on M.

We stress that there may exist trajectories which are both forward and backward regular but *not* LP-regular. However, such trajectories form a negligible set with respect to any f-invariant Borel probability measure. Note that only in some exceptional situations is *every* point in M LP-regular.[1]

Let ν be an *ergodic* f-invariant Borel measure. Since the values of Lyapunov exponents are invariant Borel functions, there exist numbers $s = s^\nu$, $\chi_i = \chi_i^\nu$, and $k_i = k_i^\nu$ for $i = 1, \dots, s$ such that

$$s(x) = s, \quad \chi_i(x) = \chi_i, \quad k_i(x) = k_i \tag{4.4}$$

for ν-almost every x. The collection of pairs

$$\operatorname{Sp} \chi(\nu) = \{(\chi_i, k_i) : 1 \le i \le s\}$$

is called the *Lyapunov spectrum* of the measure ν.

We now consider dynamical systems whose spectrum of the Lyapunov exponent does not vanish on some subset of M. More precisely, let

$$\mathcal{E} = \big\{x \in \mathcal{R} : \text{there exists } 1 \le k(x) < s(x) \\ \text{with } \chi_{k(x)}(x) < 0 \text{ and } \chi_{k(x)+1}(x) > 0\big\}. \tag{4.5}$$

This set is f-invariant. We say that f is a *dynamical system with nonzero Lyapunov exponents* if there exists an f-invariant Borel probability measure ν on M such that $\nu(\mathcal{E}) = 1$. The measure ν is called a *hyperbolic measure* for f.

Observe that every point $x \in \mathcal{E}$ is LP-regular and satisfies (4.4). Therefore, by the Multiplicative Ergodic Theorem, for every $x \in \mathcal{E}$,

$$E^s(x) = \bigoplus_{i=1}^{k} E_i(x) \quad \text{and} \quad E^u(x) = \bigoplus_{i=k+1}^{s} E_i(x).$$

The following theorem from [**68**] describes the properties of these subspaces.

Theorem 4.3. *The subspaces $E^s(x)$ and $E^u(x)$, $x \in \mathcal{E}$, have the following properties:*

(L1) *they depend Borel measurably on x;*

(L2) *they form a splitting of the tangent space, i.e. $T_xM = E^s(x) \oplus E^u(x)$;*

(L3) *they are invariant:*

$$d_xfE^s(x) = E^s(f(x)) \quad \text{and} \quad d_xfE^u(x) = E^u(f(x)).$$

Furthermore, given $x \in \mathcal{E}$, there exist $\varepsilon_0 = \varepsilon_0(x) > 0$ and functions $C(x, \varepsilon) > 0$ and $K(x, \varepsilon) > 0$ with $0 < \varepsilon \le \varepsilon_0$ such that:

[1] This is true, for example, when f is a linear hyperbolic toral automorphism (see Section 1.1).

(L4) *the subspace* $E^s(x)$ *is* stable*: if* $v \in E^s(x)$ *and* $n > 0$*, then*
$$\|d_x f^n v\| \le C(x,\varepsilon) e^{(\chi_{k(x)}(x)+\varepsilon)n} \|v\|;$$

(L5) *the subspace* $E^u(x)$ *is* unstable*: if* $v \in E^u(x)$ *and* $n < 0$*, then*
$$\|d_x f^n v\| \le C(x,\varepsilon) e^{(\chi_{k(x)+1}(x)-\varepsilon)n} \|v\|;$$

(L6)
$$\angle(E^s(x), E^u(x)) \ge K(x,\varepsilon);$$

(L7) *the functions* $C(x,\varepsilon)$ *and* $K(x,\varepsilon)$ *are Borel measurable in* x *and for every* $m \in \mathbb{Z}$*,*
$$C(f^m(x),\varepsilon) \le C(x,\varepsilon) e^{\varepsilon|m|} \quad \text{and} \quad K(f^m(x),\varepsilon) \ge K(x,\varepsilon) e^{-\varepsilon|m|}.$$

We remark that condition (L7) is crucial and is a manifestation of the LP-regularity (it is an analog of condition (3.30) in the discrete time case). Roughly speaking, it means that the estimates (L4), (L5), and (L6) may deteriorate as $|m| \to \infty$ but only with subexponential rate.[2] We stress that the rates of contraction along stable subspaces and expansion along unstable subspaces are substantially stronger.

Proof of Theorem 4.3. The first three statements are immediate consequence of Theorem 4.1.

Given $q > 0$, consider the sets
$$\mathcal{E}_q = \left\{ x \in \mathcal{E} : \chi_{k(x)}(x) < -\frac{1}{q},\ \chi_{k(x)+1}(x) > \frac{1}{q} \right\},$$
which are f-invariant, nested, i.e, $\mathcal{E}_q \subset \mathcal{E}_{q+1}$ for each $q > 0$, and exhaust $\mathcal{E}$, i.e., $\bigcup_{q>0} \mathcal{E}_q = \mathcal{E}$. It suffices to prove statements (L4)–(L7) of the theorem for every nonempty set $\mathcal{E}_q$. In what follows, we fix such a $q > 0$ and choose a sufficiently small number $\varepsilon_0 = \varepsilon_0(q)$. We need the following lemma.

Lemma 4.4. *Let* $X \subset M$ *be an* f*-invariant Borel set and let* $A(x,\varepsilon)$ *be a positive Borel function on* $X \times [0,\varepsilon_0)$*,* $0 < \varepsilon_0 < 1$*, such that for every* $\varepsilon_0 \ge \varepsilon > 0$*,* $x \in X$*, and* $m \in \mathbb{Z}$*,*
$$M_1(x,\varepsilon) e^{-\varepsilon|m|} \le A(f^m(x),\varepsilon) \le M_2(x,\varepsilon) e^{\varepsilon|m|},$$
where $M_1(x,\varepsilon)$ *and* $M_2(x,\varepsilon)$ *are Borel functions. Then one can find positive Borel functions* $B_1(x,\varepsilon)$ *and* $B_2(x,\varepsilon)$ *such that*
$$B_1(x,\varepsilon) \le A(x,\varepsilon) \le B_2(x,\varepsilon), \tag{4.6}$$
and for $m \in \mathbb{Z}$*,*
$$B_1(x,\varepsilon) e^{-2\varepsilon|m|} \le B_1(f^m(x),\varepsilon), \quad B_2(x,\varepsilon) e^{2\varepsilon|m|} \ge B_2(f^m(x),\varepsilon). \tag{4.7}$$

[2]More precisely, this means that the estimates (L4), (L5), and (L6) may deteriorate as $|m| \to \infty$ with exponential rate $e^{\varepsilon|m|}$ for *arbitrarily* small ε. However, $C(x,\varepsilon)$ may increase to ∞ and $K(x,\varepsilon)$ may decrease to zero as $\varepsilon \to 0$.

Proof of the lemma. It follows from the conditions of the lemma that there exists $m(x,\varepsilon) > 0$ such that if $m \in \mathbb{Z}$ and $|m| > m(x,\varepsilon)$, then

$$-2\varepsilon \le \frac{1}{|m|} \log A(f^m(x), \varepsilon) \le 2\varepsilon.$$

Set

$$B_1(x,\varepsilon) = \min_{-m(x,\varepsilon) \le i \le m(x,\varepsilon)} \left\{1, A(f^i(x),\varepsilon)e^{2\varepsilon|i|}\right\},$$
$$B_2(x,\varepsilon) = \max_{-m(x,\varepsilon) \le i \le m(x,\varepsilon)} \left\{1, A(f^i(x),\varepsilon)e^{-2\varepsilon|i|}\right\}.$$

The functions $B_1(x,\varepsilon)$ and $B_2(x,\varepsilon)$ are Borel functions. Moreover, if $n \in \mathbb{Z}$, then

$$B_1(x,\varepsilon)e^{-2\varepsilon|n|} \le A(f^n(x),\varepsilon) \le B_2(x,\varepsilon)e^{2\varepsilon|n|}. \tag{4.8}$$

Furthermore, if $b_1 \le 1 \le b_2$ are such that

$$b_1 e^{-2\varepsilon|n|} \le A(f^n(x),\varepsilon) \tag{4.9}$$

and

$$b_2 e^{2\varepsilon|n|} \ge A(f^n(x),\varepsilon) \tag{4.10}$$

for all $n \in \mathbb{Z}$, then $b_1 \le B_1(x,\varepsilon)$ and $b_2 \ge B_1(x,\varepsilon)$. In other words,

$$\begin{aligned} B_1(x,\varepsilon) &= \sup\{b \le 1 : \text{inequality (4.9) holds for all } n \in \mathbb{Z}\}, \\ B_2(x,\varepsilon) &= \inf\{b \ge 1 : \text{inequality (4.10) holds for all } n \in \mathbb{Z}\}. \end{aligned} \tag{4.11}$$

Inequalities (4.6) follow from (4.8) (with $n = 0$). We also have

$$A(f^{n+m}(x),\varepsilon) \le B_2(x,\varepsilon)e^{2\varepsilon|n+m|} \le B_2(x,\varepsilon)e^{2\varepsilon|n|+2\varepsilon|m|},$$

$$A(f^{n+m}(x),\varepsilon) \ge B_1(x,\varepsilon)e^{-2\varepsilon|n+m|} \ge B_1(x,\varepsilon)e^{-2\varepsilon|n|-2\varepsilon|m|}.$$

Comparing these inequalities with (4.8) written at the point $f^m(x)$ and taking (4.11) into account, we obtain (4.7). The proof of the lemma is complete. □

We apply Lemma 4.4 to construct the function $K(x,\varepsilon)$. Let $K \colon \mathcal{E}_q \to \mathbb{R}$ be a Borel function. It is said to be *tempered* at the point x if

$$\lim_{m \to \pm\infty} \frac{1}{m} \log K(f^m(x)) = 0. \tag{4.12}$$

It follows that the function $A(x,\varepsilon) = K(x)$ satisfies all the conditions of Lemma 4.4.

Fix $0 < \varepsilon < \varepsilon_0(q)$ and for $x \in \mathcal{E}_q$ consider the function

$$\gamma(x) = \angle(E^s(x), E^u(x)).$$

Exercise 4.5. Show that $\gamma(x)$ is a tempered function.

Therefore, applying Lemma 4.4, we conclude that the function $K(x,\varepsilon) = B_1(x,\frac{1}{2}\varepsilon)$ satisfies conditions (L6) and (L7) of the theorem provided the number $\varepsilon_0(q)$ is sufficiently small.

We will now show how to construct the function $C(x,\varepsilon)$. The proof is an elaboration for the discrete time case of arguments in the proof of Theorem 3.9 (see (3.30)).

Lemma 4.6. *There exist $\varepsilon_0 = \varepsilon_0(q)$ and a positive Borel function $D(x,\varepsilon)$ (where $x \in \mathcal{E}_q$ and $0 < \varepsilon < \varepsilon_0$) such that for $m \in \mathbb{Z}$ and $1 \le i \le s$,*

$$D(f^m(x),\varepsilon) \le D(x,\varepsilon)^2 e^{2\varepsilon|m|} \tag{4.13}$$

and for any $n \ge 0$,

$$\|df^n_{ix}\| \le D(x,\varepsilon)e^{(\chi_i+\varepsilon)n}, \quad \|df^{-n}_{ix}\| \ge D(x,\varepsilon)^{-1}e^{-(\chi_i+\varepsilon)n}$$

where $\chi_i = \chi_i(x)$ and $df^n_{ix} = d_x f^n|E_i(x)$.

Proof of the lemma. Let $x \in \mathcal{E}_q$. By Theorem 4.1 (we use the notation of that theorem) there exists $\varepsilon_0 = \varepsilon_0(q)$ such that for every $0 < \varepsilon < \varepsilon_0$ one can find a number $n(x,\varepsilon) \in \mathbb{N}$ such that for $n \ge n(x,\varepsilon)$,

$$\chi_i - \varepsilon \le \frac{1}{n}\log\|df^n_{ix}\| \le \chi_i + \varepsilon, \quad -\chi_i - \varepsilon \le \frac{1}{n}\log\|df^{-n}_{ix}\| \le -\chi_i + \varepsilon,$$

and

$$-\chi_i - \varepsilon \le \frac{1}{n}\log\|d'f^n_{ix}\| \le -\chi_i + \varepsilon, \quad \chi_i - \varepsilon \le \frac{1}{n}\log\|d'f^{-n}_{ix}\| \le \chi_i + \varepsilon,$$

where $d'f^n_{ix} = d'_x f^n|E^*_i(x)$ (recall that $E^*_i(x) \subset T^*_x M$ is the dual space to $E_i(x)$ and $d'_x f$ is the inverse of the codifferential). Set

$$D^+_1(x,\varepsilon) = \min_{1\le i\le s}\ \min_{0\le j\le n(x,\varepsilon)} \left\{1, \|df^j_{ix}\|e^{(-\chi_i+\varepsilon)j}, \|d'f^j_{ix}\|e^{(\chi_i+\varepsilon)j}\right\},$$

$$D^-_1(x,\varepsilon) = \min_{1\le i\le s}\ \min_{-n(x,\varepsilon)\le j\le 0} \left\{1, \|df^j_{ix}\|e^{(-\chi_i-\varepsilon)j}, \|d'f^j_{ix}\|e^{(\chi_i-\varepsilon)j}\right\},$$

$$D^+_2(x,\varepsilon) = \max_{1\le i\le s}\ \max_{0\le j\le n(x,\varepsilon)} \left\{1, \|df^j_{ix}\|e^{(-\chi_i-\varepsilon)j}, \|d'f^j_{ix}\|e^{(\chi_i-\varepsilon)j}\right\},$$

$$D^-_2(x,\varepsilon) = \max_{1\le i\le s}\ \max_{-n(x,\varepsilon)\le j\le 0} \left\{1, \|df^j_{ix}\|e^{(-\chi_i+\varepsilon)j}, \|d'f^j_{ix}\|e^{(\chi_i+\varepsilon)j}\right\},$$

and

$$\begin{aligned} D_1(x,\varepsilon) &= \min\{D^+_1(x,\varepsilon), D^-_1(x,\varepsilon)\},\\ D_2(x,\varepsilon) &= \max\{D^+_2(x,\varepsilon), D^-_2(x,\varepsilon)\},\\ D(x,\varepsilon) &= \max\{D_1(x,\varepsilon)^{-1}, D_2(x,\varepsilon)\}. \end{aligned}$$

The function $D(x,\varepsilon)$ is measurable, and if $n \ge 0$ and $1 \le i \le s$, then

$$\begin{aligned} D(x,\varepsilon)^{-1}e^{(\chi_i-\varepsilon)n} &\le \|df^n_{ix}\| \le D(x,\varepsilon)e^{(\chi_i+\varepsilon)n},\\ D(x,\varepsilon)^{-1}e^{(-\chi_i-\varepsilon)n} &\le \|df^{-n}_{ix}\| \le D(x,\varepsilon)e^{(-\chi_i+\varepsilon)n},\\ D(x,\varepsilon)^{-1}e^{(-\chi_i-\varepsilon)n} &\le \|d'f^{n}_{ix}\| \le D(x,\varepsilon)e^{(-\chi_i+\varepsilon)n},\\ D(x,\varepsilon)^{-1}e^{(\chi_i-\varepsilon)n} &\le \|d'f^{-n}_{ix}\| \le D(x,\varepsilon)e^{(\chi_i+\varepsilon)n}. \end{aligned} \tag{4.14}$$

Moreover, if $d \ge 1$ is a number for which inequalities (4.14) hold for all $n \ge 0$ and $1 \le i \le s$ with $D(x,\varepsilon)$ replaced by d, then $d \ge D(x,\varepsilon)$. Therefore,

$$\begin{aligned} D(x,\varepsilon) = \inf\{d \ge 1 : &\text{ the inequalities (4.14) hold for all } n \ge 0\\ &\text{and } 1 \le i \le s \text{ with } D(x,\varepsilon) \text{ replaced by } d\}. \end{aligned} \tag{4.15}$$

We wish to compare the values of the function $D(x,\varepsilon)$ at the points x and $f^m x$ for $m \in \mathbb{Z}$. Notice that for every $x \in M$, $v \in T_xM$, and $\varphi \in T^*_xM$ with $\varphi(v) = 1$ we have

$$(d'_x f\varphi)(d_x f v) = \varphi((d_x f)^{-1} d_x f v) = \varphi(v) = 1. \tag{4.16}$$

Using the Riemannian metric on the manifold M, we introduce the identification map $\tau_x \colon T^*_xM \to T_xM$ such that $\langle \tau_x(\varphi), v\rangle = \varphi(v)$ where $v \in T_xM$ and $\varphi \in T^*_xM$.

Let $\{v^n_k : k = 1,\dots,p\}$ be a basis of $E_i(f^n(x))$ and let $\{w^n_k : k = 1,\dots,p\}$ be the dual basis of $E^*_i(f^n(x))$. We have $\tau_{f^n(x)}(w^n_k) = v^n_k$. Denote by $A^i_{n,m}$ and $B^i_{n,m}$ the matrices corresponding to the linear maps $df^n_{if^m(x)}$ and $d'f^n_{if^m(x)}$ with respect to the above bases. It follows from (4.16) that

$$A^i_{m,0}(B^i_{m,0})^* = \mathrm{Id}$$

where * stands for matrix transposition. Hence, for every $n > 0$ the matrix corresponding to the map $df^n_{if^m(x)}$ is

$$A^i_{n,m} = A^i_{n+m,0}(A^i_{m,0})^{-1} = A^i_{n+m,0}(B^i_{m,0})^*.$$

Therefore, in view of (4.14), we obtain the following:

(1) if $n > 0$, then

$$\begin{aligned} \|df^n_{if^m(x)}\| &\le D(x,\varepsilon)^2 e^{(\chi_i+\varepsilon)(n+m)+(-\chi_i+\varepsilon)m}\\ &= D(x,\varepsilon)^2 e^{2\varepsilon m} e^{(\chi_i+\varepsilon)n},\\ \|df^n_{if^m(x)}\| &\ge D(x,\varepsilon)^{-2} e^{(\chi_i-\varepsilon)(n+m)+(-\chi_i-\varepsilon)m}\\ &= D(x,\varepsilon)^{-2} e^{-2\varepsilon m} e^{(\chi_i-\varepsilon)n}, \end{aligned}$$

(2) if $n > 0$ and $m - n \geq 0$, then

$$\begin{aligned}\|df^{-n}_{if^m(x)}\| &\leq D(x,\varepsilon)^2 e^{(\chi_i+\varepsilon)(m-n)+(-\chi_i+\varepsilon)m}\\ &= D(x,\varepsilon)^2 e^{2\varepsilon m} e^{(-\chi_i+\varepsilon)n},\\ \|df^{-n}_{if^m(x)}\| &\geq D(x,\varepsilon)^{-2} e^{(\chi_i-\varepsilon)(m-n)+(-\chi_i-\varepsilon)m}\\ &= D(x,\varepsilon)^{-2} e^{-2\varepsilon m} e^{(-\chi_i-\varepsilon)n},\end{aligned}$$

(3) if $n > 0$ and $n - m \geq 0$, then

$$\begin{aligned}\|df^{-n}_{if^m(x)}\| &\leq D(x,\varepsilon)^2 e^{(\chi_i+\varepsilon)(n-m)+(-\chi_i+\varepsilon)m}\\ &= D(x,\varepsilon)^2 e^{2\varepsilon m} e^{(-\chi_i+\varepsilon)n},\\ \|df^{-n}_{if^m(x)}\| &\geq D(x,\varepsilon)^{-2} e^{(\chi_i-\varepsilon)(n-m)+(-\chi_i-\varepsilon)m}\\ &= D(x,\varepsilon)^{-2} e^{-2\varepsilon m} e^{(-\chi_i-\varepsilon)n}.\end{aligned}$$

Similar inequalities hold for the maps $d'f^n_{if^m(x)}$ for each $n, m \in \mathbb{Z}$. Comparing this with the inequalities (4.14) applied to the point $f^m(x)$ and using (4.15), we conclude that if $m \geq 0$, then

$$D(f^m(x),\varepsilon) \leq D(x,\varepsilon)^2 e^{2\varepsilon m}. \tag{4.17}$$

Similar arguments show that if $m \leq 0$, then

$$D(f^{-m}(x),\varepsilon) \leq D(x,\varepsilon)^2 e^{-2\varepsilon m}. \tag{4.18}$$

It follows from (4.17) and (4.18) that if $m \in \mathbb{Z}$, then

$$D(f^m(x),\varepsilon) \leq D(x,\varepsilon)^2 e^{2\varepsilon|m|}.$$

This completes the proof of the lemma. □

We now proceed with the proof of the theorem. Replacing in (4.13) m by $-m$ and x by $f^m(x)$, we obtain

$$D(f^m(x),\varepsilon) \geq \sqrt{D(x,\varepsilon)}\, e^{-\varepsilon|m|}. \tag{4.19}$$

Consider two disjoint subsets $\sigma_1, \sigma_2 \subset [1, s] \cap \mathbb{N}$ and set

$$L_1(x) = \bigoplus_{i\in\sigma_1} E_i(x), \quad L_2(x) = \bigoplus_{i\in\sigma_2} E_i(x)$$

and $\gamma_{\sigma_1\sigma_2}(x) = \angle(L_1(x), L_2(x))$. By Theorem 4.1 the function $\gamma_{\sigma_1\sigma_2}$ is tempered and hence, in view of Lemma 4.4 one can find a function $K_{\sigma_1\sigma_2}(x)$ satisfying condition (L7) such that

$$\gamma_{\sigma_1\sigma_2}(x) \geq K_{\sigma_1\sigma_2}(x).$$

Set

$$T(x,\varepsilon) = \min K_{\sigma_1\sigma_2}(x),$$

where the minimum is taken over all pairs of disjoint subsets $\sigma_1, \sigma_2 \subset [1, s] \cap \mathbb{N}$. The function $T(x, \varepsilon)$ satisfies condition (L7).

Let $v \in E^s(x)$. Write $v = \sum_{i=1}^k v_i$ where $v_i \in E_i(x)$. We have

$$\|v\| \le \sum_{i=1}^k \|v_i\| \le LT^{-1}(x, \varepsilon)\|v\|,$$

where $L > 1$ is a constant. Let us set

$$C'(x, \varepsilon) = LD(x, \varepsilon)T(x, \varepsilon)^{-1}.$$

It follows from (4.13) and (4.19) that the function $C'(x, \varepsilon)$ satisfies the condition of Lemma 4.4 with

$$M_1(x, \varepsilon) = \frac{2}{\pi} L\sqrt{D(x, \varepsilon)} \quad \text{and} \quad M_2(x, \varepsilon) = LD(x, \varepsilon)^2 T(x, \varepsilon)^{-1}.$$

Therefore, there exists a function $C_1(x, \varepsilon) \ge C'(x, \varepsilon)$ for which the statements of Lemma 4.4 hold.

Applying the above arguments to the inverse map f^{-1} and the subspace $E^u(x)$, one can construct a function $C_2(x, \varepsilon)$ for which the statements of Lemma 4.4 hold. The desired function $C(x, \varepsilon)$ is defined by

$$C(x, \varepsilon) = \max\{C_1(x, \varepsilon/2), C_2(x, \varepsilon/2)\}.$$

This completes the proof of Theorem 4.3. □

4.2. Nonuniform complete hyperbolicity

In this section we introduce one of the principal concepts of smooth ergodic theory—the notion of nonuniform hyperbolicity—and we discuss its relation to dynamical systems with nonzero Lyapunov exponents introduced in the previous section.

Let $f\colon M \to M$ be a diffeomorphism of a compact smooth Riemannian manifold M of dimension p and let $Y \subset M$ be an f-invariant nonempty measurable subset. Also let $\lambda, \mu\colon Y \to (0, \infty)$ and $\varepsilon\colon Y \to [0, \varepsilon_0]$ with $\varepsilon_0 > 0$ be measurable functions satisfying

$$\lambda(f(x)) = \lambda(x), \quad \mu(f(x)) = \mu(x), \quad \varepsilon(f(x)) = \varepsilon(x) \tag{4.20}$$

(i.e., these functions are f-invariant), and

$$\lambda(x)e^{\varepsilon(x)} < 1 < \mu(x)e^{-\varepsilon(x)}, \quad x \in Y. \tag{4.21}$$

We say that the set Y is *nonuniformly (completely) hyperbolic* if there exist measurable functions $C, K\colon Y \to (0, \infty)$ such that for every $x \in Y$:

(H1) there exists a decomposition $T_xM = E^1(x) \oplus E^2(x)$, depending measurably on $x \in Y$, such that

$$d_xfE^1(x) = E^1(f(x)) \quad \text{and} \quad d_xfE^2(x) = E^2(f(x)); \tag{4.22}$$

(H2) for $v \in E^1(x)$ and $m > 0$,
$$\|d_x f^m v\| \le C(x)\lambda(x)^m e^{\varepsilon(x)m}\|v\|;$$
(H3) for $v \in E^2(x)$ and $m < 0$,
$$\|d_x f^m v\| \le C(x)\mu(x)^m e^{-\varepsilon(x)m}\|v\|;$$
(H4) $\angle(E^1(x), E^2(x)) \ge K(x)$;

(H5) for $m \in \mathbb{Z}$,
$$C(f^m(x)) \le C(x)e^{\varepsilon(x)|m|}, \quad K(f^m(x)) \ge K(x)e^{-\varepsilon(x)|m|}.$$

We stress that, in general, one should expect the functions C and K to be only (Borel) measurable but not continuous. This means that these functions may jump arbitrarily near a given point $x \in Y$ in an uncontrollable way. Condition (H5), however, provides some control over these functions along the trajectory $\{f^n(x)\}$ for $x \in Y$: the function C can increase and the function K can decrease with a small exponential rate. If ν is an invariant Borel probability measure, for which $\nu(Y) > 0$, then given $\varepsilon > 0$, there exists a subset $A \subset Y$ with $\nu(A) \ge \nu(Y) - \varepsilon > 0$ such that the function C is bounded from above on A. Moreover, due to the Poincaré recurrence theorem almost every point $x \in A$ returns to A infinitely often. Therefore, the function C indeed oscillates along the trajectory $f^n(x)$, for almost every $x \in A$, but may still become arbitrarily large. A similar observation holds for the function K.

Note that the dimensions of E^1 and E^2 are measurable f-invariant functions and hence the set Y can be decomposed into finitely many disjoint invariant measurable subsets on which the dimensions of E^1 and E^2 are constant.

Exercise 4.7. Show that if Y is nonuniformly (completely) hyperbolic, then for every $x \in Y$:

(1) $d_x f^m E^1(x) = E^1(f^m(x))$ and $d_x f^m E^2(x) = E^2(f^m(x))$;

(2) for $v \in E^a(x)$ and $m < 0$,
$$\|d_x f^m v\| \ge C(f^m(x))^{-1}\lambda(x)^m e^{\varepsilon(x)m}\|v\|;$$
(3) for $v \in E^2(x)$ and $m > 0$,
$$\|d_x f^m v\| \ge C(f^m(x))^{-1}\mu(x)^m e^{-\varepsilon(x)m}\|v\|.$$

We summarize the discussion in the previous section by saying that the set $\mathcal{E}$ (see (4.5)) of LP-regular points with nonzero Lyapunov exponents is nonuniformly (completely) hyperbolic with
$$\lambda(x) = e^{\chi_{k(x)}(x)}, \quad \mu(x) = e^{\chi_{k(x)+1}(x)},$$
$$C(x) = C(x,\varepsilon), \quad K(x) = K(x,\varepsilon)$$

for *any* fixed $0 < \varepsilon \le \varepsilon_0(x)$ with sufficiently small $\varepsilon_0(x)$ (see conditions (L1)–(L7) in Section 4.1). In fact, finding trajectories with nonzero Lyapunov exponents seems to be a universal approach in establishing nonuniform hyperbolicity.

We emphasize that the set of points with nonzero Lyapunov exponents whose regularity coefficient is sufficiently small (but may not necessarily be zero) is nonuniformly hyperbolic for some $\varepsilon > 0$.

We now introduce the notion of uniform hyperbolicity. Let $0 < \lambda < 1 < \mu$ be some numbers and let $K \subset M$ be a measurable subset. We stress that K need not be f-invariant. The set K is said to be *uniformly hyperbolic* if there exist $c > 0$ and $\gamma > 0$ such that for every $x \in K$:

(1) there exists a decomposition $T_xM = E^1(x) \oplus E^2(x)$, depending measurably on $x \in K$ and satisfying (4.22) whenever $f(x) \in K$;

(2) (a) for $v \in E^1(x)$ and $m > 0$,

$$\|d_xf^m v\| \le c\lambda^m \|v\|;$$

(b) for $v \in E^2(x)$ and $m < 0$,

$$\|d_xf^m v\| \le c\mu^m \|v\|;$$

(c) $\angle(E^1(x), E^2(x)) \ge \gamma$.

We will show below that for a nonuniformly hyperbolic set K of full measure with respect to an invariant measure there are in fact uniformly hyperbolic (noninvariant) sets $K_\delta \subset K$ of measure at least $1 - \delta$ for arbitrarily small $\delta > 0$. This observation is crucial in studying the topological and measure-theoretic properties. We stress that the "parameters" of uniform hyperbolicity, i.e., the numbers c and γ, may depend on δ approaching ∞ and 0, respectively. We will then show a crucial fact: this can occur only with a small exponential rate.

We introduce the notion of nonuniform (complete) hyperbolicity for dynamical systems with continuous time. Consider a smooth flow φ_t on a compact smooth Riemannian manifold M which is generated by a vector field $X(x)$. A measurable φ_t-invariant subset $Y \subset M$ is said to be *nonuniformly (completely) hyperbolic* if there exist measurable functions $\lambda, \mu \colon Y \to (0, \infty)$ and $\varepsilon \colon Y \to [0, \varepsilon_0]$ with $\varepsilon_0 > 0$ satisfying (4.20) and (4.21), measurable functions C, $K \colon Y \times (0, 1) \to (0, \infty)$, and subspaces $E^s(x)$ and $E^u(x)$ for each $x \in Y$, which satisfy conditions (H2)–(H5) and the following condition:

(H1′) the subspaces $E^s(x)$ and $E^u(x)$ depend measurably on x and together with the subspace $E^0(x) = \{\alpha X(x) : \alpha \in \mathbb{R}\}$ form an invariant splitting of the tangent space, i.e.,

$$T_xM = E^s(x) \oplus E^u(x) \oplus E^0(x),$$

with

$$d_x\varphi_t E^s(x) = E^s(\varphi_t(x)) \quad \text{and} \quad d_x\varphi_t E^u(x) = E^u(\varphi_t(x)).$$

We say that a dynamical system (with discrete or continuous time) is *nonuniformly (completely) hyperbolic* if it possesses an invariant nonuniformly hyperbolic subset.

4.3. Regular sets

By Luzin's theorem every measurable function on a measurable space X is "nearly" continuous with respect to a finite measure μ; that is, it is continuous outside a set of arbitrarily small measure. In other words, X can be exhausted by an increasing sequence of measurable subsets on which the function is continuous. In line with this idea, the regular sets are built to exhaust an invariant nonuniformly (completely) hyperbolic set Y by an increasing sequence of (not necessarily invariant) *uniformly* (completely) hyperbolic subsets, demonstrating that nonuniform (complete) hyperbolicity is "nearly" uniform.

Let f be a diffeomorphism of a compact smooth Riemannian manifold M and let $\lambda, \mu, \varepsilon$ be positive numbers satisfying

$$0 < \lambda e^{\varepsilon} < 1 < \mu e^{-\varepsilon}. \tag{4.23}$$

Given an integer j, $1 \le j < n$, and $\ell \ge 1$, we denote by $\Lambda^{\ell}_{\lambda\mu\varepsilon j}$ the set of points $x \in M$ for which there exists a decomposition $T_xM = E^1_x \oplus E^2_x$ such that for every $k \in \mathbb{Z}$ and $m > 0$ the following properties hold:

(1) $\dim E^a_x = j$ (and hence, $\dim E^2_x = n - j$);

(2) if $v \in d_xf^kE^a_x$, then

$$\|d_{f^k(x)}f^m v\| \le \ell\lambda^m e^{\varepsilon(m+|k|)}\|v\|$$

and

$$\|d_{f^k(x)}f^{-m} v\| \ge \ell^{-1}\lambda^{-m} e^{-\varepsilon(|k-m|+m)}\|v\|;$$

(3) if $v \in d_xf^kE^2_x$, then

$$\|d_{f^k(x)}f^{-m} v\| \le \ell\mu^{-m} e^{\varepsilon(m+|k|)}\|v\|$$

and

$$\|d_{f^k}f^{m} v\| \ge \ell^{-1}\mu^{m} e^{-\varepsilon(|k+m|+m)}\|v\|;$$

(4)

$$\angle(d_xf^kE^1_x, d_xf^kE^2_x) \ge \ell^{-1}e^{-\varepsilon|k|}.$$

The set $\Lambda^\ell_{\lambda\mu\varepsilon j}$ is called a *regular set* (or a *Pesin set*). It is a (not necessarily invariant) uniformly hyperbolic set for f. We also introduce the *level set*

$$\Lambda_{\lambda\mu\varepsilon j} := \bigcup_{\ell\geq 1} \Lambda^\ell_{\lambda\mu\varepsilon j}.$$

Exercise 4.8. Show that:

(1) $\Lambda^\ell_{\lambda\mu\varepsilon j} \subset \Lambda^{\ell+1}_{\lambda\mu\varepsilon j}$, i.e, regular sets are nested;

(2) if $m \in \mathbb{Z}$, then $f^m(\Lambda^\ell_{\lambda\mu\varepsilon j}) \subset \Lambda^{\ell'}_{\lambda\mu\varepsilon j}$, where $\ell' = \ell \exp(|m|\varepsilon)$;

(3) the set $\Lambda_{\lambda\mu\varepsilon j}$ is f-invariant;

(4) if $\varepsilon < \log(1 + 1/\ell)$, then

$$\Lambda^{\ell-1}_{\lambda\mu\varepsilon j} \subset f(\Lambda^\ell_{\lambda\mu\varepsilon j}) \subset \Lambda^{\ell+1}_{\lambda\mu\varepsilon j} \quad \text{and} \quad \Lambda^{\ell-1}_{\lambda\mu\varepsilon j} \subset f^{-1}(\Lambda^\ell_{\lambda\mu\varepsilon j}) \subset \Lambda^{\ell+1}_{\lambda\mu\varepsilon j};$$

(5) the regular sets $\Lambda^\ell = \Lambda^\ell_{\lambda\mu\varepsilon j}$ are closed (and hence compact);

(6) the subspaces E^1_x and E^2_x vary continuously with $x \in \Lambda^\ell$ (with respect to the distance in the Grassmannian bundle).

It follows that every regular set $\Lambda^\ell_{\lambda\mu\varepsilon j}$ is a (not necessarily invariant) uniformly hyperbolic set for f.

Consider a nonuniformly (completely) hyperbolic set Y for f. Given positive numbers $\lambda, \mu, \varepsilon$ satisfying (4.23) and an integer j, $1 \leq j < n$, consider the measurable set

$$Y_{\lambda\mu\varepsilon j} = \{x \in Y : \lambda(x) \leq \lambda < \mu \leq \mu(x),\ \varepsilon(x) \leq \varepsilon,\ E^1(x) = j\}.$$

Clearly, $Y_{\lambda\mu\varepsilon j}$ is invariant under f and is nonempty if the numbers $\lambda, \mu, \varepsilon$, and j are chosen appropriately. For each integer $\ell \geq 1$, consider the measurable subset

$$Y^\ell_{\lambda\mu\varepsilon j} = \{x \in Y_{\lambda\mu\varepsilon j} : C(x) \leq \ell,\ K(x) \geq \ell^{-1}\}.$$

We have

$$Y^\ell_{\lambda\mu\varepsilon j} \subset Y^{\ell+1}_{\lambda\mu\varepsilon j} \quad \text{and} \quad Y_{\lambda\mu\varepsilon j} = \bigcup_{\ell\geq 1} Y^\ell_{\lambda\mu\varepsilon j}.$$

Note that $Y^\ell_{\lambda\mu\varepsilon j}$ is a uniformly hyperbolic set for f but need not be invariant nor compact.

Exercise 4.9. Show that:

(1) $Y^\ell_{\lambda\mu\varepsilon j} \subset \Lambda^\ell_{\lambda\mu\varepsilon j}$ for every $\ell \geq 1$;

(2) $E^1_x = E^1(x)$ and $E^2_x = E^2(x)$ for every $x \in \Lambda$.

It follows that every nonuniformly (completely) hyperbolic set Y can be exhausted by a nested sequence of (not necessarily invariant) uniformly (completely) hyperbolic sets $Y^\ell_{\lambda\mu\varepsilon j}$. Moreover, to the set Y one can associate the family of nonempty f-invariant level sets

$$\{\Lambda_{\lambda\mu\varepsilon j} : \lambda, \mu, \varepsilon \text{ satisfy (4.23)}\} \tag{4.24}$$

and for each $\lambda, \mu, \varepsilon, j$ the collection of nonempty *compact* regular sets

$$\{\Lambda^\ell = \Lambda^\ell_{\lambda\mu\varepsilon j} : \ell \geq 1\}. \tag{4.25}$$

Note that f is nonuniformly (completely) hyperbolic on each level set $\Lambda_{\lambda\mu\varepsilon j}$ as well as on the set $\Lambda = \bigcup \Lambda_{\lambda\mu\varepsilon j}$ (here the union is taken over all numbers $\lambda, \mu, \varepsilon$ satisfying (4.23) and $1 \leq j < p$) that can be viewed as an "extension" of the "original" nonuniformly (completely) hyperbolic set Y. We stress that the rates of exponential contraction $\lambda(x)$ and of exponential expansion $\mu(x)$ are uniformly bounded away from 1 on each level set $\Lambda_{\lambda\mu\varepsilon j}$ but may be arbitrarily close to 1 on Λ.

4.4. Nonuniform partial hyperbolicity

In Section 4.1 we studied diffeomorphisms whose values of the Lyapunov exponent are *all* nonzero on a nonempty set $\mathcal{E}$ (with some of the values being negative and the remaining ones being positive; see (4.5)). As we saw in Section 4.2, the set $\mathcal{E}$ is nonuniformly hyperbolic and the hyperbolicity is complete. In this section we discuss the more general case of partial hyperbolicity. It deals with the situation when *some* of the values of the Lyapunov exponent are negative and some among the remaining ones may be zero.

While for dynamical systems that are nonuniformly completely hyperbolic one can obtain a sufficiently complete description of their ergodic properties (with respect to smooth invariant measures; see Chapter 9), dynamical systems that are nonuniformly partially hyperbolic may not possess "nice" ergodic properties. However, some principal results describing local behavior of systems that are nonuniformly completely hyperbolic can be extended without much extra work to systems that are only nonuniformly partially hyperbolic.[3]

Let $Z \subset M$ be an f-invariant nonempty measurable subset and let $\lambda, \mu \colon Z \to (0, \infty)$ and $\varepsilon \colon Z \to [0, \varepsilon_0]$ for some $\varepsilon_0 > 0$ be measurable functions satisfying

$$\lambda(f(x)) = \lambda(x), \quad \mu(f(x)) = \mu(x), \quad \varepsilon(f(x)) = \varepsilon(x)$$

(i.e., these functions are f-invariant), and

$$\lambda(x)e^{\varepsilon(x)} < \mu(x)e^{-\varepsilon(x)}.$$

[3]This includes constructing local stable manifolds and establishing their absolute continuity (see Remarks 7.2 and 8.13).

We say that an invariant measurable set Z is *nonuniformly partially hyperbolic in the broad sense* if there exist measurable functions $C, K\colon \Lambda \to (0,\infty)$ such that conditions (H2)–(H5) hold.

As in Section 4.3, to each set Z that is nonuniformly partially hyperbolic in the broad sense one can associate a collection of level sets $Z_{\lambda\mu\varepsilon j}$, and for each λ, μ, and ε, a collection of regular sets $Z^{\ell}_{\lambda\mu\varepsilon j}$ over $\ell > 1$. Here λ, μ, and ε are positive numbers satisfying

$$0 < \lambda e^{\varepsilon} < \mu e^{-\varepsilon}.$$

The level sets are invariant and the regular sets are nested and exhaust the set Z. The set $\Lambda = \bigcup Z_{\lambda\mu\varepsilon j}$ is nonuniformly partially hyperbolic in the broad sense and $Z \subset \Lambda$. Observe that each regular set is *uniformly* partially hyperbolic in the broad sense but not necessarily invariant.

Let $f\colon M \to M$ be a diffeomorphism of a compact smooth Riemannian manifold M of dimension p. Consider the f-invariant set

$$\mathcal{F} = \big\{x \in \mathcal{R} : \text{there exists } 1 \le k(x) < s(x) \text{ with } \chi_{k(x)}(x) < 0\big\}.$$

Repeating the arguments in the proof of Theorem 4.3, one can show that f is nonuniformly partially hyperbolic in the broad sense on $\mathcal{F}$.

4.5. Hölder continuity of invariant distributions

As we saw in Section 1.1, the stable and unstable subspaces of an Anosov diffeomorphism f depend continuously on the point in the manifold. Since these subspaces at a point x are determined by the whole positive and, respectively, negative semitrajectory through x, their dependence on the point may not be differentiable even if f is real analytic. However, one can show that they depend Hölder continuously on the point.[4]

We remind the reader of the definition of Hölder continuous distribution. A k-dimensional *distribution* E on a smooth manifold M is a family of k-dimensional subspaces $E(x) \subset T_xM$. A Riemannian metric on M naturally induces distances in TM and in the space of k-dimensional distributions on TM. The Hölder continuity of a distribution E can be defined using these distances. More precisely, for a subspace $A \subset \mathbb{R}^p$ (where $p = \dim M$) and a vector $v \in \mathbb{R}^p$, set

$$d(v, A) = \min_{w\in A} \|v - w\|.$$

In other words, $d(v, A)$ is the distance from v to its orthogonal projection on A. For subspaces A and B in $\mathbb{R}^p$, define

$$d(A, B) = \max\left\{\max_{v\in A, \|v\|=1} d(v, B), \max_{w\in B, \|w\|=1} d(w, A)\right\}.$$

[4] This result was proved by Anosov in [**2**] and is a corollary of Theorem 4.11 below.

Let $D \subset \mathbb{R}^p$ be a subset and let E be a k-dimensional distribution. The distribution E is said to be *Hölder continuous* with *Hölder exponent* $\alpha \in (0,1]$ and *Hölder constant* $L > 0$ if there exists $\varepsilon_0 > 0$ such that

$$d(E(x), E(y)) \leq L\|x-y\|^\alpha$$

for every $x, y \in D$ with $\|x-y\| \leq \varepsilon_0$.

Now let E be a continuous distribution on M. Choose a small number $\varepsilon > 0$ and an atlas $\{U_i\}$ of M. We say that E is *Hölder continuous* if the restriction $E|U_i$ is Hölder continuous for every i.

Exercise 4.10. (1) Show that if a distribution E on M is Hölder continuous with respect to an atlas $\{U_i\}$ of M, then it is also Hölder continuous with respect to any other atlas of M with the same Hölder exponent (but the Hölder constant may be different).

(2) Show that if a distribution E on M is Hölder continuous, then it remains Hölder continuous with the same Hölder exponent if the Riemannian metric is replaced by an equivalent smooth metric.

(3) Show that a distribution E on M is Hölder continuous if and only if there are positive constants C, α, and ε such that for every two points x and y with $\rho(x,y) \leq \varepsilon$ we have

$$d(E(x), \tilde{E}(x)) \leq C\rho(x,y)^\alpha,$$

where $\tilde{E}(x)$ is the subspace of T_xM that is the parallel transport of the subspace $E(y) \subset T_yM$ along the unique geodesic connecting x and y.[5]

Finally, given a subset $\Lambda \subset M$, we say that a distribution $E(x)$ on Λ is Hölder continuous if for an atlas $\{U_i\}$ of M, the restriction $E|U_i \cap \Lambda$ is Hölder continuous for every i.

If f is a diffeomorphism that is nonuniformly completely hyperbolic on an invariant subset Λ, then the stable and unstable subspaces depend only (Borel) measurably on the point in Λ. However, as we saw in Section 4.2, these subspaces vary continuously on the point in a regular set, and in this section we show that they depend Hölder continuously on the point. It should be stressed that in the case of nonuniform hyperbolicity, the Hölder continuity property requires higher regularity of the system, i.e., that f is of class $C^{1+\alpha}$. In what follows, we consider only the stable subspaces; the Hölder continuity of the unstable subspaces follows by reversing the time.

Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth manifold M and let Y be a nonuniformly (completely) hyperbolic set for f. Consider the

[5]The geodesic connecting x and y is unique if the number ε is sufficiently small.

corresponding collection $\Lambda_{\lambda\mu\varepsilon j}$ of level sets and for each λ, μ, and ε the corresponding collection of regular sets $\Lambda^\ell = \Lambda^\ell_{\lambda\mu\varepsilon j}$, $\ell \geq 1$.

Theorem 4.11. *The stable and unstable distributions $E^s(x)$ and $E^u(x)$ depend Hölder continuously on $x \in \Lambda^\ell$.*

We shall prove a more general statement of which Theorem 4.11 is an easy corollary. It applies to the cases of complete as well as of partial hyperbolicity. By the Whitney Embedding Theorem, every manifold M can be embedded in the Euclidean space $\mathbb{R}^N$ for a sufficiently large N. If M is compact, the Riemannian metric on M is equivalent to the distance $\|x-y\|$ induced by the embedding. We assume in Theorem 4.13, without loss of generality, that the manifold is embedded in $\mathbb{R}^N$.

Given a number $\kappa > 0$, we say that two subspaces $E_1, E_2 \subset \mathbb{R}^N$ are *κ-transverse* if $\|v_1 - v_2\| \geq \kappa$ for all unit vectors $v_1 \in E_1$ and $v_2 \in E_2$.

Exercise 4.12. Show that the subspaces $E^s(x)$ and $E^u(x)$ are κ-transverse for $x \in \Lambda^\ell$ and some $\kappa > 0$ which is independent of x.

Theorem 4.13. *Let M be a compact m-dimensional C^2 submanifold of $\mathbb{R}^N$ for some $m < N$, and let $f\colon M \to M$ be a $C^{1+\beta}$ map for some $\beta \in (0,1)$. Assume that there exist a set $D \subset M$ and real numbers $0 < \lambda < \mu$, $c > 0$, and $\kappa > 0$ such that for each $x \in D$ there are κ-transverse subspaces $E_1(x)$, $E_2(x) \subset T_xM$ such that:*

1. *$T_xM = E_1(x) \oplus E_2(x)$;*
2. *for every $n > 0$ and every $v_1 \in E_1(x)$, $v_2 \in E_2(x)$ we have*
$$\|d_xf^nv_1\| \leq c\lambda^n\|v_1\| \text{ and } \|d_xf^nv_2\| \geq c^{-1}\mu^n\|v_2\|.$$

Then for every $a > \max_{z\in M}\|d_zf\|^{1+\beta}$, the distribution E_1 is Hölder continuous with exponent
$$\alpha = \frac{\log\mu - \log\lambda}{\log a - \log\lambda}\beta.$$

Proof. We follow the argument in [**18**] and we begin with two technical lemmas.

Lemma 4.14. *Let A_n and B_n, for $n = 0, 1, \ldots$, be two sequences of real $N \times N$ matrices such that for some $\Delta \in (0,1)$,*
$$\|A_n - B_n\| \leq \Delta a^n$$
for every positive integer n. Assume that there exist subspaces E_A, $E_B \subset \mathbb{R}^N$ and numbers $0 < \lambda < \mu$ and $C > 1$ such that $\lambda < a$ and for each $n \geq 0$,
$$\|A_nv\| \leq C\lambda^n\|v\| \text{ if } v \in E_A; \qquad \|A_nw\| \geq C^{-1}\mu^n\|w\| \text{ if } w \in E_A{}^\perp;$$
$$\|B_nv\| \leq C\lambda^n\|v\| \text{ if } v \in E_B; \qquad \|B_nw\| \geq C^{-1}\mu^n\|w\| \text{ if } w \in E_B{}^\perp.$$

Then

$$\mathrm{dist}(E_A, E_B) \leq 3C^2 \frac{\mu}{\lambda} \Delta^{\frac{\log \mu - \log \lambda}{\log a - \log \lambda}}.$$

Proof of the lemma. Set

$$Q_A^n = \{v \in \mathbb{R}^N : \|A_n v\| \leq 2C\lambda^n \|v\|\}$$

and

$$Q_B^n = \{v \in \mathbb{R}^N : \|B_n v\| \leq 2C\lambda^n \|v\|\}.$$

For each $v \in \mathbb{R}^N$, write $v = v_1 + v_2$, where $v_1 \in E_A$ and $v_2 \in {E_A}^\perp$. If $v \in Q_A^n$, then

$$\|A_n v\| = \|A_n(v_1 + v_2)\| \geq \|A_n v_2\| - \|A_n v_1\| \geq C^{-1}\mu^n \|v_2\| - C\lambda^n \|v_1\|,$$

and hence,

$$\|v_2\| \leq C\mu^{-n}(\|A_n v\| + C\lambda^n \|v_1\|) \leq 3C^2 \left(\frac{\lambda}{\mu}\right)^n \|v\|.$$

Therefore,

$$\mathrm{dist}(v, E_A) \leq 3C^2 \left(\frac{\lambda}{\mu}\right)^n \|v\|. \tag{4.26}$$

Set $\gamma = \lambda/a < 1$. There exists a unique nonnegative integer n such that $\gamma^{n+1} < \Delta \leq \gamma^n$. If $w \in E_B$, then

$$\begin{aligned} \|A_n w\| &\leq \|B_n w\| + \|A_n - B_n\| \cdot \|w\| \\ &\leq C\lambda^n \|w\| + \Delta a^n \|w\| \\ &\leq (C\lambda^n + (\gamma a)^n)\|w\| \leq 2C\lambda^n \|w\|. \end{aligned}$$

It follows that $w \in Q_A^n$ and hence, $E_B \subset Q_A^n$. By symmetry, $E_A \subset Q_B^n$. By (4.26) and the choice of n, we obtain

$$\mathrm{dist}(E_A, E_B) \leq 3C^2 \left(\frac{\lambda}{\mu}\right)^n \leq 3C^2 \frac{\mu}{\lambda} \Delta^{\frac{\log \mu - \log \lambda}{\log a - \log \lambda}}.$$

This completes the proof of the lemma. □

For a diffeomorphism $f\colon M \to M$ of class $C^{1+\beta}$ the following result holds.

Exercise 4.15. There are positive constants $L > 0$ and $r > 0$ such that for any two points $x, y \in M$ for which $\|x - y\| < r$ we have that

$$\|d_x f - d_y f\| \leq L\|x - y\|^\beta. \tag{4.27}$$

Since M is compact, covering M by finitely many balls of radius r, we obtain that (4.27) holds for any $x, y \in M$ (with an appropriately chosen constant L). The following result extends the estimate (4.27) to powers of the map f.

Lemma 4.16. *Let $f\colon M \to M$ be a $C^{1+\beta}$ map of a compact m-dimensional C^2 submanifold $M \subset \mathbb{R}^N$. Then for every $a > \max_{z\in M} \|d_z f\|^{1+\beta}$ there exists $D > 1$ such that for every $n \in \mathbb{N}$ and every $x, y \in M$ we have*

$$\|d_x f^n - d_y f^n\| \le D a^n \|x-y\|^\beta.$$

Proof of the lemma. Let D' be such that

$$\|d_x f - d_y f\| \le D' \|x-y\|^\beta.$$

Set $b = \max_{z\in M} \|d_z f\| \ge 1$ and observe that for every x, $y \in M$,

$$\|f^n(x) - f^n(y)\| \le b^n \|x-y\|.$$

Fix $a > b$. Then the lemma holds true for $n = 1$ and any $D \ge D'$. For the inductive step we note that

$$\begin{aligned}
\|d_x f^{n+1} - d_y f^{n+1}\| &\le \|d_{f^n(x)} f\| \cdot \|d_x f^n - d_y f^n\| \\
&\quad + \|d_{f^n(x)} f - d_{f^n(y)} f\| \cdot \|d_y f^n\| \\
&\le b D a^n \|x-y\|^\beta + D' \big(b^n \|x-y\|\big)^\beta b^n \\
&\le D a^{n+1} \|x-y\|^\beta \left(\frac{b}{a} + \frac{D'}{D} \frac{(b^{1+\beta})^n}{a^{n+1}} \right).
\end{aligned}$$

If $a > b^{1+\beta}$, then there exists $D \ge D'$ for which the factor in parentheses is less than 1. □

We proceed with the proof of the theorem.

For $x \in M$, let $(T_x M)^\perp$ denote the orthogonal complement to the tangent plane $T_x M$ in $\mathbb{R}^N$. Since the distribution $(TM)^\perp$ is smooth, it is sufficient to prove that the distribution $F = E_1 \oplus (TM)^\perp$ is Hölder continuous.

Since $E_1(x)$ and $E_2(x)$ are κ-transverse and of complementary dimensions in $T_x M$, there exists $d > 1$ such that $\|d_x f^n w\| \ge d^{-1} \mu^n \|w\|$ for every $x \in D$ and $w \perp E_1(x)$.

For x, $y \in D$ and a positive integer n, let A_n and B_n be $N \times N$ matrices such that

$$A_n v = d_x f^n v \text{ if } v \in T_x M \quad \text{and} \quad A_n w = 0 \text{ if } w \in (T_x M)^\perp,$$

$$B_n v = d_y f^n v \text{ if } v \in T_y M \quad \text{and} \quad B_n w = 0 \text{ if } w \in (T_y M)^\perp.$$

By Lemma 4.16,

$$\|A_n - B_n\| \le D a^n \|x-y\|^\beta.$$

Now Theorem 4.13 follows from Lemma 4.14 with $\Delta = D\|x-y\|^\beta$, $E_A = F(x)$, $E_B = F(y)$, and $C = \max\{c, d\}$. □