Introduction

Analytic number theory is mainly devoted to finding approximate counts of number theoretical objects in situations where exact counts are out of reach. Primes, divisors, solutions of Diophantine equations, lattice points within contours, partitions of integers and ideal classes of algebraic number fields are some of the objects that have been counted. The prototypical approximate count in number theory is the Prime Number Theorem (PNT), stating that

$$\lim_{x\to+\infty} \frac{\pi(x)}{\int_2^x \frac{du}{\log(u)}} = 1$$

where $\pi(x)$ is the number of primes $p \leq x$. This was proved independently in 1896 by Jacques Hadamard and Charles de la Vallée Poussin, building on ideas of Bernhard Riemann, and applying complex analysis to the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

to establish the result. An asymptotic count like the PNT usually attracts attention with a view to improve it. As the distribution of prime numbers is one of the central topics in number theory, much effort has been expended to obtain improvements to the Prime Number Theorem. We shall prove one of the weaker ones, to the effect that there exist positive constants c, C, x_0 such that

$$\left|\pi(x) - \int_2^x \frac{du}{\log(u)}\right| \leq Cxe^{-c\log^{1/10}(x)} \quad \text{for} \quad x \geq x_0.$$

This is more precise, though also more complicated to state, than the asymptotic form of the PNT.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$d(n)$	1	2	2	3	2	4	2	4	3	4	2	6	2	4	4

Table 1. Values of the divisor function

Counting the number of divisors of positive integers leads to a difficult problem known as the Dirichlet Divisor Problem that is still unsolved today. Denoting the number of divisors of n by $d(n)$, Table 1 shows that these counts fluctuate a good deal. But much more regular behavior is revealed by averaging $d(n)$, and in fact

$$\frac{1}{x}\sum_{n\leq x} d(n) \approx \log(x) + 2\gamma - 1, \qquad \gamma = 0.5772\ldots,$$

with an absolute error that tends to zero as $x \to +\infty$. To determine how fast the error tends to zero is the divisor problem of Dirichlet.

Divisors and primes are the stuff of multiplicative number theory. But there are also interesting counting problems connected with additive questions. The eighteenth-century English algebraist Edward Waring stated that every positive integer may be expressed as a sum of a limited number of k-th powers of nonnegative integers, the number required depending on k only. We shall count the number of such representations for large integers when the number of powers allowed is sufficiently large, finding an asymptotic formula by means of the Circle Method and establishing Waring's claim. This was first achieved by David Hilbert by a method different from the one used here. The proof is the most elaborate in the book, though the prerequisites are surprisingly modest. The Circle Method is related to Fourier theory, but involves only Fourier series with finitely many terms so convergence issues do not arise.

The achievements of analytic number theory are not entirely limited to approximate counts. Some of the quantities estimated are not counting numbers, and for a few problems exact rather than approximate results have been attained. We shall cover one such case from algebraic number theory, that of the analytic class number formula

$$h_K = \frac{w_K|d_K|^{1/2}}{2^{r_1(K)+r_2(K)}\pi^{r_2(K)}R_K} \lim_{s\to 1}\frac{\zeta_K(s)}{\zeta(s)}$$

that expresses the number h_K of ideal classes of the ring of algebraic integers of a number field K in terms of other arithmetic data. This formula is due to Dirichlet and Richard Dedekind.