Introduction

Discovering a connection between two apparently disjoint areas of mathematics has always held a fascination for me. Just as a mental twist provides the punch line of a joke, a theorem giving an unsuspected link between two areas of mathematics is both enlightening and satisfying. The main goal of this book is to provide a largely self-contained, in-depth account of the linkage between nonassociative algebra and projective planes with particular emphasis on octonion planes. There are several new results and many, if not most, of the proofs are new.

A knowledge of linear algebra, basic ring theory, and basic group theory is required, as well as the ability to follow a detailed proof, but otherwise, except in Chapter 14, the development will be from first principles. Thus, a course based on this book would be accessible to most graduate students and would give them introductions to two areas which are often referenced but not often taught. Some of these students might continue in nonassociative algebra or use the geometry as a step towards research areas such as buildings or algebraic groups as indicated in Chapter 14.

The link between algebra and geometry goes back to the introduction of real coordinates in the Euclidean plane by Descartes. We also will introduce coordinates in a class of axiomatically defined geometries. The axiomatic approach to the Euclidean plane is seldom used after a high school course because a truly rigorous development is very demanding while the Cartesian product of the reals provides an easy-to-use model. However, we shall find it advantageous to start with a simple axiomatization of our geometries to set the scope of our investigation and then determine which algebraic structures can serve as coordinates. These coordinates are not limited to algebras over the reals or fields of characteristic 0, or even to nonassociative rings.

Our original axiomatization will be restricted to planar geometries for the same reason that high school students study the Euclidean plane. It is easier. However, we will find later that although the classification of higher-dimensional geometries requires more machinery, their structure is actually simpler. Indeed, the coordinates of a higher-dimensional projective geometry form an associative division ring, while the coordinates of a projective plane can be more exotic. Unlike the projective case, strictly nonassociative coordinates occur in some of the nonplanar geometries in Chapter 14. (Note that by convention "nonassociative" means "not necessarily associative", so "strictly nonassociative" is used to rule out associative rings.)

Although exceptional Lie (or algebraic) groups or Lie algebras are not mentioned explicitly except in Chapter 14, the simple group associated with the octonion plane in Chapter 12 is, in fact, of type E_6 (see [**37**, Proposition 11.20 with Proposition 12.3, Corollary 12.4, and Theorem 12.7]). Also, there are connections to physics through Lie groups and the use of projective geometries as quantum logic (see [**5**, p. 833]), although these topics will not be discussed here.

I strongly recommend that the reader have a scratch pad handy to sketch parts of the geometric proofs and to keep track of some of the nonassociative identities. The exercises present a lot of additional material not found in the main development, often with directions to the reader for supplying the proof. Each chapter has an informal preview section that introduces the reader to the coming material. We give below an overview of the contents.

We begin with affine planes which have the incidence properties of Euclidean planes, but we quickly pass to the equivalent notion of projective planes. Projective planes have the advantage that the projection of one line to another from a point is a bijection, which is not true, in general, in affine planes. Looking at automorphisms of projective planes which extend projections leads to the notion of a central automorphism.

Coordinates can be introduced into any projective plane, but, in general, the algebraic structure of the coordinates is rather weak. However, the existence of increasing sets of central automorphisms results in an increasing structure on the coordinates, ranging through Cartesian groups, Veblen-Wedderburn systems, nonassociative division rings, left Moufang division rings, alternative division rings, and associative division rings. In particular, a projective plane in which every projection extends to an automorphism has an alternative division ring as coordinates (Theorems 2.8 and 3.17).

We employ a trick using special Jordan rings to get identities in left Moufang (and hence alternative) rings. In particular, Micheev's identity shows that a left Moufang division ring is alternative (Theorem 4.2 and Corollary 4.3). This algebraic result eliminates a potential class of projective planes. The Cayley-Dickson process is a doubling construction which after several iterations results in an octonion ring, an 8-dimensional alternative algebra over its center. A major result, due independently to Skornyakov and to Bruck and Kleinfeld, is that an alternative division ring is either associative or an octonion ring.

Configuration conditions ensure that two geometric constructions give the same point (or line). Thus, configuration conditions play the same role for projective planes that identities do for nonassociative algebras. In fact, the Pappus condition is equivalent to the plane having a field for coordinates, the Desargues condition (or the quadrangle section condition) is equivalent to the plane having an associative division ring for coordinates, and the little Desargues condition (or the little quadrangle section condition) is equivalent to having an alternative division ring for coordinates.

Projective geometry is an example of "bigger is better". If the "projective dimension" is 3 or more, the coordinates are associative and the automorphism group is easily described. In order to even talk about dimension, we present an axiomatic development of dimension modeled on the dimension of a vector space and the transcendency degree of a field extension. This development is based on having the proper collection of "subobjects", e.g., the subspaces of a vector space or field extensions L/F in K/F with L algebraically closed in K. We shall see that the existence of a strong version of dimension is essentially equivalent to being a union of projective geometries (Theorems 7.2 and 7.3).

Certain algebraic machinery is needed to study octonion planes. We develop the basic properties of quadratic forms and orthogonal groups, including the Cartan-Dieudonné Theorem (Corollary 9.10). We also present an approach to homogeneous maps and their polarizations based on multilinear maps, rather than the standard use of polynomials. This allows a basis-free development which works equally well in infinite dimensions. Finally, we look at hermitian matrices $\mathcal{H}(\mathcal{C}_n)$ over a composition algebra $\mathcal{C}$ with diagonal entries from the field. There is a determinant-like norm function on $\mathcal{H}(\mathcal{C}_n)$ if and only if $n \leq 3$ or $\mathcal{C}$ is associative (Theorem 11.7). We also study the group generated by elementary matrices acting on $\mathcal{H}(\mathcal{C}_n)$ and the rank of an element of $\mathcal{H}(\mathcal{C}_n)$.

The octonion plane can be constructed using rank 1 elements in $\mathcal{H}(\mathcal{C}_3)$ and the automorphism group can be described in terms of norm semisimilarities. Moreover, this construction is valid even if the octonions are not a division ring, although the incidence geometry will not be a projective plane.

In this more general setting, the subgroup generated by transvections is simple (Lemma 12.8 and Theorem 12.10). Octonion planes and similar planes constructed from associative two-sided inverse rings are examples of projective remoteness planes, extending the notion of projective planes. Some of the results about transvections and the group they generate can be obtained in the setting of projective remoteness planes.

Finally, in Chapter 14, we assume a more extensive background and give a sketchy introduction to other geometries involving nonassociative algebras, since the complete treatment would require at least another book.

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