Preface

The present book was developed out of my course, "Applied Functional Analysis", given during the years 2007–2012 at Delft University of Technology. It provides an introduction to functional analysis on an elementary level, not presupposing, e.g., background in real analysis like metric spaces or Lebesgue integration theory. The focus lies on notions and methods that are relevant in "applied" contexts. At the same time, it should serve as a stepping stone towards more advanced texts in functional analysis.

The course (and the book) evolved over the years in a process of reflection and revision. During that process I gradually realized that I wanted the students to learn (at least):

- to view functions/sequences as *points in certain spaces*, abstracting from their internal structure;
- to treat *approximations* in a multitude of situations by virtue of the concept of an abstract *distance* (metric/norm) with its diverse instances;
- to use *approximation arguments* in order to establish properties of otherwise unwieldy objects;
- to recognize *orthogonality* and its fundamental role for series representations and distance minimization in Hilbert spaces;
- to reduce differential and integral equations to abstract fixed point or minimization problems and find solutions via approximation methods, recognizing the role of *completeness*;
- to work with *weak derivatives* in order to facilitate the search for solutions of differential equations via Hilbert space methods;

- to use *operators* as a unified tool of producing solutions to a problem with varying initial data;
- to be aware of the important role of *compactness*, in particular for eigenvalue expansions.

In this book, functional analysis is developed to an extent that serves these purposes. The included examples are of an elementary character and might appear — from the point of view of applications — a little artificial. However, with the material presented in the book at hand, students should be prepared for serious real-world applications as well as for more sophisticated theoretical functional analysis.

For the Student and the Teacher. This book can be used for self-study. Its material is divided into "mandatory" and "optional" parts. The latter are indicated by a star in front of the title; see the table of contents. By "optional" I mean that it can be omitted without affecting the "mandatory" parts. However, optional material from a later chapter may refer to optional material from an earlier one. In principle, "optional" does not necessarily mean "more advanced", although it occasionally may be like that. In effect, the optional parts can be viewed as an "honors track" amendment to the mandatory course.

In the optional parts I sometimes leave the details to the reader, something that I have tried to avoid in the mandatory part.

Being interested mainly in "applied mathematics", one may well stop with Chapter 14. Chapters 15 and 16 are more of a theoretical nature and are supposed to be a bridge towards higher functional analysis. (That, however, does not mean that they are irrelevant for applications.)

Integration Theory. A sensitive point in any introductory course on functional analysis is the use of measure-theoretic integration theory. For this book, no prior knowledge of Lebesgue theory is required. However, such ignorance has to be compensated by the will to take some things for granted and to work with some concepts even if they are only partially understood.

Chapter 7 provides the necessary information. For the later chapters one should have a vague understanding of what Lebesgue measure is and how it is connected with the notion of integral, a more thorough understanding of what a null set is and what it means that something is true almost everywhere, and a good working knowledge of the dominated convergence theorem (neglecting any measurability issues).

As unproven facts from integration theory the following results are used:

- The dominated convergence theorem (Theorem 7.16);

- The density of $\mathrm{C}[a,b]$ in $\mathrm{L}^2(a,b)$ (Theorem 7.24);
- Fubini's theorem (Section 11.1);
- The density of $\mathrm{L}^2(X) \otimes \mathrm{L}^2(Y)$ in $\mathrm{L}^2(X \times Y)$ (Theorem 11.2).

See also my "Advice for the Reader" on page 125.

Exercises. Each chapter comes with three sets of exercises — labelled Exercises A, B and C. Exercises of category A are referred to alongside the text. Some of them are utmost elementary and all of them have a direct connection to the text at the point where they are referred to. They are "simple" as far as their complexity is concerned, and the context mostly gives a hint towards the solution. One could consider these exercises as recreational pauses during a strenuous hike; pauses that allow one to look back on the distance made and to observe a little closer the passed landscape.

Exercises of category B are to deepen the understanding of the main (mandatory) text. Many of them have been posed as homework exercises or exam questions in my course. The exercises of category C either refer to the mandatory parts, but are harder; or they refer to the optional material; or they cover some additonal and more advanced topics.

Synopsis. In the following I describe shortly the contents of the individual chapters.

Chapter 1: Vector spaces of functions, linear independence of monomials, standard inner products, inner product spaces, norm associated with an inner product, polarization identity, parallelogram law, orthogonality, Pythagoras' lemma, orthonormal systems, orthogonal projections onto finite-dimensional subspaces, Gram–Schmidt procedure, the trigonometric system in $\mathrm{C}[0,1]$.

Chapter 2: Cauchy–Schwarz inequality, triangle inequality, ℓ^2, normed spaces, ℓ^1, ℓ^∞, bounded linear mappings (operators), operator norm, isometries, point evaluations, left and right shift, multiplication operators and other examples of operators. *Optional:* ℓ^p-spaces for all $1 < p < \infty$ and Hölder's inequality.

Chapter 3: Metric associated with a norm, metric spaces, discrete metric, convergence in metric spaces, uniform vs. pointwise vs. square mean convergence, mean vs. square mean convergence on $\mathrm{C}[a,b]$, closure of a subset, dense subsets, c_{00} dense in ℓ^p $(p = 1, 2)$ and in c_0, properties of the closure, Weierstrass' theorem (without proof).

Chapter 4: Open and closed sets (definition, examples, properties), continuity in metric spaces, examples (continuity of metric, norm, algebraic operations, inner product), the closure of a subspace is a subspace, continuity is equal to boundedness for linear mappings, (sequential) compactness and its consequences, Bolzano–Weierstrass theorem, stronger and weaker norms, equivalence of norms. *Optional:* separability and general compactness.

Chapter 5: Cauchy sequences, complete metric spaces, Hilbert spaces, ℓ^2 is a Hilbert space, $(\mathrm{C}[a,b], \|\cdot\|_2)$ is not complete, Banach spaces, examples (finite-dimensional spaces, $(\mathcal{B}(\Omega), \|\cdot\|_\infty)$, ℓ^∞, $(\mathrm{C}[a,b], \|\cdot\|_\infty)$, absolutely convergent series in Banach spaces.

Chapter 6 (optional): Banach's contraction principle, local existence and uniqueness of solutions to ODEs, Google's PageRank algorithm, inverse mapping theorem and implicit function theorem from many-variable calculus.

Chapter 7: Lebesgue (outer) measure, measurable sets and functions, Lebesgue integral, L^p for $p \in \{1, 2, \infty\}$, null sets, equality/convergence almost everywhere, dominated convergence theorem, completeness of L^p, Hölder's inequality, $\mathrm{C}[a,b]$ is dense in $\mathrm{L}^p(a,b)$, for $p = 1, 2$. *Optional:* L^p-spaces for general p.

Chapter 8: Best approximations, counterexamples (nonexistence and non-uniqueness), existence and uniqueness of best approximations in closed convex subsets of a Hilbert space, orthogonal projections, orthogonal decomposition, Riesz–Fréchet theorem, orthogonal series and Parseval's identity, abstract Fourier expansions and Bessel's inequality, orthonormal bases.

Chapter 9: Approximation and permanence principles, proof of Weierstrass' theorem, approximation via truncation, density of $\mathrm{C}_c^\infty(\mathbb{R})$ in $\mathrm{L}^p(\mathbb{R})$, classical Fourier series, the trigonometric system is an orthonormal basis of $\mathrm{L}^2(0,1)$, theorem of Riemann–Lebesgue. *Optional:* strong convergence lemma, Fejér's theorem, convolution operators, uniqueness theorem for Fourier series, extension of bounded linear mappings, Plancherel's theorem.

Chapter 10: Weak derivatives, Sobolev spaces $\mathrm{H}^n(a,b)$, fundamental theorem of the calculus for H^1-functions, density of $\mathrm{C}^1[a,b]$ in $\mathrm{H}^1[(a,b)$, variational method for the Poisson problem on (a,b), Poincaré's inequality for an interval. *Optional:* Poisson problem on $\Omega \subseteq \mathbb{R}^d$.

Chapter 11: Integration on product spaces, Fubini's theorem, integral operators, invertibility of operators and well-posedness of equations, Dirichlet Laplacian, Green's function, Hilbert–Schmidt integral operators, strong vs. norm convergence of operators, perturbation and Neumann series, Volterra integral equations.

Chapter 12: Operators of finite rank, compact operators, Hilbert–Schmidt operators are compact, diagonal argument, representing sesquilinear forms by operators, adjoints. *Optional:* Lax–Milgram theorem, Sturm–Liouville problems, abstract Hilbert–Schmidt operators.

Chapter 13: Eigenvalues and approximate eigenvalues, location of the spectrum, self-adjoint operators, numerical range, spectral theorem for compact self-adjoint operators, eigenvalue equation and Fredholm alternative. *Optional:* spectral theory on Banach spaces (in exercises).

Chapter 14: Eigenvalue expansion of the (one-dimensional) Dirichlet Laplacian and a Schrödinger operator, application to the associated parabolic evolution equation. *Optional:* the norm of the integration operator, best constant in the one-dimensional Poincaré inequality.

Chapter 15: Principle of nested balls, Baire's theorem, uniform boundedness principle, Banach–Steinhaus theorem, Dirichlet kernel, Du Bois-Reymond's theorem, open mapping theorem, closed graph theorem, applications, Tietze's theorem.

Chapter 16: Dual space, sublinear functionals, Hahn–Banach theorem for separable spaces, elementary duality theory, dual operators, pairings and dualities, identification of duals for c_0, ℓ^1, and $L^1[a,b]$. *Optional:* Hahn–Banach theorem for general spaces, geometric Hahn–Banach theorem (without proof), reflexivity, weak convergence, dual of ℓ^p and $L^p[a,b]$ for $1 \le p < \infty$, Riesz representation theorem, dual of $C[a,b]$.

History of Functional Analysis. Many mathematical concepts or results are named after mathematicians, contemporary or past. These names are a convenient help for our memory, but should not be mistaken as a claim about who did what first. Certainly, what I call Pythagoras' lemma in this book (Lemma 1.9) was not stated in this form by Pythagoras, and we use the name since the lemma is a generalization and modernization of a well-known theorem from Euclidean geometry that traditionally is associated with Pythagoras.

Although the taxonomy is sometimes unjustified or questionable, it is not arbitrary. There are in fact *real people* behind functional analysis, and what now appears to be a coherent and complete theory needed more than a century to find its contemporary form.

After the main text and before the appendices I have included a short account of that history with special focus on the parts that are treated in the main text. A brief historical account of the real number system is included in Appendix A.5.

What is Missing. Several topics from the classical canon of functional analysis are not covered: continuous functions on compact spaces (Urysohn's lemma, Arzelá–Ascoli, Stone–Weierstrass theorem), locally convex vector spaces, theory of distributions, Banach algebras and Gelfand theory, weak topologies, Riesz' theory of compact operators on general Banach spaces, spectral theory on Banach spaces, unbounded (symmetric or selfadjoint) operators on Hilbert spaces, the general spectral theorem, Sobolev spaces other than H^n on intervals, elliptic differential equations other than in dimension one, operator semigroups.

Further Reading. A book close in spirit to my text is the work [**GGK03**] by Gohberg, Goldberg and Kaashoek. Beyond that, I recommend the excellent works [**Che01**] by Ward Cheney and [**You88**] by Nicholas Young. These two books were a very valuable assistance during the writing.

In the direction of applications, a suitable follow-up to this book are Eberhard Zeidler's two volumes [**Zei95a, Zei95b**].

If one wants to step deeper into functional analysis there are so many possibilities that to mention just a few would do injustice to all the others. The most profound and comprehensive modern treatment that I know, and certainly a recommendation for the future expert, is Peter Lax's *opus magnum* [**Lax02**].

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