
Preface

Tropical geometry is an exciting new field at the interface between algebraic geometry and combinatorics with connections to many other areas. At its heart it is geometry over the tropical semiring, which is $\mathbb{R} \cup \{\infty\}$ with the usual operations of addition and multiplication replaced by minimum and addition, respectively. This turns polynomials into piecewise-linear functions and replaces an algebraic variety by an object from polyhedral geometry, which can be regarded as a “combinatorial shadow” of the original variety.

In this book we introduce this subject at a level that is accessible to beginners. Tropical geometry has become a large field, and only a small selection of topics can be covered in a first course. We focus on the study of tropical varieties that arise from classical algebraic varieties. Methods from commutative algebra and polyhedral geometry are central to our approach. This necessarily means that many important topics are left out. These include the systematic development of tropical geometry as an intrinsic geometry in its own right, connections to enumerative and real algebraic geometry, connections to mirror symmetry, connections to Berkovich spaces and abstract curves, and the more applied aspects of max-plus algebra. Luckily most of these topics are covered in other recent or forthcoming books, such as [BCOQ92], [But10], [Gro11], [IMS07], [Jos], [MR], and [PS05].

Prerequisites. This text is intended to be suitable for a class on tropical geometry for first-year graduate students in mathematics. We have attempted to make the material accessible to readers with a minimal background in algebraic geometry, at the level of the undergraduate text book *Ideals, Varieties, and Algorithms* by Cox, Little, and O’Shea [CLO07].

Essential prerequisites for this book are mastery of linear algebra and the material on rings and fields in a first course in abstract algebra. Since tropical geometry draws on many fields of mathematics, some additional background in geometry, topology, or number theory will be beneficial.

Polyhedra and polytopes play a fundamental role in tropical geometry, and some prior exposure to convexity and polyhedral combinatorics may help. For that we recommend Ziegler's book *Lectures on Polytopes* [Zie95].

Chapter 1 offers a friendly welcome to our readers. It has no specific prerequisites and is meant to be enjoyable for all. The first three sections in Chapter 2 cover background material in abstract algebra, algebraic geometry, and polyhedral geometry. Enough definitions and examples are given that an enthusiastic reader can fill in any gaps. All students (and their teachers) are strongly urged to explore the exercises for Chapters 1 and 2.

Some of the results and their proofs will demand more mathematical maturity and expertise. Chapters 2 and 3 require some commutative algebra. Combinatorics and multilinear algebra will be useful for studying Chapters 4 and 5. Chapter 6 assumes familiarity with modern algebraic geometry.

Overview. We begin by relearning the arithmetic operations of addition and multiplication. The rest of Chapter 1 offers tapas that can be enjoyed in any order. They show a glimpse of the past, present, and future of tropical geometry and serve as an introduction to the more formal contents of this book. In Chapter 2, the first half covers background material, while the second half develops a version of Gröbner basis theory suitable for algebraic varieties over a field with valuation. The highlights are the construction of the Gröbner complex and the resulting finiteness of tropical bases.

Chapter 3 is the heart of the book. The two main results are the Fundamental Theorem 3.2.3, which characterizes tropical varieties in seemingly different ways, and the Structure Theorem 3.3.5, which says that they are connected balanced polyhedral complexes of the correct dimension. Stable intersections of tropical varieties reveal a hint of intersection theory.

Tropical linear spaces and their parameter spaces, the Grassmannian and the Dressian, appear in Chapter 4. Matroid theory plays a foundational role. Our discussion of complete intersections includes mixed volumes of Newton polytopes and a tropical proof of Bernstein's Theorem for n equations in n variables. We also study the combinatorics of surfaces in 3-space.

Chapter 5 covers spectral theory for tropical matrices, tropical convexity, and determinantal varieties. It also showcases computations with Bergman fans of matroids and other linear spaces. Chapter 6 concerns the connection between tropical varieties and toric varieties. It introduces the tropical approach to degenerations, compactifications, and enumerative geometry.

Teaching possibilities. A one-semester graduate course could be based on Chapters 2 and 3, plus selected topics from the other chapters. One possibility is to start with two or three weeks of motivating examples selected from Chapter 1 before moving on to Chapters 2 and 3. A course for more advanced graduate students could start with Gröbner bases as presented in the second half of Chapter 2, cover Chapter 3 with proofs, and end with a sampling of topics from the later chapters. Students with an interest in combinatorics and computation might gravitate toward Chapters 4 and 5. An advanced course for students specializing in algebraic geometry would focus on Chapters 3 and 6. Covering the entire book would require a full academic year or an exceptionally well-prepared group of participants.

We have attempted to keep the prerequisites low enough to make parts of the book appropriate for self-study by a final-year undergraduate. The sections in Chapter 2 could serve as first introductions to their subject areas. A simple route through Chapter 3 is to focus in detail on the hypersurface case, and to discuss the Fundamental Theorem and Structure Theorem without proofs. The exercises suggest many possibilities for senior thesis projects.

Acknowledgments. We have drawn on the rich and ever-growing literature in tropical geometry when preparing this book. While most direct sources are mentioned, the bibliography is by no means complete. We thank the authors whose work we have drawn on for their inspiration and apologize for any omissions. Readers are encouraged to search the keywords of this book and the MSC code 14T05 to explore this beautiful subject, including the topics missing from this book.

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DIANE MACLAGAN
Mathematics Institute
University of Warwick
Coventry, CV4 7AL
United Kingdom

BERND STURMFELS
Department of Mathematics
University of California
Berkeley, CA 94720
United States