## Index

absolutely continuous, 237
almost everywhere, 77
convergence, 78
almost surely, 110
arc sine law, 26
Azuma's inequality, 232

Bayes's formula, 28
Bernoulli measure, 33
Bernstein polynomial, 132
Berry-Esseen theorem, 142
biased random walk, 35
binomial, 11
coefficient, 11
$N$ choose $m, 11$
distribution
parameters $(n, p), 34$
Borel
measurable function, 56
measurable set, 56
measure, 57
Borel-Cantelli lemma, 32, 244
branching process, 178
extinction, 178
Brownian motion, 206
Feynman-Kac formula, 269
Lévy's characterization, 263
relative to $\left\{\mathcal{F}_{t}: t \geq 0\right\}, 210$
scaling property, 209
strong law, 213
time inversion invariance, 213
centered Gaussian family, 152
central limit theorem, 140
De Moivre's, 18
Lindeberg's, 137
Chapman-Kolmogorov equation, 194
Chebychev's inequality, 44
complement, 4
concave function, 85
conditional expectation, 110
conditional probability, 27
contraction, 171
convergence
$\mu$-almost everywhere, 78
in $\mu$-measure, 79
convex
function, 85
set, 84
convolution, 121
countably additive, 57
covariance, 143
cumulant of a random variable, 51

De Morgan's law, 6
decreasing events, 5
density of a distribution, 117
difference, 4
discrete arcsine measure, 33
disjoint sets, 4
distribution, 59
having density, 117
of a random variable, 32
of a stochastic process, 193
distribution function, 70
Doeblin's
condition, 174
theorem, 173
Doob's decomposition lemma, 234
Doob's stopping time theorem
continuous parameter, 259
discrete parameter, 243
doubly stochastic, 188
empirical
mean, 145,155
variance, 145,155
empty set, 4
ergodic, 177
theory, 185
error function, 118
Euler's
Beta function, 103
Gamma function, 100
event, 3
exchangeable random variables, 249
expected value, 42, 105
exists, 42
non-negative discrete, 42
of $\mathbb{R}^{N}$-valued random variable, 143
exponential distribution, 117
Fatou's lemma, 50
Feynman-Kac formula, 269
finite measure, 57
Fubini's theorem, 90
gambler's ruin problem, 163
Gauss density, 100
Gaussian distribution, 117
concentration property, 150
Maury-Pisier estimate, 147
parameters $m$ and $\sigma^{2}, 117$
standard, 144
tail estimate, 157
with mean $m$ and covariance $C, 146$
Gaussian family, 152
centered, 152
graph, 13
complete, 13
two-colorings, 30
edges, 13
vertices, 13
Hardy-Littlewood maximal function, 229
inequality, 229
Hewitt-Savage 0-1 law, 251
Hölder's inequality, 88

Hunt's stopping time theorem
continuous parameter, 259
discrete parameter, 245
identically distributed random variables, 129
image of a measure, 59
increasing events, 5
independent
$\sigma$-algebras, 105
events, 21
random variables, 48
existence of, 109
indicator function, 24
inequality
Chebychev's, 44
Hölder's, 88
Jensen's, 85
Markov's, 43
Minkowski's, 88
Schwarz's, 92
integer part, 22
integrable, 42, 77
integral
exists, 76
of a function, 73
intersection, 3
Itô's formula, 263
Jensen's inequality, 85
for conditional expectations, 116
Kolmogorov's
0-1 law, 107
backward equation, 204
forward equation, 204
inequality, 126
strong law, 129
$L^{1}(\mu ; \mathbb{R}), 77$
$\Lambda$-system, 58
Laplace transform, 132
law of large numbers
Kolmogorov's strong law, 129
weak, 125
law of the iterated logarithm, 133
Lebesgue decomposition, 239
Lebesgue measure
on [01], 67
on $\mathbb{R}, 68$
on $\mathbb{R}^{N}, 95$
alternative construction, 103
scaling property, 70
under linear transformations, 96
Lebesgue's dominated convergence theorem, 50
limits of sets, 6
marginal distribution, 119
Markov chain, 171
renewal equation, 182
state space, 179
time-homogeneous, 171
Markov process, 194
time-homogeneous, 194
Markov property, 171
Markov's inequality, 43, 77
martingales, 225
Azuma's inequality, 232
continuous, 259
continuous parameter, 255
convergence theorem, 236, 246
discrete parameter, 225
Doob's decomposition lemma, 234
reversed, 248
upcrossing inequality, 246
mean value, $43,105,143$
measurable function, 56, 72
Borel measurable, 56
measurable space, 56
measure, 57
Borel, 57
finite, 57
non-atomic, 65
probability, 57
measure space, 57
finite, 57
probability, 57
median, 45
variational characterization, 52
vs. expectation value, 52
minimum principle, 166
Minkowski's inequality, 88
moment generating function, 47, 120
moment of a random variable, 47
monotone class, 59
monotone convergence theorem, 50
negative part, 7
nested partitions, 230
non-atomic measure, 65
normal random variable, 118
conditioning, 154
standard, 118, 144
with mean $m$ and covariance $C, 146$
optional stopping time, 271
Ornstein-Uhlenbeck process, 222
$\Pi$-system, 58
Paley-Wiener integral, 222
point mass, 174
Poisson
approximation, 36
measure, 36
process
compound, 203
simple, 199
random variable, 39,51
polar coordinates, 99
positive part, 7
probability function, 10
probability measure, 6,57
determined by $p, 10$
probability space, 57
progressively measurable, 225
continuous parameter, 254
discrete parameter, 225
Radon-Nikodym
derivative, 237
theorem, 237, 238
random variable, 32,105
$k$ th moment of, 47
exchangeable, 249
integrable, 42
mean value, 43
normal, 118
standard, 118, 144
variance of, 44
random variables
mutually independent, 107
sums of independent, 39
random walk
biased, 35
transience, 37
symmetric, 15
recurrence, 23
reflection principle
for Brownian motion, 216
for symmetric random walks, 16
renewal equation, 23,182
return time
symmetric random walk, 22
reversed
martingale, 248
submartingale, 248
right-continuous, 70
paths, 255
$\sigma$-algebra
Borel, 56
countably generated, 110
generated by, 56
generated by a random variable, 107
over $\Omega, 56$
$\sigma$-finite, 89
sample
point, 3
space, 3
Schwarz's inequality, 92
simple function, 73
simple Poisson process, 199
singular measures, 239
square integrable, 129
standard normal random variable, 144
state space, 179,193
stationary distribution, 174
for transition probability function, 200
non-existence of, 190
Stirling's formula, 101, 142
stochastic process, 159
homogeneous, independent increments, 199
stopping time, 242
continuous parameter, 257
discrete parameter, 242
old definition, 271
optional, 271
stopping time theorem
Doob's, 243, 259
Hunt's, 245, 259
strong law of large numbers, 125
sub-Gaussian, 121
submartingale, 226
continuous parameter, 255
discrete parameter, 226
reversed, 248
supermartingale, 235
surface measure on $\mathbb{S}^{N-1}, 98$
tail $\sigma$-algebra, 106
time shift map, 181, 211
time-homogeneous, 171
tournament, 14
transient, 37
transition probability, 169
doubly stochastic, 188
transition probability function, 194
backward equation, 204
forward equation, 204
translation invariant, 69, 95
translation map
on $[01), 67$
on $\mathbb{R}, 68$
triangle inequality, 92
uniform distribution, 117
uniform probability measure
on [01], 67
characterization, 68
on finite set, 10
uniformly integrable, 93
union of sets, 3
upcrossing inequality, 246
variance, 44
variation distance, 172
weak law of large numbers, 125
Weierstrass approximation theorem, 131
Wiener measure, 205

