
Preface

Overview

The present text was written for my course *Schrödinger Operators* held at the University of Vienna in winter 1999, summer 2002, summer 2005, and winter 2007. It gives a brief but rather self-contained introduction to the mathematical methods of quantum mechanics with a view towards applications to Schrödinger operators. The applications presented are highly selective; as a result, many important and interesting items are not touched upon.

Part 1 is a stripped-down introduction to spectral theory of unbounded operators where I try to introduce only those topics which are needed for the applications later on. This has the advantage that you will (hopefully) not get drowned in results which are never used again before you get to the applications. In particular, I am not trying to present an encyclopedic reference. Nevertheless I still feel that the first part should provide a solid background covering many important results which are usually taken for granted in more advanced books and research papers.

My approach is built around the spectral theorem as the central object. Hence I try to get to it as quickly as possible. Moreover, I do not take the detour over bounded operators but I go straight for the unbounded case. In addition, existence of spectral measures is established via the Herglotz rather than the Riesz representation theorem since this approach paves the way for an investigation of spectral types via boundary values of the resolvent as the spectral parameter approaches the real line.

Part 2 starts with the free Schrödinger equation and computes the free resolvent and time evolution. In addition, I discuss position, momentum, and angular momentum operators via algebraic methods. This is usually found in any physics textbook on quantum mechanics, with the only difference being that I include some technical details which are typically not found there. Then there is an introduction to one-dimensional models (Sturm–Liouville operators) including generalized eigenfunction expansions (Weyl–Titchmarsh theory) and subordinacy theory from Gilbert and Pearson. These results are applied to compute the spectrum of the hydrogen atom, where again I try to provide some mathematical details not found in physics textbooks. Further topics are nondegeneracy of the ground state, spectra of atoms (the HVZ theorem), and scattering theory (the Enß method).

Prerequisites

I assume some previous experience with Hilbert spaces and bounded linear operators which should be covered in any basic course on functional analysis. However, while this assumption is reasonable for mathematics students, it might not always be for physics students. For this reason there is a preliminary chapter reviewing all necessary results (including proofs). In addition, there is an appendix (again with proofs) providing all necessary results from measure theory.

Literature

The present book is highly influenced by the four volumes of Reed and Simon [49]–[52] (see also [16]) and by the book by Weidmann [70] (an extended version of which has recently appeared in two volumes [72], [73], however, only in German). Other books with a similar scope are, for example, [16], [17], [21], [26], [28], [30], [48], [57], [63], and [65]. For those who want to know more about the physical aspects, I can recommend the classical book by Thirring [68] and the visual guides by Thaller [66], [67]. Further information can be found in the bibliographical notes at the end.

Reader's guide

There is some intentional overlap among Chapter 0, Chapter 1, and Chapter 2. Hence, provided you have the necessary background, you can start reading in Chapter 1 or even Chapter 2. Chapters 2 and 3 are key

chapters, and you should study them in detail (except for Section 2.6 which can be skipped on first reading). Chapter 4 should give you an idea of how the spectral theorem is used. You should have a look at (e.g.) the first section, and you can come back to the remaining ones as needed. Chapter 5 contains two key results from quantum dynamics: Stone's theorem and the RAGE theorem. In particular, the RAGE theorem shows the connections between long-time behavior and spectral types. Finally, Chapter 6 is again of central importance and should be studied in detail.

The chapters in the second part are mostly independent of each other except for Chapter 7, which is a prerequisite for all others except for Chapter 9.

If you are interested in one-dimensional models (Sturm–Liouville equations), Chapter 9 is all you need.

If you are interested in atoms, read Chapter 7, Chapter 10, and Chapter 11. In particular, you can skip the separation of variables (Sections 10.3 and 10.4, which require Chapter 9) method for computing the eigenvalues of the hydrogen atom, if you are happy with the fact that there are countably many which accumulate at the bottom of the continuous spectrum.

If you are interested in scattering theory, read Chapter 7, the first two sections of Chapter 10, and Chapter 12. Chapter 5 is one of the key prerequisites in this case.

2nd edition

Several people have sent me valuable feedback and pointed out misprints since the appearance of the first edition. All of these comments are of course taken into account. Moreover, numerous small improvements were made throughout. Chapter 3 has been reworked, and I hope that it is now more accessible to beginners. Also some proofs in Section 9.4 have been simplified (giving slightly better results at the same time). Finally, the appendix on measure theory has also grown a bit: I have added several examples and some material around the change of variables formula and integration of radial functions.

Updates

The AMS is hosting a web page for this book at

<http://www.ams.org/bookpages/gsm-157/>

where updates, corrections, and other material may be found, including a link to material on my own web site:

<http://www.mat.univie.ac.at/~gerald/ftp/book-schroe/>

Acknowledgments

I would like to thank Volker Enß for making his lecture notes [20] available to me. Many colleagues and students have made useful suggestions and pointed out mistakes in earlier drafts of this book, in particular: Kerstin Ammann, Jörg Arnberger, Chris Davis, Fritz Gesztesy, Maria Hoffmann-Ostenhof, Zhenyou Huang, Helge Krüger, Katrin Grunert, Wang Lanning, Daniel Lenz, Christine Pfeuffer, Roland Möws, Arnold L. Neidhardt, Serge Richard, Harald Rindler, Alexander Sakhnovich, Robert Stadler, Johannes Temme, Karl Unterkofler, Joachim Weidmann, Rudi Weikard, and David Wimmesberger.

My thanks for pointing out mistakes in the first edition go to: Erik Makino Bakken, Alexander Beigl, Stephan Bogendörfer, Søren Fournais, Semra Demirel-Frank, Katrin Grunert, Jason Jo, Helge Krüger, Oliver Leingang, Serge Richard, Gerardo González Robert, Bob Sims, Oliver Skocek, Robert Stadler, Fernando Torres-Torija, Gerhard Tulzer, Hendrik Vogt, and David Wimmesberger.

If you also find an error or if you have comments or suggestions (no matter how small), please let me know.

I have been supported by the Austrian Science Fund (FWF) during much of this writing, most recently under grant Y330.

Gerald Teschl

Vienna, Austria
April 2014

Gerald Teschl
Fakultät für Mathematik
Oskar-Morgenstern-Platz 1
Universität Wien
1090 Wien, Austria

E-mail: Gerald.Teschl@univie.ac.at
URL: <http://www.mat.univie.ac.at/~gerald/>