Chapter 4

Extremal Metrics

Suppose that M is a compact Kähler manifold with a Kähler class $\Omega \in H^2(M, \mathbf{R})$. A natural question is to ask for a particularly nice metric representing the class Ω. In the previous section we have seen that if $c_1(M) < 0$ and $\Omega = -c_1(M)$, then M admits a unique Kähler-Einstein metric, while if $c_1(M) = 0$, then any Kähler class on M admits a unique Ricci flat metric. Extremal metrics, introduced by Calabi [**21**], are a natural generalization of these to arbitrary Kähler classes on compact Kähler manifolds. When they exist, extremal metrics are good candidates for being the "best" metrics in a given Kähler class. In this section we will introduce extremal metrics and study some of their basic properties, while later on we will study obstructions to their existence.

4.1. The Calabi functional

As above, suppose that M is a compact Kähler manifold and $\Omega \in H^2(M, \mathbf{R})$ is a Kähler class.

Definition 4.1. An extremal metric on M in the class Ω is a critical point of the functional

$$\mathrm{Cal}(\omega) = \int_M S(\omega)^2\, \omega^n,$$

for $\omega \in \Omega$, where $S(\omega)$ is the scalar curvature. This functional is called the Calabi functional.

The first important result is understanding the Euler-Lagrange equation characterizing extremal metrics. For a function $f : M \to \mathbf{R}$ on a Kähler manifold, let us write $\mathrm{grad}^{1,0} f = g^{j\bar{k}} \partial_{\bar{k}} f$. This is a section of $T^{1,0}M$, and it is (up to a factor of 2), the (1,0)-part of the Riemannian gradient of f.

Theorem 4.2. *A metric ω on M is extremal if and only if* $\mathrm{grad}^{1,0}S(\omega)$ *is a holomorphic vector field.*

Proof. First let us study the variation of the Calabi functional under variations of a Kähler metric in a fixed Kähler class. So let $\omega_t = \omega + t\sqrt{-1}\partial\overline{\partial}\varphi$, and we will compute the derivative of $\mathrm{Cal}(\omega_t)$ at $t=0$. We have

$$\frac{d}{dt}\bigg|_{t=0}\omega_t^n = n\sqrt{-1}\partial\overline{\partial}\varphi\wedge\omega^{n-1} = \Delta\varphi\,\omega^n,$$

and so

$$\frac{d}{dt}\bigg|_{t=0}\mathrm{Ric}(\omega_t) = -\sqrt{-1}\partial\overline{\partial}\Delta\varphi.$$

Using that $S(\omega_t) = g_t^{j\bar{k}}R_{t,j\bar{k}}$,where $R_{t,j\bar{k}}$ is the Ricci curvature of ω_t, we have

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}S(\omega_t) &= -g^{j\bar{q}}(\partial_p\partial_{\bar{q}}\varphi)g^{p\bar{k}}R_{j\bar{k}} - \Delta^2\varphi\\ &= -\Delta^2\varphi - R^{\bar{k}j}\partial_j\partial_{\bar{k}}\varphi.\end{aligned}$$

Writing $S = S(\omega)$ for simplicity, it follows that

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\mathrm{Cal}(\omega) &= \int_M [-2S(\Delta^2\varphi + R^{\bar{k}j}\partial_j\partial_{\bar{k}}\varphi) + S^2\Delta\varphi]\omega^n\\ &= \int_M \varphi\big[-2\Delta^2 S - 2\nabla_j\nabla_{\bar{k}}(R^{\bar{k}j}S) + \Delta(S^2)\big]\omega^n.\end{aligned}$$

Using the Bianchi identity $\nabla_{\bar{k}}R^{\bar{k}j} = g^{j\bar{k}}\nabla_{\bar{k}}S$, we have

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\mathrm{Cal}(\omega) &= \int_M \varphi\big[-2\Delta^2 S - 2\nabla_j(Sg^{j\bar{k}}\nabla_{\bar{k}}S + R^{\bar{k}j}\nabla_{\bar{k}}S) + \Delta(S^2)\big]\omega^n\\ &= \int_M \varphi\big[-2\Delta^2 S - 2\nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}S)\big]\omega^n.\end{aligned}$$

In particular if ω is an extremal metric, then this variation must vanish for every φ, so

$$\Delta^2 S + \nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}S) = 0.$$

Commuting derivatives, for any function ψ we have

$$\begin{aligned}\Delta^2\psi + \nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}\psi) &= g^{j\bar{k}}g^{p\bar{q}}\nabla_j\nabla_{\bar{k}}\nabla_p\nabla_{\bar{q}}\psi + \nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}\psi)\\ &= g^{j\bar{k}}g^{p\bar{q}}\nabla_j\nabla_p\nabla_{\bar{k}}\nabla_{\bar{q}}\psi - g^{j\bar{k}}g^{p\bar{q}}\nabla_j(R^{\bar{m}}_{\ \bar{q}p\bar{k}}\nabla_{\bar{m}}\psi)\\ &\quad + \nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}\psi)\\ &= g^{j\bar{k}}g^{p\bar{q}}\nabla_p\nabla_j\nabla_{\bar{k}}\nabla_{\bar{q}}\psi.\end{aligned}$$

It follows that if we write

$$\begin{aligned}\mathcal{D} : C^\infty(M,\mathbf{C}) &\to C^\infty(\Omega^{0,1}M\otimes\Omega^{0,1}M)\\ \psi &\mapsto \nabla_{\bar{k}}\nabla_{\bar{q}}\psi,\end{aligned}$$

then

$$\Delta^2\psi + \nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}\psi) = \mathcal{D}^*\mathcal{D}\psi,$$

where $\mathcal{D}^*$ is the formal adjoint of $\mathcal{D}$. In particular if $\mathcal{D}^*\mathcal{D}S = 0$, then

$$0 = \int_M S\mathcal{D}^*\mathcal{D}S\,\omega^n = \int_M |\mathcal{D}S|^2\,\omega^n,$$

so $\mathcal{D}S = 0$. Using the metric to identify $\Omega^{0,1}M \cong T^{1,0}M$, the operator $\mathcal{D}$ can also be thought of as

$$\mathcal{D}(\psi) = \nabla_{\bar{k}}(g^{j\bar{q}}\nabla_{\bar{q}}\psi) = \nabla_{\bar{k}}(\mathrm{grad}^{1,0}\psi) = \overline{\partial}(\mathrm{grad}^{1,0}\psi)$$

since on the holomorphic tangent bundle $T^{1,0}M$ the $(0,1)$-part of the covariant derivative coincides with the usual antiholomorphic partial derivatives. Therefore $\mathcal{D}S = 0$ is equivalent to saying that $\mathrm{grad}^{1,0}S$ is holomorphic. □

Definition 4.3. The fourth-order operator that appeared in the previous proof,

$$\begin{aligned}\mathcal{D}^*\mathcal{D}\psi &= \Delta^2\psi + \nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}\psi) \\ &= \Delta^2\psi + R^{\bar{k}j}\nabla_j\nabla_{\bar{k}}\psi + g^{j\bar{k}}\nabla_j S\nabla_{\bar{k}}\psi,\end{aligned}$$

is called the Lichnerowicz operator. We saw in the proof that on a compact Kähler manifold $\mathcal{D}^*\mathcal{D}\psi = 0$ if and only if $\mathrm{grad}^{1,0}\psi$ is holomorphic. Note that in general this is a complex operator unless S is constant. One must remember this when using the selfadjointness of $\mathcal{D}^*\mathcal{D}$. For instance for complex-valued functions f, g we have

$$\int_M (\mathcal{D}^*\mathcal{D}f)\bar{g}\,\omega^n = \int_M f\overline{\mathcal{D}^*\mathcal{D}g}\,\omega^n.$$

From the previous proof we obtain a useful description of the variation of the scalar curvature under a variation of the metric.

Lemma 4.4. *Suppose that $\omega_t = \omega + t\sqrt{-1}\partial\overline{\partial}\varphi$. Then the scalar curvature S_t of ω_t satisfies*

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} S_t &= -\mathcal{D}^*\mathcal{D}\varphi + g^{j\bar{k}}\nabla_j S\nabla_{\bar{k}}\varphi \\ &= -\overline{\mathcal{D}^*\mathcal{D}\varphi} + g^{j\bar{k}}\nabla_j\varphi\nabla_{\bar{k}}S.\end{aligned}$$

Proof. The first formula follows from the previous proof. The second one follows by taking the conjugate and noting that S_t is real. □

Example 4.5. The most important examples of extremal metrics are constant scalar curvature Kähler metrics, which we will abbreviate as cscK. In fact most compact Kähler manifolds admit no non-zero holomorphic vector fields at all, so on such manifolds an extremal metric necessarily has constant scalar curvature.

In particular Kähler-Einstein metrics have constant scalar curvature, so they are examples of extremal metrics. Conversely suppose that ω is a cscK metric and we are in a Kähler class where a Kähler-Einstein metric could exist, i.e. $c_1(M) = \lambda[\omega]$ for some λ. Then ω is in fact Kähler-Einstein. Indeed, if the scalar curvature S is constant, then

$$\overline{\partial}^* R_{j\bar{k}} = -g^{p\bar{k}}\nabla_p R_{j\bar{k}} = -\nabla_j S = 0,$$

so the Ricci form is harmonic. But $2\pi\lambda\omega$ is also a harmonic form in the same class, so we have $R_{j\bar{k}} = 2\pi\lambda g_{j\bar{k}}$.

We will see in Section 4.4 that there are also examples of extremal metrics which do not have constant scalar curvature.

Exercise 4.6. Let ω be an extremal metric on a compact Kähler manifold M. Use the implicit function theorem to show that there exists an extremal metric in every Kähler class on M which is sufficiently close to $[\omega]$. This is a theorem of LeBrun-Simanca [**71**]. At first you should assume that M has no holomorphic vector fields, which simplifies the problem substantially. For the general case it might help to study Section 8.5.

In the next section we will further study the interplay between holomorphic vector fields and extremal metrics. In the remainder of this section we will show that in the definition of extremal metrics, instead of taking the L^2-norm of the scalar curvature, we could equivalently have taken the L^2-norms of the Ricci or Riemannian curvatures. For this we first need the following.

Lemma 4.7. *Let α and β be $(1,1)$-forms, given in local coordinates by $\alpha = \sqrt{-1}\alpha_{j\bar{k}}dz^j \wedge d\bar{z}^k$ and $\beta = \sqrt{-1}\beta_{j\bar{k}}dz^j \wedge d\bar{z}^k$ such that $\alpha_{j\bar{k}}$ and $\beta_{j\bar{k}}$ are Hermitian matrices. If ω is a Kähler metric with components $g_{j\bar{k}}$, then*

$$n\alpha \wedge \omega^{n-1} = (\mathrm{tr}_\omega \alpha)\omega^n,$$
$$n(n-1)\alpha \wedge \beta \wedge \omega^{n-2} = \left[(\mathrm{tr}_\omega\alpha)(\mathrm{tr}_\omega\beta) - \langle \alpha, \beta\rangle_\omega\right]\omega^n,$$

where $\mathrm{tr}_\omega\alpha = g^{j\bar{k}}\alpha_{j\bar{k}}$ and $\langle\alpha,\beta\rangle_\omega = g^{j\bar{k}}g^{p\bar{q}}\alpha_{j\bar{q}}\beta_{p\bar{k}}$.

Proof. We will prove the second equality since the first follows by taking $\beta = \omega$. We compute in local coordinates at a point where g is the identity and α is diagonal. Then

$$\omega = \sqrt{-1}\sum_i g_{i\bar{i}} dz^i \wedge d\bar{z}^i,$$

so

$$\omega^{n-2} = (\sqrt{-1})^{n-2}(n-2)! \sum_{i<j} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge \widehat{dz^i \wedge d\bar{z}^i} \wedge \cdots \wedge \widehat{dz^j \wedge d\bar{z}^j} \wedge \cdots \wedge dz^n \wedge d\bar{z}^n,$$

where the hats mean that those terms are omitted. Also

$$\alpha \wedge \beta = (\sqrt{-1})^2 \sum_{i \neq j} \alpha_{i\bar{i}} \beta_{j\bar{j}} dz^i \wedge d\bar{z}^i \wedge dz^j \wedge d\bar{z}^j$$
$$+ (\text{terms involving } \beta_{j\bar{k}} \text{ with } j \neq k)$$

since α is diagonal. It follows that

$$\begin{aligned} n(n-1)\alpha \wedge \beta \wedge \omega^{n-2} &= (\sqrt{-1})^n n! \sum_{i \neq j} \alpha_{i\bar{i}} \beta_{j\bar{j}} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ &= \left(\sum_{i \neq j} \alpha_{i\bar{i}} \beta_{j\bar{j}} \right) \omega^n \\ &= \left(\sum_{i,j} \alpha_{i\bar{i}} \beta_{j\bar{j}} - \sum_i \alpha_{i\bar{i}} \beta_{i\bar{i}} \right) \omega^n \\ &= \left[(\mathrm{tr}_\omega \alpha)(\mathrm{tr}_\omega \beta) - \langle \alpha, \beta \rangle_\omega \right] \omega^n. \end{aligned}$$

□

We can now compare the different functionals obtained by taking the L^2-norms of the Ricci and Riemannian curvatures.

Corollary 4.8. *There are constants C_1, C_2 depending on M and the Kähler class Ω such that if $\omega \in \Omega$, then*

$$\int_M S\,\omega^n = 2n\pi c_1(M) \cup [\omega]^{n-1},$$
$$\int_M |\mathrm{Ric}|^2\,\omega^n = \int_M S^2\,\omega^n + C_1,$$
$$\int_M |\mathrm{Rm}|^2\,\omega^n = \int_M |\mathrm{Ric}|^2\,\omega^n + C_2,$$

where S, Ric*, and* Rm *are the scalar, Ricci, and Riemannian curvatures of ω.*

Proof. Let us write $\rho = \sqrt{-1} R_{j\bar{k}} dz^j \wedge d\bar{z}^k$ for the Ricci form of ω and $g_{j\bar{k}}$ for the local components of the metric ω. Applying the previous lemma, we have

$$\int_M S\,\omega^n = n \int_M \rho \wedge \omega^{n-1} = 2n\pi c_1(M) \cup [\omega]^{n-1}$$

since $\mathrm{tr}_\omega \rho = S$ and ρ is a closed form representing the cohomology class $2\pi c_1(M)$.

For the second identity we again apply the previous lemma:

$$\int_M (S^2 - |\mathrm{Ric}|^2)\omega^n = n(n-1) \int_M \rho \wedge \rho \wedge \omega^{n-2} = 4n(n-1)\pi^2 c_1(M)^2 \cup [\omega]^{n-2}$$

since $\langle \rho, \rho \rangle_\omega = |\mathrm{Ric}|^2$.

For the third equation, let us introduce the endomorphism-valued 2-form Θ_p^q defined by

$$\Theta_p^q = \sqrt{-1} R_{p\,j\bar{k}}^{\;q} dz^j \wedge d\bar{z}^k.$$

Applying the previous lemma we have

$$\begin{aligned} n(n-1)\Theta_p^q \wedge \Theta_q^p \wedge \omega^{n-2} &= \left(R_p^q R_q^p - g^{j\bar{b}} g^{a\bar{k}} R_{p\,j\bar{k}}^{\;q} R_{q\,a\bar{b}}^{\;p} \right) \omega^n \\ &= \left(|\mathrm{Ric}|^2 - |\mathrm{Rm}|^2 \right) \omega^n. \end{aligned}$$

The (2,2)-form $\Theta_p^q \wedge \Theta_q^p$ is a closed form whose cohomology class is independent of the metric (in fact it is the characteristic class $4\pi^2 c_1(M)^2 - 8\pi^2 c_2(M)$), and therefore

$$\int_M (|\mathrm{Ric}|^2 - |\mathrm{Rm}|^2)\, \omega^n = C_2. \qquad \square$$

For us the most important point from the previous result is that the average scalar curvature

$$\hat{S} = \frac{2n\pi c_1(M) \cup [\omega]^{n-1}}{[\omega]^n}$$

only depends on M and the Kähler class $[\omega]$. Since

$$\int_M S(\omega)^2\, \omega^n = \int_M (S(\omega) - \hat{S})^2\, \omega^n + \int_M \hat{S}^2\, \omega^n,$$

if a cscK metric exists in a Kähler class, then it minimizes the Calabi functional. It turns out that more generally extremal metrics minimize the Calabi functional in their respective Kähler classes, but this is much harder to prove. See Donaldson [**46**] and Exercise 7.24 for the case of projective manifolds and Chen [**30**] for Kähler manifolds.

Remark 4.9. An important consequence of the previous result is that if ω is an extremal metric, then we have an estimate for the L^2-norm of the curvature of ω. This can be exploited to understand how a family of extremal metrics could degenerate in certain cases. See for example Chen-LeBrun-Weber [**35**] for an existence result based on a careful analysis of the possible "blow-up" behaviors.

4.2. Holomorphic vector fields and the Futaki invariant

As before, M is a compact Kähler manifold with Kähler metric ω. A holomorphic vector field is a holomorphic section of $T^{1,0}M$. We will focus our attention on those vector fields which can be written as $v^j = g^{j\bar{k}} \partial_{\bar{k}} f$ for a function f. It is natural to allow complex-valued functions too. Let us define

$$\mathfrak{h} := \{\text{holomorphic sections } v \text{ of } T^{1,0}M$$
$$\text{such that } v^j = g^{j\bar{k}}\partial_{\bar{k}}f \text{ for some } f : M \to \mathbf{C}\}.$$

We have seen that $v^j = g^{j\bar{k}}\partial_{\bar{k}}f \in \mathfrak{h}$ if and only if $\mathcal{D}^*\mathcal{D}f = 0$ and v^j determines f up to the addition of a constant. We call f a *holomorphy potential* for v. We can identify $\mathfrak{h}$ with the functions in $\ker \mathcal{D}^*\mathcal{D}$ which have integral zero. The space $\mathfrak{h}$ is independent of the choice of metric in the Kähler class $[\omega]$ because of the following.

Lemma 4.10. *Let us write $g_{\varphi,j\bar{k}} = g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi$ for some φ. If $v \in \mathfrak{h}$ and $v^j = g^{j\bar{k}}\partial_{\bar{k}}f$, then*

$$v^j = g_\varphi^{j\bar{k}}\partial_{\bar{k}}(f + v(\varphi)),$$

where $v(\varphi) = v^i\partial_i\varphi$ is the derivative of φ along v.

Proof. We have

$$g_{\varphi,j\bar{p}}v^j = (g_{j\bar{p}} + \partial_j\partial_{\bar{p}}\varphi)g^{j\bar{k}}\partial_{\bar{k}}f = \partial_{\bar{p}}f + \partial_{\bar{p}}(v^j\partial_j\varphi),$$

where we used that $\nabla_{\bar{p}}v^j = \partial_{\bar{p}}v^j = 0$ since v is holomorphic. Multiplying this equation by the inverse of g_φ we get the required result. □

Exercise 4.11. Show that the space $\mathfrak{h}$ is closed under the Lie bracket.

Remark 4.12. It turns out that $\mathfrak{h}$ consists of precisely those holomorphic vector fields which have a zero somewhere (see LeBrun-Simanca [**72**]), so $\mathfrak{h}$ does not even depend on the choice of Kähler class. We will also see this in Exercise 4.15 which gives yet another characterization of $\mathfrak{h}$ amongst the holomorphic vector fields.

Exercise 4.13. Show that if $c_1(M) = 0$, then $\mathfrak{h} = \{0\}$.

Exercise 4.14. Give an example of a compact Kähler manifold M and a holomorphic section v of $T^{1,0}M$ such that $v \notin \mathfrak{h}$.

Exercise 4.15. Let v be a holomorphic vector field. Show that $v \in \mathfrak{h}$ if and only if $\alpha(v) = 0$ for all holomorphic $(1,0)$-forms α.

Exercise 4.16. Suppose that M is a Fano manifold, i.e. $c_1(M) > 0$. Show that then $\mathfrak{h}$ is the space of all holomorphic vector fields on M.

Remark 4.17. It is often useful to think of sections of $T^{1,0}M$ as real vector fields. This can be achieved by identifying $T^{1,0}M$ with the real tangent bundle TM, mapping a vector field of type (1,0) to its real part. In local coordinates $z^i = x^i + \sqrt{-1}y^i$. In view of equation (1.4), this means that

$$\frac{\partial}{\partial z^i} \mapsto \frac{1}{2}\frac{\partial}{\partial x^i}, \qquad \sqrt{-1}\frac{\partial}{\partial z^i} \mapsto \frac{1}{2}\frac{\partial}{\partial y^i}.$$

We can then calculate that if $f = u + \sqrt{-1}v$ is the decomposition of f into its real and imaginary parts, then

$$g^{j\bar{k}}\partial_{\bar{k}} f \mapsto \frac{1}{2}(\mathrm{grad}\, u + J\mathrm{grad}\, v),$$

where grad is the usual Riemannian gradient and J is the complex structure. We will see in Section 5.1 that $J\mathrm{grad}\, v$ is the Hamiltonian vector field corresponding to v with respect to the symplectic form ω. It follows that if $v \in \mathfrak{h}$ has a purely imaginary holomorphy potential, then the real part of v is a Killing field. Conversely, if the real part of v is a Killing field, then $v^j = g^{j\bar{k}}\partial_{\bar{k}} f$ for a purely imaginary function f.

In view of the previous remark, let us denote by $\mathfrak{k} \subset \mathfrak{h}$ the vector fields in $\mathfrak{h}$ which correspond to Killing vector fields under the identification $T^{1,0}M = TM$. The following is a basic result about the Lie algebra $\mathfrak{h}$ on a cscK manifold (see Lichnerowicz [**76**]).

Proposition 4.18. *Suppose that ω is a cscK metric on M. Then $\mathfrak{h} = \mathfrak{k} \oplus J\mathfrak{k}$.*

Proof. Let $\ker_0 \mathcal{D}^*\mathcal{D}$ denote the elements in the kernel with zero integral. Under the identification

$$\begin{aligned} \ker_0 \mathcal{D}^*\mathcal{D} &\xrightarrow{\cong} \mathfrak{h} \\ f &\mapsto g^{j\bar{k}}\partial_{\bar{k}} f, \end{aligned}$$

the subspace $\mathfrak{k}$ corresponds to the purely imaginary functions. On the other hand, when ω has constant scalar curvature, then

$$\mathcal{D}^*\mathcal{D} = \Delta^2 + R^{\bar{k}j}\nabla_j\nabla_{\bar{k}}$$

is a real operator, and so $u + \sqrt{-1}v \in \ker \mathcal{D}^*\mathcal{D}$ for real functions u, v if and only if $u, v \in \ker \mathcal{D}^*\mathcal{D}$. □

Remark 4.19. Since $\mathfrak{k}$ generates a compact group of automorphisms, this result implies that if M admits a cscK metric, then the Lie algebra $\mathfrak{h}$ is reductive. This can be used to give examples of manifolds which do not admit a cscK metric. For example if $M = \mathrm{Bl}_p\mathbf{CP}^2$ is the blow-up of the projective plane at one point, then using Exercise 4.16 and Exercise 8.1, the Lie algebra $\mathfrak{h}$ can be identified with the holomorphic vector fields on $\mathbf{CP}^2$ which vanish at p. This latter Lie algebra can be identified with the 3×3 matrices of the form

$$\begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

which is not reductive. It follows that M does not admit cscK metrics in any Kähler class. In Exercise 4.32 we will see that M does admit extremal metrics in every Kähler class.

Remark 4.20. Studying the automorphism group can also be used to find manifolds which do not admit extremal metrics in any Kähler class. The proposition shows that if M admits a cscK metric and $\mathfrak{h}$ is non-trivial, then the group of holomorphic automorphisms of M must contain a compact subgroup. The same holds if M admits an extremal metric since then $J\mathrm{grad}\, S$ is a holomorphic Killing field. Amongst other examples, Levine [**74**] showed that if M is a suitable 4-point blow-up of $\mathbf{CP}^1 \times \mathbf{CP}^1$ (blowing up the points $(p,p),(p,q),(p,r),(q,p)$ with p,q,r any three points on $\mathbf{CP}^1$), then the identity component of the automorphism group of M is $\mathbf{C}$, and so it has no compact subgroup. It follows that M cannot admit an extremal metric in any Kähler class.

The main point in Proposition 4.18 was that when ω is a cscK metric, then $\mathcal{D}^*\mathcal{D}$ is a real operator. In general we have

$$\big(\mathcal{D}^*\mathcal{D} - \overline{\mathcal{D}^*\mathcal{D}}\big)\varphi = g^{j\bar{k}}\big(\nabla_j S \nabla_{\bar{k}}\varphi - \nabla_j\varphi\nabla_{\bar{k}} S\big).$$

If ω is an extremal metric, then $v_{\mathbf{s}} = g^{j\bar{k}}\partial_{\bar{k}} S$ is holomorphic and if in addition $v_f = g^{j\bar{k}}\partial_{\bar{k}} f \in \mathfrak{h}$, then we can compute that

$$[v_{\mathbf{s}}, v_f] = g^{p\bar{q}}\nabla_{\bar{q}} g^{j\bar{k}}\big(\nabla_j\varphi\nabla_{\bar{k}} S - \nabla_j S\nabla_{\bar{k}}\varphi\big).$$

Denote by $\mathfrak{h}_{\mathbf{s}} \subset \mathfrak{h}$ the subalgebra commuting with $v_{\mathbf{s}}$, and note that elements in $\mathfrak{k}$ commute with $v_{\mathbf{s}}$ since they correspond to Killing fields. Then the same proof as in Proposition 4.18 can be used to show that $\mathfrak{h}_{\mathbf{s}} = \mathfrak{k} \oplus J\mathfrak{k}$. A further refinement of this result is given in Calabi [**22**].

The following theorem, due to Futaki [**55**], gives an obstruction to finding cscK metrics in a Kähler class. It will turn out to be a first glimpse into the obstruction given by K-stability.

Theorem 4.21. *Let (M,ω) be a compact Kähler manifold. Let us define the functional $F : \mathfrak{h} \to \mathbf{C}$, called the Futaki invariant, by*

$$F(v) = \int_M f(S - \hat{S})\,\omega^n, \tag{4.1}$$

where f is a holomorphy potential for v and $\hat{S}$ is the average of the scalar curvature S. This functional is independent of the choice of metric in the Kähler class $[\omega]$. In particular if $[\omega]$ admits a cscK metric, then $F(v) = 0$ for all $v \in \mathfrak{h}$.

Proof. Suppose that $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$ is another Kähler metric in $[\omega]$, and write $\omega_t = \omega + t\sqrt{-1}\partial\overline{\partial}\varphi$. Let

$$F_t(v) = \int_M f_t(S_t - \hat{S})\omega_t^n,$$

where f_t is a holomorphy potential for v with respect to ω_t and S_t is the scalar curvature of ω_t. Note that by Corollary 4.8 the average $\hat{S}$ is independent of t. It is enough to show that the derivative of $F_t(v)$ at $t = 0$ vanishes. By Lemma 4.10, we can choose f_t so that

$$\left.\frac{d}{dt}\right|_{t=0} f_t = v^j \partial_j \varphi = g^{j\bar{k}} \partial_{\bar{k}} f \partial_j \varphi,$$

and from the proofs of Theorem 4.2 and Lemma 4.4 we have

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \omega_t^n &= \Delta\varphi\, \omega^n, \\ \left.\frac{d}{dt}\right|_{t=0} S_t &= -\overline{\mathcal{D}^*\mathcal{D}\varphi} + g^{j\bar{k}} \partial_j \varphi \partial_{\bar{k}} S. \end{aligned}$$

It follows that

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} F_t(v) &= \int_M \big[g^{j\bar{k}} \partial_{\bar{k}} f \partial_j \varphi (S - \hat{S}) - f(\overline{\mathcal{D}^*\mathcal{D}\varphi} - g^{j\bar{k}} \partial_j \varphi \partial_{\bar{k}} S) \\ &\qquad\qquad + f(S - \hat{S})\Delta\varphi \big]\, \omega^n \\ &= \int_M -f \overline{\mathcal{D}^*\mathcal{D}\varphi}\, \omega^n \\ &= -\int_M \varphi \mathcal{D}^*\mathcal{D} f\, \omega^n, \end{aligned}$$

after writing $\Delta\varphi = g^{j\bar{k}} \partial_{\bar{k}} \partial_j \varphi$ and integrating by parts. Using that f is a holomorphy potential, we have $\mathcal{D}^*\mathcal{D} f = 0$, so the result follows. □

To compute the Futaki invariant using the defining formula directly is impractical if not impossible in all but the simplest cases. Instead, it is possible to use a localization formula to compute $F(v)$ for a holomorphic vector field by studying the zero set of v (see Tian [**113**]). A third approach, which will be fundamental in the later developments, is that if M is a projective manifold, then the Futaki invariant can be computed algebro-geometrically. We will explain this in Section 7.4.

A useful corollary to the previous theorem is the following.

Corollary 4.22. *Suppose that ω is an extremal metric on a compact Kähler manifold M. If the Futaki invariant vanishes (relative to the Kähler class $[\omega]$), then ω has constant scalar curvature.*

Proof. Since ω is an extremal metric, the vector field $v^j = g^{j\bar{k}} \partial_{\bar{k}} S$ is in $\mathfrak{h}$. It follows that

$$0 = F(v) = \int_M S(S - \hat{S})\omega^n = \int_M (S - \hat{S})^2\, \omega^n,$$

so we must have $S = \hat{S}$, i.e. S is constant. □

4.3. The Mabuchi functional and geodesics

In this section we will see that cscK metrics have an interesting variational characterization, discovered by Mabuchi [**80**], which is different from being critical points of the Calabi functional. Moreover this variational point of view gives insight into when we can expect a cscK metric to exist.

As before, let (M, ω) be a compact Kähler manifold. Let us write

$$\mathcal{K} = \{\varphi : M \to \mathbf{R} \,|\, \varphi \text{ is smooth and } \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0\},$$

for the space of Kähler potentials for Kähler metrics in the class $[\omega]$. For any $\varphi \in \mathcal{K}$ we will write

$$\omega_\varphi = \omega + \sqrt{-1}\partial\overline{\partial}\varphi$$

for the corresponding Kähler metric, and we will put a φ subscript on other geometric quantities to indicate that they refer to this metric. The tangent space $T_\varphi\mathcal{K}$ at φ can be identified with the smooth real-valued functions $C^\infty(M)$. We can therefore define a 1-form α on $\mathcal{K}$ by letting

$$\alpha_\varphi(\psi) = \int_M \psi(\hat{S} - S_\varphi)\, \omega_\varphi^n.$$

We can check that this 1-form is closed. This boils down to differentiating $\alpha_\varphi(\psi)$ with respect to φ and showing that the resulting 2-tensor is symmetric. More precisely we need to compute

$$\left.\frac{d}{dt}\right|_{t=0} \alpha_{\varphi+t\psi_2}(\psi_1)$$

and show that it is symmetric in ψ_1 and ψ_2. We have

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} \alpha_{\varphi+t\psi_2}(\psi_1) \\
&\qquad = \int_M \left[\psi_1(\mathcal{D}_\varphi^*\mathcal{D}_\varphi\psi_2 - g_\varphi^{j\bar{k}}\partial_j S_\varphi \partial_{\bar{k}}\psi_2) + \psi_1(\hat{S} - S_\varphi)\Delta_\varphi\psi_2\right] \omega_\varphi^n \\
&\qquad = \int_M \left[\psi_1\mathcal{D}_\varphi^*\mathcal{D}_\varphi\psi_2 - (\hat{S} - S_\varphi) g^{j\bar{k}}\partial_j\psi_1\partial_{\bar{k}}\psi_2\right] \omega_\varphi^n.
\end{aligned}$$

Switching ψ_1 and ψ_2 amounts to taking the conjugate of the whole expression (using selfadjointness of the complex operator $\mathcal{D}^*\mathcal{D}$). The left-hand side of the equation is real, so it follows that the expression is symmetric in ψ_1 and ψ_2.

Since α is a closed form and $\mathcal{K}$ is contractible, there exists a function $\mathcal{M} : \mathcal{K} \to \mathbf{R}$ such that $d\mathcal{M} = \alpha$ which we can normalize so that $\mathcal{M}(0) = 0$. We could get a more explicit formula by integrating α along straight lines, but the variation of $\mathcal{M}$ is more transparent. To summarize, we have the following.

Proposition 4.23. *There is a functional $\mathcal{M} : \mathcal{K} \to \mathbf{R}$ such that the variation of $\mathcal{M}$ along a path $\varphi_t = \varphi + t\psi$ is given by*

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{M}(\varphi_t) = \int_M \psi(\hat{S} - S_\varphi)\omega_\varphi^n, \tag{4.2}$$

where S_φ is the scalar curvature of the metric $\omega_\varphi = \omega + \sqrt{-1}\partial\overline{\partial}\varphi$. This is called the Mabuchi functional or the K-energy.

Exercise 4.24. Suppose that we define the Mabuchi functional as follows. For any $\varphi \in \mathcal{K}$ let φ_t be a path in $\mathcal{K}$ such that $\varphi_0 = 0$ and $\varphi_1 = \varphi$, and define

$$\mathcal{M}(\varphi) = \int_0^1 \int_M \dot{\varphi}_t(\hat{S} - S_t)\omega_t^n \, dt,$$

where $\omega_t = \omega + \sqrt{-1}\partial\overline{\partial}\varphi_t$ and S_t is the scalar curvature of ω_t. Check directly that this is well-defined, i.e. the integral is independent of the path φ_t that we choose connecting 0 and φ.

Note that since the variation of $\mathcal{M}$ in the direction of the constant functions vanishes, $\mathcal{M}$ actually descends to a functional on the space of Kähler metrics in $[\omega]$. Moreover it is clear that critical points of $\mathcal{M}$ are given by constant scalar curvature metrics. A modification of the Mabuchi functional has been introduced by Guan [**60**] whose critical points are extremal metrics.

Next we will show that $\mathcal{M}$ is a convex function on $\mathcal{K}$ if we endow $\mathcal{K}$ with a natural Riemannian metric, introduced by Mabuchi [**81**] (see also Semmes [**94**] and Donaldson [**43**]). Given two elements $\psi_1, \psi_2 \in T_\varphi\mathcal{K}$ in the tangent space at $\varphi \in \mathcal{K}$, we can define the inner product

$$\langle \psi_1, \psi_2 \rangle_\varphi = \int_M \psi_1\psi_2 \, \omega_\varphi^n.$$

This defines a Riemannian metric on the infinite-dimensional space $\mathcal{K}$. Let us first compute the equation satisfied by geodesics.

Proposition 4.25. *A path $\varphi_t \in \mathcal{K}$ is a (constant speed) geodesic if and only if*

$$\ddot{\varphi}_t - |\partial\dot{\varphi}_t|_t^2 = \ddot{\varphi}_t - g_t^{j\bar{k}}\partial_j\dot{\varphi}_t\partial_{\bar{k}}\dot{\varphi}_t = 0,$$

where the dots mean t-derivatives and g_t is the metric $\omega + \sqrt{-1}\partial\overline{\partial}\varphi_t$.

Proof. A constant speed geodesic is a critical point of the energy of a path. The energy of the path φ_t for $t \in [0, 1]$, say, is

$$E(\varphi_t) = \int_0^1 \int_M \dot{\varphi}_t^2 \, \omega_t^n \, dt.$$

Under a variation $\varphi_t + \varepsilon\psi_t$, where ψ_t vanishes at $t = 0$ and $t = 1$, we have

$$\begin{aligned}
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E(\varphi_t + \varepsilon\psi_t) &= \int_0^1 \int_M (2\dot{\varphi}_t\dot{\psi}_t + \dot{\varphi}_t^2\Delta_t\psi_t)\,\omega_t^n\,dt \\
&= \int_0^1 \int_M (2\dot{\varphi}_t\dot{\psi}_t + \Delta(\dot{\varphi}_t^2)\psi_t)\,\omega_t^n\,dt \\
&= \int_0^1 \int_M \big[-2\ddot{\varphi}_t\psi_t - 2\dot{\varphi}_t\psi_t\Delta_t\dot{\varphi}_t + \Delta_t(\dot{\varphi}_t^2)\psi_t\big]\,\omega_t^n\,dt \\
&= \int_0^1 \int_M -2\psi_t\big[\ddot{\varphi}_t - g_t^{j\bar{k}}\partial_j\dot{\varphi}_t\partial_{\bar{k}}\dot{\varphi}_t\big]\,\omega_t^n\,dt,
\end{aligned}$$

where we integrated by parts on the manifold and also with respect to t (the $\Delta_t\dot{\varphi}_t$ term in the third line comes from differentiating ω_t^n with respect to t). The required expression for the geodesic equation follows. □

Example 4.26. A useful family of geodesics arises as follows. Suppose that $v \in \mathfrak{h}$ has holomorphy potential $u : M \to \mathbf{R}$ and $v_{\mathbf{R}}$ is the real part of v, thought of as a section of TM. Then $v_{\mathbf{R}} = \frac{1}{2}\mathrm{grad}u$, and $v_{\mathbf{R}}$ is a real holomorphic vector field; i.e. the one-parameter group of diffeomorphisms $f_t : M \to M$ generated by $v_{\mathbf{R}}$ preserves the complex structure of M. We can then define the path of metrics

$$\omega_t = f_t^*(\omega),$$

and we can check that

$$\omega_t = \omega + \sqrt{-1}\partial\overline{\partial}\varphi_t,$$

where

$$\dot{\varphi}_t = f_t^* u.$$

A good exercise is to check that φ_t defines a geodesic line in $\mathcal{K}$. The derivative of the Mabuchi functional along this line is given by

$$\begin{aligned}
\frac{d}{dt}\mathcal{M}(\varphi_t) &= \int_M \dot{\varphi}_t(\hat{S} - S_t)\,\omega_t^n \\
&= \int_M f_t^* u\,(\hat{S} - f^*S(\omega))\,f^*(\omega^n) \\
&= \int_M u(\hat{S} - S(\omega))\,\omega^n \\
&= -F(v),
\end{aligned}$$

where $F(v)$ is the Futaki invariant of v. In other words the Mabuchi functional is linear along this geodesic line, with derivative given by the Futaki invariant.

Proposition 4.27. *The Mabuchi functional $\mathcal{M} : \mathcal{K} \to \mathbf{R}$ is convex along geodesics.*

Proof. Suppose that φ_t defines a geodesic and let us compute the second derivative of $\mathcal{M}(\varphi_t)$. By definition

$$\frac{d}{dt}\mathcal{M}(\varphi_t) = \int_M \dot\varphi_t(\hat S - S_t)\,\omega_t^n,$$

so

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{M}(\varphi_t) &= \int_M \big[\ddot\varphi_t(\hat S - S_t) + \dot\varphi_t(\mathcal{D}_t^*\mathcal{D}_t\dot\varphi_t - g_t^{j\bar k}\partial_j S_t \partial_{\bar k}\dot\varphi_t) \\ &\qquad\qquad + \dot\varphi_t(\hat S - S_t)\Delta_t\dot\varphi_t\big]\,\omega_t^n \\ &= \int_M \Big[|\mathcal{D}_t\dot\varphi_t|_t^2 + (\hat S - S_t)(\ddot\varphi_t - |\partial\dot\varphi_t|_t^2)\Big]\,\omega_t^n \\ &= \int_M |\mathcal{D}_t\dot\varphi_t|_t^2\,\omega_t^n \geqslant 0.\end{aligned} \tag{4.3}$$

Therefore $\mathcal{M}$ is convex along the path φ_t. □

From this result a very appealing picture arises. We have a convex functional $\mathcal{M} : \mathcal{K} \to \mathbf{R}$ whose critical points are the cscK metrics in the class $[\omega]$. We can therefore at least heuristically expect a cscK metric to exist if and only if as we approach the "boundary" of $\mathcal{K}$, the derivative of $\mathcal{M}$ becomes positive. Since we are on an infinite-dimensional space, it is hard to make this picture rigorous, but we will find that the notion of K-stability can be seen as an attempt to encode this behavior "at infinity" of the functional $\mathcal{M}$.

Unfortunately it is difficult to construct geodesics in $\mathcal{K}$, and in fact it is possible to construct pairs of potentials in $\mathcal{K}$ on any Kähler manifold which are not joined by a smooth geodesic (see Lempert-Vivas [**73**], Darvas [**38**]). Nevertheless it is possible to show the existence of non-smooth geodesics with enough regularity that geometric conclusions can be drawn (see Chen [**29**], Chen-Tian [**36**]). In particular Chen and Tian showed that extremal metrics, if they exist, are unique up to isometry in a Kähler class.

Exercise 4.28. Suppose that ω_1, ω_2 are two cscK metrics in the same Kähler class on M. Assuming that there is a geodesic path connecting ω_1 and ω_2, prove that there is a biholomorphism $f : M \to M$ such that $f^*\omega_2 = \omega_1$.

To conclude this section, we briefly mention that when $\omega \in c_1(M)$, then there is another natural functional on $\mathcal{K}$ whose critical points are Kähler-Einstein metrics, introduced by Ding [**40**]. To define it, for any $\varphi \in \mathcal{K}$ define the Ricci potential h_φ by the equation

$$\mathrm{Ric}(\omega_\varphi) - \omega_\varphi = \sqrt{-1}\partial\bar\partial h_\varphi,$$

together with the normalization

$$\int_M e^{h_\varphi}\omega_\varphi^n = \int_M \omega_\varphi^n.$$

The variation of the Ding functional $\mathcal{F} : \mathcal{K} \to \mathbf{R}$ along a path $\varphi_t = \varphi + t\psi$ is given by

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}(\varphi_t) = \int_M \psi(e^{h_\varphi} - 1)\omega_\varphi^n.$$

Exercise 4.29. Show that if $\int_M f e^{h_\varphi}\omega_\varphi^n = 0$, then

$$\int_M f^2 e^{h_\varphi}\,\omega_\varphi^n \leqslant \int_M |\nabla f|^2 e^{h_\varphi}\omega_\varphi^n.$$

Exercise 4.30. Show that a functional $\mathcal{F}$ exists with the variational formula above and that $\mathcal{F}$ is convex along smooth geodesics in $\mathcal{K}$.

The advantage of the Ding functional $\mathcal{F}$ over the Mabuchi functional is that it can be defined for metrics with less regularity. In particular the convexity of $\mathcal{F}$ can be established along geodesics in $\mathcal{K}$ with very low regularity, and this leads to results on the uniqueness of Kähler-Einstein metrics, even ones with certain singularities, as shown by Berndtsson [**15**]. Note that the uniqueness of smooth Kähler-Einstein metrics up to isometry has previously been established by Bando-Mabuchi [**11**] without the use of geodesics.

4.4. Extremal metrics on a ruled surface

In this section we will describe the construction of explicit extremal metrics on a ruled surface, due to Tønnesen-Friedman [**116**]. We will only do the calculation in a special case, but much more general results along these lines can be found in the work of Apostolov–Calderbank–Gauduchon–Tønnesen-Friedman [**2**].

Let Σ be a genus 2 curve, and let ω_Σ be a Kähler metric on Σ with constant scalar curvature -2. By the Gauss-Bonnet theorem the area of Σ is 2π with this metric. Let L be a degree -1 holomorphic line bundle on Σ (i.e. $c_1(L)[\Sigma] = -1$), and let h be a metric on L with curvature form $F(h) = -\omega_\Sigma$.

We will construct metrics on the projectivization $X = \mathbf{P}(L \oplus \mathcal{O})$ over Σ, where $\mathcal{O}$ is the trivial line bundle. Thus X is a $\mathbf{CP}^1$-bundle over Σ. We will follow the method of Hwang-Singer [**65**]. First we construct metrics on the complement of the zero section in the total space of L and then describe what is necessary to complete the metrics across the zero and infinity sections of X.

We will consider metrics of the form

$$\omega = p^*\omega_\Sigma + \sqrt{-1}\partial\overline{\partial} f(s), \tag{4.4}$$

where $p : L \to \Sigma$ is the projection map, $s = \log|z|_h^2$, and f is a strictly convex function which makes ω positive definite. Let us compute the metric ω in local coordinates. Choose a local holomorphic coordinate z on Σ and a fiber coordinate w for L, corresponding to a holomorphic trivialization around z. The fiberwise norm is then given by $|(z,w)|_h^2 = |w|^2 h(z)$ for some function h, and so our coordinate s is given by

$$s = \log|w|^2 + \log h(z).$$

Let us work at a point (z_0, w_0), in a trivialization such that $d\log h(z_0) = 0$. Then at this point

$$\begin{aligned} \sqrt{-1}\partial\bar\partial f(s) &= f'(s)\sqrt{-1}\partial\bar\partial \log h + f''(s)\sqrt{-1}\frac{dw\wedge d\bar w}{|w|^2} \\ &= f'(s)p^*\omega_\Sigma + f''(s)\sqrt{-1}\frac{dw\wedge d\bar w}{|w|^2}, \end{aligned} \tag{4.5}$$

where we used that $-\sqrt{-1}\partial\bar\partial\log h$ is the curvature of L. It follows that

$$\omega = (1+f'(s))p^*\omega_\Sigma + f''(s)\sqrt{-1}\frac{dw\wedge d\bar w}{|w|^2}, \tag{4.6}$$

and so

$$\omega^2 = \frac{1}{|w|^2}(1+f'(s))f''(s)p^*\omega_\Sigma\wedge(\sqrt{-1}dw\wedge d\bar w).$$

We can check that if we now use a different trivialization for the line bundle in which $\tilde w = g(z)w$ for a holomorphic function z, then the same formula for ω^2 holds, so this formula holds at every point. It follows that the Ricci form of ω is

$$\begin{aligned} \rho &= -\sqrt{-1}\partial\bar\partial\log\left(\frac{1}{|w|^2}(1+f'(s))f''(s)\right) + p^*\rho_\Sigma \\ &= -\sqrt{-1}\partial\bar\partial\log\Big[(1+f'(s))f''(s)\Big] - 2p^*\omega_\Sigma, \end{aligned} \tag{4.7}$$

where $\rho_\Sigma = -2\omega_\Sigma$ is the Ricci form of Σ. We could at this point compute the scalar curvature of ω, but it is more convenient to change coordinates. From (4.6) we know that for ω to be positive, f must be strictly convex. We can therefore take the Legendre transform of f. The Legendre transform F is defined in terms of the variable $\tau = f'(s)$ by the formula

$$f(s) + F(\tau) = s\tau.$$

If $I \subset \mathbf{R}$ is the image of f', then F is a strictly convex function defined on I. The momentum profile of the metric is defined to be $\varphi : I \to \mathbf{R}$, where

$$\varphi(\tau) = \frac{1}{F''(\tau)}.$$

The following relations can be verified:

$$s = F'(\tau), \quad \frac{ds}{d\tau} = F''(\tau), \quad \varphi(\tau) = f''(s).$$

Using (4.6) and (4.7) we have

$$\begin{aligned} \omega &= (1+\tau)p^*\omega_\Sigma + \varphi(\tau)\frac{\sqrt{-1}dw \wedge d\bar{w}}{|w|^2}, \\ \rho &= -\sqrt{-1}\partial\bar{\partial}\log\big[(1+\tau)\varphi(\tau)\big] - 2p^*\omega_\Sigma. \end{aligned} \tag{4.8}$$

A calculation now shows that the scalar curvature is given by

$$S(\tau) = -\frac{2}{1+\tau} - \frac{1}{1+\tau}\big[(1+\tau)\varphi\big]'', \tag{4.9}$$

where the primes mean derivatives with respect to τ.

We still need to understand when we can complete the metric across the zero and infinity sections. We will just focus on the metric in the fiber directions, which according to (4.6) is given by

$$f''(s)\frac{\sqrt{-1}dw \wedge d\bar{w}}{|w|^2}.$$

Let us write $r = |w|$, so $s = 2\log r$. The condition that this metric extends across $w = 0$ is that f'' has the form

$$f''(s) = c_2 r^2 + c_3 r^3 + c_4 r^4 + \cdots .$$

Then, since $d/ds = \frac{r}{2}d/dr$, we have

$$f'''(s) = c_2 r^2 + \frac{3}{2}c_3 r^3 + 2c_4 r^4 + \cdots ,$$

and since $f''(s) = \varphi(\tau)$ and $f'''(s) = \varphi'(\tau)\varphi(\tau)$, we have

$$\varphi'(\tau) = 1 + O(r).$$

In particular if the range of τ is an interval (a, b), then

$$\lim_{\tau\to a}\varphi(\tau) = 0, \qquad \lim_{\tau\to a}\varphi'(\tau) = 1.$$

A similar computation can be done as $w \to \infty$ by changing coordinates to w^{-1}, showing that

$$\lim_{\tau\to b}\varphi(\tau) = 0, \qquad \lim_{\tau\to b}\varphi'(\tau) = -1.$$

Note also that by (4.8) the metric will be positive definite as long as $(1+\tau)$ and $\varphi(\tau)$ are positive on $[a, b]$. For simplicity we can take the interval $[0, m]$ for some $m > 0$. The value of m determines the Kähler class of the resulting metric. Viewing X as a $\mathbf{CP}^1$-bundle over Σ, the space $H^2(X, \mathbf{R})$ is generated by Poincaré duals of a fiber C and the infinity section S_∞, which is the image of the subbundle $L \oplus \{0\} \subset L \oplus \mathcal{O}$ under the projection map to the projectivization $X = \mathbf{P}(L \oplus \mathcal{O})$. We have the following intersection formulas:

$$C \cdot C = 0, \quad S_\infty \cdot S_\infty = 1, \quad C \cdot S_\infty = 1.$$

The Kähler class of the metric can then be determined by computing the areas of C and S_∞. The area of C is given by

$$\int_{\mathbf{C}\setminus\{0\}} f''(s)\frac{\sqrt{-1}dw \wedge d\bar{w}}{|w|^2} = 2\pi(\lim_{s\to\infty} f'(s) - \lim_{s\to-\infty} f'(s)) = 2\pi m,$$

while the area of the infinity section S_∞ is

$$(1+m)\int_\Sigma \omega_\Sigma = 2\pi(1+m).$$

It follows that if we denote by $\mathcal{L}_m$ the Poincaré dual to the Kähler class of ω, then

$$\mathcal{L}_m = 2\pi(C + mS_\infty).$$

The final thing to check is when the metric is extremal, i.e. when $\mathrm{grad}^{1,0}S(\tau)$ is holomorphic. We can compute that

$$\mathrm{grad}^{1,0}S(\tau) = S'(\tau)w\frac{\partial}{\partial w},$$

which is a holomorphic vector field if and only if $S'(\tau)$ is constant. So ω is extremal if and only if $S''(\tau) = 0$.

The end result is the following theorem, which follows from the more general results in Hwang-Singer [**65**].

Theorem 4.31. *Suppose that $\varphi : [0, m] \to \mathbf{R}$ is a smooth function which is positive on $(0, m)$ and satisfies the boundary conditions*

$$\varphi(0) = \varphi(m) = 0, \quad \varphi'(0) = 1, \quad \varphi'(m) = -1. \tag{4.10}$$

Then by the above construction we obtain a metric on X in the Kähler class Poincaré dual to $\mathcal{L}_m = 2\pi(C + mS_\infty)$ whose scalar curvature is given by

$$S(\tau) = -\frac{2}{1+\tau} - \frac{1}{1+\tau}\big[(1+\tau)\varphi\big]''.$$

The metric is extremal if and only if $S''(\tau) = 0$.

We can now construct extremal metrics by solving the ODE $S''(\tau) = 0$ for $\varphi : [0, m] \to \mathbf{R}$ satisfying the boundary conditions (4.10). The equation to be solved is

$$\frac{-1}{1+\tau}\Big(2 + \big[(1+\tau)\varphi\big]''\Big) = A + B\tau,$$

for some A, B. This equation can easily be integrated using the boundary conditions, and we obtain

$$\varphi(\tau) = \frac{\tau(m-\tau)}{m(m^2+6m+6)(1+\tau)}\big[\tau^2(2m+2) + \tau(-m^2+4m+6) + m^2+6m+6\big].$$

This will only give rise to a metric if $\varphi(\tau) > 0$ for all $\tau \in (0, m)$. This happens only if $m < k_1$, where $k_1 \approx 18.889$ is the positive root of $m^4 - 16m^3 - 52m^2 - 48m - 12$. We have therefore constructed extremal metrics with non-constant scalar curvature on the $\mathbf{CP}^1$-bundle X in the Kähler classes Poincaré dual to $\mathcal{L}_m$ for $m < k_1$.

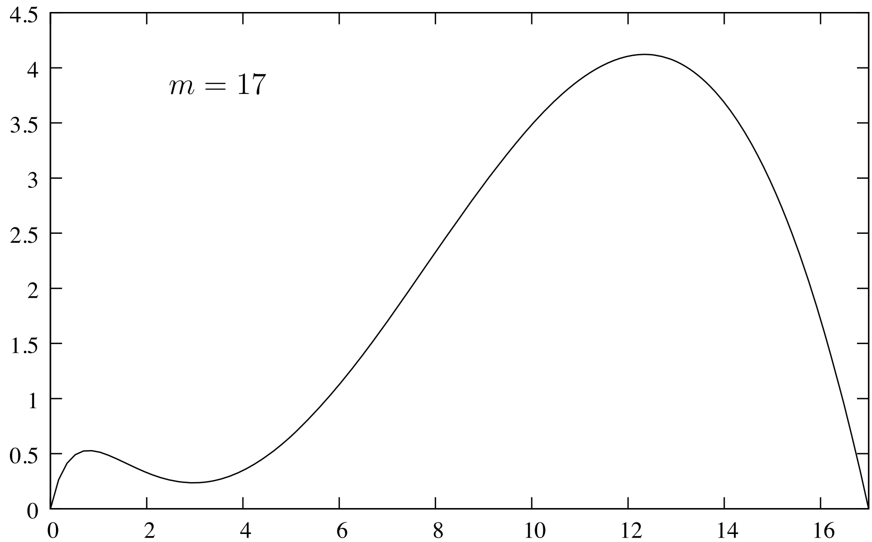

Figure 4.1. The momentum profile for the extremal metric when $m = 17$.

It is interesting to see what happens as $m \to k_1$. At $m = k_1$ the solution $\varphi(\tau)$ acquires a zero in $(0, m)$ (Figure 4.1 shows the graph of φ when $m = 17$). Geometrically this corresponds to the fiber metrics degenerating in such a way that the diameter becomes unbounded but the area remains bounded. In other words the fibers break up into two pieces, each with an end asymptotic to a hyperbolic cusp. We will see in Section 6.5 that X is not relatively K-stable when $m \geqslant k_1$, and it follows that it does not admit an extremal metric for these Kähler classes.

Exercise 4.32. Show that the blow-up $\mathrm{Bl}_p\mathbf{CP}^2$ of the projective plane in one point admits an extremal metric in every Kähler class. Use the fact that we can write $\mathrm{Bl}_p\mathbf{CP}^2 = \mathbf{P}(\mathcal{O}(1) \oplus \mathcal{O})$ as a $\mathbf{CP}^1$-bundle over $\mathbf{CP}^1$, and so we can use the method above. An alternative approach is to exploit the fact that $\mathrm{Bl}_p\mathbf{CP}^2$ is a toric manifold and use the calculations in the next section. This is an example due to Calabi [**21**].

4.5. Toric manifolds

Toric manifolds are a fertile testing ground for many ideas in algebraic and symplectic geometry, and it turns out that the study of extremal metrics on them is also very fruitful. The basic Kähler geometry of toric manifolds was worked out by Guillemin [**61**], and the study of extremal metrics on them was initiated by Abreu [**1**]. This was then taken considerably further by a sequence of works by Donaldson [**44**], [**45**], [**48**], [**49**], culminating in a general existence result for cscK metrics on K-stable toric surfaces, with a further extension to the extremal case by Chen-Li-Sheng [**28**]. In this section we will discuss Kähler metrics on toric manifolds and Abreu's formula for their scalar curvature.

There are many different descriptions of toric manifolds from the point of view of algebraic geometry and symplectic geometry. In this section, the main point for us is that an n-dimensional toric manifold M contains a dense open set biholomorphic to $\mathbf{T}_{\mathbf{C}} = (\mathbf{C}^*)^n$, and the action of this complex torus on itself extends in a smooth way to an action on all of M.

Similarly to what we did in Section 4.4 we will be interested in Kähler metrics on M which on $(\mathbf{C}^*)^n$ can be written as

$$\omega = \sqrt{-1}\partial\overline{\partial}f(x), \tag{4.11}$$

where $x \in \mathbf{R}^n$ has components $x^i = \log|z^i|^2$ for $(z_1, \ldots, z_n) \in (\mathbf{C}^*)^n$. In terms of local complex coordinates $w^i = \log z^i$, we can compute

$$\sqrt{-1}\partial\overline{\partial}f(x) = \sqrt{-1}\frac{\partial^2 f}{\partial x^i \partial x^j} dw^i \wedge d\bar{w}^j, \tag{4.12}$$

so ω is a Kähler metric whenever $f : \mathbf{R}^n \to \mathbf{R}$ is strictly convex. The function f needs to satisfy certain conditions at infinity for this metric to extend to M, but as in the previous section, it is useful to first take a Legendre transform. We thus introduce the variable

$$y = \nabla f(x)$$

and define the function u by

$$f(x) + u(y) = x \cdot y. \tag{4.13}$$

The function u is called the *symplectic potential* of the metric, and it is a convex function on a subset $P \subset \mathbf{R}^n$ given by the range of ∇f. The set P turns out to be the interior of a polytope, and ∇f is a moment map for the action of $(S^1)^n$; see Example 5.7 for more details. For this reason P is called the *moment polytope*. The scalar curvature of ω has a particularly nice form in terms of u due to Abreu [**1**].

Proposition 4.33. *The scalar curvature of ω as a function of the y^i is given by*

$$S(\omega) = -\sum_{j,k} \frac{u^{jk}}{\partial y^j \partial y^k},$$

where u^{jk} is the inverse of the Hessian of u.

Proof. From expression (4.12), the Ricci form of ω in terms of the w^i is given by

$$\begin{aligned} R_{j\bar{k}} &= -\partial_j \partial_{\bar{k}} \log\det(f_{ab}) \\ &= -\frac{\partial^2}{\partial x^j \partial x^k} \log\det(f_{ab}), \end{aligned}$$

where f_{ab} denotes the Hessian of f in the x^i variables. From the definition of the Legendre transform we have

$$\frac{\partial}{\partial x^j} = \frac{\partial y^p}{\partial x^j} \frac{\partial}{\partial y^p}$$

and also

$$f_{ab}(x) = u^{ab}(y),$$

where u^{ab} is the inverse of the Hessian of u in the y^i variables. It then follows that

$$\begin{aligned} S(\omega) &= f^{jk} \frac{\partial y^p}{\partial x^j} \frac{\partial}{\partial y^p} \left(\frac{\partial y^q}{\partial x^k} u^{ab} \frac{\partial}{\partial y^q} u_{ab} \right) \\ &= u_{jk} u^{pj} \frac{\partial}{\partial y^p} \left(u^{kq} u^{ab} \frac{\partial}{\partial y^a} u_{qb} \right) \\ &= -\frac{\partial}{\partial y^k} \frac{\partial}{\partial y^a} u^{ak}, \end{aligned}$$

where we sum over repeated indices. This formula is what we wanted to prove. □

In order to decide when ω is an extremal metric, we need to know when $\mathrm{grad}^{1,0} h$ is holomorphic for a function h of the variables y^i.

Lemma 4.34. *The vector field $\mathrm{grad}^{1,0} h(y)$ is holomorphic if and only if h is an affine linear function of y.*

Proof. Suppose that $v^j = g^{j\bar{k}}\partial_{\bar{k}}h$ in terms of the variables w^i. Then

$$\begin{aligned}\partial_{\bar{p}}v^j &= \partial_{\bar{p}}(g^{j\bar{k}}\partial_{\bar{k}}h) \\ &= \frac{\partial}{\partial x^p}\left(f^{jk}\frac{\partial h}{\partial x^k}\right) \\ &= u^{pq}\frac{\partial}{\partial y^q}\left(u_{jk}u^{kl}\frac{\partial h}{\partial y^l}\right) \\ &= u^{pq}\frac{\partial^2 h}{\partial y^q \partial y^j}.\end{aligned}$$

It follows that v^j is holomorphic if and only if h is affine linear. □

We will now briefly discuss the question of when the metric ω extends to M from $\mathbf{T_C}$. The points in $M \setminus \mathbf{T_C}$ have non-trivial stabilizer, and we can classify them according to the dimension of the stabilizer. Let us focus on a fixed point $p \in M$ of the torus action, where the stabilizer is the whole n-dimensional torus. We can choose local coordinates z^i centered at p such that the action of $\mathbf{T_C}$ is given by componentwise multiplication. Suppose for simplicity that ω is given by

$$\omega = \sqrt{-1}\sum_j dz^j \wedge d\overline{z}^j$$

in a neighborhood of p. In terms of the w^i and x^i we have

$$\begin{aligned}\omega &= \sqrt{-1}\sum_j e^{2\mathrm{Re}\, w^j} dw^j \wedge d\overline{w}^j \\ &= \sqrt{-1}\sum_j e^{x^j} dw^j \wedge d\overline{w}^j,\end{aligned}$$

and so up to the addition of an affine linear function we have

$$f(x) = e^{x^1} + \cdots + e^{x^n}.$$

Taking the Legendre transform we obtain

$$u(y) = \sum_j \left(y^j \ln y^j - y^j\right), \tag{4.14}$$

where $y^j = e^{x^j}$. The point p corresponds to $y^j = 0$, and the domain of u is a neighborhood of the origin in the positive orthant. Modifying f by an affine linear function amounts to a translation in the y variables, while choosing a different integral basis for the torus transforms the x and y variables by an element of $SL(n, \mathbf{Z})$.

More generally we can work at a point $p \in M$ where the stabilizer is a k-dimensional torus. We can then choose coordinates z^i in which

$$p = (\overbrace{0, \ldots, 0}^{k}, \overbrace{1, \ldots, 1}^{n-k}),$$

and the torus action is still given by componentwise multiplication. Still using the Euclidean metric as above, if we take the Legendre transform, then we obtain the function u as in (4.14), but now p corresponds to the point $y = (0, \ldots, 0, 1, \ldots, 1)$. The domain of u will be a neighborhood of this point inside the positive orthant and once again different choices of coordinates, and modifying f by a linear function amounts to an $SL(n, \mathbf{Z})$ transformation and a translation.

From expression (4.12) for the metric in terms of f we can see that if g is an $(S^1)^n$-invariant function on $\mathbf{T_C}$ which extends smoothly to M, then in terms of the x variables we have $\nabla g \to 0$ at infinity. It follows that the image of ∇f, i.e. the domain of y, does not change when we modify our metric ω by a globally defined Kähler potential. This means that the information that we obtained above about the possible domains of y near points in $M \setminus \mathbf{T_C}$ applies even if we do not use the Euclidean metric. Piecing this information together we see that the domain of y is the interior of a convex polytope $P \subset \mathbf{R}^n$ and a neighborhood of each vertex of P is equivalent to a neighborhood of the origin in the positive orthant, under a translation and the action of $SL(n, \mathbf{Z})$. This means that P satisfies the following Delzant condition.

Definition 4.35. Let $P \subset \mathbf{R}^n$ be a convex polytope defined by a set of inequalities

$$l_i(y) \geqslant c_i, \tag{4.15}$$

where the l_i are linear functions with coprime integral coefficients and $c_i \in \mathbf{R}$. We say that P satisfies the *Delzant condition* if n faces meet at each vertex $p \in P$, given by equations

$$l_1(y) = c_1, \ldots, l_n(y) = c_n,$$

where the l_i generate the dual space $(\mathbf{Z}^n)^*$ over $\mathbf{Z}$.

Based on the observations above, one has the following result (see Abreu [**1**] for details of the proof).

Theorem 4.36. *Let (M, ω) be a toric Kähler manifold with moment polytope $P \subset \mathbf{R}^n$ defined by inequalities* (4.15). *Then every $(S^1)^n$-invariant Kähler metric in the class $[\omega]$ has a symplectic potential $u : P \to \mathbf{R}$ of the* *form*

$$u = \sum_j (l_i(y) - c_i)\ln(l_i(y) - c_i) + v, \tag{4.16}$$

where v is smooth up to the boundary of P, while u is strictly convex on the interior of P and its restriction to each facet of P is strictly convex on the interior of that facet.

Let us denote by $\mathcal{S}$ the set of functions $u : P \to \mathbf{R}$ of the form (4.16) satisfying the conditions in the theorem. In summary we see that to find a torus invariant extremal metric on M in a given Kähler class, we need to find $u \in \mathcal{S}$ such that $S(u)$ is affine linear, where

$$S(u) = -\sum_{j,k} \frac{\partial^2 u^{jk}}{\partial y^j \partial y^k}.$$

To conclude this section we will examine what the Futaki invariant, the Mabuchi functional, and geodesics correspond to in terms of symplectic potentials. The following basic integration by parts formula can be found in Donaldson [**44**].

Lemma 4.37. *Suppose that $u \in \mathcal{S}$, and let $g : P \to \mathbf{R}$ be a continuous convex function that is smooth on the interior of P. Then*

$$\int_P u^{jk} g_{jk}\, d\mu = \int_{\partial P} g\, d\sigma - \int_P gS(u)\, d\mu,$$

where $d\mu$ is the Lebesgue measure on P, while $d\sigma$ is a positive measure on the boundary ∂P normalized so that on a face defined by $l_i(y) = c_i$ as in (4.15) *we have $d\sigma \wedge dl_i = \pm d\mu$. In addition u^{jk} and g_{jk} are the inverse Hessian of u and the Hessian of g, respectively.*

Note that in terms of the variables x^i, θ^i, where $w_i = \frac{1}{2}x^i + \sqrt{-1}\theta^i$, the volume form of the metric ω in (4.11) is

$$\frac{\omega^n}{n!} = \det(f_{jk}) dx^1 \wedge d\theta^1 \wedge \cdots \wedge dx^n \wedge d\theta^n,$$

which after transforming to the y^i variables becomes

$$\frac{\omega^n}{n!} = dy^1 \wedge d\theta^1 \wedge \cdots \wedge dy^n \wedge d\theta^n.$$

It follows that the integral of a function $g(y)$ on (M, ω) is simply the integral of g on P up to a factor of $(2\pi)^n$. Applied to $g = 1$, Lemma 4.37 then implies that the average of the scalar curvature $S(u)$ is the constant

$$a = \frac{\int_P S(u)\, d\mu}{\int_P d\mu} = \frac{\mathrm{Vol}(\partial P, d\sigma)}{\mathrm{Vol}(P, d\mu)}.$$

We have already seen that the holomorphy potentials on (M, ω) correspond to affine linear functions h on P. Formula (4.1) defining the Futaki invariant then gives

$$\begin{aligned}(2\pi)^{-n} F(h) &= \int_P h(S(u) - a)\, d\mu \\ &= \int_{\partial P} h\, d\sigma - a \int_P h\, d\mu.\end{aligned}$$

The Futaki invariant vanishes for all vector fields in $\mathfrak{h}$ when $F(h) = 0$ for all affine linear functions h. This is equivalent to saying that the center of mass of $(P, a\, d\mu)$ equals the center of mass of $(\partial P, d\sigma)$.

Note that Lemma 4.37 implies a simple necessary condition for the existence of a symplectic potential $u \in \mathcal{S}$ with constant scalar curvature. Indeed, if $S(u)$ is constant, then necessarily $S(u) = a$, and so for every convex smooth function g on P which is not affine linear, we have

$$\int_{\partial P} g\, d\sigma - a \int_P g\, d\mu = \int_P u^{jk} g_{jk}\, d\mu > 0.$$

More generally, if $S(u) = A$ for an affine linear function A, then the same argument implies that

$$\int_{\partial P} g\, d\sigma - \int_P A g\, d\mu > 0, \tag{4.17}$$

for all non-affine linear convex functions g. We will relate this condition to stability in Section 6.7.

Let us turn now to the Mabuchi functional. We have the following from [**44**].

Proposition 4.38. *With a suitable normalization by adding a constant, the Mabuchi functional evaluated at $u \in \mathcal{S}$ is given, up to a factor of $(2\pi)^n$, by*

$$\mathcal{M}(u) = -\int_P \log\det(u_{ab})\, d\mu + \int_{\partial P} u\, d\sigma - a \int_P u\, d\mu. \tag{4.18}$$

Proof. To see this, it is enough to check that the variation of the functional given by (4.18) matches up with the definition of the Mabuchi functional in (4.2). If $u_t = u + tv \in \mathcal{S}$, then we have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \mathcal{M}(u_t) &= -\int_P u^{ab} v_{ab}\, d\mu + \int_{\partial P} v\, d\sigma - a \int_P v\, d\mu \\ &= \int_P v(S(u) - a)\, d\mu.\end{aligned} \tag{4.19}$$

At the same time, if the $f_t(x)$ are the Legendre transforms of $u_t(y)$, then differentiating formula (4.13) we get

$$\frac{d}{dt}\bigg|_{t=0} f_t(x) = -\frac{d}{dt}\bigg|_{t=0} u_t(y).$$

It follows that the variation of $\mathcal{M}$ in (4.18) matches up with the variation of the Mabuchi functional in (4.2). □

We now turn to geodesics of toric Kähler metrics.

Proposition 4.39. *A family $u_t \in \mathcal{S}$ of symplectic potentials corresponds to a (constant speed) geodesic of Kähler metrics if and only if $\frac{d^2}{dt^2}u_t = 0$.*

Proof. Rather than rewriting the geodesic equation from Proposition 4.25 in terms of symplectic potentials, we will derive the equation again from the energy of a path. The key point is that in terms of the y variables the volume form is fixed. The energy of the path u_t, for $t \in [0,1]$, say, is

$$E(u_t) = \int_0^1 \int_P \dot{u}_t^2 \, d\mu \, dt,$$

so given a variation $u_t + \varepsilon v_t$ with $v_0 = v_1 = 0$, we have

$$\begin{aligned} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E(u_t + \varepsilon v_t) &= \int_0^1 \int_P 2\dot{u}_t \dot{v}_t \, d\mu \, dt \\ &= \int_0^1 \int_P -2\ddot{u}_t v_t \, d\mu \, dt. \end{aligned}$$

It follows that critical points of E satisfy $\ddot{u}_t = 0$. □

In particular any two symplectic potentials u_0, u_1 can be joined by a smooth geodesic using linear interpolation. The following uniqueness result is a simple consequence of this.

Proposition 4.40. *Suppose that $S(u_0) = S(u_1) = a$ for symplectic potentials $u_0, u_1 \in \mathcal{S}$. Then $u_0 - u_1$ is an affine linear function.*

Proof. Consider the geodesic $u_t = u_0 + tv$, where $v = u_1 - u_0$. We have

$$\frac{d}{dt}\mathcal{M}(u_t) = \int_P v(S(u_t) - a) \, d\mu,$$

so by our assumption the derivative of $\mathcal{M}(u_t)$ vanishes for $t = 0$ and $t = 1$. At the same time, from Proposition 4.27 we know that $\mathcal{M}(u_t)$ is convex. It follows that $\mathcal{M}(u_t)$ must be a constant and so from equation (4.3) we see that $v = u_1 - u_0$ must be a holomorphy potential, i.e. an affine linear function. □

We conclude this section with an example.

Example 4.41. Let $M = \mathbf{CP}^2$ with homogeneous coordinates $[Z^0 : Z^1 : Z^2]$. The points $[1 : z^1 : z^2]$ for $(z^1, z^2) \in (\mathbf{C}^*)$ define a dense complex torus, and the natural multiplication action extends as

$$(z^1, z^2) \cdot [Z^0 : Z^1 : Z^2] = [Z^0 : z_1 Z^1 : z_2 Z^2].$$

This action has three fixed points, $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$, corresponding to vertices p_0, p_1, p_2 of the moment polytope P. We can work out what the moment polytope looks like near these fixed points as follows. Near the point $[1 : 0 : 0]$ the action is standard, relative to the coordinates

$$z^1 = \frac{Z^1}{Z^0}, \qquad z^2 = \frac{Z^2}{Z^0},$$

so a neighborhood of p_0 is a translation of a neighborhood of the origin in the first quadrant. Near $[0 : 1 : 0]$ and $[0 : 0 : 1]$ we have the following coordinates:

$$\text{near } [0:1:0] : \; \frac{Z^2}{Z^1} = \frac{z^2}{z^1}, \; \frac{Z^0}{Z^1} = \frac{1}{z^1},$$
$$\text{near } [0:0:1] : \; \frac{Z^0}{Z^2} = \frac{1}{z^2}, \; \frac{Z^1}{Z^2} = \frac{z^1}{z^2}.$$

These are related to the basis z^1, z^2 of the torus by the $SL(2, \mathbf{Z})$ matrices

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

which transform the "standard corner" spanned by the vectors $(1, 0), (0, 1)$ into corners spanned by $(-1, 1), (-1, 0)$ and $(0, -1), (1, -1)$. It follows that P is a right triangle with two equal sides parallel to the x and y axes. The size of the triangle is determined by the Kähler class, while its location in the plane is determined by the choice of normalization for the Kähler potential on $\mathbf{T_C}$, or equivalently a choice of moment map for the $(S^1)^n$-action.

Suppose that our Kähler class is chosen in such a way that P has vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Then a symplectic potential on P is given by

$$u = x \ln x + y \ln y + (1 - x - y) \ln(1 - x - y).$$

A straightforward although tedious calculation shows that $S(u) = 6$, so u corresponds to a cscK metric. In fact it is the Fubini-Study metric on $\mathbf{CP}^2$.

In general it is a difficult problem to find symplectic potentials giving rise to extremal metrics, and a complete existence theory has so far only been worked out in the 2-dimensional case. We will discuss this briefly in Section 6.7, where we study the algebro-geometric side of the problem.