Introduction

Background

Linear algebra plays a key role in the theory of dynamical systems, and concepts from dynamical systems allow the study, characterization and generalization of many objects in linear algebra, such as similarity of matrices, eigenvalues, and (generalized) eigenspaces. The most basic form of this interplay can be seen as a quadratic matrix A gives rise to a discrete time dynamical system $x_{k+1} = Ax_k,\ k = 0, 1, 2, \ldots$ and to a continuous time dynamical system via the linear ordinary differential equation $\dot{x} = Ax$.

The (real) Jordan form of the matrix A allows us to write the solution of the differential equation $\dot{x} = Ax$ explicitly in terms of the matrix exponential, and hence the properties of the solutions are intimately related to the properties of the matrix A. Vice versa, one can consider properties of a linear flow in $\mathbb{R}^d$ and infer characteristics of the underlying matrix A. Going one step further, matrices also define (nonlinear) systems on smooth manifolds, such as the sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, the Grassmannian manifolds, the flag manifolds, or on classical (matrix) Lie groups. Again, the behavior of such systems is closely related to matrices and their properties.

Since A.M. Lyapunov's thesis [**97**] in 1892 it has been an intriguing problem how to construct an appropriate linear algebra for time-varying systems. Note that, e.g., for stability of the solutions of $\dot{x} = A(t)x$ it is not sufficient that for all $t \in \mathbb{R}$ the matrices $A(t)$ have only eigenvalues with negative real part (see, e.g., Hahn [**61**], Chapter 62). Classical Floquet theory (see Floquet's 1883 paper [**50**]) gives an elegant solution for the periodic case, but it is not immediately clear how to build a linear algebra around Lyapunov's 'order numbers' (now called Lyapunov exponents) for more general time dependencies. The key idea here is to write the time dependency as a dynamical system with certain recurrence properties. In this way, the multiplicative ergodic theorem of Oseledets from 1968 [**109**] resolves the basic issues for measurable linear systems with stationary time dependencies, and the Morse spectrum together with Selgrade's theorem [**124**] goes a long way in describing the situation for continuous linear systems with chain transitive time dependencies.

A third important area of interplay between dynamics and linear algebra arises in the linearization of nonlinear systems about fixed points or arbitrary trajectories. Linearization of a differential equation $\dot{y} = f(y)$ in $\mathbb{R}^d$ about a fixed point $y_0 \in \mathbb{R}^d$ results in the linear differential equation $\dot{x} = f'(y_0)x$ and theorems of the type Grobman-Hartman (see, e.g., Bronstein and Kopanskii [**21**]) resolve the behavior of the flow of the nonlinear equation locally around y_0 up to conjugacy, with similar results for dynamical systems over a stochastic or chain recurrent base.

These observations have important applications in the natural sciences and in engineering design and analysis of systems. Specifically, they are the basis for stochastic bifurcation theory (see, e.g., Arnold [**6**]), and robust stability and stabilizability (see, e.g., Colonius and Kliemann [**29**]). Stability radii (see, e.g., Hinrichsen and Pritchard [**68**]) describe the amount of perturbation the operating point of a system can sustain while remaining stable, and stochastic stability characterizes the limits of acceptable noise in a system, e.g., an electric power system with a substantial component of wind or wave based generation.

Goal

This book provides an introduction to the interplay between linear algebra and dynamical systems in continuous time and in discrete time. There are a number of other books emphasizing these relations. In particular, we would like to mention the book [**69**] by M.W. Hirsch and S. Smale, which always has been a great source of inspiration for us. However, this book restricts attention to autonomous equations. The same is true for other books like M. Golubitsky and M. Dellnitz [**54**] or F. Lowenthal [**96**], which is designed to serve as a text for a first course in linear algebra, and the relations to linear autonomous differential equations are established on an elementary level only.

Our goal is to review the autonomous case for one $d \times d$ matrix A via induced dynamical systems in $\mathbb{R}^d$ and on Grassmannians, and to present the main nonautonomous approaches for which the time dependency $A(t)$ is given via skew-product flows using periodicity, or topological (chain recurrence) or ergodic properties (invariant measures). We develop generalizations of (real parts of) eigenvalues and eigenspaces as a starting point for a linear algebra for classes of time-varying linear systems, namely periodic, random, and perturbed (or controlled) systems. Several examples of (low-dimensional) systems that play a role in engineering and science are presented throughout the text.

Originally, we had also planned to include some basic concepts for the study of genuinely nonlinear systems via linearization, emphasizing invariant manifolds and Grobman-Hartman type results that compare nonlinear behavior locally to the behavior of associated linear systems. We decided to skip this discussion, since it would increase the length of this book considerably and, more importantly, there are excellent treatises of these problems available in the literature, e.g., Robinson [**117**] for linearization at fixed points, or the work of Bronstein and Kopanskii [**21**] for more general linearized systems.

Another omission is the rich interplay with the theory of Lie groups and semigroups where many concepts have natural counterparts. The monograph [**48**] by R. Feres provides an excellent introduction. We also do not treat nonautonomous differential equations via pullback or other fiberwise constructions; see, e.g., Crauel and Flandoli [**37**], Schmalfuß [**123**], and Rasmussen [**116**]; our emphasis is on the treatment of families of nonautonomous equations. Further references are given at the end of the chapters.

Finally, it should be mentioned that all concepts and results in this book can be formulated in continuous and in discrete time. However, sometimes results in discrete time may be easier to state and to prove than their analogues in continuous time, or vice versa. At times, we have taken the liberty to pick one convenient setting, if the ideas of a result and its proof are particularly intuitive in the corresponding setup. For example, the results in Chapter 5 on induced systems on Grassmannians are formulated and derived only in continuous time. More importantly, the proof of the multiplicative ergodic theorem in Chapter 11 is given only in discrete time (the formulation and some discussion are also given in continuous time). In contrast, Selgrade's Theorem for topological linear dynamical systems in Chapter 9 and the results on Morse decompositions in Chapter 8, which are used for its proof, are given only in continuous time.

Our aim when writing this text was to make 'time-varying linear algebra' in its periodic, topological and ergodic contexts available to beginning graduate students by providing complete proofs of the major results in at least one typical situation. In particular, the results on the Morse spectrum in Chapter 9 and on multiplicative ergodic theory in Chapter 11 have detailed proofs that, to the best of our knowledge, do not exist in the current literature.

Prerequisites

The reader should have basic knowledge of real analysis (including metric spaces) and linear algebra. No previous exposure to ordinary differential equations is assumed, although a first course in linear differential equations certainly is helpful. Multilinear algebra shows up in two places: in Section 5.2 we discuss how the volumes of parallelepipeds grow under the flow of a linear autonomous differential equation, which we relate to chain recurrent sets of the induced flows on Grassmannians. The necessary elements of multilinear algebra are presented in Section 5.1. In Chapter 11 the proof of the multiplicative ergodic theorem requires further elements of multilinear algebra which are provided in Section 11.3. Understanding the proofs in Chapter 10 on ergodic theory and Chapter 11 on random linear dynamical systems also requires basic knowledge of σ-algebras and probability measures (actually, a detailed knowledge of Lebesgue measure should suffice). The results and methods in the rest of the book are independent of these additional prerequisites.

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