Chapter 1

Affine and Projective Planes

1.1. Preview

Although geometry is a wide-ranging concept, it always has its roots in Euclidean geometry. We shall see that projective geometry is no exception. Euclidean geometry includes a rich variety of concepts such as point, line, incidence, parallel lines, angle, distance between points, congruence of segments, and congruence of angles. It is often useful to isolate these concepts and study them on their own merits. For example, the study of metric spaces is the study of points and distance.

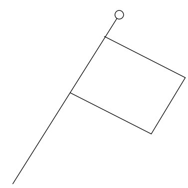
Projective geometry starts with a very practical problem in Euclidean geometry. Suppose that you are an artist wishing to realistically record a scene on you canvas. You would mentally draw a line from your eye to a point in the scene, say the top of a tree. You would then color the point where the line intersects the canvas with the color of the tree and repeat for the entire scene. A later viewer standing in front of the canvas as you did would have essentially the same visual input that you had when you viewed the original scene. This perspective picture was obtained by projecting the real scene onto the canvas. A similar example of projection is the projection of a slide or film onto a screen.

Although a perspective painting evokes the original scene, under projection much is changed. Distance and even congruence of segments and angles is not preserved by projection. Generally, projection maps points to points, lines to lines, and preserves the incidence relation (point lying on a line). However, as the following example shows, there is an important exception. Suppose that the scene that you are painting is a straight railroad track on a vast flat desert. The two rails are parallel in reality, but on your canvas they are represented by two lines which approach each other and appear to meet at the horizon. This occurs because as you look farther and farther down the track, the line from your eye approaches a line parallel to the desert floor. The limit line intersects the canvas at the point on the horizon where the rails meet. Thus, the rails meet on the canvas, but not in reality. Moreover, the horizon line and the meeting point of the rails on the canvas do not correspond to an incident point and line in reality. We shall see that we can avoid these problems by augmenting Euclidean geometry with fictitious points "at infinity", thereby creating a projective geometry.

Rather than working with the Euclidean geometry, we restrict our attention to the concepts involved in projection, namely point, line, and incidence. We add to this incidence geometry some basic axioms of the Euclidean plane involving only points, lines, and incidence. The resulting structure is called an affine plane. Adding in points at infinity and a line at infinity yields a projective plane. Conversely, deleting a line and its points from a projective plane yields an affine plane. Although one can pass back and forth between affine and projective planes, projective planes have certain advantages. Besides accommodating projection much better, projective planes unify certain affine concepts (e.g., intersecting lines and parallel lines) into a single projective concept (intersecting lines). Also, it turns out that the roles of points and lines are completely symmetric in projective planes; i.e., reversing the roles of points and lines gives another projective plane, called the dual plane. Duality provides a unification of concepts and proofs giving another advantage of projective planes.

1.2. Incidence geometry

An **incidence geometry** $\mathcal{G} = \mathcal{G}(\mathcal{P}, \mathcal{L}, I)$ consists of two disjoint sets $\mathcal{P} = \mathcal{P}(\mathcal{G})$ and $\mathcal{L} = \mathcal{L}(\mathcal{G})$ and a relation I between $\mathcal{P}$ and $\mathcal{L}$; i.e., $I \subset \mathcal{P} \times \mathcal{L}$. We usually call elements of $\mathcal{P}$ **points** and elements of $\mathcal{L}$ **lines**. Collectively, elements of $\mathcal{P} \cup \mathcal{L}$ are **objects**. If $(P, l) \in I$, we say the point P is **incident** to the line l and write $P \mid l$ or $l \mid P$. We shall freely use synonyms for incidence such as P lies on l or l passes through P. If a set of points all lie on a line, we say they are **collinear**. If a set of lines all pass through a point, we say they are **concurrent** or **meet at the point**. If x is an object, let $I_x = \{y : y \mid x\}$. An incident point-line pair $(P, l) \in I$ is sometimes called a **flag**. This terminology, which has more general usage, is suggested by the picture of an incident point-line-plane, so "flag pole" would be more accurate for a point and a line.



Flag

We say $\mathcal{G}(\mathcal{P}', \mathcal{L}', I')$ is a **subgeometry** of $\mathcal{G}(\mathcal{P}, \mathcal{L}, I)$ if $\mathcal{P}' \subset \mathcal{P}$, $\mathcal{L}' \subset \mathcal{L}$, and $I' = I \cap (\mathcal{P}' \times \mathcal{L}')$. Since $\mathcal{G}(\mathcal{P}', \mathcal{L}', I')$ is determined by $\mathcal{P}' \cup \mathcal{L}'$, we often just say that $\mathcal{P}' \cup \mathcal{L}'$ is a subgeometry. If $\mathcal{G} = \mathcal{G}(\mathcal{P}, \mathcal{L}, I)$ is an incidence geometry, then the **opposite geometry** is obtained by reversing the roles of points and lines; i.e., $\mathcal{G}^{op} = \mathcal{G}(\mathcal{L}, \mathcal{P}, I^{op})$ where $I^{op} = \{(l, P) : (P, l) \in I\}$.

A **homomorphism** from $\mathcal{G} = \mathcal{G}(\mathcal{P}, \mathcal{L}, I)$ to $\widetilde{\mathcal{G}} = \mathcal{G}(\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}}, \widetilde{I})$ is a map $\phi : \mathcal{P} \cup \mathcal{L} \to \widetilde{\mathcal{P}} \cup \widetilde{\mathcal{L}}$ with $\phi(\mathcal{P}) \subset \widetilde{\mathcal{P}}$, with $\phi(\mathcal{L}) \subset \widetilde{\mathcal{L}}$, and with $P \mid l$ implying $\phi(P) \mid \phi(l)$. If ϕ is bijective and both ϕ and ϕ^{-1} are homomorphisms, then ϕ is an **isomorphism**. A **monomorphism** from $\mathcal{G}$ to $\widetilde{\mathcal{G}}$ is an isomorphism of $\mathcal{G}$ to a subgeometry of $\widetilde{\mathcal{G}}$.

Sometimes an incidence geometry $\mathcal{G}$ is presented by specifying the lines to be certain subsets of the set $\mathcal{P}$ of points and defining $P \mid l$ if $P \in l$. Clearly, any incidence geometry in which each line l is uniquely determined by I_l is isomorphic to one presented this way. A homomorphism between two incidence geometries presented this way can be viewed as the map $\mathcal{P} \to \widetilde{\mathcal{P}}$ which maps each collinear set to a collinear set. The name **collineation** is often used instead of our name isomorphism. We prefer to use a terminology which is standard in other parts of mathematics and which does not depend on presenting the lines as subsets of points.

1.3. Affine planes

We define an **affine plane** to be an incidence geometry satisfying the following:

(AP1) Any two distinct points lie on a unique line.

(AP2) Through a point P not on a line m, there is a unique line l which does not meet m.

(AP3) There are three points which are not collinear.

Clearly, the Euclidean plane is an affine plane. Note that in an affine plane, two distinct lines pass through at most one point by (AP1). We say that lines l and m are **parallel**, written $l \parallel m$, if either $l = m$ or l does not meet m.

Lemma 1.1. *If $\mathcal{G}(\mathcal{P}, \mathcal{L}, I)$ is an affine plane, then parallelism is an equivalence relation on $\mathcal{L}$.*

Proof. We only need to check that $l \parallel m$ and $m \parallel n$ implies that $l \parallel n$. This is obvious if l, m, n are not distinct. If they are distinct, then m meets neither l nor n. If l meets n at P, then clearly $P \nmid m$, which gives a contradiction to (AP2). □

1.4. Projective planes

We will quickly shift our attention from affine planes to the essentially equivalent incidence geometries known as projective planes. Affine planes capture the incidence properties of the Euclidean plane and are very natural geometries to study, but projective planes have a distinct advantage in the formulation of concepts and the statement and proof of theorems. We will first give the axioms for a projective plane and the connection between affine and projective planes. Later, we will point out the advantages of the projective plane formulation.

A **projective plane** is an incidence geometry satisfying the following:

(PP1) Distinct points P and Q lie on a unique line, denoted PQ.

(PP2) Distinct lines l and m pass through a unique point, denoted lm.

(PP3) There are four points, no three of which are collinear.

We remark that (PP2) is a stronger assumption than is necessary. If two distinct lines pass through any point, then the point is unique by (PP1).

Suppose $\mathcal{A} = \mathcal{G}(\mathcal{P}, \mathcal{L}, I)$ is an affine plane. We have seen in Lemma 1.1 that parallelism gives an equivalence relation on lines. We shall denote equivalence classes of parallel lines by Ω, Θ, etc. Let $\bar{\mathcal{L}}$ be the set of all equivalence classes. We define the **extended plane** to be the incidence geometry

$$\mathcal{E}(\mathcal{A}) = \mathcal{G}(\mathcal{P} \cup \bar{\mathcal{L}}, \mathcal{L} \cup \{\infty\}, I'),$$

where we assume that ∞ is not in $\mathcal{L}$ and that I' is given by the following table:

	l	∞
P	if $P \mid l$ in $\mathcal{A}$	never
Ω	if $l \in \Omega$	always

Note that parallel lines l, m in $\mathcal{A}$ "meet at ∞" in $\mathcal{E}(\mathcal{A})$, i.e., at the point $\Omega = \{n \in \mathcal{L} : n \parallel l\}$ on ∞. On the other hand, if $\mathcal{G} = \mathcal{G}(\mathcal{P}, \mathcal{L}, I)$ is a projective plane and l is a line, we define the **deleted plane** to be the subgeometry $\mathcal{D}(\mathcal{G}, l) = (\mathcal{P} \setminus I_l) \cup (\mathcal{L} \setminus \{l\})$.

Theorem 1.2. *If $\mathcal{A} = \mathcal{G}(\mathcal{P}, \mathcal{L}, I)$ is an affine plane, then the extended plane $\mathcal{E}(\mathcal{A})$ is a projective plane. If $\mathcal{G} = \mathcal{G}(\mathcal{P}, \mathcal{L}, I)$ is a projective plane and l is a line, then the deleted plane $\mathcal{D}(\mathcal{G}, l)$ is an affine plane. Moreover, $\mathcal{D}(\mathcal{E}(\mathcal{A}), \infty) = \mathcal{A}$ and $\mathcal{E}(\mathcal{D}(\mathcal{G}, l)) \cong \mathcal{G}$.*

Proof. Assume that $\mathcal{A}$ is an affine plane. We shall say that a point $P \in \mathcal{P}$ is an **ordinary point** and an equivalence class Ω is an **ideal point**. Similarly, $l \in \mathcal{L}$ is an **ordinary line**. We first verify (PP1) for $\mathcal{E}(\mathcal{A})$ by considering various cases.

If $P \neq Q$ are ordinary points, then they are not incident to ∞ and there is a unique ordinary line l incident to both. Thus, l is the unique line in $\mathcal{E}(\mathcal{A})$ incident to both.

If P is ordinary and Ω is ideal, let $m \in \Omega$. If $P \mid m$, set $l = m$. If $P \nmid m$, let l be given by (AP2). In either case, we have $P \mid l$ and $l \in \Omega$. We claim l is unique with these properties. Indeed, if l' is another such ordinary line, then $l \parallel l'$ but l meets l' at P, so $l = l'$. Since $P \nmid \infty$, l is the unique line in $\mathcal{E}(\mathcal{A})$ incident to both P and Ω.

If $\Omega \neq \Theta$ are ideal points, then Ω and Θ are disjoint equivalence classes. Thus, Ω and Θ cannot lie on the same ordinary line. We see $\infty = \Omega\Theta$.

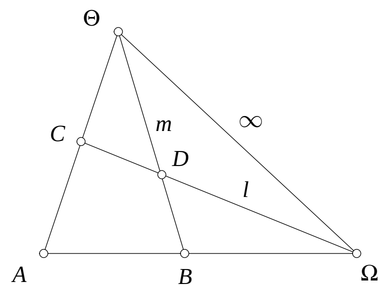
We now verify (PP2), recalling that we do not need to verify uniqueness.

If $l \neq m$ are ordinary lines, either l meets m at an ordinary point P, or l and m are parallel and lie in the same equivalence class Ω. Thus, either $lm = P$ or $lm = \Omega$.

If l is ordinary and Ω is the equivalence class containing l, then $l\infty = \Omega$.

Finally, we verify (PP3). Let A, B, C be three ordinary points given by (AP3). Clearly, $A \neq B$ and $AB \neq \infty$, so we can form $\Omega = (AB)\infty$ and $l = C\Omega$. Since $C \nmid AB$ and $l \parallel AB$, we see that neither A nor B lie on l.

Similarly, we have $\Theta = (AC)\infty$ and $m = B\Theta$ with $A, C \nmid m$. Clearly, $l \neq m$ and we can form $D = lm$.



Verifying (PP3)

We claim that A, B, C, D satisfy (PP3). Because A, B, C are not collinear, it suffices to show that D does not lie on any of AB, AC, BC. Clearly, D is distinct from A, B, C. If $D \mid AB$, then A lies on $AB = BD = m$, a contradiction. The cases $D \mid AC$ or $D \mid BC$ are similar.

Conversely, we now assume that $\mathcal{G}$ is a projective plane with line l. If P, Q are distinct points not on l, then $PQ \neq l$ satisfies (AP1) for $\mathcal{D}(\mathcal{G}, l)$. Suppose that P and m are in $\mathcal{D}(\mathcal{G}, l)$ and that P is not on m. A line $n \mid P$ does not meet m in $\mathcal{D}(\mathcal{G}, l)$ if and only if $nm \mid l$ in $\mathcal{G}$; i.e., $nm = lm$. We see that $n = P(lm)$, showing (AP2). Let A, B, C, D be four points in $\mathcal{G}$ with no three collinear. Clearly, (AP3) holds unless l passes through two of the four points. Thus, we may assume $l = AB$. Let $E = (AC)(BD)$. We claim that C, D, E are not collinear in $\mathcal{D}(\mathcal{G}, l)$. Clearly, C and D are not on AB. Now $E \mid AB$ would imply that $A = (AB)(AC) = E$ is on BD, a contradiction. Similarly, $E \nmid CD$.

The relations $\mathcal{D}(\mathcal{E}(\mathcal{A}), \infty) = \mathcal{A}$ and $\mathcal{E}(\mathcal{D}(\mathcal{G}, l)) \cong \mathcal{G}$ are immediate. □

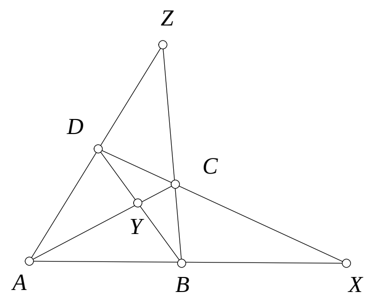
We shall now indicate why working within the framework of projective planes is generally easier than working with affine planes. We first observe that if $\mathcal{G}$ is a projective plane, then we get an affine plane $\mathcal{D}(\mathcal{G}, l)$ for each choice of line l. In general, $\mathcal{D}(\mathcal{G}, l)$ and $\mathcal{D}(\mathcal{G}, m)$ are not isomorphic, so various nonisomorphic affine planes can be contained in the same projective plane. Secondly, a property or definition that is fairly easy to state in projective language can require several cases to state in affine language. Before giving an example, we need to make some definitions.

A **triangle** in a projective plane consists of three noncollinear points A, B, C called **vertices** and the three lines $a = BC$, $b = AC$, $c = AB$ called **sides**. Note that A, B, C and a, b, c are distinct. Conversely, if A, B, C are distinct points and a, b, c are distinct lines with

$$A \mid b \mid C \mid a \mid B \mid c \mid A,$$

then A, B, C are noncollinear, so A, B, C, a, b, c form a triangle.

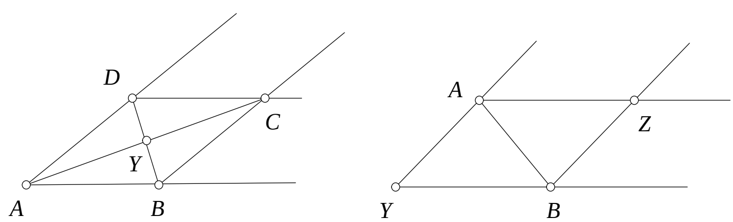
Four points A, B, C, D in a projective plane with no three collinear are the **vertices** of a **quadrangle**. The quadrangle has six **sides**, AB, AC, AD, BC, BD, CD, and three **diagonal points**, $X = (AB)(CD)$, $Y = (AC)(BD)$, and $Z = (AD)(BC)$.



Quadrangle

It is easy to verify that the sides of a quadrangle are distinct and that the vertices and diagonal points are distinct.

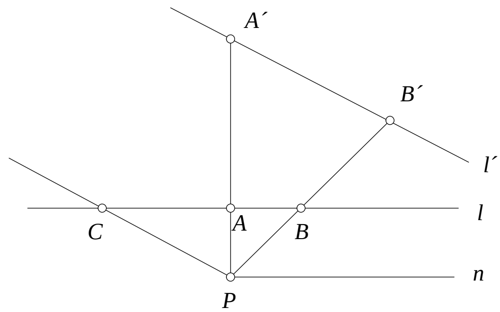
We now consider a quadrangle in an extended affine plane. If the vertices and diagonal points are all ordinary points, then the above figure still applies. However, the line ∞ could pass through some of the vertices and diagonal points. We indicate two possibilities below. In the first, X and Z are ideal points, while in the second, D, C, and X are ideal points. We see that the notion of a quadrangle in projective language covers several cases in affine language.



Quadrangles in an extended affine plane

As a final indication of the advantages of projective planes, we consider a projective plane $\mathcal{G}$ and $P \nmid l$. It is easy to see that $Q \to PQ$ and $m \to lm$ are inverse maps between I_l and I_P. If P lies on neither l nor l', then $Q \to l'(PQ)$ is a bijection from I_l to $I_{l'}$ called the **projection of** l **to** l' **from** P. Projections in affine planes are not as easy to handle, since there is not a bijection between I_l and I_P for $P \nmid l$. Indeed, $Q \to PQ$

and $m \to lm$ are inverse maps between I_l and $I_P \setminus \{n\}$ where n is the line through P parallel to l. In the figure below, we indicate the difficulties with projection in an affine plane. While A and B project to A' and B' on l', there is no projection of C since $PC \parallel l'$. Also, n corresponds to a point on l' but not to a point on l.



Affine projection

1.5. Duality

We now consider another property of projective planes which greatly facilitates their study. Recall that the opposite geometry of $\mathcal{G}(\mathcal{P}, \mathcal{L}, I)$ is $\mathcal{G}^{op} = \mathcal{G}(\mathcal{L}, \mathcal{P}, I^{op})$ where $I^{op} = \{(l, P) : (P, l) \in I\}$.

Proposition 1.3. *If $\mathcal{G}$ is a projective plane, then so is $\mathcal{G}^{op}$.*

Proof. (PP1) for $\mathcal{G}$ is (PP2) for $\mathcal{G}^{op}$ and vice versa. Thus, we only need to verify (PP3) for $\mathcal{G}^{op}$ or equivalently, verify the following for $\mathcal{G}$:

(PP3′) There are four lines, no three of which are concurrent.

Let A, B, C, D be the vertices of a quadrangle given by (PP3). We claim the four distinct sides AB, BC, CD, DA satisfy (PP3′). Indeed, if the first three lines are concurrent at P, then $P = (AB)(BC) = B$ and $P = (BC)(CD) = C$, a contradiction. The other cases are handled by cyclically permuting the roles of A, B, C, D. □

If $\mathcal{G}$ is a projective plane, we call the opposite geometry the **dual** of $\mathcal{G}$ and write $\mathcal{G}^{dual} = \mathcal{G}^{op}$. Although the notions of dual geometry and opposite geometry coincide for projective planes, we will see in Chapter 7 that they are distinct concepts for higher-dimensional projective geometries, which is why we have employed the two terminologies.

If S is any statement or definition for a projective plane, the statement or definition S^{dual} obtained by interchanging the roles of points and lines is called the **dual** of S. We have already observed that (PP1) and (PP2) are dual statements. Also, statements (PP3) and (PP3′) are dual.

The dual of a triangle is a **trilateral**, i.e., three nonconcurrent lines (called sides) a, b, c and three **vertices** $A = bc$, $B = ac$, $C = ab$. In a projective plane, the notions of triangle and trilateral coincide, so triangles are **self-dual**. The dual of a quadrangle is a **quadrilateral**, i.e., four lines (called **sides)** a, b, c, d, no three of which are concurrent, six **vertices** ab, ac, ad, bc, bd, cd, and three **diagonal lines** $(ab)(cd)$, $(ac)(bd)$, $(ad)(bc)$.

We can now formulate a pseudo-theorem about projective planes. The **principle of duality** asserts that a statement S holds for $\mathcal{G}$ if and only if S^{dual} holds for $\mathcal{G}^{dual}$. Similarly, a set of objects satisfies the definition S in $\mathcal{G}$ if and only if it satisfies S^{dual} in $\mathcal{G}^{dual}$.

Frequently, after having shown that a statement S holds for a projective plane $\mathcal{G}$, we will assert that S^{dual} holds **by duality**. We mean by this that the same proof applies to $\mathcal{G}^{dual}$ or equivalently the dual proof applies to $\mathcal{G}$. Of course, to employ duality in this way, we implicitly assume that the properties of $\mathcal{G}$ used to prove S also hold in $\mathcal{G}^{dual}$.

If $\mathcal{G}$ is a projective plane, a homomorphism $\phi : \mathcal{G} \to \mathcal{G}^{dual}$ is called a **dual homomorphism.** A dual isomorphism is just called a **duality**. Equivalently, a duality is a bijection interchanging points and lines of $\mathcal{G}$ with $P \mid l$ if and only if $\phi(l) \mid \phi(P)$. A duality ϕ with $\phi^2 = \mathrm{Id}$ is a **polarity**.

Please note that the terms dual and duality are employed in several ways. There is the dual plane, the dual of a statement or definition, the principle of duality, a proof by duality, and a duality map.

Lemma 1.4. *An incidence geometry satisfies the statement*

(X) *two distinct points are incident to at most one line*

if and only if it is a subgeometry of a projective plane.

Proof. If $\mathcal{G}$ satisfies (X), we first extend $\mathcal{G}$ to $\mathcal{G}_1$ by adding four points which are incident to no lines. Since incidence in $\mathcal{G}_1$ involves only objects from $\mathcal{G}$, (X) still holds in $\mathcal{G}_1$. Moreover, the four new points satisfy (PP3). We recursively extend $\mathcal{G}_k$ to $\mathcal{G}_{k+1}$ adding a line l_{PQ} for each pair P, Q of distinct noncollinear points in $\mathcal{G}_k$ with l_{PQ} incident to only P and Q in $\mathcal{G}_{k+1}$, and dually for points.

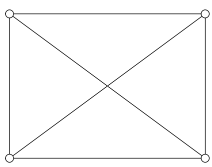
We claim that each $\mathcal{G}_k$ satisfies (X) and (PP3). Suppose that in $\mathcal{G}_{k+1}$ there are $P \neq Q$ lying on $l \neq m$. If $l \notin \mathcal{G}_k$, then it is incident in $\mathcal{G}_{k+1}$ to precisely two points and these are noncollinear in $\mathcal{G}_k$. They must be P and Q, so $l = l_{PQ}$. Now $m \notin \mathcal{G}_k$ implies $m = l_{PQ}$, a contradiction, while $m \in \mathcal{G}_k$ implies P and Q are collinear in $\mathcal{G}_k$, also a contradiction. Thus, using a dual argument, each of P, Q, l, m are in $\mathcal{G}_k$. Similarly, three points in $\mathcal{G}_k$ which are collinear in $\mathcal{G}_{k+1}$ are, in fact, collinear in $\mathcal{G}_k$. The claims now follow by induction.

Let $\tilde{\mathcal{G}} = \bigcup_k \mathcal{G}_k$ and define incidence by $P \mid l$ in $\tilde{\mathcal{G}}$ if $P \mid l$ in some $\mathcal{G}_k$. Since any finite set of objects from $\tilde{\mathcal{G}}$ lies in some $\mathcal{G}_k$, it is easy to see that $\mathcal{G}$ satisfies (X) and (PP3) and that any two points lie on a line. Since (X) implies its dual, we see that (PP1) and (PP2) hold and $\tilde{\mathcal{G}}$ is a projective plane.

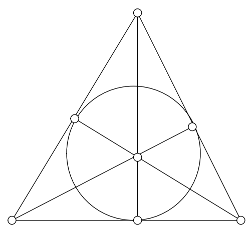
The converse is clear. □

1.6. Exercises

Exercise 1.1. Show that the incidence geometry of four points and six lines in the figure labeled *Affine* below is an affine plane $\mathcal{A}$. Show that $\mathcal{E}(\mathcal{A})$ is isomorphic to the incidence geometry of seven points and seven lines (the circle is also a line) in the figure labeled *Projective*.



Affine



Projective

Exercise 1.2. Let Δ be an associative division ring and let $\mathcal{A}(\Delta)$ be the incidence geometry whose lines are the subsets of $\mathcal{P} = \Delta \times \Delta$ given by

$$\begin{aligned} [m, b] &= \{(x, y) \in \Delta \times \Delta : y = mx + b\}, \\ [a] &= \{(x, y) \in \Delta \times \Delta : x = a\} \end{aligned}$$

for any $m, b, a \in \Delta$. Show that $\mathcal{A}(\Delta)$ is an affine plane.

Exercise 1.3. If $\mathbb{Z}_2$ is the field with two elements, show that $\mathcal{A}(\mathbb{Z}_2)$ is isomorphic to the affine plane in Exercise 1.1.

Exercise 1.4. Show that the ideal points in $\mathcal{E}(\mathcal{A}(\Delta))$ can be labeled by (n) with $n \in \Delta \cup \{\infty\}$. Writing $[\infty]$ for the line ∞, show that incidence in $\mathcal{E}(\mathcal{A}(\Delta))$ is given by

	$[m, b]$	$[a]$	$[\infty]$
(x, y)	if $y = mx + b$	if $x = a$	never
(n)	if $n = m$	never	always
(∞)	never	always	always

Exercise 1.5. Let Δ^n denote the right vector space of column vectors of length n over an associative division ring Δ. Let $\mathcal{G}(\Delta^3) = \mathcal{G}(\mathcal{P}, \mathcal{L}, I)$ where $\mathcal{P}$ consists of all subspaces of dimension 1 in Δ^3, where $\mathcal{L}$ consists of all

subspaces of dimension 2 in Δ^3, and where $(P, l) \in I$ if $P \subset l$. Show that $\mathcal{G}(\Delta^3)$ is a projective plane.

Exercise 1.6. Show that $\mathcal{G}(\mathbb{Z}_2^3)$ is isomorphic to the projective plane in Exercise 1.1.

Exercise 1.7. Let ${}^n\Delta$ denote the left vector space of all row vectors of length n over an associative division ring Δ. Show that $\Delta u \to \{v \in \Delta^3 : uv = 0\}$ is a bijection from the set $\mathcal{L}'$ of all subspaces of dimension 1 in ${}^3\Delta$ to the set $\mathcal{L}$ of all subspaces of dimension 2 in Δ^3. Use this to show that $\mathcal{G}(\Delta^3) \cong \mathcal{G}(\mathcal{P}, \mathcal{L}', I')$ where $(v\Delta, \Delta u) \in I'$ if $uv = 0$.

Exercise 1.8. Use Exercises 1.4 and 1.7 to show that $\mathcal{E}(\mathcal{A}(\Delta)) \cong \mathcal{G}(\Delta^3)$ via an isomorphism extending the map

$$(x, y) \to \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \Delta.$$

Exercise 1.9. Recall that the **opposite ring** Δ^{op} is the set Δ with the same addition and with product $a \cdot b = ba$. Also, recall that a right vector space over Δ can be viewed as a left vector space over Δ^{op} with $a \cdot v = va$. The transpose map $v \to v^t$ is an isomorphism of Δ^n viewed as a left vector space over Δ^{op} with ${}^n(\Delta^{op})$. Use these facts and Exercise 1.7 to show that $(\mathcal{G}(\Delta^3))^{dual} \cong \mathcal{G}((\Delta^{op})^3)$.

Exercise 1.10. Draw figures representing all possible affine formulations in the Euclidean plane of the projective notion of quadrangle.

Exercise 1.11. If P is a point and l is a line in a projective plane $\mathcal{G}$, show that $\text{card}(I_P) = \text{card}(I_l) \geq 3$ (note that $\text{card}(S)$ is the cardinality of the set S). If $\text{card}(I_P) = n + 1$ is finite, we say that n is the **order** of $\mathcal{G}$.

Exercise 1.12. If $\mathcal{G}$ is a projective plane of order n, show $\text{card}(\mathcal{P}) = \text{card}(\mathcal{L}) = n^2 + n + 1$.

Exercise 1.13. If $\mathcal{G}$ and $\mathcal{G}'$ are projective planes of order 2 and A, B, C in $\mathcal{G}$ and A', B', C' in $\mathcal{G}'$ are two sets of noncollinear points, show that there is a unique isomorphism from $\mathcal{G}$ to $\mathcal{G}'$ mapping A to A', B to B', and C to C'.

Exercise 1.14. A projective plane satisfies the **Fano condition** if the diagonal points of every quadrilateral are collinear. Show that if $\mathcal{G}$ satisfies the Fano condition, then so does $\mathcal{G}^{dual}$. What is the condition dual to the Fano condition?

Exercise 1.15. Let A, B, C and A', B', C' be two sets of distinct collinear points in a projective plane. Show that there is a composition of at most three projections mapping A to A', B to B', and C to C'.

Exercise 1.16. Let $\mathcal{G}$ be a projective plane. Show that all of the deleted planes $\mathcal{D}(\mathcal{G}, l)$, $l \in \mathcal{L}$, are isomorphic if and only if $\mathrm{Aut}(\mathcal{G})$ is transitive on $\mathcal{L}$.

Exercise 1.17. Let $\mathcal{A}$ be an affine plane and suppose that $\mathrm{Aut}(\mathcal{E}(\mathcal{A}))$ is transitive on $\mathcal{L}(\mathcal{E}(\mathcal{A}))$. Show that the following two statements are equivalent for $\mathcal{A}$.

(a) If a, b, c are distinct parallel lines with $A_i \mid a$, $B_i \mid b$, and $C_i \mid c$ for $i = 1, 2$ and if $A_1B_1 \parallel A_2B_2$ and $B_1C_1 \parallel B_2C_2$, then $A_1C_1 \parallel A_2C_2$.

(b) If a, b, l are distinct parallel lines with $A_i \mid a$, $B_i \mid b$ for $i = 1, 2$ and $X, Y \mid l$ and if $A_iY \parallel B_iX$ for $i = 1, 2$, then A_1B_1, A_2B_2, l are concurrent.

Hint: Show that (a) for $\mathcal{A}$ is equivalent to (b) for $\mathcal{D}(\mathcal{E}(\mathcal{A}), c)$ with $l = \infty$ in $\mathcal{E}(\mathcal{A})$.

Chapter 2

Central Automorphisms of Projective Planes

2.1. Preview

The symmetries or automorphisms of a mathematical structure often reveal a great deal about the structure. We shall see that we can distinguish projective planes by the types of automorphisms they possess. First, consider the following basic automorphisms of the Euclidean plane with its usual Cartesian coordinate system: σ is the reflection about the x-axis, τ is the vertical translation by one unit, and δ is the dilation $\delta : (x, y) \to (2x, 2y)$. These extend uniquely to automorphisms (also denoted σ, τ, δ) of the extended plane. We see that σ fixes all points on the x-axis and all vertical lines, and therefore it fixes the point V at infinity determined by the vertical lines. Similarly, τ fixes all vertical lines and V. Also, τ maps a line to a parallel line, so τ fixes all points at infinity and hence the line ∞ at infinity. Finally, δ fixes the origin and all lines through the origin and maps a line to a parallel line, so again δ fixes ∞ and all points on it. In each case, the automorphism has a line of fixed points (called the axis) and point of fixed lines (called the center). If the center lies on the axis, the automorphism is called a transvection and otherwise it is called a dilatation. Thus, translations are transvections while reflections and dilations are dilatations. This again shows a unification of affine concepts in the extended plane.

We can classify projective planes by the transvections and dilatations that they admit. For example, a vertical translation of the Euclidean plane permutes the points on the y-axis. Moreover, given P, Q on the y-axis, there is a unique vertical translation sending P to Q. Thus, the extended plane admits all potential vertical translations, i.e., all transvections with center V and axis ∞. In fact, the Euclidean plane admits all potential transvections and all potential dilatations. For a general projective plane $\mathcal{G}$, we consider the family $Tflag(\mathcal{G})$ of flags (C, a) such that $\mathcal{G}$ admits all potential transvections with center C and axis a. $Tflag(\mathcal{G})$ is a measure of the degree of symmetry in $\mathcal{G}$ and can be used to distinguish projective planes. For planes with more transvections, it is convenient to also consider the subgeometry $Tgeom(\mathcal{G})$ consisting of points C such that $\mathcal{G}$ admits all potential transvections with axis C (and any axis) and dually for lines. We shall classify the possibilities for $Tgeom(\mathcal{G})$ and the possibilities for $Tflag(\mathcal{G})$ if $Tgeom(\mathcal{G}) = \emptyset$.

If ϕ is an automorphism of a projective plane and if P, Q are distinct points, then $\phi(PQ)$ is the unique line through both $\phi(P)$ and $\phi(Q)$. Thus, $\phi(PQ) = \phi(P)\phi(Q)$ and dually. We shall use this repeatedly in our study of transvections and dilatations.

2.2. Projections and automorphisms

Let C be a point in a projective plane $\mathcal{G}$ which does not lie on either l or l'. Recall that the projection π of l to l' from C maps Q on l to $Q' = (CQ)l'$ on l'. Composing projections would be greatly facilitated if they were defined on all of $\mathcal{G}$ rather than just on points lying on some line. Suppose now that π does extend to an automorphism ϕ of $\mathcal{G}$ fixing C; i.e., $\phi(Q) = \pi(Q) = Q'$ for $Q \mid l$ and $\phi(C) = C$. If $m \mid C$, then $m = CQ$ where $Q = ml$ is on l. Thus,

$$\phi(m) = \phi(C)\phi(Q) = CQ' = CQ = m,$$

i.e., ϕ fixes all lines through C. In general, an automorphism of $\mathcal{G}$ fixing all lines through some point C is called a **central automorphism** with **center** C.

Conversely, suppose ϕ is any central automorphism with center C and $l \nmid C$. Let $l' = \phi(l)$ and note that for $Q \mid l$, we have

$$\phi(Q) = \phi((CQ)l) = \phi(CQ)\phi(l) = (CQ)l'.$$

Thus, ϕ restricts to a projection of l to l' from C. We see that central automorphisms are just extensions of projections.

2.3. Transvections and dilatations

The dual of the notion of a central automorphism is an **axial automorphism** with **axis** a, i.e., an automorphism fixing all points on a. In fact, the two notions are equivalent.

Lemma 2.1. *An automorphism of a projective plane is central if and only if it is axial. A central automorphism* $\phi \neq \mathrm{Id}$ *has a unique center* C *and a unique axis* a. *Moreover, the set of fixed points of* ϕ *is* $\{C\} \cup I_a$ *and the set of fixed lines is* $\{a\} \cup I_C$.

Proof. Let ϕ be a central automorphism with center C. Suppose that $l \nmid C$ and that P lies on both l and $\phi(l)$; i.e., $P \mid l$ if $\phi(l) = l$ and $P = l\phi(l)$ if $\phi(l) \neq l$. Since

$$\phi(P) = \phi((CP)l) = (CP)\phi(l) = P$$

and since C itself is fixed, we see that every line passes through a fixed point. Moreover, any fixed line which does not pass through C is an axis. If P, Q, R are noncollinear fixed points, one of the fixed lines PQ, PR, QR does not pass through C and is an axis. On the other hand, if all fixed points lie on some line a, then a is an axis. Indeed, every point on a can be written as al for some line l and must be the fixed point on l. Suppose that ϕ has distinct axes a, b. If $P \neq C, ab$, then we can choose $l \mid P$ which does not pass through C or ab. Now $l = (la)(lb)$ is fixed and hence an axis. Thus, ϕ fixes P, hence all points, and therefore all lines; i.e., $\phi = \mathrm{Id}$. Since $\phi \neq \mathrm{Id}$ has a unique axis a, the fixed lines must be $\{a\} \cup I_C$. The rest follows by duality. □

We let $\mathrm{Cent}(C, a)$ denote the subgroup of the group of automorphisms $\mathrm{Aut}(\mathcal{G})$ consisting of all central automorphisms with center C and axis a. Also, if x is an object of $\mathcal{G}$ (i.e., x is a point or a line), we let $\mathrm{Cent}(x)$ denote the subgroup consisting of all central automorphisms with center or axis x. We now observe that an element of $\mathrm{Cent}(C, a)$ is uniquely determined by the image of a single point.

Corollary 2.2. *If* $\phi, \theta \in \mathrm{Cent}(C, a)$ *and* $\phi(P) = \theta(P)$ *for some* $P \notin \{C\} \cup I_a$, *then* $\phi = \theta$.

Proof. Since $\theta^{-1}\phi \in \mathrm{Cent}(C, a)$ fixes P, we have that $\theta^{-1}\phi = \mathrm{Id}$. □

There are two types of central automorphisms depending on whether or not the center lies on the axis. If $\phi \in \mathrm{Cent}(C, a)$ with $C \mid a$, we say that ϕ is a (C, a)-**transvection** or simply a **transvection**. If $C \nmid a$, we say that ϕ is a (C, a)-**dilatation** or simply a **dilatation**. Note that we regard $\phi = \mathrm{Id}$ as both a transvection and a dilatation. A transvection is sometimes called an **elation** or a **translation**, and a dilatation is also called a **homology** or a **dilation**.

2.4. Transitivity properties

We first note the following properties of conjugates $\alpha\phi\alpha^{-1}$ and commutators $(\phi, \theta) = \phi\theta\phi^{-1}\theta^{-1}$ of central automorphisms.

Lemma 2.3. *If $\alpha \in \mathrm{Aut}(\mathcal{G})$ and x is an object of $\mathcal{G}$, then*

$$\alpha(\mathrm{Cent}(x))\alpha^{-1} = \mathrm{Cent}(\alpha(x)).$$

If $\phi \in \mathrm{Cent}(C)$ and $\theta \in \mathrm{Cent}(a)$ with $C \mid a$, then the commutator

$$(\phi, \theta) \in \mathrm{Cent}(C, a).$$

Proof. The first statement follows from the fact that $\phi \in \mathrm{Aut}(\mathcal{G})$ fixes y if and only if $\alpha\phi\alpha^{-1}$ fixes $\alpha(y)$. For the second, we note that

$$\begin{aligned} \phi\theta\phi^{-1} &\in \mathrm{Cent}(\phi(a)) = \mathrm{Cent}(a), \\ \theta\phi^{-1}\theta^{-1} &\in \mathrm{Cent}(\theta(C)) = \mathrm{Cent}(C), \end{aligned}$$

so $(\phi, \theta) \in \mathrm{Cent}(C) \cap \mathrm{Cent}(a) = \mathrm{Cent}(C, a)$. □

Clearly, $\mathrm{Cent}(C)$ permutes the lines $l \nmid C$. If $\mathrm{Cent}(C)$ acts transitively on this set, we say that $\mathcal{G}$ is C**-transitive**.

Lemma 2.4. *If $\mathcal{G}$ is C-transitive for all points C, then $\mathrm{Aut}(\mathcal{G})$ is transitive on the set I of flags.*

Proof. We first show that $\mathrm{Aut}(\mathcal{G})$ is transitive on points. Given distinct points P, Q, let $C \mid PQ$ be distinct from P and Q, and let $D \nmid PQ$. There exists $\phi \in \mathrm{Cent}(C)$ with $\phi(PD) = QD$, so

$$\phi(P) = \phi((PQ)(PD)) = (PQ)(QD) = Q.$$

Given flags $P \mid l$ and $Q \mid m$, we can use the transitivity on points to assume that $P = Q$. Choosing C not on l or m and $\phi \in \mathrm{Cent}(C)$ with $\phi(l) = m$, we also have

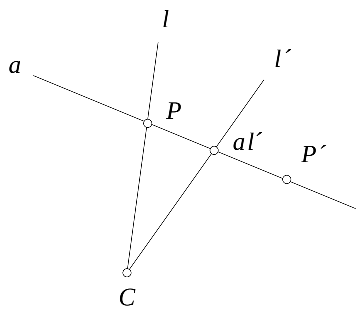
$$\phi(P) = \phi((CP)l) = (CP)m = P,$$

as desired. □

The group $\mathrm{Cent}(C, a)$ acts on the set of points $I_l \setminus \{al, C\}$ for any $l \mid C$ with $l \neq a$ and, dually, on the set of lines $I_P \setminus \{CP, a\}$ for any $P \mid a$ with $P \neq C$. These actions are closely related.

Lemma 2.5. *Let* $l \in I_C \backslash \{a\}$ *and* $P \in I_a \backslash \{C\}$. *The action of* $\mathrm{Cent}(C, a)$ *on* $I_l \backslash \{al, C\}$ *is transitive if and only if the action on* $I_P \backslash \{CP, a\}$ *is transitive. In this case, the action of* $\mathrm{Cent}(C, a)$ *on either set is sharply transitive.*

Proof. If $P \nmid l$, then the map $Q \to PQ$ is a bijection from $I_l \backslash \{al, C\}$ to $I_P \backslash \{CP, a\}$, which commutes with the actions of $\mathrm{Cent}(C, a)$ since $\phi(PQ) = P\phi(Q)$ for $\phi \in \mathrm{Cent}(C, a)$. The result is now obvious if $P \nmid l$. If $P \mid l$, then $C \nmid a$, for otherwise $C = al = P$. Let $l' \neq l$ be another line through C and let P' be a point on a distinct from P and al'. The result is true for the pairs $P \nmid l'$, $P' \nmid l'$, $P' \nmid l$, and hence for $P \mid l$.



Transfering the action

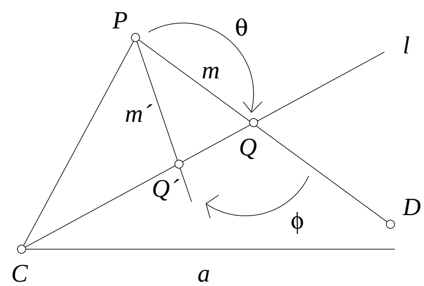
To show that the $\mathrm{Cent}(C, a)$ action is sharply transitive, we note that if $Q, Q' \in I_l \backslash \{al, C\}$, then $\phi \in \mathrm{Cent}(C, a)$ with $\phi(Q) = Q'$ is unique by Corollary 2.2. □

Since the transitivity in Lemma 2.5 does not depend on the choice of l or P, we shall say that $\mathcal{G}$ is (C, a)**-transitive** if $\mathrm{Cent}(C, a)$ is transitive on either set. If $\mathcal{G}$ is (C, a)-transitive for all flags $C \mid a$, we say that $\mathcal{G}$ is a **transvection plane**. If $\mathcal{G}$ is (C, a)-transitive for all pairs $C \nmid a$, then $\mathcal{G}$ is a **dilatation plane**. Roughly speaking, a transvection plane has all possible transvections and a dilatation plane has all possible dilatations.

Lemma 2.6. *Let* $C \mid a$. *If* $\mathcal{G}$ *is* C*-transitive and* $\mathrm{Cent}(D, a) \neq \{\mathrm{Id}\}$ *for some* $D \neq C$, *then* $\mathcal{G}$ *is* (C, a)*-transitive.*

Proof. Let $\mathrm{Id} \neq \theta \in \mathrm{Cent}(D, a)$. Let $l \mid C$ with $l \neq a, CD$, and suppose that $Q, Q' \in I_l \backslash \{C\}$. Let $P = \theta^{-1}(Q)$, and note that $P \neq Q$ by Lemma 2.1 since $Q \notin I_a \cup \{D\}$. In fact, $P \nmid l$, since otherwise $P = \theta^{-1}(QD)l =$

$(QD)l = Q$. Thus, we may form $m = PQ = QD$ and $m' = PQ'$. Since $l = CQ = CQ'$, we see that $m, m' \nmid C$, so there is $\phi \in \mathrm{Cent}(C)$ with $\phi(m) = m'$.



Action of θ and ϕ

Now $(\phi, \theta) \in \mathrm{Cent}(C, a)$ by Lemma 2.3 and

$$\begin{aligned}(\phi,\theta)(Q) &= \phi\theta\phi^{-1}(P) = \phi\theta\phi^{-1}((PC)m') \\ &= \phi\theta((PC)m) = \phi\theta(P) \\ &= \phi(Q) = \phi(lm) = lm' = Q'.\end{aligned}$$

Thus, $\mathcal{G}$ is (C, a)-transitive. □

Corollary 2.7. *Let $C \mid a$. If $\mathcal{G}$ is C-transitive and a-transitive, then $\mathcal{G}$ is (C, a)-transitive.*

Proof. Let P and Q be distinct points not on a with $Q \nmid PC$. Let $\theta \in \mathrm{Cent}(a)$ with $\theta(P) = Q$. Clearly, $\theta \neq \mathrm{Id}$ and by Lemma 2.1 θ has center $D \neq C$. □

Theorem 2.8. *The following are equivalent for a projective plane $\mathcal{G}$:*

(a) *Any projection from any point C extends to an automorphism fixing C.*

(b) *$\mathcal{G}$ is C-transitive for all points C.*

(c) *$\mathcal{G}$ is a transvection plane.*

Proof. We have already seen that the projection of l to l' from C extends to an automorphism fixing C if and only if there is a central automorphism with center C mapping l to l'. Thus, (a) and (b) are equivalent. Assume now that (b) holds and $C \mid a$. Let $\mathrm{Id} \neq \theta \in \mathrm{Cent}(D', a')$ be any nontrivial central automorphism and let $C' \mid a'$ with $C' \neq D'$. Using Lemmas 2.3 and 2.4, we can replace θ by a conjugate to assume that $\theta \in \mathrm{Cent}(D, a)$ and $C \neq D$. Thus, by Lemma 2.6, $\mathcal{G}$ is (C, a)-transitive, showing (c). Finally, assume that (c) holds. Given $C \nmid l, l'$ with $l \neq l'$, let $P = ll'$ and $a = CP$. Since $l, l' \in I_P \backslash \{a\}$, there exists $\phi \in \mathrm{Cent}(C, a)$ with $\phi(l) = l'$, showing (b). □

Corollary 2.9. *A dilatation plane is a transvection plane.*

Proof. If $\operatorname{card}(I_P) < 4$ for some point P in a projective plane $\mathcal{G}$, then Exercises 1.12 and 1.13 show that $\mathcal{G}$ is unique up to isomorphism. We leave this case (i.e., $\mathcal{G}$ has order 2) to Exercise 2.2 and assume that every point lies on at least four lines. If $\mathcal{G}$ is a dilatation plane and if l, l' are distinct lines not through C, let $P = ll'$ and let $a \mid P$ distinct from l, l', and CP. There is a dilatation $\phi \in \operatorname{Cent}(C, a)$ mapping l to l', so $\mathcal{G}$ is C-transitive and hence a transvection plane. □

Lemma 2.10. *Let a be a line in a projective plane $\mathcal{G}$ and let C, D be distinct points on a.*

(a) *The set* $\operatorname{Trans}(a) = \bigcup_{E|a} \operatorname{Cent}(E, a)$ *of all transvections with axis a is a subgroup of* $\operatorname{Cent}(a)$.

(b) *If* $\operatorname{Cent}(C, a) \neq \{\mathrm{Id}\}$ *and* $\operatorname{Cent}(D, a) \neq \{\mathrm{Id}\}$, *then* $\operatorname{Trans}(a)$ *is abelian.*

(c) *If $\mathcal{G}$ is both (C, a)- and (D, a)-transitive, then*

$$\operatorname{Trans}(a) = \operatorname{Cent}(C, a)\operatorname{Cent}(D, a)$$

and $\mathcal{G}$ is (E, a)-transitive for all $E \mid a$. In particular, $\operatorname{Trans}(a)$ *is transitive on all $P \nmid a$.*

Proof. First suppose C, D are distinct points on a and let $\sigma \in \operatorname{Cent}(C, a)$ and $\tau \in \operatorname{Cent}(D, a)$. If $\sigma\tau$ fixes some $P \nmid a$, then $\tau(P) = \sigma^{-1}(P)$ lies on $\tau(PD) = PD$ and on $\sigma^{-1}(PC) = PC$. Thus, $\tau(P) = \sigma^{-1}(P) = P$, and $\sigma = \tau = \mathrm{Id}$ by Lemma 2.1. In particular, if $\sigma\tau \neq \mathrm{Id}$, then Lemma 2.1 shows that $\sigma\tau$ has a center which must lie on a. Thus,

$$\operatorname{Cent}(C, a)\operatorname{Cent}(D, a) \subset \operatorname{Trans}(a).$$

Since $\operatorname{Cent}(C, a)$ is a subgroup, (a) holds.

Using Lemma 2.3, we see that

$$(\sigma, \tau) = (\tau, \sigma)^{-1} \in \operatorname{Cent}(C, a) \cap \operatorname{Cent}(D, a) = \{\mathrm{Id}\},$$

so $\operatorname{Cent}(C, a)$ commutes with $\operatorname{Cent}(D, a)$. We also claim that if $\operatorname{Cent}(D, a) \neq \{\mathrm{Id}\}$, then $\operatorname{Cent}(C, a)$ is abelian. Indeed, if $\sigma, \sigma' \in \operatorname{Cent}(C, a)$ and $\mathrm{Id} \neq \tau \in \operatorname{Cent}(D, a)$, then we have seen that $\sigma\tau \in \operatorname{Cent}(E, a)$ for some $E \mid a$. If $E = C$, then $\tau = \sigma^{-1}(\sigma\tau) \in \operatorname{Cent}(C, a)$, but τ has a unique center D. Thus, $E \neq C$, so σ' commutes with $\sigma\tau$ and with τ^{-1}, and hence with σ. In (b), if $E \mid a$, then using the claim for either $E \neq D$ or $E \neq C$, we see $\operatorname{Cent}(E, a)$ is abelian for all $E \mid a$ and (b) follows.

For (c), let $E \mid a$, $P \nmid a$, and $P' \in I_{PE} \backslash \{E\}$. Set $Q = (PD)(P'C)$ and let $\tau \in \operatorname{Cent}(D, a)$ with $\tau(P) = Q$ and $\sigma \in \operatorname{Cent}(C, a)$ with $\sigma(Q) = P'$.

Thus, $\sigma\tau \in \mathrm{Trans}(a)$ maps P to P'. If $P \neq P'$ and E' is the center of $\sigma\tau$, then $P' \mid PE'$, so $E' = a(PP') = E$. Thus, $\mathcal{G}$ is (E, a)-transitive for all $E \mid a$. Moreover, we have shown that

$$\mathrm{Cent}(E, a) \subset \mathrm{Cent}(C, a)\mathrm{Cent}(D, a),$$

so

$$\mathrm{Trans}(a) = \mathrm{Cent}(C, a)\mathrm{Cent}(D, a). \qquad \square$$

If a projective plane $\mathcal{G}$ is (C, a)-transitive with $C \mid a$, we say the flag (C, a) is a **transvection flag**. We also say that a line a is a **transvection line** if $\mathcal{G}$ is (C, a)-transitive for all $C \mid a$, with a dual definition of **transvection point**. Let $Tflag(\mathcal{G})$ be the set of all transvection flags and let $Tgeom(\mathcal{G})$ be the subgeometry of $\mathcal{G}$ consisting of all transvection points and lines. Clearly, Lemma 2.3 shows that any automorphism of $\mathcal{G}$ permutes the transvection flags and is an automorphism of $Tgeom(\mathcal{G})$. Our goal will be to determine the possibilities for $Tgeom(\mathcal{G})$ and $Tflag(\mathcal{G})$.

Lemma 2.11. *Let $\mathcal{G}$ be a projective plane.*

(a) *If (C, a) and (D, b) are transvection flags with $C \neq D \mid a$, then a is a transvection line.*

(b) *If $C \neq D$ are transvection points, then CD is a transvection line and all $E \mid CD$ are transvection points.*

(c) *If P is a transvection point and l is a transvection line with $P \nmid l$, then $Tgeom(\mathcal{G}) = \mathcal{G}$.*

Proof. If $a = b$, then (a) is just Lemma 2.10(c). If $a \neq b$, let $E \in I_a \backslash \{C, D\}$ and let $\varphi \in \mathrm{Cent}(D, b)$ with $\varphi(C) = E$. We see (E, a) is a transvection flag, so a is a transvection line by the first case. In (b), we can apply (a) with $a = CD$ and any $a \neq b \mid D$ to see that CD is a transvection line. Moreover, with E and φ as above, we see $E = \varphi(C)$ is a transvection point. In (c), since $\mathrm{Trans}(l)$ is transitive on points $Q \nmid l$ by Lemma 2.10(c), we see that all $Q \nmid l$ are transvection points. Thus, using (b), we see that all lines are transvection lines and dually all points are transvection points. $\square$

Theorem 2.12. *If $\mathcal{G}$ is a projective plane, then $Tgeom(\mathcal{G})$ is one of the following:*

(a) *the empty set,*

(b) *a single point or a single line,*

(c) *a flag,*

(d) *a single line and all points incident to that line or the dual*[1],

(e) *the entire plane* $\mathcal{G}$.

Proof. We can assume that $Tgeom(\mathcal{G})$ is not one of (a), (b), or (e). Thus, $Tgeom(\mathcal{G})$ contains at least one point, at least one line, and by Lemma 2.11(c), every point and line in $Tgeom(\mathcal{G})$ are incident. If $Tgeom(\mathcal{G})$ has a single point and a single line, then (c) holds. If C, D are distinct transvection points, then Lemma 2.11(b) shows that (d) holds for the single line CD. Similarly, the dual holds if there are distinct transvection lines. □

Theorem 2.13. *Let* $\mathcal{G}$ *be a projective plane. If* $Tgeom(\mathcal{G}) = \emptyset$*, then* $Tflag(\mathcal{G})$ *is one of the following:*

(a) *the empty set,*

(b) *a single flag,*

(c) $I \cap (I_m \times I_Q)$ *for some* $Q \nmid m$.

If $Tgeom(\mathcal{G}) \neq \emptyset$*, then*

$$Tflag(\mathcal{G}) = \{(C, a) \in I : \textit{either } C \textit{ or } a \textit{ is in } Tgeom(\mathcal{G})\}.$$

Proof. Suppose $Tgeom(\mathcal{G}) = \emptyset$ and $Tflag(\mathcal{G})$ is not (a) or (b). Let (C, a) and (D, b) be distinct transvection flags. Since either $C \neq D$ or $a \neq b$, Lemma 2.11(a) and its dual show that $D \nmid a$ and $C \nmid b$. In particular, $a \neq b$ and $C \neq D$. Let $Q = ab$, $m = CD$, and note that $Q \nmid m$. Suppose $(P, l) \in I \cap (I_m \times I_Q)$ with $P \neq C$. Since $\text{Cent}(C, a)$ is transitive on $I_m \backslash \{C\}$, there exists $\varphi \in \text{Cent}(C, a)$ mapping $(D, b) = (D, DQ)$ to $(P, l) = (P, PQ)$. Thus, (P, l) is a transvection flag, so $I \cap (I_m \times I_Q) \subset Tflag(\mathcal{G})$. Conversely, if (R, s) is a transvection flag, let l be a line through Q and R (i.e., $l = QR$ or any line through $Q = R$). Since (ml, l) is a transition flag, we must have

$$(R, s) = (ml, l) \in I \cap (I_m \times I_Q),$$

for otherwise we have seen that $R \nmid l$, a contradiction. Thus, $Tgeom(\mathcal{G}) = \emptyset$ implies (a), (b), or (c).

Now assume that $Tgeom(\mathcal{G}) \neq \emptyset$. Using duality, we can assume that there is a transvection point D. If (C, a) is a transvection flag, then either $C = D \in Tgeom(\mathcal{G})$ or taking $b = CD$, Lemma 2.11(a) shows a is a transvection line. The converse is immediate. □

[1] An algebraic proof in Chapter 4 will show that this option does not actually occur. See Theorem 3.17 and Corollary 4.3.

2.5. Exercises

Exercise 2.1. Let $\mathcal{G}$ be a projective plane with noncollinear points A, B, C such that $\mathcal{G}$ is (P, l)-transitive for $(P, l) = (A, AB)$, (B, AB), (B, BC), (C, BC), and (C, AC), and let T be the group generated by all transvections. Show that

(a) T is transitive on ordered triples of noncollinear points,

(b) T is transitive on flags,

(c) $\mathcal{G}$ is a transvection plane.

Exercise 2.2. Suppose that $\mathcal{G}$ is a projective plane of order 2. Show that $\mathcal{G}$ is a dilatation plane, although the only dilatation is $\phi = \mathrm{Id}$. Show that $\mathcal{G}$ is a transvection plane. (See Exercise 1.13).

Exercise 2.3. A dilatation of order 2 is a **reflection**. Show that if P, Q, R, S are vertices of a quadrangle and σ is a reflection with $\sigma(P) = Q$ and $\sigma(R) = S$, then one of the diagonal points is the center of σ and the other two diagonal points lie on the axis. Show that if every quadrangle in a projective plane has collinear diagonal points, then there are no reflections.

Exercise 2.4. Show that if $l \neq l'$ and $C \nmid l, l'$, then there is at most one reflection σ with center C and $\sigma(l) = l'$. Hint: Use Exercise 2.3 to find the axis of σ.

Exercise 2.5. Show that if σ_1, σ_2 are reflections with center C and axes $a_1 \neq a_2$, then $\tau = \sigma_1\sigma_2$ is a transvection with center C and axis $C(a_1a_2)$. Hint: If $\tau(l) = l$, then $\sigma_1(l) = \sigma_2(l)$. Show that σ_1 is the only reflection with center C and axis a_1 by considering $\sigma_1\sigma_1' = \sigma_1\sigma_2\sigma_2\sigma_1'$.

Exercise 2.6. Suppose that C is a point in a projective plane and that for every $a \nmid C$ there is a reflection with center C and axis a. Show that given $C \nmid l, l'$ and a reflection σ_1 with center C, there is a reflection σ_2 with center C with $\tau(l) = l'$ for the transvection $\tau = \sigma_1\sigma_2$. (Hint: Show that if $l \neq l'$, we may replace l, l' with another pair of lines to assume that $ll' \nmid a_1$, the axis of σ_1. In this case, show that one may form $Q = (C(ll'))a_1$, $R = l\sigma_1(l')$, and $a_2 = QR$ and use $l = (ll')R$.) Show that C is a transvection point.

Exercise 2.7. A projective plane having a reflection with center C and axis a for every $C \nmid a$ is a **reflection plane**. Show that in a reflection plane if $l \neq l'$ and $C \nmid l, l'$, then there is a reflection σ with center C and $\sigma(l) = l'$. Show that a reflection plane is a transvection plane and that some quadrangle has noncollinear diagonal points. The converse is Exercise 5.7.