Chapter 2

Linear Dynamical Systems in $\mathbb{R}^d$

In this chapter we will introduce the general notion of dynamical systems in continuous and in discrete time and discuss some of their properties. Linear autonomous differential equations $\dot{x} = Ax$ where $A \in gl(d, \mathbb{R})$ generate linear continuous-time dynamical systems or flows in $\mathbb{R}^d$. Similarly, linear autonomous difference equations $x_{n+1} = Ax_n$ generate linear discrete-time dynamical systems in $\mathbb{R}^d$. A standard concept for the classification of these dynamical systems are conjugacies mapping solutions into solutions. It turns out that these classifications of dynamical systems are closely related to classifications of matrices. For later purposes, the definitions for dynamical systems are given in the abstract setting of metric spaces, instead of $\mathbb{R}^d$.

Section 2.1 introduces continuous-time dynamical systems on metric spaces and conjugacy notions. Section 2.2 determines the associated equivalence classes for linear flows in $\mathbb{R}^d$ in terms of matrix classifications. Section 2.3 presents analogous results in discrete time.

2.1. Continuous-Time Dynamical Systems or Flows

In this section we introduce general continuous-time dynamical systems or flows on metric spaces and the notions of conjugacy.

Recall that a metric space is a set X with a distance function $d : X \times X \to [0, \infty)$ satisfying $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$, and $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Definition 2.1.1. A continuous dynamical system or flow over the 'time set' $\mathbb{R}$ with state space X, a metric space, is defined as a continuous map $\Phi : \mathbb{R} \times X \longrightarrow X$ with the properties

(i) $\Phi(0, x) = x$ for all $x \in X$,

(ii) $\Phi(s + t, x) = \Phi(s, \Phi(t, x))$ for all $s, t \in \mathbb{R}$ and all $x \in X$.

For each $x \in X$ the set $\{\Phi(t, x) \mid t \in \mathbb{R}\}$ is called the orbit (or trajectory) of the system through the point x. For each $t \in \mathbb{R}$ the time-t map is defined as $\varphi_t = \Phi(t, \cdot) : X \longrightarrow X$. Using time-$t$ maps, properties (i) and (ii) above can be restated as $\varphi_0 = id$, the identity map on X, and $\varphi_{s+t} = \varphi_s \circ \varphi_t$ for all $s, t \in \mathbb{R}$.

More precisely, a system Φ as above is a continuous dynamical system in continuous time. For simplicity, we just talk about continuous dynamical systems in the following. Note that we have defined a dynamical system over the (two-sided) time set $\mathbb{R}$. This immediately implies invertibility of the time-t maps.

Proposition 2.1.2. *Each time-t map φ_t has the inverse $(\varphi_t)^{-1} = \varphi_{-t}$, and $\varphi_t : X \longrightarrow X$ is a homeomorphism, i.e., a continuous bijective map with continuous inverse. Denote the set of time-t maps again by $\Phi = \{\varphi_t \mid t \in \mathbb{R}\}$. A dynamical system is a group in the sense that $(\Phi, \circ)$, with $\circ$ denoting composition of maps, satisfies the group axioms, and $\varphi : (\mathbb{R}, +) \longrightarrow (\Phi, \circ)$, defined by $\varphi(t) = \varphi_t$ is a group homomorphism.*

Systems defined over the one-sided time set $\mathbb{R}^+ := \{t \in \mathbb{R} \mid t \geq 0\}$ satisfy the corresponding semigroup property and their time-t maps need not be invertible. Standard examples for continuous dynamical systems are given by solutions of differential equations.

Example 2.1.3. For $A \in gl(d, \mathbb{R})$ the solutions of a linear differential equation $\dot{x} = Ax$ form a continuous dynamical system with time set $\mathbb{R}$ and state space $X = \mathbb{R}^d$. Here $\Phi : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is defined by $\Phi(t, x_0) = x(t, x_0) = e^{At}x_0$. This follows from Corollary 1.1.2.

Also, many nonlinear differential equations define dynamical systems. Since we do not need this general result, we just state it.

Example 2.1.4. Suppose that the function f in the initial value problem (1.4.2) is locally Lipschitz continuous and for all $x_0 \in \mathbb{R}^d$ there are solutions $\varphi(t, x_0)$ defined for all $t \in \mathbb{R}$. Then $\Phi(t, x_0) = \varphi(t, x_0)$ defines a dynamical system $\Phi : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$. In this case, the vector field f is called complete.

Two specific types of orbits will play an important role in this book, namely fixed points and periodic orbits.

Definition 2.1.5. A fixed point (or equilibrium) of a dynamical system Φ is a point $x \in X$ with the property $\Phi(t,x) = x$ for all $t \in \mathbb{R}$.

A solution $\Phi(t,x), t \in \mathbb{R}$, of a dynamical system Φ is called periodic if there exists $S > 0$ such that $\Phi(S+s,x) = \Phi(s,x)$ for all $s \in \mathbb{R}$. The infimum T of the numbers S with this property is called the period of the solution and the solution is called T-periodic.

Since a solution is continuous in t, the period T satisfies $\Phi(T+s,x) = \Phi(s,x)$ for all $s \in \mathbb{R}$. Note that a solution of period 0 is a fixed point. For a periodic solution we also call the orbit $\{\Phi(t,x) \mid t \in \mathbb{R}\}$ periodic. If the system is given by a differential equation as in Example 2.1.4, the fixed points are easily characterized, since we assume that solutions of initial value problems are unique: A point $x_0 \in X$ is a fixed point of the dynamical system Φ associated with a differential equation $\dot{x} = f(x)$ if and only if $f(x_0) = 0$.

For linear differential equations as in Example 2.1.3 we can say a little more.

Proposition 2.1.6. *(i) A point $x_0 \in \mathbb{R}^d$ is a fixed point of the dynamical system Φ associated with the linear differential equation $\dot{x} = Ax$ if and only if $x_0 \in \ker A$, the kernel of A.*

(ii) The solution for $x_0 \in \mathbb{R}^d$ is T-periodic if and only if x_0 is in the eigenspace of the eigenvalue 1 of e^{AT}. This holds, in particular, if x_0 is in the eigenspace of an imaginary eigenvalue pair $\pm\imath\nu \neq 0$ of A and $T = \frac{2\pi}{\nu}$.

Proof. Assertion (i) and the first assertion in (ii) are obvious from direct constructions of solutions. The second assertion in (ii) follows from Example 1.3.6 and the fact that the eigenvalues $\pm\imath\nu$ of A are mapped onto the eigenvalue 1 of $e^{A\frac{2\pi}{\nu}}$. The reader is asked to prove this in detail in Exercise 2.4.5. □

Example 2.1.7. The converse of the second assertion in (ii) is not true: Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

with eigenvalues $\{0, \pm\imath\}$. The initial value $x_0 = (1,1,0)^\top$ (which is not in an eigenspace of A) leads to the 2π-periodic solution $e^{At}x_0 = (1, \cos t, \sin t)^\top$.

Conjugacy

A fundamental topic in the theory of dynamical systems concerns comparison of two systems, i.e., how can we tell that two systems are 'essentially the same'? In this case, they should have similar properties. For example, fixed points and periodic solutions should correspond to each other. This idea can be formalized through conjugacies, which we define next.

Definition 2.1.8. (i) Two continuous dynamical systems Φ, $\Psi : \mathbb{R} \times X \longrightarrow X$ on a metric space X are called C^0-conjugate or topologically conjugate if there exist a homeomorphism $h : X \to X$ such that

$$h(\Phi(t, x)) = \Psi(t, h(x)) \text{ for all } x \in X \text{ and } t \in \mathbb{R}. \tag{2.1.1}$$

(ii) Let Φ, $\Psi : \mathbb{R} \times X \longrightarrow X$ be C^k-maps, $k \geq 1$, on an open subset X of $\mathbb{R}^d$. They are called C^k-**conjugate** if there exists a C^k diffeomorphism $h : X \to X$ with (2.1.1). Then h is also called a smooth conjugacy.

The conjugacy property (2.1.1) can be illustrated by the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Phi(t,\cdot)} & X \\ h\big\downarrow & & \big\downarrow h \\ X & \xrightarrow{\Phi(t,\cdot)} & X\ . \end{array}$$

Note that while this terminology is standard in dynamical systems, the term *conjugate* is used differently in linear algebra. (Smooth) *conjugacy* as used here is related to matrix similarity (compare Theorem 2.2.1), not to matrix conjugacy. Topological conjugacies preserve many properties of dynamical systems. The next proposition shows some of them.

Proposition 2.1.9. *Let $h : X \to X$ be a topological conjugacy of two dynamical systems Φ, $\Psi : \mathbb{R} \times X \longrightarrow X$ on a metric state space X. Then*

(i) the point $p \in X$ is a fixed point of Φ if and only if $h(p)$ is a fixed point of Ψ;

(ii) the solution $\Phi(\cdot, p)$ is T-periodic if and only if $\Psi(\cdot, h(p))$ is T-periodic.

(iii) Let, in addition, $g : Y \to Y$ be a topological conjugacy of two dynamical systems $\Phi_1, \Psi_1 : \mathbb{R} \times Y \longrightarrow Y$ on a metric space Y. Then the product flows $\Phi \times \Phi_1$ and $\Psi \times \Psi_1$ on $X \times Y$ are topologically conjugate via $h \times g : X \times Y \longrightarrow X \times Y$.

Proof. The proof of assertions (i) and (ii) is deferred to Exercise 2.4.1. Assertion (iii) follows, since $h \times g$ is a homeomorphism and for $x \in X, y \in Y$, and $t \in \mathbb{R}$ one has

$$\begin{aligned}(h \times g)(\Phi \times \Phi_1)(x, y) &= (h(\Phi(t, x), g(\Phi_1(t, y))) = (\Psi(t, h(x)), \Psi_1(t, g(y))) \\ &= (\Psi \times \Psi_1)\,(t, (h \times g)(x, y)).\end{aligned}$$

□

2.2. Conjugacy of Linear Flows

For linear flows associated with linear differential equations as introduced in Example 2.1.3, conjugacy can be characterized directly in terms of the matrix A. We start with smooth conjugacies.

Theorem 2.2.1. *For two linear flows Φ (associated with $\dot{x} = Ax$) and Ψ (associated with $\dot{x} = Bx$) in $\mathbb{R}^d$, the following are equivalent:*

(i) Φ and Ψ are C^k-conjugate for $k \geq 1$,

(ii) Φ and Ψ are linearly conjugate, i.e., the conjugacy map h is an invertible linear operator on $\mathbb{R}^d$,

(iii) A and B are similar, i.e., $A = SBS^{-1}$ for some $S \in Gl(d, \mathbb{R})$.

Each of these statements is equivalent to the property that A and B have the same Jordan form. Thus the C^k-conjugacy classes are exactly the real Jordan form equivalence classes in $gl(d, \mathbb{R})$.

Proof. Properties (ii) and (iii) are obviously equivalent and imply (i). Suppose that (i) holds, and let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a C^k-conjugacy. Thus for all $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$,

$$h(\Phi(t, x)) = h(\mathrm{e}^{At}x) = \mathrm{e}^{Bt}h(x) = \Psi(h(x)).$$

Differentiating with respect to x and using the chain rule we find

$$Dh(\mathrm{e}^{At}x)\mathrm{e}^{At} = \mathrm{e}^{Bt}Dh(x).$$

Evaluating this at $x = 0$ we get with $H := Dh(0)$,

$$H\mathrm{e}^{At} = \mathrm{e}^{Bt}H \text{ for all } t \in \mathbb{R}.$$

Differentiation with respect to t in $t = 0$ finally gives $HA = BH$. Since h is a diffeomorphism, the linear map $H = Dh(0)$ is invertible and hence defines a linear conjugacy. □

In particular we obtain conjugacy for the linear dynamical systems induced by a matrix A and its real Jordan form $J^{\mathbb{R}}$.

Corollary 2.2.2. *For each matrix $A \in gl(d, \mathbb{R})$ its associated linear flow in $\mathbb{R}^d$ is C^k-conjugate for all $k \geq 1$ to the dynamical system associated with the Jordan form $J^{\mathbb{R}}$.*

Theorem 2.2.1 clarifies the structure of two matrices that give rise to conjugate flows under C^k-diffeomorphisms with $k \geq 1$. The eigenvalues and the dimensions of the Jordan blocks remain invariant, while the eigenspaces and generalized eigenspaces are mapped onto each other.

For homeomorphisms, i.e., for $k = 0$, the situation is quite different and somewhat surprising. To explain the corresponding result we first need to introduce the concept of hyperbolicity.

Definition 2.2.3. The matrix $A \in gl(d, \mathbb{R})$ is hyperbolic if it has no eigenvalues on the imaginary axis.

The set of hyperbolic matrices in $gl(d, \mathbb{R})$ is rather 'large' in $gl(d, \mathbb{R})$ (which may be identified with $\mathbb{R}^{d^2}$ and hence carries the corresponding topology.)

Proposition 2.2.4. *The set of hyperbolic matrices is open and dense in $gl(d, \mathbb{R})$ and for every hyperbolic matrix A there is a neighborhood $U \subset gl(d, \mathbb{R})$ of A such that the dimension of the stable subspace is constant for $B \in U$.*

Proof. Let A be hyperbolic. Then also for all matrices in a neighborhood of A all eigenvalues have nonvanishing real parts, since the eigenvalues depend continuously on the matrix entries. Hence openness and the last assertion follow. Concerning density, transform an arbitrary matrix $A \in gl(d, \mathbb{R})$ via a matrix T into a matrix $T^{-1}AT$ in Jordan normal form. For such a matrix it is clear that one finds arbitrarily close matrices B which are hyperbolic. Transforming them back into TBT^{-1} one obtains hyperbolic matrices arbitrarily close to A. □

With these preparations we can formulate the characterization of C^0-conjugacies of linear flows:

Theorem 2.2.5. *(i) If A and B are hyperbolic, then the associated linear flows Φ and Ψ in $\mathbb{R}^d$ are topologically conjugate if and only if the dimensions of the stable subspaces (and hence the dimensions of the unstable subspaces) of A and B agree.*

(ii) A matrix A is hyperbolic if and only if its flow is structurally stable, i.e., there exists a neighborhood $U \subset gl(d, \mathbb{R})$ of A such that for all $B \in U$ the associated linear flows are conjugate to the flow of A.

Observe that assertion (ii) is an immediate consequence of (i) and Proposition 2.2.4. The proof of assertion (i) is complicated and needs some preparation. Consider first asymptotically stable differential equations $\dot{x} = Ax$ and $\dot{x} = Bx$. In our construction of a topological conjugacy h, we will first consider the unit spheres and then extend h to $\mathbb{R}^d$. This requires that trajectories intersect the unit sphere exactly once. In general, this is not true, since asymptotic stability only guarantees that for all $\kappa > \max\{\operatorname{Re}\lambda \mid \lambda \in \operatorname{spec}(A)\}$ there is $c \geq 1$ with

$$\left\| e^{At}x \right\| \leq c e^{\kappa t} \left\| x \right\| \text{ for all } x \in \mathbb{R}^d,\ t \geq 0.$$

In the following simple example the Euclidean norm, $\|x\|_2 = \sqrt{x_1^2 + \ldots + x_d^2}$, does not decrease monotonically along solutions.

Example 2.2.6. Consider in $\mathbb{R}^2$,

$$\dot{x} = -x - y,\ \dot{y} = 4x - y.$$

The eigenvalues of $A = \left[\begin{smallmatrix} -1 & -1 \\ 4 & -1 \end{smallmatrix}\right]$ are the zeros of $\det(\lambda I - A) = (\lambda + 1)^2 + 4$, hence they are equal to $-1 \pm 2\imath$. The solutions are

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{-t} \begin{bmatrix} \cos 2t & -\frac{1}{2}\sin 2t \\ 2\sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

The origin is asymptotically stable, but the Euclidean distance to the origin does not decrease monotonically.

The next proposition shows that monotonicity always holds in a norm which is adapted to the matrix A.

Proposition 2.2.7. *For $\dot{x} = Ax$ with $A \in gl(d, \mathbb{R})$ the following properties are equivalent:*

(i) For every eigenvalue μ of A one has $\operatorname{Re}\mu < 0$.

(ii) For every norm $\|\cdot\|$ on $\mathbb{R}^d$ there are $a > 0$ and $c \geq 1$ with

$$\left\|e^{At}x\right\| \leq c\, e^{-at} \|x\| \text{ for } t \geq 0.$$

(iii) There is a norm $\|\cdot\|_A$ on $\mathbb{R}^d$, called an adapted norm, such that for some $a > 0$ and for all $x \in \mathbb{R}^d$,

$$\left\|e^{At}x\right\|_A \leq e^{-at} \|x\|_A \text{ for } t \geq 0.$$

Proof. (iii) implies (ii), since all norms on $\mathbb{R}^d$ are equivalent. Property (ii) is equivalent to (i) by Theorem 1.4.4. It remains to show that (ii) implies (iii). Let $b \in (0, a)$. Then (ii) (with any norm) implies for $t \geq 0$,

$$\left\|e^{At}x\right\| \leq c\, e^{-at} \|x\| = c\, e^{(b-a)t} e^{-bt} \|x\|.$$

Hence there is $\tau > 0$ such that $c\, e^{(b-a)t} < 1$ for all $t \geq \tau$ and therefore

$$\left\|e^{At}x\right\| \leq e^{-bt} \|x\|. \tag{2.2.1}$$

Then

$$\|x\|_A := \int_0^\tau e^{bs} \left\|e^{As}x\right\| ds,\ x \in \mathbb{R}^d,$$

defines a norm, since $\|x\|_A = 0$ if and only if $e^{bs}\left\|e^{As}x\right\| = 0$ for $s \in [0, \tau]$ if and only if $x = 0$, and

$$\|x + y\|_A = \int_0^\tau e^{bs} \left\|e^{As}(x + y)\right\| ds \leq \|x\|_A + \|y\|_A.$$

This norm has the desired monotonicity property: For $t \geq 0$ write $t = n\tau + T$ with $0 \leq T < \tau$ and $n \in \mathbb{N}_0$. Then

$$\begin{aligned}
\left\| e^{At}x \right\|_A &= \int_0^\tau e^{bs} \left\| e^{As}e^{At}x \right\| ds \\
&= \int_0^{\tau - T} e^{bs} \left\| e^{An\tau}e^{A(T+s)}x \right\| ds + \int_{\tau-T}^{\tau} e^{bs} \left\| e^{A(n+1)\tau}e^{A(T-\tau+s)}x \right\| ds \\
&\leq \int_T^\tau e^{b(\sigma - T)} \left\| e^{An\tau}e^{A\sigma}x \right\| d\sigma + \int_0^T e^{b(\sigma-T+\tau)} \left\| e^{A(n+1)\tau}e^{A\sigma}x \right\| d\sigma
\end{aligned}$$

with $\sigma := T + s$ and $\sigma := T - \tau + s$, respectively. We can use (2.2.1) to estimate the second summand from above, since $(n+1)\,\tau \geq \tau$. If $n = 0$, we leave the first summand unchanged, otherwise we can also apply (2.2.1). In any case we obtain

$$\begin{aligned}
&\leq \int_T^\tau e^{b(\sigma - T - n\tau)} \left\| e^{A\sigma}x \right\| d\sigma + \int_0^T e^{b(\sigma - T + \tau - (n+1)\tau)} \left\| e^{A\sigma}x \right\| d\sigma \\
&= e^{-bt} \int_0^\tau e^{b\sigma} \left\| e^{A\sigma}x \right\| d\sigma = e^{-bt} \left\| x \right\|_A
\end{aligned}$$

and hence (ii) implies (iii). □

We show the assertion of Theorem 2.2.5 first in the asymptotically stable case.

Proposition 2.2.8. *Let $A, B \in gl(d, \mathbb{R})$. If all eigenvalues of A and of B have negative real parts, then the flows e^{At} and e^{Bt} are topologically conjugate.*

Proof. Let $\|\cdot\|_A$ and$\|\cdot\|_B$ be corresponding adapted norms. Hence with constants $a, b > 0$,

$$\left\| e^{At}x \right\|_A \leq e^{-at} \left\| x \right\|_A \text{ and } \left\| e^{Bt}x \right\|_B \leq e^{-bt} \left\| x \right\|_B \text{ for } t \geq 0 \text{ and } x \in \mathbb{R}^d.$$

Then for $t \leq 0$,

$$\left\| e^{At}x \right\|_A \geq e^{a(-t)} \left\| x \right\|_A \text{ and } \left\| e^{Bt}x \right\|_B \geq e^{b(-t)} \left\| x \right\|_B,$$

by applying the inequality above to $e^{At}x$ and $-t$. Thus backwards in time, the norms of the solutions are strictly increasing. Consider the corresponding unit spheres

$$S_A = \{x \in \mathbb{R}^d \mid \|x\|_A = 1\} \text{ and } S_B = \{x \in \mathbb{R}^n \mid \|x\|_B = 1\}.$$

They are fundamental domains of the flows e^{At} and e^{Bt}, respectively (every nontrivial trajectory intersects them). Define a homeomorphism $h_0 : S_A \to S_B$ by

$$h_0(x) := \frac{x}{\|x\|_B} \text{ with inverse } h_0^{-1}(y) = \frac{y}{\|y\|_A}.$$

In order to extend this map to $\mathbb{R}^d$ observe that (by the intermediate value theorem and by definition of the adapted norms) there is for every $x \neq 0$ a unique time $\tau(x) \in \mathbb{R}$ with $\left\| e^{A\tau(x)}x \right\|_A = 1$. This immediately implies $\tau(e^{At}x) = \tau(x) - t$. The map $x \mapsto \tau(x)$ is continuous: If $x_n \to x$, then the assumption that $\tau(x_{n_k}) \to \sigma \neq \tau(x)$ for a subsequence implies $\|\varphi(\sigma, x)\| = \|\varphi(\tau(x), x)\| = 1$ contradicting uniqueness of $\tau(x)$. Now define $h : \mathbb{R}^d \to \mathbb{R}^d$ by

$$h(x) = \begin{cases} e^{-B\tau(x)} h_0(e^{A\tau(x)}x) & \text{for} \quad x \neq 0, \\ 0 & \text{for} \quad x = 0. \end{cases}$$

Then h is a conjugacy, since

$$\begin{aligned} h(e^{At}x) &= e^{-B\tau(e^{At}x)} h_0(e^{A\tau(e^{At}x)} e^{At}x) = e^{-B[\tau(x)-t]} h_0(e^{A[\tau(x)-t]} e^{At}x) \\ &= e^{Bt} e^{-B\tau(x)} h_0(e^{A\tau(x)}x) = e^{Bt} h(x). \end{aligned}$$

The map h is continuous in $x \neq 0$, since e^{At} and e^{Bt} as well as $\tau(x)$ are continuous. In order to prove continuity in $x = 0$, consider a sequence $x_j \to 0$. Then $\tau_j := \tau(x_j) \to -\infty$. Let $y_j := h_0(e^{\tau_j} x_j)$. Then $\|y_j\|_B = 1$ and hence

$$\|h(x_j)\|_B = \left\| e^{-B\tau_j} y_j \right\|_B \leq e^{b\tau_j} \to 0 \text{ for } j \to \infty.$$

The map is injective: Suppose $h(x) = h(z)$. The case $x = 0$ is clear. Hence suppose that $x \neq 0$. Then $h(x) = h(z) \neq 0$, and with $\tau := \tau(x)$ the conjugation property implies

$$h(e^{A\tau}x) = e^{B\tau} h(x) = e^{B\tau} h(z) = h(e^{A\tau}z).$$

Thus $h(e^{A\tau}z) = h(e^{A\tau}x) \in S_B$. Since h maps only S_A to S_B, it follows that $e^{A\tau}z \in S_A$ and hence $\tau = \tau(x) = \tau(z)$. By

$$h_0(e^{A\tau}x) = h(e^{A\tau}x) = h(e^{A\tau}z) = h_0(e^{A\tau}z)$$

and injectivity of h_0 we find

$$e^{A\tau}x = e^{A\tau}z, \text{ and hence } x = z.$$

Exchanging the roles of A and B we see that h^{-1} exists and is continuous. □

Proof of Theorem 2.2.5. If the dimensions of the stable subspaces coincide, there are topological conjugacies

$$h^s : E_A^s \to E_B^s \text{ and } h^u : E_A^u \to E_B^u$$

between the restrictions to the stable and the unstable subspaces of e^{At} and e^{Bt}, respectively. With the projections

$$\pi^s : \mathbb{R}^n \to E_A^s \text{ and } \pi^u : \mathbb{R}^n \to E_A^u,$$

a topological conjugacy is defined by

$$h(x) := h^s(\pi^s(x)) + h^u(\pi^u(x)).$$

Conversely, any topological conjugacy homeomorphically maps the stable subspace onto the stable subspace. This implies that the dimensions coincide (invariance of domain theorem, Massey [**101**, Chapter VIII, Theorem 6.6 and Exercise 6.5]). □

2.3. Linear Dynamical Systems in Discrete Time

The purpose of this section is to analyze properties of autonomous linear difference equations from the point of view of dynamical systems. First we define dynamical systems in discrete time, and then we classify the conjugacy classes of the dynamical systems generated by autonomous linear difference equations of the form $x_{n+1} = Ax_n$ with $A \in Gl(d, \mathbb{R})$.

In analogy to the continuous-time case, Definition 2.1.1, we define dynamical systems in discrete time in the following way.

Definition 2.3.1. A continuous dynamical system in discrete time over the 'time set' $\mathbb{Z}$ with state space X, a metric space, is defined as a continuous map $\Phi : \mathbb{Z} \times X \longrightarrow X$ with the properties

(i) $\Phi(0, x) = x$ for all $x \in X$;

(ii) $\Phi(m + n, x) = \Phi(m, \Phi(n, x))$ for all $m, n \in \mathbb{Z}$ and all $x \in X$.

For each $x \in X$ the set $\{\Phi(n, x) \mid n \in \mathbb{Z}\}$ is called the orbit (or trajectory) of the system through the point x. For each $n \in \mathbb{Z}$ the time-n map is defined as $\varphi_n = \Phi(n, \cdot) : X \longrightarrow X$. Using time-$n$ maps, properties (i) and (ii) above can be restated as $\varphi_0 = id$, the identity map on X, and $\varphi_{m+n} = \varphi_n \circ \varphi_m$ for all $m, n \in \mathbb{Z}$.

It is an immediate consequence of the definition, that a dynamical system in discrete time is completely determined by its time-1 map, also called its generator.

Proposition 2.3.2. *For every $n \in \mathbb{Z}$, the time-n map φ_n is given by $\varphi_n = (\varphi_1)^n$. In particular, each time-n map φ_n has an inverse $(\varphi_n)^{-1} = \varphi_{-n}$, and $\varphi_n : X \longrightarrow X$ is a homeomorphism. A dynamical system in discrete time is a group in the sense that $(\{\varphi_n \mid n \in \mathbb{Z}\}, \circ)$, with $\circ$ denoting composition of maps, satisfies the group axioms, and $\varphi : (\mathbb{Z}, +) \longrightarrow (\{\varphi_n \mid n \in \mathbb{Z}\}, \circ)$, defined by $\varphi(n) = \varphi_n$ is a group homomorphism.*

This proposition also shows that every homeomorphism f defines a continuous dynamical system in discrete time by $\varphi_n := f^n, n \in \mathbb{Z}$. In particular, this holds if f is given by a matrix $A \in Gl(d, \mathbb{R})$.

Remark 2.3.3. It is worth mentioning that a major difference to the continuous-time case comes from the fact that, contrary to e^{At}, the matrix A may not be invertible or, equivalently, that 0 may be an eigenvalue of A. In this case, one only obtains a map

$$\Phi : \mathbb{N}_0 \times \mathbb{R}^d \to \mathbb{R}^d,\ \Phi(n,x) := A^n x,$$

which is linear in the second argument and satisfies property (ii) in Definition 2.3.1 only for $m, n \geq 0$. If A is invertible, the map Φ can be extended to a dynamical system $\Phi : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$. We only consider the invertible case with $A \in Gl(d,\mathbb{R})$. Naturally, one may also define general systems Φ over the one-sided time set $\mathbb{N}_0$ satisfying the corresponding semigroup property; their time-n maps need not be invertible.

Continuous dynamical systems in discrete time may be classified up to conjugacies, in analogy to the case in continuous time. Formally, we define the following.

Definition 2.3.4. Let $\Phi, \Psi : \mathbb{Z} \times X \to X$ be continuous dynamical systems generated by homeomorphisms $f, g : X \to X$, respectively. These systems (and also f and g) are called topologically conjugate if there exists a homeomorphism $h : X \to X$ such that $h(\Phi(n,x)) = \Psi(n, h(x))$ for all $n \in \mathbb{Z}$ and all $x \in X$.

By induction, one sees that this is equivalent to the requirement that $h \circ f = g \circ h$.

Remark 2.3.5. We remark that the smooth conjugacy problem for linear systems in discrete time is trivial: For a C^k-conjugacy h with $k \geq 1$ differentiation of the equation $h(Ax) = Bh(x)$ in $x = 0$ yields the linear conjugacy or matrix similarity

$$Dh(0)A = BDh(0).$$

Two dynamical systems Φ_A and Φ_B in discrete time generated by matrices $A, B \in Gl(d,\mathbb{R})$, respectively, are topologically conjugate, if there is a homeomorphism $h : \mathbb{R}^d \to \mathbb{R}^d$ such that for all $n \in \mathbb{Z}$ and all $x \in \mathbb{R}^d$ one has $h(Ax) = Bh(x)$. We will discuss the topological conjugacy classes using arguments which are analogous to the continuous-time case. As seen in Section 1.5, the stability properties of this dynamical system are again determined by the eigenvalues of the matrix A. Here the role of the imaginary axis in continuous time is taken over by the unit circle: For example, an eigenvector v for a real eigenvalue μ of A satisfies

$$A^n v = \mu^n v \to 0 \text{ if and only if } |\mu| < 1.$$

First we introduce adapted norms for discrete-time dynamical systems.

Proposition 2.3.6. *For $x_{n+1} = Ax_n$ with $A \in Gl(d,\mathbb{R})$ the following properties are equivalent:*

(i) There is a norm $\|\cdot\|_A$ *on* $\mathbb{R}^d$*, called an adapted norm, such that for some* $0 < a < 1$ *and for all* $x \in \mathbb{R}^d$*,*

$$\|A^n x\|_A \le a^n \|x\|_A \text{ for all } n \ge 0.$$

(ii) For every norm $\|\cdot\|$ *on* $\mathbb{R}^d$ *there are* $0 < a < 1$ *and* $c \ge 1$ *such that for all* $x \in \mathbb{R}^d$*,*

$$\left\|A^k x\right\| \le c\, a^n \|x\| \text{ for all } n \ge 0.$$

(iii) For every eigenvalue μ *of* A *one has* $|\mu| < 1$*.*

Proof. The proof is analogous to the continuous-time case, Proposition 2.2.7. Property (i) implies (ii), since all norms on $\mathbb{R}^d$ are equivalent. Property (ii) (with $a \in (\max|\mu|, 1))$ implies asymptotic stability which is equivalent to (iii), by Theorem 1.5.11. It remains to show that (ii) implies (i). Let $b \in (a, 1)$. Then (ii) implies for $n \ge 0$

$$\|A^n x\| \le c\, a^n \|x\| = c\left(\frac{a}{b}\right)^n b^n \|x\|.$$

There is $N \in \mathbb{N}$ such that $c\left(\frac{a}{b}\right)^n < 1$ for all $n \ge N$, hence $\|A^n x\| \le b^n \|x\|$. Then one shows that

$$\|x\|_A = \sum_{j=0}^{N-1} b^{-j} \left\|A^j x\right\|, \ x \in \mathbb{R}^d, \tag{2.3.1}$$

defines an adapted norm (Exercise 2.4.4). □

A matrix A satisfying the properties in Proposition 2.3.6(i) or, equivalently, $|\mu| < 1$ for all eigenvalues, is called a linear contraction.

Suppose that all eigenvalues μ of A satisfy $|\mu| > 1$. Then all eigenvalues of A^{-1} have modulus less than 1, since they are the inverses of the eigenvalues of A. An adapted norm for A^{-1} yields for some $a \in (0, 1)$, all $x \in \mathbb{R}^d$ and all $n \ge 0$

$$\left\|A^{-n} x\right\|_A \le a^n \|x\|_A.$$

In particular, this holds for $x = A^n y, y \in \mathbb{R}^d$, and hence for all $n \ge 0$,

$$\|y\|_A = \left\|A^{-n} x\right\|_A \le a^n \|x\|_A = a^n \|A^n y\|_A$$

implying

$$\|A^n y\|_A \ge b^n \|y\|_A \text{ with } b := a^{-1} > 1.$$

A matrix A with this property is called a linear expansion.

We turn to a result on topological conjugacy of linear contractions, i.e., of autonomous linear dynamical systems in discrete time which are asymptotically stable. Here the discrete-time case is more complicated than the continuous-time case treated in Proposition 2.2.8 for two reasons: The space $Gl(d, \mathbb{R})$ of invertible $d \times d$-matrices is not connected, since the image of the determinant, which is a continuous function, has two path connected components corresponding to the sign of the determinant (in fact, this determines the two connected components of $Gl(d,\mathbb{R})$; cf. Remark 2.3.8). A second difficulty in the proof occurs, since not every orbit intersects the unit sphere; this was an essential ingredient in the proof of Proposition 2.2.7. Hence we have to blow up the sphere to a ring (an 'annulus') in order to guarantee that every orbit intersects this ring.

Theorem 2.3.7. *Let $A, B \in Gl(d,\mathbb{R})$ be invertible linear contractions in the same path connected component of the set of linear contractions, i.e., one finds linear contractions $A_t, t \in [0,1]$, depending continuously on t with $A_0 = B$ and $A_1 = A$. Then the generated dynamical systems Φ_A and Φ_B are topologically conjugate.*

Proof. Let $A_t, t \in [0,1]$, be a curve in $Gl(d,\mathbb{R})$ connecting A and B, $A_0 = B$ and $A_1 = A$. For corresponding adapted norms $\|\cdot\|_A$ and $\|\cdot\|_B$ consider the unit disc and sphere,

$$D_A := \{x \in \mathbb{R}^d \mid \|x\|_A < 1\} \text{ and } S_A := \{x \in \mathbb{R}^d \mid \|x\|_A = 1\},$$

and analogously for B. The following rings or annuli

$$F_A := \mathrm{cl}\,(D_A \setminus AD_A) \text{ and } F_B = \mathrm{cl}\,(D_B \setminus BD_B)$$

are called fundamental domains for the associated dynamical systems, since for all $x \neq 0$ there is $j = j_A \in \mathbb{Z}$ with $A^j x \in F_A$. In fact, by the definition of adapted norms, if $\|x\|_A > 1$, there is $j \in \mathbb{N}$ with $\left\|A^{j-1}x\right\|_A > 1$ and $\left\|A^j x\right\|_A \leq 1$, hence $A^j x \in \mathrm{cl}(D_A \setminus AD_A)$. Observe also that the 'outer' boundary of F_A equals S_A and the 'inner' boundary equals AS_A. Analogous statements hold for B. First we will construct a conjugating homeomorphism $h_0 : F_A \to F_B$, hence $h_0(Ax) = Bh_0(x), x \in F_A$, and then extend it to $\mathbb{R}^d$. The idea for the construction is to map the outer and inner boundary of F_A to the outer and inner boundary of F_B, respectively. On the outer boundary, h_0 will be the radial projection of S_A to S_B, and on the inner boundary, h_0 will essentially be equal to BA^{-1} (plus radial projection to $B(S_B)$). Then it will be easy to see that h_0 becomes a conjugacy. This construction separates the radial component from the angular component in $\mathbb{S}^{d-1}$.

For the radial component we will first define h_A, h_B on the standard ring $[0,1] \times \mathbb{S}^{d-1}$ with values in F_A and F_B, respectively. Then we define $H : [0,1] \times \mathbb{S}^{d-1} \to [0,1] \times \mathbb{S}^{d-1}$ using the path from B to A. Here the t-values remain preserved and on $\mathbb{S}^{d-1}$ we use $A_t A^{-1}$. This yields the identity for $t = 1$ and BA^{-1} for $t = 0$.

Let us make this program precise. Define maps

$$\tau_A, h_A : [0,1] \times \mathbb{S}^{d-1} \to F_A \text{ by } h_A(t,x) = \tau_A(t,x)x,$$

where τ_A is the map which is affine in t and determined by $\tau_A(1,x) = \|x\|_A^{-1}$ and $\tau_A(0,x) = 1/\left\|A^{-1}x\right\|_A$. Then $h_A(1,x) = x/\|x\|_A \in S_A$, the outer boundary of F_A, and $h_A(0,x)$ is on the inner boundary of F_A, since

$$h_A(0,x) = \tau_A(0,x)x = x\left\|A^{-1}x\right\|_A = A\left(A^{-1}x/\left\|A^{-1}x\right\|_A\right) \in AS_A.$$

Since τ_A is affine in t, it follows that

$$\tau_A(t,x) = \frac{t}{\|x\|_A} + \frac{1-t}{\|A^{-1}x\|_A},\ t \in [0,1].$$

We find for all $y \in S_A$ (with $x = Ay/\|Ay\|$),

$$h_A\left(0, \frac{Ay}{\|Ay\|}\right) = \frac{Ay}{\|Ay\|}\frac{\|Ay\|}{\|y\|_A} = \frac{Ay}{\|y\|_A} = Ay, \text{ hence } h_A^{-1}(Ay) = \left(0, \frac{Ay}{\|Ay\|}\right).$$

Analogously, define τ_B and $h_B : [0,1] \times \mathbb{S}^{d-1} \to F_B$ to obtain

$$h_B\left(0, \frac{Bx}{\|Bx\|}\right) = \frac{Bx}{\|x\|_B} \text{ and } h_B\left(1, \frac{x}{\|x\|}\right) = \frac{x}{\|x\|_B}.$$

Now we use the path A_t in $Gl(d,\mathbb{R})$ with $A_0 = B$, $A_1 = A$ to construct $H : [0,1] \times \mathbb{S}^{d-1} \to [0,1] \times \mathbb{S}^{d-1}$ such that the 'radius' $t \in [0,1]$ is preserved and the 'angle' in $\mathbb{S}^{d-1}$ changes continuously: Define

$$H(t,x) := \left(t, \frac{A_t A^{-1}x}{\|A_t A^{-1}x\|}\right).$$

Then $H(1,x) = (1,x)$ and

$$H(0,x) = \left(0, \frac{BA^{-1}x}{\|BA^{-1}x\|}\right), \text{ hence } H\left(0, \frac{Ax}{\|Ax\|}\right) = \left(0, \frac{Bx}{\|Bx\|}\right).$$

The map h_0 defined by

$$h_0 : F_A \to F_B,\ h_0 := h_B \circ H \circ h_A^{-1},$$

is a composition of injective maps, hence injective. It is a conjugacy (here we only have to consider the case $x, Ax \in F_A$, in particular, $x \in S_A$), since

$$Bh_0(x) = Bh_B \circ H \circ h_A^{-1}(x) = Bh_B \circ H\left(1, \frac{x}{\|x\|}\right) = \frac{Bx}{\|x\|_B}$$

and

$$\begin{aligned} h_0(Ax) &= h_B \circ H \circ h_A^{-1}(Ax) = h_B \circ H\left(0, \frac{Ay}{\|Ay\|}\right) \\ &= h_B\left(0, \frac{Bx}{\|Bx\|}\right) = \frac{Bx}{\|x\|_B} = Bh_0(x). \end{aligned}$$

Now extend h_0 to $\mathbb{R}^d$ by $h(0) := 0$ and $h(x) = B^{-j(x)}h_0(A^{j(x)}x)$ for $x \neq 0$, where $j(x) \in \mathbb{N}$ is taken such that $A^{j(x)} \in F_A$. If in addition to $A^j x \in F_A$ also $A^{j+1}x \in F_A$, the conjugation property implies

$$B^{-j-1}h_0(A^{j+1}x) = B^{-j-1}h_0(A\ A^j x) = B^{-j-1}Bh_0(A^j x) = B^{-j}h_0(A^j x).$$

Thus the map h is well defined. The map h is obviously continuous on $\mathbb{R}^d \setminus \{0\}$. It is also continuous in 0, since $x_n \to 0$ implies $j(x_n) \to -\infty$; now use that B^{-1} is a linear expansion and $\left\| h_0(A^{j(x_n)} x_n) \right\| = 1$ to show

$$h(x_n) = B^{-j(x_n)} h_0(A^{j(x_n)} x_n) \to 0.$$

Exchanging the roles of A and B one finds a continuous inverse h^{-1} proving that h is a homeomorphism. □

Remark 2.3.8. Hyperbolic systems in discrete time are given by matrices which do not have an eigenvalue on the unit circle in $\mathbb{C}$. The statement of Theorem 2.2.5 for hyperbolic systems also holds in the discrete-time case, with an analogous proof. Additionally, one has to take into account that the subset of contractions in $Gl(d, \mathbb{R})$ has exactly two path connected components determined by $\det A < 0$ and $\det A > 0$, respectively. A proof is sketched in Robinson [**117**, Chapter IV, Theorem 9.6].

2.4. Exercises

Exercise 2.4.1. Prove parts (i) and (ii) of Proposition 2.1.9: Let $h : X \to X$ be a topological conjugacy for dynamical systems $\Phi, \Psi : \mathbb{R} \times X \longrightarrow X$ on a metric state space X. Then (i) the point $p \in X$ is a fixed point of Φ if and only if $h(p)$ is a fixed point of Ψ; (ii) the solution $\Phi(\cdot, p)$ is periodic with period T if and only if $\Psi(\cdot, h(p))$ is periodic with period T.

Exercise 2.4.2. Construct explicitly a topological conjugacy $h : \mathbb{R} \to \mathbb{R}$ between the systems $\dot{x} = -x$ and $\dot{y} = -2y$.

Exercise 2.4.3. Construct explicitly a topological conjugacy for the linear differential equations determined by

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Exercise 2.4.4. Work out the details of the proof of Proposition 2.3.6 by showing that formula (2.3.1) defines an adapted norm in discrete time.

Exercise 2.4.5. Prove the second part of Proposition 2.1.6(ii): Suppose that x_0 is in the eigenspace of an imaginary eigenvalue pair $\pm\imath\nu \neq 0$ of A and $T = \frac{2\pi}{\nu}$. Then the solution for $x_0 \in \mathbb{R}^d$ is periodic with period T.

2.5. Orientation, Notes and References

Orientation. The linear dynamical systems considered in this chapter are generated by linear autonomous differential equations $\dot{x} = Ax$ or difference equations $x_{n+1} = Ax_n$. In continuous time one has flows $\Phi : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ over the time domain $\mathbb{R}$ satisfying $\Phi(t, \alpha x + \beta y) = \alpha\Phi(t, x) + \beta\Phi(t, y)$ for all $x, y \in \mathbb{R}^d, \alpha, \beta \in \mathbb{R}$ and $t \in \mathbb{R}$, and analogously in discrete time over the time domain $\mathbb{Z}$. A natural question to ask is, which properties of the systems are preserved under transformations of the system, i.e., conjugacies. Theorem 2.2.1 and Theorem 2.3.7 show that C^k-conjugacies with $k \geq 1$ reduce to linear conjugacies, thus they preserve the Jordan normal form of the generator A. As seen in Chapter 1 this means that practically all dynamical properties are preserved. On the other hand, mere topological conjugacies only fix the dimensions of the stable and the unstable subspaces. Hence both classifications do not characterize the Lyapunov spaces which determine the exponential growth rates of the solutions. Smooth conjugacies are too fine if one is interested in exponential growth rates, and topological conjugacies are too rough. Hence important features of matrices and their associated linear differential or difference equations cannot be described by these conjugacies in $\mathbb{R}^d$.

Recall that the exponential growth rates and the associated Lyapunov spaces are determined by the real parts of the eigenvalues of the matrix generator A; cf. Definition 1.4.1 and Theorem 1.4.3 (or by the logarithms of the moduli of the eigenvalues) and the generalized eigenspaces. In Chapter 4 we will take a different approach by looking not at conjugacies in order to characterize the Lyapunov spaces. Instead we analyze induced nonlinear systems in projective space and analyze them topologically. The next chapter, Chapter 3, introduces some concepts and results necessary for the analysis of nonlinear dynamical systems. We will use them in Chapters 4 and 5 to characterize the Lyapunov spaces, hence obtain additional information on the connections between matrices and dynamical systems given by autonomous linear differential and difference equations.

Notes and references. The ideas and results of this chapter can be found, e.g., in Robinson [**117**]; in particular, our construction of conjugacies for linear systems follows the exposition in [**117**, Chapter 4]. Continuous dependence of eigenvalues on the matrix is proved, e.g., in Hinrichsen and Pritchard [**68**, Corollary 4.2.1] as well as in Kato [**74**] and Baumgärtel [**17**].

Example 2.1.4 can be generalized to differentiable manifolds: Suppose that X is a C^k-differentiable manifold and f a C^k-vector field on X such that the differential equation $\dot{x} = f(x)$ has unique solutions $x(t, x_0), t \in \mathbb{R}$, with $x(0, x_0) = x_0$ for all $x_0 \in X$. Then $\Phi(t, x_0) = x(t, x_0)$ defines a dynamical system $\Phi : \mathbb{R} \times X \longrightarrow X$. Similarly, C^k-conjugacies can be defined in this setting.

The characterization of matrices via invariance properties of the associated linear autonomous differential and difference equations under smooth and continuous conjugacies may be viewed as part of Klein's Erlanger Programm in the nineteenth century defining geometries by groups of transformations. This point of view is emphasized by McSwiggen and Meyer in

[**105**] who also discuss invariance properties under Lipschitz and Hölder conjugacies; see also Kawan and Stender [**76**] for a classification under Lipschitz conjugacies. Conjugacies are not the only way to classify flows: If one looks at the trajectories, the parametrization by time does not play a role, except for the orientation. This leads to the notion of C^k-equivalence, $k \geq 0$. For $k \geq 1$, the flows for $\dot{x} = Ax$ and $\dot{y} = By$ are C^k-equivalent if and only if there are a real number $\alpha > 0$ and $T \in Gl(d, \mathbb{R})$ with $A = \alpha TBT^{-1}$; cf. Ayala, Kliemann, and Colonius [**12**] for a proof.

The topological conjugacy problem for nonhyperbolic systems in the continuous-time case is treated by Kuiper [**87**]; cf. Ladis [**92**] for topological equivalence. The discrete-time case is much more complicated; cf. Kuiper and Robbin [**89**] and the references given in Ayala and Kawan [**14**].

Chapter 4

Linear Systems in Projective Space

In this chapter we return to matrices $A \in gl(d, \mathbb{R})$ and the dynamical systems defined by them. Geometrically, the invertible linear map e^{At} on $\mathbb{R}^d$ associated with A maps k-dimensional subspaces onto k-dimensional subspaces. In particular, the flow $\Phi_t = e^{At}$ induces a dynamical system on projective space, i.e., the set of all one-dimensional subspaces, and, more generally, on every Grassmannian, i.e., the set of all k-dimensional subspaces, $k = 1, \ldots, d$. As announced at the end of Chapter 2, we will characterize certain properties of A through these associated systems. More precisely, we will show in the present chapter that the Lyapunov spaces uniquely correspond to the chain components of the induced dynamical system on projective space. Chapter 5 will deal with the technically more involved systems on the Grassmannians.

Section 4.1 shows for continuous-time systems that the chain components in projective space characterize the Lyapunov spaces. Section 4.2 proves an analogous result in discrete time.

4.1. Linear Flows Induced in Projective Space

This section shows that the projections of the Lyapunov spaces coincide with the chain components in projective space.

We start with the following motivating observations. Consider the system in $\mathbb{R}^2$ given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{4.1.1}$$

The nontrivial trajectories consist of circles around the origin (this is the linear oscillator $\ddot{x} = -x$.) The slope along a trajectory is $k(t) := \frac{x_2(t)}{x_1(t)}$. Using the quotient rule, one finds that it satisfies the differential equation

$$\frac{d}{dt}k(t) = \dot{k} = \frac{\dot{x}_2 x_1 - x_2 \dot{x}_1}{x_1^2} = -\frac{x_1^2}{x_1^2} - \frac{x_2^2}{x_1^2} = -1 - k^2,$$

as long as $x_1(t) \neq 0$. For $x_1(t) \to 0$ one finds $k(t) \to \infty$. Thus this nonlinear differential equation, a Riccati equation, has solutions with a bounded interval of existence. Naturally, this can also be seen by the solution formula for $k(t)$ with initial condition $k(0) = k_0$,

$$k(t) = \tan(-t + \arctan k_0),\ t \in \left(-\arctan k_0 - \frac{\pi}{2}, -\arctan k_0 + \frac{\pi}{2}\right).$$

Geometrically, this Riccati differential equation describes the evolution of a one-dimensional subspace (determined by the slope) under the flow of the differential equation (4.1.1). Note that for $x_1 \neq 0$ the points (x_1, x_2) and $(1, \frac{x_2}{x_1})$ generate the same subspace. The Riccati equation can describe this evolution only on a bounded time interval, since it uses the parametrization of the subspaces given by the slope, which must be different from $\pm\infty$, i.e., it breaks down on the x_2-axis. The analysis in projective space will avoid the artificial problem resulting from parametrizations.

These considerations are also valid in higher dimensions. Consider for a solution of $\dot{x} = Ax(t)$ with $x_1(t) \neq 0$ the vector $K(t) := \left[\frac{x_2(t)}{x_1(t)}, \ldots, \frac{x_d(t)}{x_1(t)}\right]^\top \in \mathbb{R}^{d-1}$. Partition $A = (a_{ij}) \in gl(d, \mathbb{R})$ in

$$A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{12} = (a_{12}, \ldots, a_{1d}), A_{21} = (a_{21}, \ldots, a_{d1})^\top$ and $A_{22} \in gl(d-1, \mathbb{R})$. Then the function $K(\cdot)$ satisfies the Riccati differential equation

$$\dot{K} = A_{21} + A_{22}K - Ka_{11} - KA_{12}K. \tag{4.1.2}$$

In fact, one finds from

$$\dot{x}_1 = a_{11}x_1 + (a_{12}, \ldots, a_{1d}) \begin{bmatrix} x_2 \\ \vdots \\ x_d \end{bmatrix} \text{ and } \begin{bmatrix} \dot{x}_2 \\ \vdots \\ \dot{x}_d \end{bmatrix} = \begin{bmatrix} a_{21} \\ \vdots \\ a_{d1} \end{bmatrix} x_1 + A_{22} \begin{bmatrix} x_2 \\ \vdots \\ x_d \end{bmatrix}$$

the expression

$$\dot{K} = \begin{bmatrix} \dot{x}_2 \\ \vdots \\ \dot{x}_d \end{bmatrix} \frac{x_1}{x_1^2} - \begin{bmatrix} x_2 \\ \vdots \\ x_d \end{bmatrix} \frac{\dot{x}_1}{x_1^2} = A_{21} + A_{22}K - Ka_{11} - KA_{12}K.$$

Conversely, the same computations show that for any solution $K(t) = (k_2(t), \dots, k_d(t))^\top$ of the Riccati equation (4.1.2) (as long as it exists), the solution of $\dot{x} = Ax$ with initial condition

$$x_1(0) = 1, x_j(0) = k_j(0), j = 2, \dots, d.$$

satisfies $K(t) = \left[\frac{x_2(t)}{x_1(t)}, \dots, \frac{x_d(t)}{x_1(t)}\right]^\top$. Hence the vectors $K(t), t \in \mathbb{R}$, determine the curve in projective space which describes the evolution of the one-dimensional subspace spanned by $x(0)$, as long as the first coordinate is different from 0.

This discussion shows that the behavior of lines in $\mathbb{R}^d$ under the flow e^{At} is locally described by a certain Riccati equation (as in the linear oscillator case, one may use different parametrizations when $x_1(t)$ approaches 0). If one wants to discuss the limit behavior as time tends to infinity, this local description is not adequate and one should consider a compact state space.

For the diagonal matrix $A = \operatorname{diag}(1, -1)$ in Example 3.1.2 one obtains two one-dimensional Lyapunov spaces, each corresponding to two opposite points on the unit circle. These points are chain components of the flow on the unit circle. Opposite points should be identified in order to get a one-to-one correspondence between Lyapunov spaces and chain components in this simple example. Thus, in fact, the space of lines, i.e., projective space, is better suited for the analysis than the unit sphere.

The projective space $\mathbb{P}^{d-1}$ for $\mathbb{R}^d$ can be constructed in the following way. Introduce an equivalence relation on $\mathbb{R}^d \setminus \{0\}$ by saying that x and y are equivalent, $x \sim y$, if there is $\alpha \neq 0$ with $x = \alpha y$. The quotient space $\mathbb{P}^{d-1} := \mathbb{R}^d \setminus \{0\}/\sim$ is the projective space. Clearly, it suffices to consider only vectors x with Euclidean norm $\|x\| = 1$. Thus, geometrically, projective space is obtained by identifying opposite points on the unit sphere $\mathbb{S}^{d-1}$ or it may be considered as the space of lines through the origin. We write $\mathbb{P} : \mathbb{R}^d \setminus \{0\} \to \mathbb{P}^{d-1}$ for the projection and usually, denote the elements of $\mathbb{P}^{d-1}$ by $p = \mathbb{P}x$, where $0 \neq x \in \mathbb{R}^d$ is any element in the corresponding equivalence class. A metric on $\mathbb{P}^{d-1}$ is given by

$$\text{(4.1.3)} \qquad \mathrm{d}(\mathbb{P}x, \mathbb{P}y) := \min\left(\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|, \left\|\frac{x}{\|x\|} - \frac{-y}{\|y\|}\right\|\right).$$

Note that for a point x in the unit sphere $\mathbb{S}^{d-1}$ and a subspace W of $\mathbb{R}^d$ one has

(4.1.4)

$$\operatorname{dist}(x, W \cap \mathbb{S}^{d-1}) = \inf_{y \in W \cap \mathbb{S}^{d-1}} \|x - y\| = \min_{y \in W} \mathrm{d}(\mathbb{P}x, \mathbb{P}y) =: \operatorname{dist}(\mathbb{P}x, \mathbb{P}W).$$

Any matrix in $Gl(d, \mathbb{R})$ (in particular, matrices of the form e^{At}) induces an invertible map on the projective space $\mathbb{P}^{d-1}$. The flow properties of

$\Phi_t = e^{At}, t \in \mathbb{R}$, are inherited by the induced maps and we denote by $\mathbb{P}\Phi_t$ the induced dynamical system on projective space. More precisely, the projection $\mathbb{P}$ is a semiconjugacy, i.e., it is a continuous surjective map satisfying for every $t \in \mathbb{R}$ the conjugacy property

$$\begin{array}{ccc} \mathbb{R}^d \setminus \{0\} & \xrightarrow{\Phi_t} & \mathbb{R}^d \setminus \{0\} \\ \mathbb{P}\downarrow & & \downarrow\mathbb{P} \\ \mathbb{P}^{d-1} & \xrightarrow{\mathbb{P}\Phi_t} & \mathbb{P}^{d-1} \end{array}.$$

We will not need that the projective flow $\mathbb{P}\Phi$ is generated by a differential equation on projective space which, in fact, is a $(d-1)$-dimensional differentiable manifold. Instead, we only need that projective space is a compact metric space and that $\mathbb{P}\Phi$ is a continuous flow; in Exercise 4.3.1, the reader is asked to verify this in detail. Nevertheless, the following differential equation in $\mathbb{R}^d$ leaving the unit sphere $\mathbb{S}^{d-1}$ invariant is helpful to understand the properties of the flow in projective space.

Lemma 4.1.1. *For $A \in gl(d, \mathbb{R})$ let $\Phi_t = e^{At}, t \in \mathbb{R}$, be its linear flow in $\mathbb{R}^d$. The flow Φ projects onto a flow on $\mathbb{S}^{d-1}$, given by the differential equation*

$$\dot{s} = h(s, A) = (A - s^\top As\ I)s, \text{ with } s \in \mathbb{S}^{d-1}.$$

Proof. Exercise 4.3.2. □

Naturally, the flow on the unit sphere also projects to the projective flow $\mathbb{P}\Phi$. In order to determine the global behavior of the projective flow we first show that points outside of the Lyapunov spaces $L_j := L(\lambda_j)$ are not chain recurrent; cf. Definition 1.4.1.

Lemma 4.1.2. *Let $\mathbb{P}\Phi_t$ be the projection to $\mathbb{P}^{d-1}$ of a linear flow $\Phi_t = e^{At}$. If $x \notin \bigcup_{j=1}^{\ell} L(\lambda_j)$, then $\mathbb{P}x$ is not chain recurrent for the induced projective flow.*

Proof. We may suppose that A is given in real Jordan form, since a linear conjugacy in $\mathbb{R}^d$ yields a topological conjugacy in projective space which preserves the chain transitive sets by Proposition 3.1.15. The following construction shows that for $\varepsilon > 0$ small enough there is no (ε, T)-chain from $\mathbb{P}x$ to $\mathbb{P}x$. It may be viewed as a generalization of Example 3.2.1 where a scalar system was considered.

Recall the setting of Theorem 1.4.4. The Lyapunov exponents are ordered such that $\lambda_1 > \ldots > \lambda_\ell$ with associated Lyapunov spaces $L_j = L(\lambda_j)$. Then

$$V_j = L_\ell \oplus \ldots \oplus L_j \text{ and } W_j = L_j \oplus \ldots \oplus L_1$$

define flags of subspaces

$$\{0\} = V_{\ell+1} \subset V_\ell \subset \ldots \subset V_1 = \mathbb{R}^d, \{0\} = W_0 \subset W_1 \subset \ldots \subset W_\ell = \mathbb{R}^d.$$

For $x \notin \bigcup_{j=1}^{\ell} L(\lambda_j)$ there is a minimal j such that $x \in V_j \setminus V_{j+1}$ and hence there are unique $x_i \in L(\lambda_i)$ for $i = j, \ldots, \ell$ with

$$x = x_\ell + \ldots + x_j.$$

Here $x_j \neq 0$ and at least one $x_i \neq 0$ for some $i \geq j+1$. Hence $x \notin W_j = L(\lambda_j) \oplus \ldots \oplus L(\lambda_1)$ and $V_{j+1} \cap W_j = \{0\}$. We may suppose that x is on the unit sphere $\mathbb{S}^{d-1}$ and has positive distance $\delta > 0$ to the intersection $W_j \cap \mathbb{S}^{d-1}$. By (4.1.4) it follows that $\delta > 0$ is the distance of $\mathbb{P}x$ to the projection $\mathbb{P}W_j$.

The solution formulas show that for all $0 \neq y \in \mathbb{R}^d$,

$$\frac{e^{At}y}{\|e^{At}y\|} = \frac{e^{At}y_\ell}{\|e^{At}y\|} + \ldots + \frac{e^{At}y_j}{\|e^{At}y\|} + \ldots + \frac{e^{At}y_1}{\|e^{At}y\|}.$$

If $y \notin V_{j+1}$ one has for $i \geq j+1$ that $\frac{e^{At}y_i}{\|e^{At}y\|} \to 0$ for $t \to \infty$. Also for some $i \leq j$ one has $y_i \neq 0$ and $\frac{e^{At}y_i}{\|e^{At}y\|} \in L(\lambda_i)$. This implies that for $t \to \infty$,

$$\operatorname{dist}\left(\mathbb{P}\Phi_t\left(y\right), \mathbb{P}W_j\right) = \operatorname{dist}\left(\frac{e^{At}y}{\|e^{At}y\|}, W_j \cap \mathbb{S}^{d-1}\right) \to 0\ .$$

There is $0 < 2\varepsilon < \delta$ such that the 2ε-neighborhood N of $\mathbb{P}W_j$ has void intersection with $\mathbb{P}V_{j+1}$. We may take $T > 0$ large enough such that for all initial values $\mathbb{P}y$ in the compact set $\operatorname{cl} N$ and all $t \geq T$,

$$\operatorname{dist}\left(\mathbb{P}\Phi_t\left(y\right), \mathbb{P}W_j\right) < \varepsilon.$$

Now consider an (ε, T) chain starting in $\mathbb{P}x_0 = \mathbb{P}x \notin \bigcup_{j=1}^{\ell} L(\lambda_j)$ and let $T_0 > T$ such that $\operatorname{dist}\left(\mathbb{P}\Phi_{T_0}\left(x_0\right), \mathbb{P}W_j\right) < \varepsilon$. The next point $\mathbb{P}x_1$ of the chain has distance less than ε to $\mathbb{P}\Phi_{T_0}\left(x_0\right)$, hence

$$\operatorname{dist}(\mathbb{P}x_1, \mathbb{P}W_j) \leq d(\mathbb{P}x_1, \mathbb{P}\Phi_{T_0}\left(x_0\right)) + \operatorname{dist}\left(\mathbb{P}\Phi_{T_0}\left(x_0\right), \mathbb{P}W_j\right) < 2\varepsilon < \delta.$$

Thus $\mathbb{P}x_1 \in N$ and it follows that $\operatorname{dist}\left(\mathbb{P}\Phi_t\left(x_1\right), \mathbb{P}W_j\right) < \varepsilon$ for all $t \geq T$. Repeating this construction along the (ε, T)-chain, one sees that the final point $\mathbb{P}x_n$ has distance less than δ from $\mathbb{P}W_j$ showing, by definition of δ, that $\mathbb{P}x_n \neq \mathbb{P}x_0 = \mathbb{P}x$. □

The characteristics of the projected flow $\mathbb{P}\Phi$ are summarized in the following result. In particular, it shows that the topological properties of this projected flow determine the decomposition of $\mathbb{R}^d$ into the Lyapunov spaces; cf. Definition 1.4.1.

Theorem 4.1.3. *Let $\mathbb{P}\Phi$ be the projection onto $\mathbb{P}^{d-1}$ of a linear flow $\Phi_t(x) = e^{At}x$. Then the following assertions hold.*

(i) $\mathbb{P}\Phi$ has ℓ chain components $\mathcal{M}_1, \ldots, \mathcal{M}_\ell$, where ℓ is the number of Lyapunov exponents $\lambda_1 > \ldots > \lambda_\ell$.

(ii) One can number the chain components such that $\mathcal{M}_j = \mathbb{P}L(\lambda_j)$, the projection onto $\mathbb{P}^{d-1}$ of the Lyapunov space $L_j = L(\lambda_j)$ corresponding to the Lyapunov exponent λ_j.

(iii) The sets

$$\mathbb{P}^{-1}\mathcal{M}_j := \{x \in \mathbb{R}^d \mid \ x = 0 \ \textit{or} \ \mathbb{P}x \in \mathcal{M}_j\}$$

coincide with the Lyapunov spaces and hence yield a decomposition of $\mathbb{R}^d$ into linear subspaces

$$\mathbb{R}^d = \mathbb{P}^{-1}\mathcal{M}_1 \oplus \ \ldots \ \oplus \mathbb{P}^{-1}\mathcal{M}_\ell.$$

Proof. We may assume that A is given in Jordan canonical form $J^{\mathbb{R}}$, since coordinate transformations map the real generalized eigenspaces and the chain transitive sets into each other. Lemma 4.1.2 shows that points outside of a Lyapunov space L_j cannot project to a chain recurrent point. Hence it remains to show that the flow $\mathbb{P}\Phi$ restricted to a projected Lyapunov space $\mathbb{P}L_j$ is chain transitive. Then assertion (iii) is an immediate consequence of the fact that the L_i are linear subspaces. We may assume that the corresponding Lyapunov exponent, i.e., the common real part of the eigenvalues, is zero. First, the proof will show that the projected sum of the corresponding eigenspaces is chain transitive. Then the assertion is proved by analyzing the projected solutions in the corresponding generalized eigenspaces.

Step 1: The projected eigenspace for the eigenvalue 0 is chain transitive, since it is connected and consists of equilibria; see Proposition 3.2.5.

Step 2: For a complex conjugate eigenvalue pair $\mu, \bar{\mu} = \pm i\nu, \nu > 0$, an element $x_0 \in \mathbb{R}^d$ with coordinates $(y_0, z_0)^\top$ in the real eigenspace satisfies

$$y(t, x_0) = y_0 \cos \nu t - z_0 \sin \nu t, \ \ z(t, x_0) = z_0 \cos \nu t + y_0 \sin \nu t.$$

Thus it defines a $\frac{2\pi}{\nu}$-periodic solution on $\mathbb{R}^d$ and together they form a two-dimensional subspace of periodic solutions. The projection to $\mathbb{P}^{d-1}$ is also periodic and hence chain transitive. The same is true for the whole eigenspace of $\pm i\nu$.

Step 3: Now consider for $k = 1, \ldots, m$ a collection of eigenvalue pairs $\pm i\nu_k, \nu_k > 0$ such that all ν_k are rational, i.e., there are $p_k, q_k \in \mathbb{N}$ with $\nu_k = \frac{p_k}{q_k}$. Then the corresponding eigensolutions have periods $\frac{2\pi}{\nu_k} = 2\pi\frac{q_k}{p_k}$. It follows that these solutions have the common (nonminimal) period $2\pi q_1 \ldots q_m$. Then the projected sum of the eigenspaces consists of periodic solutions and

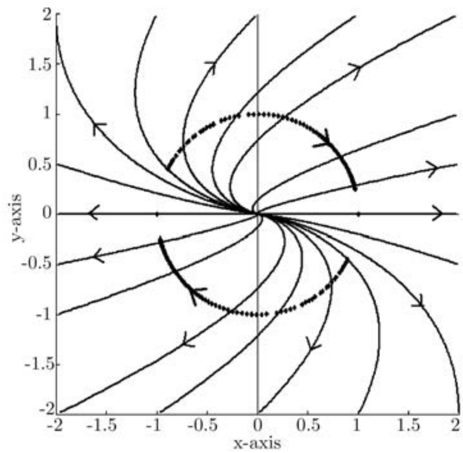

Figure 4.1. The flow for a two-dimensional Jordan block

hence is chain transitive. If the ν_k are arbitrary real numbers, we can approximate them by rational numbers $\tilde{\nu}_k$. This can be used to construct (ε, T)-chains, where, by Proposition 3.1.10, it suffices to construct (ε, T)-chains with jump times $T_i \in (1, 2]$. Replacing in the matrix the ν_k by $\tilde{\nu}_k$, one obtains matrices which are arbitrarily close to the original matrix. By Corollary 1.1.2(ii), for every $\varepsilon > 0$ one may choose the $\tilde{\nu}_k$ such that for every $x \in \mathbb{R}^d$ the corresponding solution $\tilde{\Phi}_t x$ satisfies

$$\left\| \Phi_t x - \tilde{\Phi}_t x \right\| < \varepsilon \text{ for all } t \in [0, 2].$$

This also holds for the distance in projective space showing that the projected sum of all eigenspaces for complex conjugate eigenvalue pairs is chain transitive. Next, we may also add the eigenspace for the eigenvalue 0 and see that the projected sum of all real eigenspaces is chain transitive. This follows, since the component of the solution in the eigenspace for 0 is constant (cf. Proposition 2.1.6(i)).

Step 4: Call the subspaces of $\mathbb{R}^d$ corresponding to the Jordan blocks Jordan subspaces. Consider first initial values in a Jordan subspace corresponding to a real eigenvalue, i.e., by assumption to the eigenvalue zero. The projective eigenvector p (i.e., an eigenvector projected on $\mathbb{P}^{d-1}$) is an equilibrium for $\mathbb{P}\Phi$. For all other initial values the projective solutions tend to p for $t \to \pm\infty$, since for every initial value the component corresponding to the eigenvector has the highest polynomial growth; cf. the solution formula (1.3.2). This shows that the projective Jordan subspace is chain transitive. Figures 4.1 and 4.2 illustrate the situation for a two-dimensional and a three-dimensional Jordan block, respectively. In Figure 4.1 solutions of the linear system in $\mathbb{R}^2$ (with positive real part of the eigenvalues) and their projections to the unit circle are indicated, while Figure 4.2 shows projected solutions on the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ (note that here the eigenspace is the vertical axis). The analogous statement holds for Jordan subspaces corresponding to a complex-conjugate pair of eigenvalues.

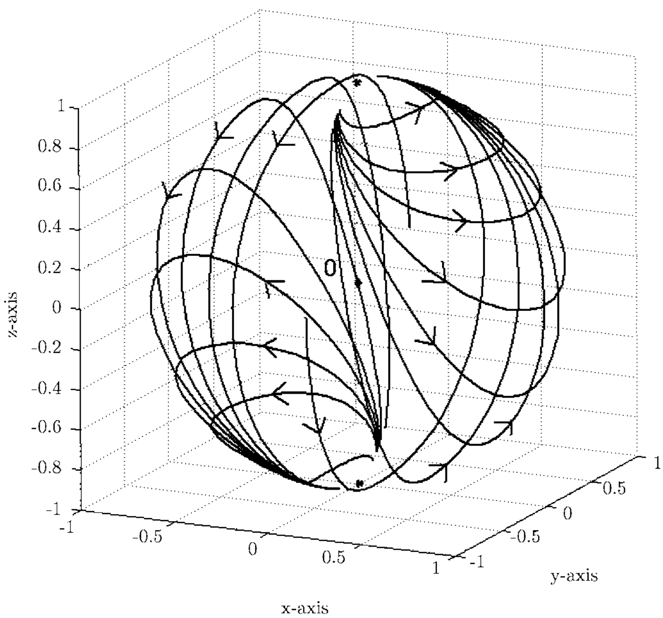

Figure 4.2. The projected flow for a three-dimensional Jordan block

Step 5: It remains to show that the projected sum of all Jordan subspaces is chain transitive. By **Step 4** the components in every Jordan subspace converge for $t \to \pm\infty$ to the corresponding real eigenspace, and hence the sum converges to the sum of the real eigenspaces. By **Step 3** the projected sum of the real eigenspaces is chain transitive. This finally proves that the Lyapunov spaces project to chain transitive sets in projective space. □

Remark 4.1.4. Theorem 4.1.3 shows that the Lyapunov spaces are characterized topologically by the induced projective flow. Naturally, the magnitude of the Lyapunov exponents is not seen in projective space, only their order. The proof of Lemma 4.1.2 also shows that the chain components $\mathcal{M}_j$ corresponding to the Lyapunov exponents λ_j are ordered in the same way by a property of the flow in projective space: Two Lyapunov exponents satisfy $\lambda_i < \lambda_j$, if and only if there exists a point p in projective space with $\alpha(p) \subset \mathcal{M}_i$ and $\omega(p) \subset \mathcal{M}_j$; cf. Exercise 4.3.3.

Surprisingly enough, one can reconstruct the actual values of the Lyapunov exponents from the behavior on the unit sphere based on the differential equation given in Lemma 4.1.1. This is shown in Exercise 4.3.2.

The chain components are preserved under conjugacies of the flows on projective space.

Corollary 4.1.5. *For A, $B \in gl(d,\mathbb{R})$ let $\mathbb{P}\Phi$ and $\mathbb{P}\Psi$ be the associated flows on $\mathbb{P}^{d-1}$ and suppose that there is a topological conjugacy h of $\mathbb{P}\Phi$ and $\mathbb{P}\Psi$. Then the chain components $\mathcal{N}_1,\ldots,\mathcal{N}_\ell$ of $\mathbb{P}\Psi$ are of the form $\mathcal{N}_i = h(\mathcal{M}_i)$, where $\mathcal{M}_i$ is a chain component of $\mathbb{P}\Phi$. In particular, the number of Lyapunov spaces of Φ and Ψ agrees.*

Proof. By Proposition 3.1.15(iii) the maximal chain transitive sets, i.e., the chain components, are preserved by topological conjugacies. The second assertion follows by Theorem 4.1.3. □

4.2. Linear Difference Equations in Projective Space

In this section it is shown that for linear difference equations the projections of the Lyapunov spaces coincide with the chain components in projective space.

Consider a linear difference equation of the form

$$x_{n+1} = Ax_n, n \in \mathbb{Z},$$

where $A \in Gl(d,\mathbb{R})$. According to the discussion in Section 2.3, A generates a continuous dynamical system Φ in discrete time with time-1 map $\varphi_1 = \Phi(1,\cdot) = A$. By linearity, this induces a dynamical system $\mathbb{P}\Phi$ in discrete time on projective space $\mathbb{P}^{d-1}$ with time-1 map $\mathbb{P}\varphi = \mathbb{P}\Phi(1,\cdot)$ given by

$$p \mapsto \mathbb{P}(Ax) \text{ for any } x \text{ with } \mathbb{P}x = p.$$

This can also be obtained by first considering the induced map on the unit sphere $\mathbb{S}^{d-1}$ and then identifying opposite points. The system on the unit sphere projects to the projective flow $\mathbb{P}\Phi$. The characteristics of the projected dynamical system $\mathbb{P}\Phi$ are summarized in the following result. In particular, it shows that the topological properties of this projected flow determine the decomposition of $\mathbb{R}^d$ into Lyapunov spaces (recall Definition 1.5.4.)

Theorem 4.2.1. *Let $\mathbb{P}\Phi$ be the projection onto $\mathbb{P}^{d-1}$ of a linear dynamical system $\Phi(n,x) = A^n x, n \in \mathbb{Z}, x \in \mathbb{R}^d$, associated with $x_{n+1} = Ax_n$. Then the following assertions hold.*

(i) $\mathbb{P}\Phi$ has ℓ chain components $\mathcal{M}_1,\ldots,\mathcal{M}_\ell$, where ℓ is the number of Lyapunov exponents $\lambda_1 > \ldots > \lambda_\ell$.

(ii) One can number the chain components such that $\mathcal{M}_j = \mathbb{P}L(\lambda_j)$, the projection onto $\mathbb{P}^{d-1}$ of the Lyapunov space $L(\lambda_j)$ corresponding to the Lyapunov exponent λ_j.

(iii) The sets

$$\mathbb{P}^{-1}\mathcal{M}_j := \{x \in \mathbb{R}^d \mid \ x = 0 \text{ or } \mathbb{P}x \in \mathcal{M}_j\}$$

coincide with the Lyapunov spaces and hence yield a decomposition of $\mathbb{R}^d$ into linear subspaces

$$\mathbb{R}^d = \mathbb{P}^{-1}\mathcal{M}_1 \oplus \ \dots \ \oplus \mathbb{P}^{-1}\mathcal{M}_\ell.$$

Proof. We may assume that A is given in Jordan canonical form, since coordinate transformations map the generalized eigenspaces and the chain transitive sets into each other.

Analogously to Lemma 4.1.2 and its proof one sees that points outside of the Lyapunov spaces are not chain recurrent. This follows from Theorem 1.5.8. Hence it remains to show that the system $\mathbb{P}\Phi$ restricted to a projected Lyapunov space $\mathbb{P}L(\lambda_j)$ is chain transitive. Then assertion (iii) is an immediate consequence of the fact that the L_i are linear subspaces. We go through the same steps as for the proof of Theorem 4.1.3. Here we may assume that all eigenvalues have modulus 1.

Step 1: The projected eigenspace for a real eigenvalue μ is chain transitive, since it is connected and consists of equilibria; see Proposition 3.3.5(iii).

Step 2: Consider a complex conjugate eigenvalue pair $\mu, \bar{\mu} = \alpha \pm i\beta, \beta > 0$, with $|\mu| = |\bar{\mu}| = 1$. Then an element $x_0 \in \mathbb{R}^d$ with coordinates $(y_0, z_0)^\top$ in the real eigenspace satisfies

$$\varphi(n, x_0) = A^n x_0 = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}^n \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}.$$

This means that we apply n times a rotation by the angle β, i.e., a single rotation by the angle $n\beta$. If $\frac{2\pi}{\beta}$ is rational, there are $p, q \in \mathbb{N}$ with $\frac{2\pi}{\beta} = \frac{p}{q}$, and hence $p\beta = 2\pi q$. Then $\varphi(p, x_0) = x_0$ and hence x_0 generates a p-periodic solution in $\mathbb{R}^d$. These solutions form a two-dimensional subspace of periodic solutions. The projections are also periodic and hence, by Proposition 3.3.5(iii), one obtains a chain transitive set. The same is true for the whole real eigenspace of μ.

Now consider for $k = 1, \dots, m$ a collection of eigenvalue pairs $\mu_k, \bar{\mu}_k = \alpha_k \pm i\beta_k, \beta_k > 0$ such that all $\frac{2\pi}{\beta_k}$ are rational, i.e., there are $p_k, q_k \in \mathbb{N}$ with $\frac{2\pi}{\beta_k} = \frac{p_k}{q_k}$. Then the corresponding eigensolutions have periods p_k. It follows that these solutions have the common (not necessarily minimal) period $2\pi p_1 \dots p_m$. Hence the projected sum of the real eigenspaces is chain transitive.

If the β_k are arbitrary real numbers, we can approximate them by rational numbers $\tilde{\beta}_k$. This can be used to construct ε-chains: Replacing in the matrix the β_k by $\tilde{\beta}_k$, one obtains matrices $\tilde{A}$ which are close to the original matrix. The matrices $\tilde{A}$ may be chosen such that $\left\| Ax - \tilde{A}x \right\| < \varepsilon$ for every $x \in \mathbb{R}^d$ with $\|x\| = 1$. This also holds for the distance in projective space showing that the projected sum of all real eigenspaces for complex conjugate eigenvalue pairs is chain transitive.

Step 3: By Steps 1 and 2 and using similar arguments one shows that the projected sum of all real eigenspaces is chain transitive.

Step 4: Call the subspaces of $\mathbb{R}^d$ corresponding to the Jordan blocks Jordan subspaces. Consider first initial values in a Jordan subspace corresponding to a real eigenvalue. The projective eigenvector p (i.e., an eigenvector projected on $\mathbb{P}^{d-1}$) is an equilibrium for $\mathbb{P}\Phi$. For all other initial values the projective solutions tend to p for $n \to \pm\infty$, since they induce the highest polynomial growth in the component corresponding to the eigenvector. This shows that the projective Jordan subspace is chain transitive. The analogous statement holds for Jordan subspaces corresponding to a complex-conjugate pair of eigenvalues.

Step 5: It remains to show that the projected sum of all Jordan subspaces is chain transitive. This follows, since for $n \to \pm\infty$ the components in every Jordan subspace converge to the corresponding eigenspace, and hence the sum converges to the sum of the eigenspaces. The same is true for the projected sum of all generalized eigenspaces. This, finally, shows that the Lyapunov spaces project to chain transitive sets in projective space. □

Theorem 4.2.1 shows that the Lyapunov spaces are characterized topologically by the induced projective system. Naturally, the magnitudes of the Lyapunov exponents are not seen in projective space, only their order. Furthermore the chain components $\mathcal{M}_j$ corresponding to the Lyapunov exponents λ_j are ordered in the same way by a property of the flow in projective space: Two Lyapunov exponents satisfy $\lambda_i < \lambda_j$, if and only if there exists a point p in projective space with $\alpha(p) \subset \mathcal{M}_i$ and $\omega(p) \subset \mathcal{M}_j$.

How do the chain components behave under conjugacy of the flows on $\mathbb{P}^{d-1}$?

Corollary 4.2.2. *For A, $B \in Gl(d, \mathbb{R})$ let $\mathbb{P}\Phi$ and $\mathbb{P}\Psi$ be the associated dynamical systems on $\mathbb{P}^{d-1}$ and suppose that there is a topological conjugacy h of $\mathbb{P}\Phi$ and $\mathbb{P}\Psi$. Then the chain components $\mathcal{N}_1, \ldots, \mathcal{N}_\ell$ of $\mathbb{P}\Psi$ are of the form $\mathcal{N}_i = h(\mathcal{M}_i)$, where $\mathcal{M}_i$ is a chain component of $\mathbb{P}\Phi$. In particular, the number of chain components of $\mathbb{P}\Phi$ and $\mathbb{P}\Psi$ agree.*

Proof. This is a consequence of Theorem 3.3.7. □

4.3. Exercises

Exercise 4.3.1. (i) Prove that the metric (4.1.3) is well defined and turns the projective space $\mathbb{P}^{d-1}$ into a compact metric space. (ii) Show that the linear flow $\Phi_t(x) = e^{At}x, x \in \mathbb{R}^d, t \in \mathbb{R}$, induces a continuous flow $\mathbb{P}\Phi$ on projective space.

Exercise 4.3.2. Let $x(t, x_0)$ be a solution of $\dot{x} = Ax$ with $A \in gl(d, \mathbb{R})$. Write $s(t) = \frac{x(t,x_0)}{\|x(t,x_0)\|}, t \in \mathbb{R}$, for the projection to the unit sphere in the Euclidean norm. (i) Show that $s(t)$ is a solution of the differential equation

$$\dot{s}(t) = [A - s(t)^\top As(t) \cdot I]s(t).$$

Observe that this is a differential equation in $\mathbb{R}^d$ which leaves the unit sphere invariant. Give a geometric interpretation! Use this equation to show that eigenvectors corresponding to real eigenvalues give rise to fixed points on the unit sphere. (ii) Prove the following formula for the Lyapunov exponents:

$$\lambda(x_0) = \lim_{t\to\infty} \frac{1}{t} \int_0^t s(\tau)^\top As(\tau)d\tau$$

by considering the 'polar decomposition' $\mathbb{S}^{d-1} \times (0, \infty)$.

Exercise 4.3.3. Consider the chain components given in Theorem 4.1.3. Show that there is $p \in \mathbb{P}^{d-1}$ with $\alpha(p) \subset \mathcal{M}_i$ and $\omega(p) \subset \mathcal{M}_j$ if and only if $\lambda_i < \lambda_j$.

Exercise 4.3.4. Consider the linear difference equation in $\mathbb{R}^2$ given by

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}$$

and determine the eigenvalues and the eigenspaces. Show that the line through the initial point $x_0 = 0, y_0 = 1$ converges under the flow to the line with slope $\left(1 + \sqrt{5}\right)/2$, the golden mean. Explain the relation to the Fibonacci numbers given by the recursion $f_{k+1} = f_k + f_{k-1}$ with initial values $f_0 = 0, f_1 = 1$.

Exercise 4.3.5. Consider the method for calculating $\sqrt{2}$ which was proposed by Theon of Smyrna in the second century B.C.: Starting from $(1, 1)$, iterate the transformation $x \mapsto x + 2y, y \mapsto x + y$. Explain why this gives a method to compute $\sqrt{2}$.
Hint: Argue similarly as in Exercise 4.3.4.

4.4. Orientation, Notes and References

Orientation. This chapter has characterized the Lyapunov spaces of linear dynamical systems by a topological analysis of the induced systems on projective space. Theorems 4.1.3 and 4.2.1 show that the projections of the

Lyapunov spaces $L(\lambda_j)$ to projective space coincide with the chain components of the projected flow. It is remarkable that these topological objects in fact have a 'linear structure'. The proofs are based on the explicit solution formulas and the structure in $\mathbb{R}^d$ provided by the Lyapunov exponents and the Lyapunov spaces. The insight gained in this chapter will be used in the second part of this book in order to derive decompositions of the state space into generalized Lyapunov spaces related to generalized Lyapunov exponents. More precisely, in Chapter 9 we will analyze a general class of linear dynamical systems (in continuous time) and construct a decomposition into generalized Lyapunov spaces. Here the line of proof will be reversed, since no explicit solution formulas are available: first the chain components yielding a linear decomposition are constructed and then associated exponential growth rates are determined.

In the next chapter, a generalization to flows induced on the space of k-dimensional subspaces, the k-Grassmannian, will be given. This requires some notions and facts from mulitilinear algebra, which are collected in Section 5.1. An understanding of the results in this chapter is not needed for the rest of this book, with the exception of some facts from multilinear algebra. They can also be picked up later, when they are needed (in Chapter 11 in the analysis of random dynamical systems).

Notes and references. The characterization of the Lyapunov spaces as the chain components in projective space is folklore (meaning that it is well known to the experts in the field, but it is difficult to find explicit statements and proofs). The differential equation on the unit sphere given in Lemma 4.1.1 is also known as Oja's flow (Oja [**108**]) and plays an important role in principal component analysis in neural networks where dominant eigenvalues are to be extracted. But the idea of using the $\mathbb{S}^{d-1} \times (0, \infty)$ coordinates (together with explicit formulas in Lemma 4.1.1 and Exercise 4.3.2) to study linear systems goes back at least to Khasminskii [**78, 79**].

Theorems 2.2.5 and 2.3.7 characterize the equivalence classes of linear differential and difference equations in $\mathbb{R}^d$ up to topological conjugacy. Thus it is natural to ask for a characterization of the topological conjugacy classes in projective space. Corollaries 4.1.5 and 4.2.2 already used such topological conjugacies of the projected linear dynamical systems in continuous and discrete time. However, the characterization of the corresponding equivalence classes is surprisingly difficult and has generated a number of papers. A partial result in the general discrete-time case has been given by Kuiper [**88**]; Ayala and Kawan [**14**] give a complete solution for continuous-time systems (and a correction to Kuiper's proof) and discuss the literature.

Exercises 4.3.4 and 4.3.5 are taken from Chatelin [**25**, Examples 3.1.1 and 3.1.2].