Preface

There are a myriad of books about probability theory already on the market. Nonetheless, a few years ago Sergei Gelfand asked if I would write a probability theory book for advanced undergraduate and beginning graduate students who are interested in mathematics. He had in mind an updated version of the first volume of William Feller's renowned *An Introduction to Probability Theory and Its Applications* [**3**]. Had I been capable of doing so, I would have loved to oblige, but, unfortunately, I am neither the mathematician that Feller was nor have I his vast reservoir of experience with applications. Thus, shortly after I started the project, I realized that I would not be able to produce the book that Gelfand wanted. In addition, I learned that there already exists a superb replacement for Feller's book. Namely, for those who enjoy combinatorics and want to see how probability theory can be used to obtain combinatorial results, it is hard to imagine a better book than N. Alon and J. H. Spencer's *The Probabilistic Method* [**1**]. For these reasons, I have written instead a book that is a much more conventional introduction to the ideas and techniques of modern probability theory. I have already authored such a book, *Probability Theory, An Analytic View* [**9**], but that book makes demands on the reader that this one does not. In particular, that book assumes a solid grounding in analysis, especially Lebesgue's theory of integration. In the hope that it will be appropriate for students who lack that background, I have made this one much more self-contained and developed the measure theory that it uses.

Chapter 1 contains my attempt to explain the basic concepts in probability theory unencumbered by measure-theoretic technicalities. After introducing the terminology, I devote the rest of the chapter to probability theory on finite and countable sample spaces. In large part because I am such a poor combinatorialist myself, I have emphasized computations that do not require a mastery of counting techniques. Most of the examples involve Bernoulli trials. I have not shied away from making the same computations several times, each time employing a different line of reasoning. My hope is that in this way I will have made it clear to the reader why concepts like independence and conditioning have been developed.

Many of the results in Chapter 1 are begging for the existence of a probability measure on an uncountable sample space. For example, when discussing random walks in Chapter 1, only computations involving a finite number of steps can be discussed. Thus, answers to questions about recurrence were deficient. Using this deficiency as motivation, in Chapter 2 I first introduce the fundamental ideas of measure theory and then construct the Bernoulli measures on $\{0,1\}^{\mathbb{Z}^+}$. Once I have the Bernoulli measures, I obtain Lebesgue measure as the image of the symmetric Bernoulli measure and spend some time discussing its translation invariance properties. The remainder of Chapter 2 gives a brief introduction to Lebesgue's theory of integration.

With the tools developed in Chapter 2 at hand, Chapter 3 explains how Kolmogorov fashioned those tools into what has become the standard mathematical model of probability theory. Specifically, Kolmogorov's formulations of independence and conditioning are given, and the chapter concludes with his strong law of large numbers.

Chapter 4 is devoted to Gaussian distributions and normal random variables. It begins with Lindeberg's derivation of the central limit theorem and then moves on to explain some of the transformation properties of multi-dimensional normal random variables. The final topic here is centered Gaussian families.

In the first section of Chapter 5 I revisit the topic that I used to motivate the contents of Chapter 2. That is, I do several computations of quantities that require the Bernoulli measures constructed in § 2.2. I then turn to a somewhat cursory treatment of Markov chains, concluding with a discussion of their ergodic properties when the state space is finite or countable.

Chapter 6 begins with Markov processes that are the continuous parameter analog of Markov chains. Here I also introduce transition probability functions and discuss some properties of general continuous parameter Markov processes. The second part of this chapter contains Lévy's construction of Brownian motion and proves a few of its elementary path properties. The chapter concludes with a brief discussion of the Ornstein–Uhlenbeck process.

Martingale theory is the subject of Chapter 7. The first three sections give the discrete parameter theory, and the continuous parameter theory is given in the final section. In both settings, I try to emphasize the contributions that martingale theory can make to topics treated earlier. In particular, in the final section, I discuss the relationship between continuous martingales and Brownian motion and give some examples that indicate how martingales provide a bridge between differential equations and probability theory.

In conclusion, it is clear that I have not written the book that Gelfand asked for, but, if I had written that book, it undoubtedly would have looked pale by comparison to Feller's. Nonetheless, I hope that, for those who read it, the book that I have written will have some value. I will be posting an errata file on www.ams.org/bookpages/gsm-149. I expect that this file will grow over time.

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