Preface

Hilbert's fifth problem, from his famous list of twenty-three problems in mathematics from 1900, asks for a topological description of Lie groups, without any direct reference to smooth structure. As with many of Hilbert's problems, this question can be formalised in a number of ways, but one commonly accepted formulation asks whether any locally Euclidean topological group is necessarily a Lie group. This question was answered affirmatively by Montgomery and Zippin [**MoZi1952**] and Gleason [**Gl1952**]; see Theorem 1.1.9. As a byproduct of the machinery developed to solve this problem, the structure of locally compact groups was greatly clarified, leading in particular to the very useful *Gleason-Yamabe theorem* (Theorem 1.1.13) describing such groups. This theorem (and related results) have since had a number of applications, most strikingly in Gromov's celebrated theorem [**Gr1981**] on groups of polynomial growth (Theorem 1.3.1), and in the classification of finite approximate groups (Theorem 1.2.12). These results in turn have applications to the geometry of manifolds, and on related topics in geometric group theory.

In the fall of 2011, I taught a graduate topics course covering these topics, developed the machinery needed to solve Hilbert's fifth problem, and then used it to classify approximate groups and then finally to develop applications such as Gromov's theorem. Along the way, one needs to develop a number of standard mathematical tools, such as the Baker-Campbell-Hausdorff formula relating the group law of a Lie group to the associated Lie algebra, the Peter-Weyl theorem concerning the representation-theoretic structure of a compact group, or the basic facts about ultrafilters and ultraproducts that underlie nonstandard analysis.

This text is based on the lecture notes from that course, as well as from some additional posts on my blog at `terrytao.wordpress.com` on further topics related to Hilbert's fifth problem. Part 1 of this text can thus serve as the basis for a one-quarter or one-semester advanced graduate course, depending on how much of the optional material one wishes to cover. The material here assumes familiarity with basic graduate real analysis (such as measure theory and point set topology), as covered for instance in my texts [**Ta2011**], [**Ta2010**], and including topics such as the Riesz representation theorem, the Arzelá-Ascoli theorem, Tychonoff's theorem, and Urysohn's lemma. A basic understanding of linear algebra (including, for instance, the spectral theorem for unitary matrices) is also assumed.

The core of the text is Part 1. The first part of this section of the book is devoted to the theory surrounding Hilbert's fifth problem, and in particular in fleshing out the long road from locally compact groups to Lie groups. First, the theory of Lie groups and Lie algebras is reviewed, and it is shown that a Lie group structure can be built from a special type of metric known as a *Gleason metric*, thanks to tools such as the Baker-Campbell-Hausdorff formula. Some representation theory (and in particular, the Peter-Weyl theorem) is introduced next, in order to classify compact groups. The two tools are then combined to prove the fundamental Gleason-Yamabe theorem, which among other things leads to a positive solution to Hilbert's fifth problem.

After this, the focus turns from the "soft analysis" of locally compact groups to the "hard analysis" of approximate groups, with the useful tool of *ultraproducts* serving as the key bridge between the two topics. By using this bridge, one can start imposing approximate Lie structure on approximate groups, which ultimately leads to a satisfactory classification of approximate groups as well. Finally, Part 1 ends with applications of this classification to geometric group theory and the geometry of manifolds, and in particular in reproving Gromov's theorem on groups of polynomial growth.

Part 2 contains a variety of additional material that is related to one or more of the topics covered in Part 1, but which can be omitted for the purposes of teaching a graduate course on the subject.

Notation

For reasons of space, we will not be able to define every single mathematical term that we use in this book. If a term is italicised for reasons other than emphasis or for definition, then it denotes a standard mathematical object, result, or concept, which can be easily looked up in any number of references. (In the blog version of the book, many of these terms were linked to their Wikipedia pages, or other on-line reference pages.)

Given a subset E of a space X, the *indicator function* $1_E : X \to \mathbf{R}$ is defined by setting $1_E(x)$ equal to 1 for $x \in E$ and equal to 0 for $x \notin E$.

The cardinality of a finite set E will be denoted $|E|$. We will use the asymptotic notation $X = O(Y)$, $X \ll Y$, or $Y \gg X$ to denote the estimate $|X| \leq CY$ for some absolute constant $C > 0$. In some cases we will need this constant C to depend on a parameter (e.g., d), in which case we shall indicate this dependence by subscripts, e.g., $X = O_d(Y)$ or $X \ll_d Y$. We also sometimes use $X \sim Y$ as a synonym for $X \ll Y \ll X$. (Note though that once we deploy the machinery of nonstandard analysis in Chapter 7, we will use a closely related, but slightly different, asymptotic notation.)

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`terrytao.wordpress.com/category/teaching/254a-hilberts-fifth-problem/`

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