Chapter 1

Tropical Islands

In tropical algebra, the sum of two numbers is their minimum and the product of two numbers is their sum. This algebraic structure is known as the *tropical semiring* or as the min-plus algebra. With minimum replaced by maximum we get the isomorphic max-plus algebra. The adjective "tropical" was coined by French mathematicians, notably Jean-Eric Pin [**Pin98**], to honor their Brazilian colleague Imre Simon [**Sim88**], who pioneered the use of min-plus algebra in optimization theory. There is no deeper meaning to the adjective "tropical". It simply stands for the French view of Brazil.

The origins of algebraic geometry lie in the study of zero sets of systems of multivariate polynomials. These objects are algebraic varieties, and they include familiar examples such as plane curves and surfaces in three-dimensional space. It makes perfect sense to define polynomials and rational functions over the tropical semiring. These functions are piecewise linear. Algebraic varieties can also be defined in the tropical setting. They are now subsets of $\mathbb{R}^n$ that are composed of convex polyhedra. Thus tropical algebraic geometry is a piecewise-linear version of algebraic geometry.

This chapter serves as a friendly welcome to tropical mathematics. We present the basic concepts concerning the tropical semiring, we discuss some of the historical origins of tropical geometry, and we show by way of elementary examples how tropical methods can be used to solve problems in algebra, geometry, and combinatorics. Proofs, precise definitions, and the general theory will be postponed to later chapters. Our primary objective here is to show the reader that the tropical approach is both useful and fun.

The chapter title stands for our view of a day at the beach. The sections are disconnected but island hopping between them should be quick and easy.

1.1. Arithmetic

Our basic object of study is the *tropical semiring* $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$. As a set this is just the real numbers $\mathbb{R}$, together with an extra element ∞ which represents infinity. In this semiring, the basic arithmetic operations of addition and multiplication of real numbers are redefined as follows:

$$x \oplus y \;:=\; \min(x,y) \qquad \text{and} \qquad x \odot y \;:=\; x+y.$$

In words, the *tropical sum* of two numbers is their minimum, and the *tropical product* of two numbers is their usual sum. Here are some examples of how to do arithmetic in this exotic number system. The tropical sum of 4 and 9 is 4. The tropical product of 4 and 9 equals 13. We write this as follows:

$$4 \oplus 9 \;=\; 4 \qquad \text{and} \qquad 4 \odot 9 \;=\; 13.$$

Many of the familiar axioms of arithmetic remain valid in tropical mathematics. For instance, both addition and multiplication are *commutative*:

$$x \oplus y \;=\; y \oplus x \qquad \text{and} \qquad x \odot y \;=\; y \odot x.$$

These two arithmetic operations are also associative, and the times operator $\odot$ takes precedence when plus $\oplus$ and times $\odot$ occur in the same expression.

The *distributive law* holds for tropical addition and multiplication:

$$x \odot (y \oplus z) \;=\; x \odot y \;\oplus\; x \odot z.$$

Here is a numerical example to show distributivity:

$$\begin{aligned} 3 \odot (7 \oplus 11) &= 3 \odot 7 = 10, \\ 3 \odot 7 \;\oplus\; 3 \odot 11 &= 10 \oplus 14 = 10. \end{aligned}$$

Both arithmetic operations have an identity element. Infinity is the *identity element* for addition and zero is the *identity element* for multiplication:

$$x \oplus \infty = x \qquad \text{and} \qquad x \odot 0 = x.$$

We also note the following identities involving the two identity elements:

$$x \odot \infty = \infty \qquad \text{and} \qquad x \oplus 0 = \begin{cases} 0 & \text{if } x \geq 0, \\ x & \text{if } x < 0. \end{cases}$$

Elementary school students prefer tropical arithmetic because the multiplication table is easier to memorize, and even long division becomes easy.

Here is a tropical *addition table* and a tropical *multiplication table*:

$\oplus$	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2
3	1	2	3	3	3	3	3
4	1	2	3	4	4	4	4
5	1	2	3	4	5	5	5
6	1	2	3	4	5	6	6
7	1	2	3	4	5	6	7

$\odot$	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10
4	5	6	7	8	9	10	11
5	6	7	8	9	10	11	12
6	7	8	9	10	11	12	13
7	8	9	10	11	12	13	14

An essential feature of tropical arithmetic is that there is no subtraction. There is no real number x that we can call "13 minus 4" because the equation $4 \oplus x = 13$ has no solution x at all. Tropical division is defined to be classical subtraction, so $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ satisfies all ring axioms (and indeed field axioms) except for the existence of an additive inverse. Such algebraic structures are called *semirings*, whence the name tropical semiring.

It is extremely important to remember that "0" is the multiplicative identity element. For instance, the tropical *Pascal's triangle* looks like this:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 0 & & 0 & & 0 & & 0 \\ \cdots & \cdots & & \cdots & & \cdots & & \cdots & \cdots \end{array}$$

The rows of Pascal's triangle are the coefficients appearing in the *Binomial Theorem*. For instance, the third row in the triangle represents the identity

$$\begin{aligned} (x \oplus y)^3 &= (x \oplus y) \odot (x \oplus y) \odot (x \oplus y) \\ &= 0 \odot x^3 \ \oplus \ 0 \odot x^2 y \ \oplus \ 0 \odot x y^2 \ \oplus \ 0 \odot y^3. \end{aligned}$$

Of course, the zero coefficients can be dropped in this identity:

$$(x \oplus y)^3 \quad = \quad x^3 \ \oplus \ x^2 y \ \oplus \ x y^2 \ \oplus \ y^3.$$

Moreover, the *Freshman's Dream* holds for all powers in tropical arithmetic:

$$(x \oplus y)^3 \quad = \quad x^3 \ \oplus \ y^3.$$

The validity of the three displayed identities is easily verified by noting that the following equations hold in classical arithmetic for all $x, y \in \mathbb{R}$:

$$3 \cdot \min\{x, y\} \quad = \quad \min\{3x, 2x + y, x + 2y, 3y\} \quad = \quad \min\{3x, 3y\}.$$

The linear algebra operations of adding and multiplying vectors and matrices make sense over the tropical semiring. For instance, the tropical scalar product in $\mathbb{R}^3$ of a row vector with a column vector is the scalar

$$\begin{aligned}(u_1, u_2, u_3) \odot (v_1, v_2, v_3)^{\mathrm{T}} &= u_1 \odot v_1 \;\oplus\; u_2 \odot v_2 \;\oplus\; u_3 \odot v_3 \\ &= \min\{u_1 + v_1,\, u_2 + v_2,\, u_3 + v_3\}.\end{aligned}$$

Here is the product of a column vector and a row vector of length 3:

$$\begin{aligned}&(u_1, u_2, u_3)^T \odot (v_1, v_2, v_3) \\ &\quad = \begin{pmatrix} u_1 \odot v_1 & u_1 \odot v_2 & u_1 \odot v_3 \\ u_2 \odot v_1 & u_2 \odot v_2 & u_2 \odot v_3 \\ u_3 \odot v_1 & u_3 \odot v_2 & u_3 \odot v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 & u_1 + v_2 & u_1 + v_3 \\ u_2 + v_1 & u_2 + v_2 & u_2 + v_3 \\ u_3 + v_1 & u_3 + v_2 & u_3 + v_3 \end{pmatrix}.\end{aligned}$$

Any matrix which can be expressed as such a product has *tropical rank* 1. See Section 5.3 for three different definitions of the rank of a tropical matrix.

Here are a few more examples of arithmetic with vectors and matrices:

$$2 \odot (3, -7, 6) \;=\; (5, -5, 8)\;, \quad (\infty, 0, 1) \odot (0, 1, \infty)^T \;=\; 1\,,$$

$$\begin{pmatrix} 3 & 3 \\ 0 & 7 \end{pmatrix} \oplus \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}, \text{ and } \begin{pmatrix} 3 & 3 \\ 0 & 7 \end{pmatrix} \odot \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 4 & 1 \end{pmatrix}.$$

Given a $d \times n$-matrix A, we might be interested in computing its image $\{A \odot \mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$ and in solving the linear systems $A \odot \mathbf{x} = \mathbf{b}$ for various right-hand sides $\mathbf{b}$. We will discuss the relevant geometry in Section 5.2. For an introduction to tropical linear systems and their applications we recommend the books *Synchronization and Linearity* by Baccelli, Cohen, Olsder, and Quadrat [**BCOQ92**] and *Max-linear Systems* by Butkovič [**But10**].

Students of computer science and discrete mathematics may encounter tropical matrix multiplication in algorithms for shortest paths in graphs and networks. The general framework for such algorithms is known as *dynamic programming*. We shall explore this connection in the next section.

Let $x_1, x_2, \ldots, x_n$ be variables which represent elements in the tropical semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$. A *monomial* is any product of these variables, where repetition is allowed. Throughout this book, we generally allow negative integer exponents. By commutativity, we can sort the product and write monomials in the usual notation, with the variables raised to exponents:

$$x_2 \odot x_1 \odot x_3 \odot x_1 \odot x_4 \odot x_2 \odot x_3 \odot x_2 \quad = \quad x_1^2 x_2^3 x_3^2 x_4.$$

A monomial represents a function from $\mathbb{R}^n$ to $\mathbb{R}$. When evaluating this function in classical arithmetic, what we get is a linear function:

$$x_2 + x_1 + x_3 + x_1 + x_4 + x_2 + x_3 + x_2 \quad = \quad 2x_1 + 3x_2 + 2x_3 + x_4.$$

Remark 1.1.1. Every linear function with integer coefficients arises in this way, so tropical monomials are linear functions with integer coefficients.

A *tropical polynomial* is a finite linear combination of tropical monomials:

$$p(x_1,\ldots,x_n) \quad = \quad a\odot x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} \;\oplus\; b\odot x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n} \;\oplus\; \cdots .$$

Here the coefficients $a, b, \ldots$ are real numbers and the exponents $i_1, j_1, \ldots$ are integers. Every tropical polynomial represents a function $\mathbb{R}^n \to \mathbb{R}$. When evaluating this function in classical arithmetic, what we get is the minimum of a finite collection of linear functions, namely

$$p(x_1,\ldots,x_n) = \min\bigl(a + i_1x_1 + \cdots + i_nx_n\,,\; b + j_1x_1 + \cdots + j_nx_n\,,\; \ldots\bigr).$$

This function $p : \mathbb{R}^n \to \mathbb{R}$ has the following three important properties:

- p is continuous,
- p is piecewise linear with a finite number of pieces, and
- p is concave: $p\bigl(\frac{1}{2}(\mathbf{x}+\mathbf{y})\bigr) \geq \frac{1}{2}\bigl(p(\mathbf{x}) + p(\mathbf{y})\bigr)$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Every function which satisfies these three properties can be represented as the minimum of a finite set of linear functions; see Exercise 1.9(4). We conclude:

Lemma 1.1.2. *The tropical polynomials in n variables $x_1, \ldots, x_n$ are precisely the piecewise-linear concave functions on $\mathbb{R}^n$ with integer coefficients.*

It is instructive to examine tropical polynomials and the functions they define even for polynomials in one variable. For instance, consider the general cubic polynomial in one variable x:

$$p(x) \quad = \quad a\odot x^3 \;\oplus\; b\odot x^2 \;\oplus\; c\odot x \;\oplus\; d. \tag{1.1.1}$$

To graph this function we draw four lines in the (x, y) plane: $y = 3x + a$, $y = 2x + b$, $y = x + c$, and the horizontal line $y = d$. The value of $p(x)$ is the smallest y-value such that (x, y) is on one of these four lines; the graph of $p(x)$ is the lower envelope of the lines. All four lines actually contribute if

$$b - a \;\leq\; c - b \;\leq d - c. \tag{1.1.2}$$

These three values of x are the breakpoints where $p(x)$ fails to be linear, and the cubic has a corresponding factorization into three linear factors:

$$p(x) \quad = \quad a\odot(x\oplus(b-a))\odot(x\oplus(c-b))\odot(x\oplus(d-c)).$$

The three breakpoints (1.1.2) of the graph are called the *roots* of the cubic polynomial $p(x)$. The graph and its breakpoints are shown in Figure 1.1.1.

Every tropical polynomial function can be written uniquely as a tropical product of tropical linear functions; the *Fundamental Theorem of Algebra* holds tropically (see Exercise 1.9(2)). In this statement we must underline the word "function". Distinct polynomials can represent the same function $p\colon \mathbb{R} \to \mathbb{R}$. We are not claiming that every polynomial factors into linear functions. What we are claiming is that every polynomial can be replaced

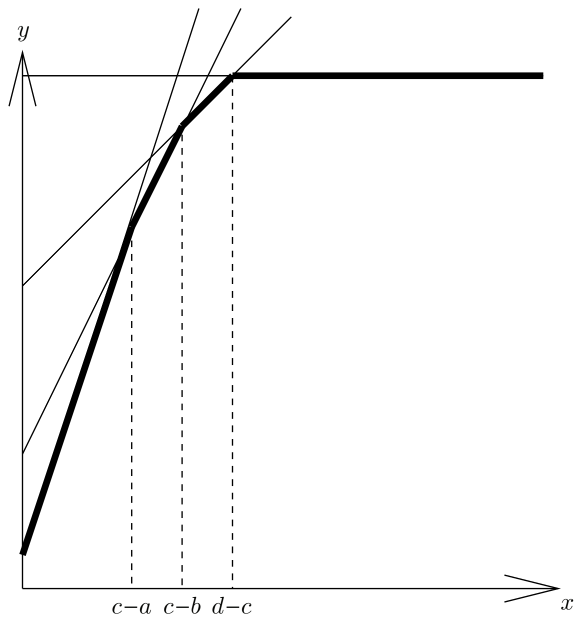

Figure 1.1.1. The graph of a cubic polynomial and its roots.

by an equivalent polynomial, representing the same function, that can be factored into linear factors. Here is an example of a quadratic polynomial function and its unique factorization into linear polynomial functions:

$$x^2 \oplus 17 \odot x \oplus 2 \quad = \quad x^2 \oplus 1 \odot x \oplus 2 \quad = \quad (x \oplus 1)^2.$$

Unique factorization of tropical polynomials holds in one variable, but it no longer holds in two or more variables. What follows is a simple example of a bivariate polynomial that has two distinct irreducible factorizations:

$$(x \oplus 0) \odot (y \oplus 0) \odot (x \odot y \oplus 0) \\ = (x \odot y \oplus x \oplus 0) \odot (x \odot y \oplus y \oplus 0). \tag{1.1.3}$$

Here is a geometric way of interpreting this identity.

Definition 1.1.3. Let $f(x, y)$ be a polynomial in two variables, in either classical or tropical arithmetic. Its *Newton polygon* $\mathrm{Newt}(f)$ is defined as the convex hull in $\mathbb{R}^2$ of all points (i, j) such that $x^i y^j$ appears in the expansion of $f(x, y)$. For more information see Definition 2.3.4 and Figure 2.3.5.

The Newton polygon of the polynomial in (1.1.3) is a hexagon. The identity means that the hexagon is the Minkowski sum of three line segments and also the Minkowski sum of two triangles. We refer to (2.3.1) and (2.3.3) for precise definitions of the relevant concepts in arbitrary dimensions.

1.2. Dynamic Programming

To see why tropical arithmetic might be relevant for computer science, let us consider the problem of finding shortest paths in a weighted directed graph. We fix a directed graph G with n nodes that are labeled by $1, 2, \ldots, n$. Every directed edge (i, j) in G has an associated length d_{ij} which is a nonnegative real number. If (i, j) is not an edge of G, then we set $d_{ij} = +\infty$.

We represent the weighted directed graph G by its $n \times n$-adjacency matrix $D_G = \big(d_{ij}\big)$ whose off-diagonal entries are the edge lengths d_{ij}. The diagonal entries of D_G are zero: $d_{ii} = 0$ for all i. The matrix D_G need not be symmetric; it may well happen that $d_{ij} \neq d_{ji}$ for some i, j. However, if G is an undirected graph with edge lengths, then we can represent G as a directed graph with two directed edges (i, j) and (j, i) for each undirected edge $\{i, j\}$. In that special case, D_G is a symmetric matrix, and we can think of $d_{ij} = d_{ji}$ as the distance between node i and node j. For a general directed graph G, the adjacency matrix D_G will not be symmetric.

Consider the $n \times n$-matrix with entries in $\mathbb{R}_{\geq 0} \cup \{\infty\}$ that results from tropically multiplying the given adjacency matrix D_G with itself $n-1$ times:

$$D_G^{\odot(n-1)} \quad = \quad D_G \odot D_G \odot \cdots \odot D_G. \tag{1.2.1}$$

Proposition 1.2.1. *Let G be a weighted directed graph on n nodes with $n \times n$-adjacency matrix D_G. The entry of the matrix $D_G^{\odot(n-1)}$ in row i and column j equals the length of a shortest path from node i to node j in G.*

Proof. Let $d_{ij}^{(r)}$ denote the minimum length of any path from node i to node j which uses at most r edges in G. We have $d_{ij}^{(1)} = d_{ij}$ for any two nodes i and j. Since the edge weights d_{ij} were assumed to be nonnegative, a shortest path from node i to node j visits each node of G at most once. In particular, any shortest path in the directed graph G uses at most $n-1$ directed edges. Hence the length of a shortest path from i to j equals $d_{ij}^{(n-1)}$.

For $r \geq 2$ we have a recursive formula for the length of a shortest path:

$$d_{ij}^{(r)} \quad = \quad \min\big\{d_{ik}^{(r-1)} + d_{kj} \,:\, k = 1, 2, \ldots, n\big\}. \tag{1.2.2}$$

Using tropical arithmetic, this formula can be rewritten as follows:

$$\begin{aligned} d_{ij}^{(r)} \quad &= \quad d_{i1}^{(r-1)} \odot d_{1j} \;\oplus\; d_{i2}^{(r-1)} \odot d_{2j} \;\oplus\; \cdots \;\oplus\; d_{in}^{(r-1)} \odot d_{nj} \\ &= \quad (d_{i1}^{(r-1)}, d_{i2}^{(r-1)}, \ldots, d_{in}^{(r-1)}) \odot (d_{1j}, d_{2j}, \ldots, d_{nj})^T. \end{aligned}$$

From this it follows, by induction on r, that $d_{ij}^{(r)}$ coincides with the entry in row i and column j of the $n \times n$-matrix $D_G^{\odot r}$. Indeed, the right-hand side of the recursive formula is the tropical product of row i of $D_G^{\odot(r-1)}$ and column

j of D_G, which is the (i,j) entry of $D_G^{\odot r}$. In particular, $d_{ij}^{(n-1)}$ coincides with the entry in row i and column j of $D_G^{\odot(n-1)}$. This proves the claim. □

The iterative evaluation of the formula (1.2.2) is *Floyd's algorithm* for finding shortest paths in a weighted digraph. This algorithm and its running time are featured in undergraduate textbooks on discrete mathematics. For us, running that algorithm means performing the matrix multiplication

$$D_G^{\odot r} \quad = \quad D_G^{\odot(r-1)} \odot D_G \qquad \text{for } r = 2, \ldots, n-1.$$

Example 1.2.2. Let G be the weighted directed graph on $n = 4$ nodes, with no loops, that is defined by the adjacency matrix

$$D_G \quad = \quad \begin{pmatrix} 0 & 1 & 3 & 7 \\ 2 & 0 & 1 & 3 \\ 4 & 5 & 0 & 1 \\ 6 & 3 & 1 & 0 \end{pmatrix}.$$

The first and second tropical power of this matrix are

$$D_G^{\odot 2} \quad = \quad \begin{pmatrix} 0 & 1 & 2 & 4 \\ 2 & 0 & 1 & 2 \\ 4 & 4 & 0 & 1 \\ 5 & 3 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad D_G^{\odot 3} \quad = \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 1 & 2 \\ 4 & 4 & 0 & 1 \\ 5 & 3 & 1 & 0 \end{pmatrix}.$$

The entries in $D_G^{\odot 3}$ are the lengths of the shortest paths in the digraph G.

The tropical computation mirrors the following matrix computation in ordinary arithmetic. Let ϵ denote an indeterminate that represents a very small positive real number, and let $A_G(\epsilon)$ be the $n \times n$-matrix whose entry in row i and column j is the monomial $\epsilon^{d_{ij}}$. In our example we have

$$A_G(\epsilon) \quad = \quad \begin{pmatrix} 1 & \epsilon^1 & \epsilon^3 & \epsilon^7 \\ \epsilon^2 & 1 & \epsilon^1 & \epsilon^3 \\ \epsilon^4 & \epsilon^5 & 1 & \epsilon^1 \\ \epsilon^6 & \epsilon^3 & \epsilon^1 & 1 \end{pmatrix}.$$

Now, we compute the third power of this matrix in ordinary arithmetic:

$$A_G(\epsilon)^3 = \begin{pmatrix} 1 + 3\epsilon^3 + \cdots & 3\epsilon + \epsilon^4 + \cdots & 3\epsilon^2 + 3\epsilon^3 + \cdots & \epsilon^3 + 6\epsilon^4 + \cdots \\ 3\epsilon^2 + 4\epsilon^5 + \cdots & 1 + 3\epsilon^3 + \cdots & 3\epsilon + \epsilon^3 + \cdots & 3\epsilon^2 + 3\epsilon^3 + \cdots \\ 3\epsilon^4 + 2\epsilon^6 + \cdots & 3\epsilon^4 + 6\epsilon^5 + \cdots & 1 + 3\epsilon^2 + \cdots & 3\epsilon + \epsilon^3 + \cdots \\ 6\epsilon^5 + 3\epsilon^6 + \cdots & 3\epsilon^3 + \epsilon^5 + \cdots & 3\epsilon + \epsilon^3 + \cdots & 1 + 3\epsilon^2 + \cdots \end{pmatrix}.$$

The entry of the classical matrix power $A_G(\epsilon)^3$ in row i and column j is a polynomial in ϵ which represents the lengths of all paths from node i to node j using at most three edges. The lowest exponent appearing in this polynomial is the (i,j)-entry in the tropical matrix power $D_G^{\odot 3}$. ◊

This is a general phenomenon, summarized informally as follows:

$$\text{tropical} \quad = \quad \lim_{\epsilon \to 0} \log_\epsilon\big(\text{classical}(\epsilon)\big). \tag{1.2.3}$$

This process of passing from classical arithmetic to tropical arithmetic is referred to as *tropicalization*. Equation (1.2.3) is not a mathematical statement. To make this rigorous we use the algebraic notion of *valuations* which will be developed in our introductory discussion of fields in Section 2.1.

We shall give two more examples of how tropical arithmetic ties in naturally with algorithms in discrete mathematics. The first example concerns the dynamic programming approach to *integer linear programming*. The integer linear programming problem can be stated as follows. Let $A = (a_{ij})$ be a $d \times n$-matrix of nonnegative integers, let $\mathbf{w} = (w_1, \ldots, w_n)$ be a row vector with real entries, and let $\mathbf{b} = (b_1, \ldots, b_d)$ be a column vector with nonnegative integer entries. Our task is to find a nonnegative integer column vector $\mathbf{u} = (u_1, \ldots, u_n)$ which solves the following optimization problem:

$$\text{Minimize } \mathbf{w} \cdot \mathbf{u} \text{ subject to } \mathbf{u} \in \mathbb{N}^n \text{ and } A\mathbf{u} = \mathbf{b}. \tag{1.2.4}$$

Let us assume that all columns of the matrix A sum to the same number α and that $b_1 + \cdots + b_d = m\alpha$. This assumption is convenient because it ensures that all feasible solutions $\mathbf{u} \in \mathbb{N}^n$ of (1.2.4) satisfy $u_1 + \cdots + u_n = m$.

We can solve the integer programming problem (1.2.4) using tropical arithmetic as follows. Let $x_1, \ldots, x_d$ be variables and consider the expression

$$w_1 \odot x_1^{a_{11}} \odot x_2^{a_{21}} \odot \cdots \odot x_d^{a_{d1}} \;\oplus\; \cdots \;\oplus\; w_n \odot x_1^{a_{1n}} \odot x_2^{a_{2n}} \odot \cdots \odot x_d^{a_{dn}}. \tag{1.2.5}$$

Proposition 1.2.3. *The optimal value of* (1.2.4) *is the coefficient of the monomial* $x_1^{b_1} x_2^{b_2} \cdots x_d^{b_d}$ *in the* m*th power of the tropical polynomial* (1.2.5).

The proof of this proposition is not difficult and is similar to that of Proposition 1.2.1. The process of taking the mth power of the tropical polynomial (1.2.5) can be regarded as solving the shortest path problem in a certain graph. This is the dynamic programming approach to (1.2.4). This approach furnishes a polynomial-time algorithm for integer programming in fixed dimension under the assumption that the integers in A are bounded.

Example 1.2.4. Let $d = 2$, let $n = 5$, and consider the instance of (1.2.4) with

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}, \quad \text{and} \quad \mathbf{w} = (2, 5, 11, 7, 3).$$

Here we have $\alpha = 4$ and $m = 3$. The matrix A and the cost vector $\mathbf{w}$ are encoded by a tropical polynomial as in (1.2.5):

$$p \quad = \quad 2x_1^4 \oplus 5x_1^3 x_2 \oplus 11 x_1^2 x_2^2 \oplus 7x_1 x_2^3 \oplus 3x_2^4.$$

The third power of this polynomial, evaluated tropically, is equal to

$$p \odot p \odot p \;=\; 6x_1^{12} \oplus 9x_1^{11}x_2 \oplus 12x_1^{10}x_2^2 \oplus 11x_1^9x_2^3 \oplus 7x_1^8x_2^4 \oplus 10x_1^7x_2^5 \oplus 13x_1^6x_2^6 \oplus 12x_1^5x_2^7 \oplus 8x_1^4x_2^8 \oplus 11x_1^3x_2^9 \oplus 17x_1^2x_2^{10} \oplus 13x_1x_2^{11} \oplus 9x_2^{12}.$$

The coefficient 12 of $x_1^5x_2^7$ in $p \odot p \odot p$ is the optimal value. An optimal solution to this integer programming problem is $\mathbf{u} = (1,0,0,1,1)^{\mathrm{T}}$. ♢

Our final example concerns the notion of the determinant of an $n \times n$-matrix $X = (x_{ij})$. As there is no negation in tropical arithmetic, the *tropical determinant* is the same as the *tropical permanent*, namely, the sum over the diagonal products obtained by taking all $n!$ permutations π of $\{1,2,\ldots,n\}$:

$$\text{(1.2.6)} \qquad \operatorname{trop}\det(X) \;:=\; \bigoplus_{\pi \in S_n} x_{1\pi(1)} \odot x_{2\pi(2)} \odot \cdots \odot x_{n\pi(n)}.$$

Here S_n is the *symmetric group* of permutations of $\{1,2,\ldots,n\}$. Evaluating the tropical determinant means solving the classical *assignment problem* of combinatorial optimization. Imagine a company that has n jobs and n workers, and each job needs to be assigned to exactly one of the workers. Let x_{ij} be the cost of assigning job i to worker j. The company wishes to find the cheapest assignment $\pi \in S_n$. The optimal total cost is the minimum:

$$\min\bigl\{x_{1\pi(1)} + x_{2\pi(2)} + \cdots + x_{n\pi(n)} \;:\; \pi \in S_n\bigr\}.$$

This number is precisely the tropical determinant of the matrix $Q = (x_{ij})$:

Remark 1.2.5. The tropical determinant solves the assignment problem.

In the assignment problem we seek the minimum over $n!$ quantities. This appears to require exponentially many operations. However, there is a well-known polynomial-time algorithm for solving this problem. It was developed by Harold Kuhn in 1955 who called it the *Hungarian Assignment Method* [**Kuh55**]. This algorithm maintains a price for each job and an (incomplete) assignment of workers and jobs. At each iteration, an unassigned worker is chosen and a shortest augmenting path from this person to the set of jobs is chosen. The total number of arithmetic operations is $O(n^3)$.

In classical arithmetic, the evaluation of determinants and the evaluation of permanents are in different complexity classes. The determinant of an $n \times n$-matrix can be computed in $O(n^3)$ steps, namely by *Gaussian elimination*, while computing the permanent of an $n \times n$-matrix is a fundamentally harder problem. A famous theorem due to Leslie Valiant says that computing the (classical) permanent is *#P-complete*. In tropical arithmetic, computing the permanent is easier, thanks to the Hungarian Assignment Method. We can think of that method as a certain tropicalization of Gaussian elimination.

For an example, consider a 3×3-matrix $A(\epsilon)$ whose entries are polynomials in the unknown ϵ. For each entry we list the term of lowest order:

$$A(\epsilon) \quad = \quad \begin{pmatrix} a_{11}\epsilon^{x_{11}} + \cdots & a_{12}\epsilon^{x_{12}} + \cdots & a_{13}\epsilon^{x_{13}} + \cdots \\ a_{21}\epsilon^{x_{21}} + \cdots & a_{22}\epsilon^{x_{22}} + \cdots & a_{23}\epsilon^{x_{23}} + \cdots \\ a_{31}\epsilon^{x_{31}} + \cdots & a_{32}\epsilon^{x_{32}} + \cdots & a_{33}\epsilon^{x_{33}} + \cdots \end{pmatrix}.$$

Suppose that the a_{ij} are sufficiently general integers, so that no cancellation occurs in the lowest-order coefficient when we expand the determinant of $A(\epsilon)$. Writing X for the 3×3-matrix with entries x_{ij}, we have

$$\det(A(\epsilon)) \quad = \quad \alpha \cdot \epsilon^{\mathrm{trop}\det(X)} + \cdots \qquad \text{for some } \alpha \in \mathbb{R}\backslash\{0\}.$$

Thus the tropical determinant of X can be computed from this expression by taking the logarithm and letting ϵ tend to zero, as suggested by (1.2.3).

The material in this section is closely related to Chapter 2 in the book *Algebraic Statistics for Computational Biology* by Lior Pachter and Bernd Sturmfels [**PS05**]. The connection to computational biology arises because many algorithms in that field (e.g., for sequence alignment and gene prediction) are based on dynamic programming. These algorithms can be interpreted as the evaluation of a tropical polynomial. The book [**PS05**] and the paper [**PS04**] that preceded it argue that the tropical interpretation of dynamic programming algorithms is useful for statistical inference.

Readers who enjoyed this section might like to take a peek at Section 5.1. That section concerns the eigenvalue and eigenvectors of a square matrix.

1.3. Plane Curves

A tropical polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ is given as the minimum of a finite set of linear functions. We define the *hypersurface* $V(p)$ of p to be the set of all points $\mathbf{w} \in \mathbb{R}^n$ at which this minimum is attained at least twice. Equivalently, a point $\mathbf{w} \in \mathbb{R}^n$ lies in $V(p)$ if and only if p is not linear at $\mathbf{w}$.

For instance, let $n = 1$ and fix the univariate tropical polynomial

$$p(x) \quad = \quad a \odot x^3 \;\oplus\; b \odot x^2 \;\oplus\; c \odot x \;\oplus\; d$$

seen in (1.1.1). If the assumption $b - a \leq c - b \leq d - c$ of (1.1.2) holds, then

$$V(p) \quad = \quad \big\{\, b - a,\, c - b,\, d - c \,\big\}.$$

Thus the hypersurface $V(p)$ is the set of "roots" of the polynomial $p(x)$.

For an example of a tropical polynomial in many variables consider the determinant function $p = \mathrm{trop}\det$ from (1.2.6). Its hypersurface $V(p)$ consists of all $n \times n$-matrices that are *tropically singular*. A square matrix being tropically singular means that the optimal solution to the assignment problem discussed in the previous section is not unique, so among all $n!$ ways of assigning n workers to n jobs, there are at least two assignments both of which minimize the total cost. For further information see Example 3.1.11.

In this section we study the geometry of a polynomial in two variables:

$$p(x,y) \quad = \quad \bigoplus_{(i,j)} c_{ij} \odot x^i \odot y^j.$$

The corresponding tropical hypersurface $V(p)$ is a *plane tropical curve*. The following proposition summarizes the salient features of such a curve.

Proposition 1.3.1. *The curve $V(p)$ is a finite graph that is embedded in the plane $\mathbb{R}^2$. It has both bounded and unbounded edges, all edge slopes are rational, and this graph satisfies a* balancing condition *around each node.*

This result is a consequence of the Structure Theorem for tropical varieties, which is our Theorem 3.3.5. Balancing refers to the following geometric condition: Consider any node (x, y) of the graph. The edges adjacent to this node lie on lines with rational slopes. Translate (x, y) to the origin $(0, 0)$. In the direction of each edge, now consider the first nonzero lattice vector on that line. *Balancing* at (x, y) means that the sum of these vectors is zero.

In general, it will be necessary to assign a positive integer *multiplicity* to each edge of $V(p)$, in order for this balancing condition to hold. These multiplicities will make informal appearances throughout this chapter. The precise definition will be given, for arbitrary dimensions, in Section 3.4.

Our first example is a *line* in the plane. It is defined by a polynomial

$$(1.3.1) \qquad p(x,y) \quad = \quad a \odot x \ \oplus \ b \odot y \ \oplus \ c, \qquad \text{where } a, b, c \in \mathbb{R}.$$

The tropical curve $V(p)$ consists of all points (x, y) where the function

$$p \ : \ \mathbb{R}^2 \to \mathbb{R}\,, \qquad (x,y) \ \mapsto \ \min\big(a + x,\, b + y,\, c\big)$$

is not linear. It consists of three half-rays emanating from the point $(x, y) = (c - a, c - b)$ into the northern, eastern, and southwestern directions.

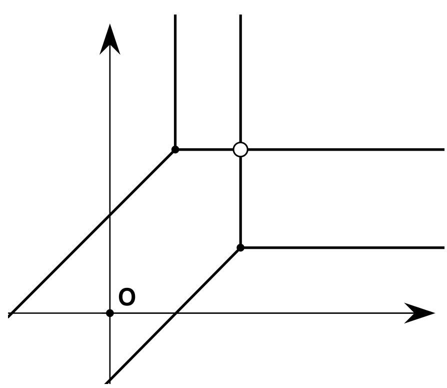

Figure 1.3.1. Two lines in the tropical plane meet in one point.

Two lines in the tropical plane will always meet in one point. This is shown in Figure 1.3.1. When the lines are in special position, it can happen that the set-theoretic intersection is a half-ray. In that case the notion of stable intersection discussed below is used to get a unique intersection point.

Let p be any tropical polynomial in x and y, and consider any term $\gamma \odot x^i \odot y^j$ appearing in p. In classical arithmetic this represents the linear function $(x,y) \mapsto \gamma + ix + jy$. The tropical polynomial function $p : \mathbb{R}^2 \to \mathbb{R}$ is given by the minimum of these linear functions. The graph of p is concave and piecewise linear. It looks like a tent over the plane $\mathbb{R}^2$. The tropical curve $V(p)$ is the set of all points in $\mathbb{R}^2$ at which the function is not linear.

As an example we consider the general quadratic polynomial

$$p(x,y) \quad = \quad a \odot x^2 \;\oplus\; b \odot xy \;\oplus\; c \odot y^2 \;\oplus\; d \odot y \;\oplus\; e \;\oplus\; f \odot x.$$

Suppose that the coefficients $a, b, c, d, e, f \in \mathbb{R}$ satisfy the inequalities

$$b + f < a + d, \qquad d + f < b + e, \qquad b + d < c + f.$$

Then the graph of $p : \mathbb{R}^2 \to \mathbb{R}$ is the lower envelope of six planes in $\mathbb{R}^3$. This is shown in Figure 1.3.2, where each linear piece of the graph is labeled by the corresponding linear function. Below this "tent" lies the tropical quadratic curve $V(p) \subset \mathbb{R}^2$. This curve has four vertices, three bounded edges, and six half-rays (two northern, two eastern, and two southwestern).

If $p(x,y)$ is a tropical polynomial, then its curve $V(p)$ can be constructed from its Newton polygon $\mathrm{Newt}(p)$, which we recall from Definition 1.1.3. Namely, the planar graph dual to $V(p)$ is a subdivision of $\mathrm{Newt}(p)$ into smaller polygons. This subdivision is determined by the coefficients of p. Typically, these smaller polygons are triangles, in which case the subdivision is a *triangulation*. The triangulation is *unimodular* if

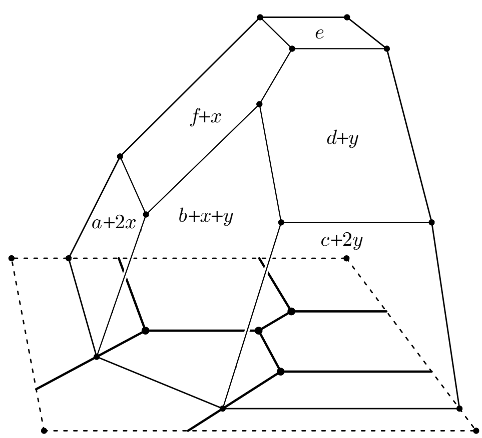

Figure 1.3.2. The graph and the curve defined by a quadratic polynomial.

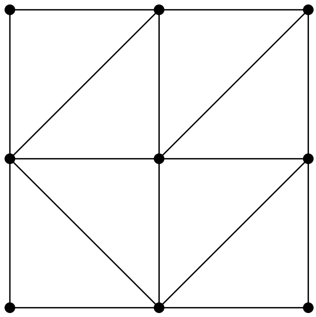

Figure 1.3.3. Two subdivisions of the Newton polygon of a biquadratic curve. Their planar duals are the curves in Figure 1.3.4.

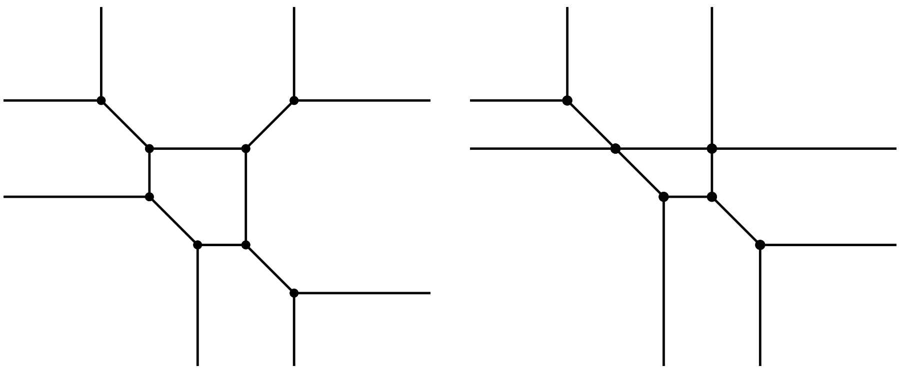

Figure 1.3.4. Two tropical biquadratic curves. The curve on the left is smooth.

each cell is a lattice triangle of unit area 1/2. In this case we call $V(p)$ a *smooth tropical curve*. The adjective "smooth" will be justified in Proposition 4.5.1. For our subdivisions and triangulations in arbitrary dimensions, see Definition 2.3.8.

The unbounded rays of a tropical curve $V(p)$ are perpendicular to the edges of the Newton polygon. For example, if p is a biquadratic polynomial, then $\mathrm{Newt}(p)$ is the square with vertices $(0,0)$, $(0,2)$, $(2,0)$, $(2,2)$. Here, $V(p)$ has two unbounded rays for each of the four edges of the square. Figure 1.3.3 shows two subdivisions. The corresponding tropical curves are shown in Figure 1.3.4. The curve on the left is smooth. It has genus one. The unique cycle corresponds to the interior lattice point of $\mathrm{Newt}(p)$. This is an example of a *tropical elliptic curve*. The curve on the right is not smooth.

If we draw tropical curves in the plane, then we discover that they intersect and interpolate just as algebraic curves do. In particular, we observe

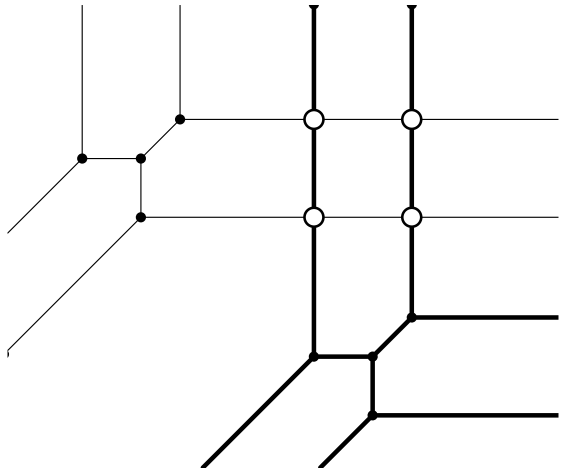

Figure 1.3.5. Bézout's Theorem: Two quadratic curves meet in four points.

the following:

- Two general lines meet in one point (Figure 1.3.1).
- Two general points lie on a unique line.
- A general line and quadric meet in two points (Figure 1.3.6).
- Two general quadrics meet in four points (Figures 1.3.5 and 1.3.7).
- Five general points lie on a unique quadric.

A classical result from algebraic geometry, known as *Bézout's Theorem*, holds in tropical algebraic geometry as well. In order to state this theorem, we need the multiplicities that were mentioned after Proposition 1.3.1. In addition to that, we assign a positive integer to any two lines with distinct rational slopes in $\mathbb{R}^2$. If their primitive direction vectors are $(u_1, u_2) \in \mathbb{Z}^2$ and $(v_1, v_2) \in \mathbb{Z}^2$, respectively, then the intersection multiplicity of the two lines at their unique common point is $|u_1v_2 - u_2v_1|$. We multiply that number with the product of the multiplicities of the two edges determining the lines.

We now focus on tropical curves whose Newton polygons are the standard triangles with vertices $(0,0)$, $(0,d)$, and $(d,0)$. We refer to such a curve as a *curve of degree* d. A curve of degree d has d rays, possibly counting multiplicities, perpendicular to each of the three edges of its Newton triangle. Suppose that C and D are two tropical curves in $\mathbb{R}^2$ that intersect transversally, that is, every common point lies in the relative interior of a unique edge in C and also in D. The multiplicity of that point is the product of the multiplicities of the edges times the intersection multiplicity $|u_1v_2 - u_2v_1|$.

Theorem 1.3.2 (Bézout). *Consider two tropical curves C and D of degree c and d in $\mathbb{R}^2$. If the two curves intersect transversally, then the number of intersection points, counted with multiplicities as above, is equal to cd.*

Just as in classical algebraic geometry, it is possible to remove the restriction "intersect transversally" from the statement of Bézout's Theorem. In fact, the situation is even better here because of the following important phenomenon, which is false in classical geometry. The intersection points depend continuously on the coefficients of the two tropical polynomials. These continuous functions are well defined on the entire space of coefficients, even at locations when the two polynomials are very special.

We explain this for the intersection of two curves C and D of degrees c and d in $\mathbb{R}^2$. Suppose the intersection of C and D is not transverse or not even finite. Pick *any* nearby curves C_ϵ and D_ϵ such that C_ϵ and D_ϵ intersect transversely in finitely many points. Then, according to the refined count of Theorem 1.3.2, the intersection $C_\epsilon \cap D_\epsilon$ is a multiset of cardinality cd.

Theorem 1.3.3 (Stable Intersection Principle). *The limit of the point configuration $C_\epsilon \cap D_\epsilon$ is independent of the choice of perturbations. It is a well-defined multiset of cd points contained in the intersection $C \cap D$.*

Here the limit is taken as ϵ tends to 0. Multiplicities add up when points collide. The limit is a finite configuration of points in $\mathbb{R}^2$ with multiplicities, where the sum of the multiplicities is cd. We call this limit the *stable intersection* of the curves C and D. This is a multiset of points, denoted by

$$C \cap_{\mathrm{st}} D \;=\; \lim_{\epsilon \to 0} (C_\epsilon \cap D_\epsilon).$$

Hence we can strengthen the statement of Bézout's Theorem as follows.

Corollary 1.3.4. *Any two curves of degrees c and d in $\mathbb{R}^2$, no matter how special they might be, intersect stably in a well-defined multiset of cd points.*

The Stable Intersection Principle is illustrated in Figures 1.3.6 and 1.3.7. In Figure 1.3.6 we see the intersection of a tropical line with a tropical quadric, moving from general position to special position. In the diagram on the right, the set-theoretic intersection of the two curves is infinite, but the stable intersection is well defined. It consists of two points A and B.

Figure 1.3.7 shows an even more dramatic situation. In that picture, a quadric is intersected stably with itself. For any small perturbation of the coefficients of the two tropical polynomials, we obtain four intersection points near the four nodes of the original quadric. This shows that the stable intersection of a quadric with itself consists precisely of the four nodes.

We refer to Section 3.6 for a thorough treatment of stable intersections.

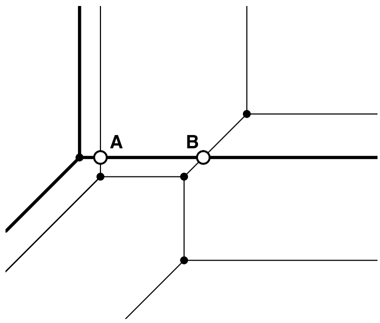

Figure 1.3.6. The stable intersection of a line and a quadric.

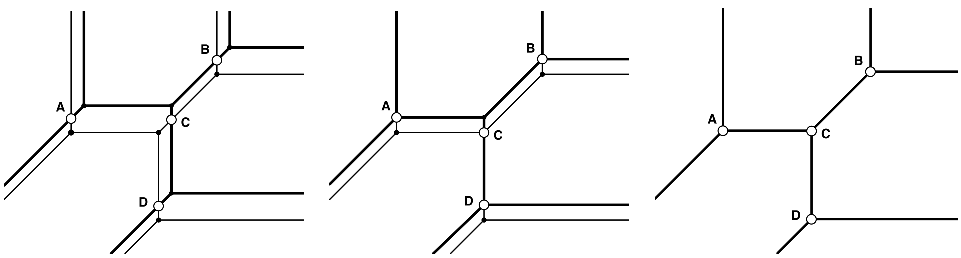

Figure 1.3.7. The stable intersection of a quadric with itself.

1.4. Amoebas and their Tentacles

One early source in tropical algebraic geometry is a 1971 paper on *the logarithmic limit-set of an algebraic variety* by George Bergman [**Ber71**]. With hindsight, the structure introduced by Bergman is the same as the tropical variety arising from a subvariety in a complex algebraic torus $(\mathbb{C}^*)^n$. Here $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ denotes the multiplicative group of nonzero complex numbers.

The *amoeba* of such a variety is its image under taking the coordinate-wise logarithm of the absolute value of any point on the variety. The term "amoeba" was coined by Gel'fand, Kapranov, and Zelevinsky in their monograph *Discriminants, Resultants, and Multidimensional Determinants* [**GKZ08**]. Bergman's logarithmic limit set arises from the amoeba as the set of all tentacle directions. In this section we discuss these and related topics.

Let I be an ideal in the Laurent polynomial ring $S=\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. Its algebraic variety is the common zero set of all Laurent polynomials in I:

$$V(I) \;=\; \bigl\{\, \mathbf{z} \in (\mathbb{C}^*)^n \;:\; f(\mathbf{z}) = 0 \text{ for all } f \in I \,\bigr\}.$$

Note that this is well defined because $0 \notin \mathbb{C}^*$. The *amoeba* of the ideal I is the subset of $\mathbb{R}^n$ defined as image of the coordinate-wise logarithm map:

$$\mathcal{A}(I) = \bigl\{ \bigl(\log(|z_1|), \log(|z_2|), \ldots, \log(|z_n|)\bigr) \in \mathbb{R}^n \,:\, \mathbf{z} = (z_1, \ldots, z_n) \in V(I) \bigr\}.$$

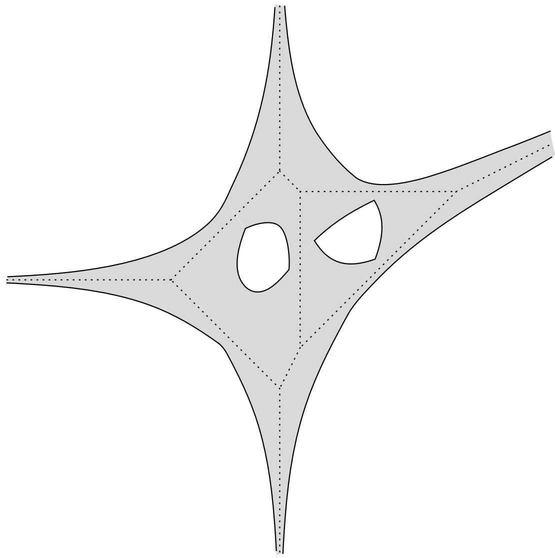

Figure 1.4.1. The amoeba of a plane curve and its spine.

If $n = 1$ and I is a proper ideal in $S = \mathbb{C}[x, x^{-1}]$, then I is a principal ideal. It is generated by a single polynomial $f(x)$ that factors over $\mathbb{C}$:

$$f(x) \;=\; (u_1 + iv_1 - x)(u_2 + iv_2 - x)\,\cdots\,(u_m + iv_m - x).$$

Here $u_1, v_1, \ldots, u_m, v_m \in \mathbb{R}$ are the real and imaginary parts of the various roots of $f(x)$, and the amoeba is the following set of at most m real numbers:

$$\mathcal{A}(I) = \mathcal{A}(f) = \left\{ \log\left(\sqrt{u_1^2 + v_1^2}\,\right),\, \log\left(\sqrt{u_2^2 + v_2^2}\,\right),\, \ldots,\, \log\left(\sqrt{u_m^2 + v_m^2}\,\right) \right\}.$$

It is instructive to draw some amoebas for $n = 2$. Let $I = \langle f(x_1, x_2) \rangle$ be the ideal of a curve $\{f(x_1, x_2) = 0\}$ in $(\mathbb{C}^*)^2$. The amoeba $\mathcal{A}(f)$ of that curve is a closed subset of $\mathbb{R}^2$ whose boundary is described by analytic functions. It has finitely many tentacles that emanate toward infinity, and the directions of these tentacles are precisely the directions perpendicular to the edges of the Newton polygon $\mathrm{Newt}(f)$. The complement $\mathbb{R}^2 \backslash \mathcal{A}(f)$ of the amoeba is a finite union of open convex subsets of the plane $\mathbb{R}^2$.

We refer to work of Passare and his collaborators [**PR04, PT05**] for foundational results on amoebas of hypersurfaces in $(\mathbb{C}^*)^n$, and to the article by Theobald [**The02**] for methods for computing and drawing amoebas. An interesting Nullstellensatz for amoebas was established by Purbhoo [**Pur08**].

Example 1.4.1. Figure 1.4.1 shows the complex amoeba of the curve

$$f(z, w) \;=\; 1 + 5zw + w^2 - z^3 + 3z^2w - z^2w^2.$$

Note the two bounded convex components in the complement of

$$\mathcal{A}(f) \;=\; \bigl\{ (\log(|z|), \log(|w|)) \in \mathbb{R}^2 \,:\, z, w \in \mathbb{C}^* \text{ and } f(z,w) = 0 \bigr\}.$$

They correspond to the two interior lattice points of the Newton polygon of f. The tentacles of the amoeba converge to four rays in $\mathbb{R}^2$. Up to sign, the union of these rays is the plane curve $V(p)$ defined by the tropical polynomial

$$p \;=\; \mathrm{trop}(f) \;=\; 0 \oplus u \odot v \oplus v^2 \oplus u^3 \oplus u^2 \odot v \oplus u^2 \odot v^2.$$

This expression is the tropicalization of f, to be defined formally in (2.4.1). All coefficients of p are zero because the coefficients of f are real numbers.

Note that in our definition of amoeba (and in Figure 1.4.1) the max-convention was used. (Mikael Passare always preferred max-convention because of his son's first name: it is Max and not Min). Inside the amoeba of Figure 1.4.1, we see the curve defined by a tropical polynomial of the form

$$q \;=\; c_1 \oplus c_2 \odot u \odot v \oplus c_3 \odot v^2 \oplus c_4 \odot u^3 \oplus c_5 \odot u^2 \odot v \oplus c_6 \odot u^2 \odot v^2.$$

The tropical curve $V(q)$ is a canonical deformation retract of $-\mathcal{A}(f)$. It is known as the *spine* of the amoeba. The coefficients c_i are defined below. ♢

There are three different ways in which tropical varieties arise from amoebas. We associate the name of a mathematician with each of them.

The Passare Construction: Every complex hypersurface amoeba $\mathcal{A}(f)$ has a *spine* which is a canonical tropical hypersurface contained in $\mathcal{A}(f)$. Suppose $f = f(z,w)$ is a polynomial in two variables. Then its *Ronkin function* is

$$N_f(u,v) \;=\; \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(u,v)} \log|f(z,w)| \frac{dz}{z} \wedge \frac{dw}{w}.$$

Passare and Rullgård [**PR04**] showed that this function is convex and that it is linear on each connected component of the complement of $\mathcal{A}(f)$. Let $q(u,v)$ denote the negated maximum of these affine-linear functions, one for each component in the amoeba complement. Then $q(u,v)$ is a tropical polynomial function (a piecewise-linear concave function) which satisfies $N_f(u,v) \geq -q(u,v)$ for all $(u,v) \in \mathbb{R}^2$. Its tropical curve $V(q)$ is the spine.

The Maslov Construction: Tropical varieties arise as limits of amoebas as one changes the base of the logarithm and makes it either very large or very small. This limit process is also known as *Maslov dequantization*, and it can be made precise as follows. Given $h > 0$, we redefine arithmetic as follows:

$$x \oplus_h y \;=\; h \cdot \log\Bigl(\exp\bigl(\frac{x}{h}\bigr) + \exp\bigl(\frac{y}{h}\bigr)\Bigr) \qquad \text{and} \qquad x \odot_h y \;=\; x + y.$$

This is what happens to ordinary addition and multiplication of positive real numbers under the coordinate transformation $\mathbb{R}_+ \to \mathbb{R}, x \mapsto h \cdot \log(x)$.

We now consider a polynomial $f_h(z,w)$ whose coefficients are rational functions of the parameter h. For each $h > 0$, we take the amoeba $\mathcal{A}_h(f_h)$ of f_h with respect to scaled logarithm map $(z,w) \mapsto h\cdot\big(\log(|z|), \log(|w|)\big)$. The limit in the Hausdorff topology of the set $-\mathcal{A}_h(f_h)$ as $h \to 0+$ is a tropical hypersurface $V(q)$. For details see [**Mik04**]. The coefficients of the tropical polynomials q are the orders (of poles or zeros) of the coefficients at $h = 0$. This process can be thought of as a sequence of amoebas converging to their spine, but it is different from the construction using Ronkin functions.

The Bergman Construction: Our third connection between amoebas on tropical varieties arises by examining their tentacles. Here we disregard the interior structure of $\mathcal{A}(f)$, such as the bounded convex regions in the complement. We focus only on the asymptotic directions. This makes sense for any subvariety of $(\mathbb{C}^*)^n$, so our input now is an ideal $I \subset S$ as above.

We denote the unit sphere by $\mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}|| = 1\}$. For any real number $M > 0$, we consider the following set:

$$\mathcal{A}_M(I) \;=\; -\frac{1}{M}\mathcal{A}(I) \cap \mathbb{S}^{n-1}.$$

The *logarithmic limit set* $\mathcal{A}_\infty(I)$ is the set of points $\mathbf{v}$ on the sphere $\mathbb{S}^{n-1}$ such that there exists a sequence of points $\mathbf{v}_M \in \mathcal{A}_M(I)$ converging to $\mathbf{v}$:

$$\lim_{M\to\infty}\mathbf{v}_M \;=\; \mathbf{v}.$$

We next exhibit the relationship to the *tropical variety* $\mathrm{trop}(V(I))$ of I. Here $\mathrm{trop}(V(I))$ is defined to be the intersection of the tropical hypersurfaces $V(p)$ where $p = \mathrm{trop}(f)$ is the tropicalization of any polynomial $f \in I$.

Theorem 1.4.2. *The tropical variety of I coincides with the cone over the logarithmic limit set $\mathcal{A}_\infty(I)$, i.e., a nonzero vector $\mathbf{w} \in \mathbb{R}^n$ lies in $\mathrm{trop}(V(I))$ if and only if the corresponding unit vector $\frac{1}{||\mathbf{w}||}\mathbf{w}$ lies in $\mathcal{A}_\infty(I)$.*

When the ideal I is principal, this appears in [**MR01**]. For a proof of Theorem 1.4.2 and connections of amoebas to Berkovich spaces, see [**Jon14**].

The tropical variety $\mathrm{trop}(V(I))$ is the principal actor in this book. It will be studied in great detail in Chapter 3. We shall see in Corollary 3.5.5 that $\mathrm{trop}(V(I))$ has the structure of a polyhedral fan, and we shall establish various properties of that fan. Theorem 1.4.2 and the fan property of $\mathrm{trop}(V(I))$ imply that $\mathcal{A}_\infty(I)$ is a spherical polyhedral complex in $\mathbb{S}^{n-1}$.

It is interesting to see the motivation behind the paper [**Ber71**]. Bergman introduced tropical varieties in order to prove a conjecture of Zalessky concerning the multiplicative action of $\mathrm{GL}(n,\mathbb{Z})$ on the Laurent polynomial ring S. Here, an integer matrix $g = (g_{ij})$ acts on S as the ring homomorphism that maps each variable x_i to the Laurent monomial $\prod_{j=1}^n x_j^{g_{ij}}$.

If I is a proper ideal in S, then we consider its stabilizer subgroup:

$$\text{Stab}(I) = \{ g \in \text{GL}(n, \mathbb{Z}) : gI = I \}.$$

The following result from [**Ber71**, Theorem 1] answers Zalessky's question:

Corollary 1.4.3. *The stabilizer* $\text{Stab}(I) \subset \text{GL}(n, \mathbb{Z})$ *of a proper ideal* $I \subset S$ *has a subgroup of finite index that stabilizes a nontrivial sublattice of* $\mathbb{Z}^n$.

Proof. The tropical variety of $V(I)$ has the structure of a proper polyhedral fan in $\mathbb{R}^n$. Let $\mathcal{U}$ be the finite set of linear subspaces of $\mathbb{R}^n$ that are spanned by the maximal cones in $V(I)$. While the fan structure is not unique, the set $\mathcal{U}$ of linear subspaces of $\mathbb{R}^n$ is uniquely determined by I. The set $\mathcal{U}$ does not change under refinement or coarsening of the fan structure on $\text{trop}(V(I))$.

The group $\text{Stab}(I)$ acts by linear transformations on $\mathbb{R}^n$, and it leaves the tropical variety of I invariant. This implies that it acts by permutations on the finite set $\mathcal{U}$ of subspaces in $\mathbb{R}^n$. Fix one particular subspace $U \in \mathcal{U}$, and let G be the subgroup of all elements $g \in \text{Stab}(I)$ that fix U. Then G has finite index in $\text{Stab}(I)$ and it stabilizes the sublattice $U \cap \mathbb{Z}^n$ of $\mathbb{Z}^n$. $\square$

A counterpart to the amoeba $\mathcal{A}(I)$ is the *co-amoeba*, which records the phases of the coordinates of all points in a complex variety $V(I)$. An analogue of Bergman's logarithmic limit set for co-amoebas is the *phase limit set* of $V(I)$. See [**NS13**] for recent results and references on these topics.

1.5. Implicitization

An algebraic variety can be represented either as the image of a rational map or as the zero set of some multivariate polynomials. The latter representation exists for all algebraic varieties while the former representation requires that the variety be *unirational*, which is a very special property in algebraic geometry. The transition between two representations is a basic problem in computer algebra. *Implicitization* is the problem of passing from the first representation to the second, that is, given a rational map Φ, one seeks to determine the prime ideal of all polynomials that vanish on the image of Φ.

In this section we examine the simplest instance, namely, we consider the case of a plane curve in $\mathbb{C}^2$ that is given by a rational parameterization:

$$\Phi : \mathbb{C} \to \mathbb{C}^2, \;\; t \mapsto (\phi_1(t), \phi_2(t)). \tag{1.5.1}$$

To make the map Φ actually well defined, here we tacitly assume that the poles of ϕ_1 and ϕ_2 have been removed from the domain $\mathbb{C}$. The implicitization problem is to compute the unique (up to scaling) irreducible polynomial $f(x, y)$ vanishing on the curve $\text{Image}(\Phi) = \{(\phi_1(t), \phi_2(t)) \in \mathbb{C}^2 : t \in \mathbb{C}\}$.

Example 1.5.1. Consider the plane curve defined parametrically by

$$\Phi(t) \;=\; \left(\frac{t^3+4t^2+4t}{t^2-1}, \frac{t^3-t^2-t+1}{t^2} \right).$$

The implicit equation of this curve equals

$$f(x,y) \;=\; x^3y^2 - x^2y^3 - 5x^2y^2 - 2x^2y - 4xy^2 - 33xy + 16y^2 + 72y + 81.$$

This irreducible polynomial vanishes on all points $(x,y) = \Phi(t)$ for $t \in \mathbb{C}$. ♢

Two standard methods used in computer algebra for solving implicitization problems are Gröbner bases and resultants. These methods are explained in the textbook by Cox, Little, and O'Shea [**CLO07**]. Specifically, the desired polynomial $f(x,y)$ equals the Sylvester resultant of the numerator of $x - \phi_1(t)$ and the numerator of $y - \phi_2(t)$ with respect to the variable t. For instance, the implicit equation in Example 1.5.1 is easily found by

$$f(x,y) \;=\; \text{resultant}_t\big(\,t^3+4t^2+4t-(t^2-1)x\,,\,t^3-t^2-t+1-t^2y\big).$$

For larger problems in higher dimensions, Gröbner bases and resultants often do not perform well enough or do not give enough geometric insight. This is where the approach to implicitization using tropical geometry comes in. We shall explain the basic idea behind this approach for rational plane curves.

Suppose we are given the parameterization Φ, and wish to compute the implicit equation $f(x,y)$. Tropical geometry allows us to compute the Newton polygon $\text{Newt}(f)$ first, directly from the parameterization Φ, without knowing $f(x,y)$. This is the content of Theorem 1.5.2. Once the Newton polygon $\text{Newt}(f)$ is known, we recover the desired polynomial $f(x,y)$ by a linear algebra computation. The next paragraph explains that computation.

Pretend that the polynomial $f(x,y)$ in Example 1.5.1 is unknown and impossible to compute using resultants or Gröbner bases. Suppose further that we are given its Newton polygon. According to Definition 1.1.3, this is

$$\text{(1.5.2)} \qquad \text{Newt}(f) \;=\; \text{conv}\left\{ \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix} \right\}.$$

This pentagon has four additional lattice points in its interior, so $\text{Newt}(f)$ contains precisely nine lattice points. That information reveals

$$f(x,y) \;=\; c_1x^3y^2 + c_2x^2y^3 + c_3x^2y^2 + c_4x^2y + c_5xy^2 + c_6xy + c_7y^2 + c_8y + c_9,$$

where the coefficients $c_1, c_2, \ldots, c_9$ are unknown parameters. At this point we can set up a linear system of equations as follows. For any choice of complex number τ, the equation $f(\phi_1(\tau), \phi_2(\tau)) = 0$ holds. This equation translates into one linear equation for the nine unknowns c_i. Eight of such linear equations will determine the coefficients uniquely (up to scaling). For instance, in our example, if we take $\tau = \pm 2, \pm 3, \pm 4, \pm 5$, then we get eight linear equations which stipulate that the vector $(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9)^{\mathrm{T}}$ lies in the kernel of the following 8×9-matrix of rational numbers

$$
\begin{array}{c}
\tau \\ -5 \\ -4 \\ -3 \\ -2 \\ 2 \\ 3 \\ 4 \\ 5
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc}
x^3y^2 & x^2y^3 & x^2y^2 & x^2y & xy^2 & xy & y^2 & y & 1
\end{array} \\
\begin{pmatrix}
-\frac{2187}{10} & -\frac{419904}{625} & \frac{2916}{25} & -\frac{81}{4} & -\frac{7776}{125} & \frac{54}{5} & \frac{20736}{625} & -\frac{144}{25} & 1 \\
-\frac{80}{3} & -\frac{1875}{16} & 25 & -\frac{16}{3} & -\frac{375}{16} & 5 & \frac{5625}{256} & -\frac{75}{16} & 1 \\
-\frac{2}{3} & -\frac{512}{81} & \frac{16}{9} & -\frac{1}{2} & -\frac{128}{27} & \frac{4}{3} & \frac{1024}{81} & -\frac{32}{9} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{81}{16} & -\frac{9}{4} & 1 \\
\frac{2048}{3} & 48 & 64 & \frac{256}{3} & 6 & 8 & \frac{9}{16} & \frac{3}{4} & 1 \\
\frac{15625}{6} & \frac{40000}{81} & \frac{2500}{9} & \frac{625}{4} & \frac{800}{27} & \frac{50}{3} & \frac{256}{81} & \frac{16}{9} & 1 \\
\frac{34992}{5} & \frac{32805}{16} & 729 & \frac{1296}{5} & \frac{1215}{16} & 27 & \frac{2025}{256} & \frac{45}{16} & 1 \\
\frac{235298}{15} & \frac{3687936}{625} & \frac{38416}{25} & \frac{2401}{6} & \frac{18816}{125} & \frac{196}{5} & \frac{9216}{625} & \frac{96}{25} & 1
\end{pmatrix}
\end{array}.
$$

This matrix has rank 8, so its kernel is one dimensional. Any generator of that kernel translates into (a scalar multiple of) the desired polynomial $f(x, y)$.

While the implicit equation $f(x, y)$ of a parametric curve can always be recovered from its Newton polygon by solving linear equations, the relevant matrices tend to be dense and ill conditioned. It is a nontrivial challenge to recover the coefficients numerically when $f(x, y)$ has thousands of terms.

By contrast, some mathematicians can rightfully consider the implicitization problem to be solved once the Newton polygon has been found. Thus, in what follows, we study the following alternative version of implicitization:

Tropical implicitization problem: Given two rational functions $\phi_1(t)$ and $\phi_2(t)$, compute the Newton polygon $\mathrm{Newt}(f)$ of the implicit equation $f(x, y)$.

We shall present the solution to the tropical implicitization problem for plane curves. By the Fundamental Theorem of Algebra, the two given rational functions are products of linear factors over the complex numbers $\mathbb{C}$:

$$
\begin{aligned}
\phi_1(t) &= (t - \alpha_1)^{u_1}(t - \alpha_2)^{u_2} \cdots (t - \alpha_m)^{u_m}, \\
\phi_2(t) &= (t - \alpha_1)^{v_1}(t - \alpha_2)^{v_2} \cdots (t - \alpha_m)^{v_m}.
\end{aligned}
\tag{1.5.3}
$$

Here the α_i are the zeros and poles of either of the two functions ϕ_1 and ϕ_2. It may occur that u_i is zero while v_i is nonzero or vice versa.

For what follows we do not need the algebraic numbers α_i but only the exponents u_i and v_j occurring in the factorizations. These can be found by symbolic computation. For instance, it suffices to factor $\phi_1(t)$ and $\phi_2(t)$ over their field of definition, say, the rational numbers $\mathbb{Q}$. No field extensions or floating point computations are needed to obtain the integers u_i and v_j.

We abbreviate $u_0 = -u_1 - u_2 - \cdots - u_m$ and $v_0 = -v_1 - v_2 - \cdots - v_m$, and we consider the following collection of $m+1$ integer vectors in the plane:

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \ldots, \begin{pmatrix} u_m \\ v_m \end{pmatrix}. \tag{1.5.4}$$

We consider the rays spanned by these $m+1$ vectors. Each ray has a natural multiplicity, namely the sum of the lattice lengths of all vectors $(u_i, v_i)^T$ lying on that ray. Since the vectors in (1.5.4) sum to zero, this configuration of rays satisfies the balancing condition: it is a tropical curve in the plane $\mathbb{R}^2$.

The following result can be derived from the Fundamental Theorem 3.2.3. We will ask for a proof in Exercise 5.6(26). A higher-dimensional generalization of Theorem 1.5.2 is presented in Theorem 5.5.1. As stated, Theorem 1.5.2 and Corollary 1.5.3 need the hypothesis that the map Φ is one-to-one. Otherwise, one first divides (1.5.4) by the degree of Φ.

Theorem 1.5.2. *The tropical curve* $V(\mathrm{trop}(f))$ *defined by the unknown polynomial* f *equals the tropical curve determined by the vectors in* (1.5.4).

The Newton polygon $\mathrm{Newt}(f)$ can be recovered from the tropical curve $V(f)$ as follows. The first step is to rotate our vectors by 90 degrees:

$$\begin{pmatrix} v_0 \\ -u_0 \end{pmatrix}, \begin{pmatrix} v_1 \\ -u_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ -u_2 \end{pmatrix}, \ldots, \begin{pmatrix} v_m \\ -u_m \end{pmatrix}. \tag{1.5.5}$$

Since these vectors sum to zero, there exists a convex polygon P whose edges are translates of these vectors. We construct P by sorting the vectors by increasing slope and then simply concatenating them. The polygon P is unique up to translation. Hence there exists a unique translate P^+ of the polygon P that lies in the nonnegative orthant $\mathbb{R}^2_{\geq 0}$ and that has nonempty intersection with both the x-axis and the y-axis. The latter requirements are necessary (and sufficient) for a lattice polygon to arise as the Newton polygon of an irreducible polynomial in $\mathbb{C}[x, y]$. We conclude:

Corollary 1.5.3. *The polygon* P^+ *equals the Newton polygon* $\mathrm{Newt}(f)$ *of the defining irreducible polynomial of the parameterized curve* $\mathrm{Image}(\Phi)$.

This solves the tropical implicitization problem for plane curves over $\mathbb{C}$. We illustrate this solution for our running example.

Example 1.5.4. Write the map of Example 1.5.1 in factored form (1.5.3):

$$\begin{aligned} \phi_1(t) &= (t-1)^{\mathbf{-1}}\, t^{\mathbf{1}}\, (t+1)^{\mathbf{-1}}\, (t+2)^{\mathbf{2}}, \\ \phi_2(t) &= (t-1)^{\mathbf{2}}\, t^{\mathbf{-2}}\, (t+1)^{\mathbf{1}}\, (t+2)^{\mathbf{0}}. \end{aligned}$$

The derived configuration of five vectors as in (1.5.4) equals

$$\begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

We form their rotations as in (1.5.5), and we order them by increasing slope:

$$\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}-2\\-1\end{pmatrix}, \begin{pmatrix}0\\-2\end{pmatrix}.$$

We concatenate these vectors starting at the origin. The resulting edges all remain in the nonnegative orthant. The result is the pentagon P^+ in Corollary 1.5.3. As predicted, it coincides with the pentagon in (1.5.2). ♢

The technique of tropical implicitization can be used, in principle, to compute the tropicalization of any parametrically presented algebraic variety. The details are more complicated than the simple curve case discussed here. A proper treatment requires toric geometry and concepts from resolution of singularities. The proof of Theorem 6.5.16 demonstrates this point. For further reading on tropical implicitization, we refer to [**STY07, SY08**].

1.6. Group Theory

One of the origins in tropical geometry is the work of Bieri, Groves, Strebel, and Neumann in group theory [**BG84, BS80, BNS87**]. Starting in the late 1970s, these authors associate polyhedral fans to certain classes of discrete groups, and they establish remarkable results concerning generators, relations and higher cohomology of these groups in terms of their fans. This part of our tropical island is more secluded and offers breathtaking vistas.

We begin with an easy illustrative example. Fix a nonzero real number ξ, and let G_ξ denote the group generated by the two invertible 2×2-matrices

$$A = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix}1 & 0\\0 & \xi\end{pmatrix}. \tag{1.6.1}$$

What relations do these two generators satisfy? In particular, is the group G_ξ finitely presented? Does this property depend on the number ξ?

To answer these questions, we explore some basic computations such as

$$X^u A^c X^{-u} X^v A^d X^{-v} \;=\; \begin{pmatrix}1 & c\,\xi^{-u} + d\,\xi^{-v}\\0 & 1\end{pmatrix}. \tag{1.6.2}$$

Here u, v, c, and d can be arbitrary integers. This identity shows that the two matrices $X^u A^c X^{-u}$ and $X^v A^d X^{-v}$ commute, and this commutation relation is a valid relation among the two generators of G_ξ. If the number ξ is not algebraic over $\mathbb{Q}$, then the set of all such commutation relations constitutes a complete presentation of G_ξ, and in this case the group G_ξ is never finitely presented. On the other hand, if ξ is an algebraic number, then additional relations can be derived from the irreducible minimal polynomial $f \in \mathbb{Z}[x]$ of ξ. To show how this works, we consider the explicit example $\xi = \sqrt{2} + \sqrt{3}$.

The minimal polynomial of this algebraic number is $f(x) = x^4 - 10x^2 + 1$. This polynomial translates into the matrix identity

$$(1.6.3) \qquad (X^{-4}A^1X^4)\cdot(X^{-2}A^{-10}X^2)\cdot(X^0A^1X^0) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The left-hand side gives the word $X^{-4}AX^2A^{-10}X^2A$ in the generators A and X. That word is a relation in G_ξ. Our question is whether the group of all such relations is finitely generated. It turns out that the answer is affirmative for $\xi = \sqrt{2} + \sqrt{3}$, and we shall list the generators in Example 1.6.10.

In general, finite presentation is characterized by the following result:

Theorem 1.6.1. *The group $G_\xi = \langle A, X \rangle$ is finitely presented if and only if either the real number ξ or its reciprocal $1/\xi$ is an algebraic integer over $\mathbb{Q}$.*

The condition that either ξ or $1/\xi$ is an algebraic integer says that either the highest term or the lowest term of $f(x)$ has coefficient $+1$ or -1. This is equivalent to saying that either the highest or the lowest term of the minimal polynomial $f(x)$ is a unit in $\mathbb{Z}[x, x^{-1}]$. It is precisely this condition on leading terms that underlies the tropical thread in geometric group theory.

Bieri and Strebel introduced tropical varieties over $\mathbb{Z}$ in their 1980 paper on metabelian groups [**BS80**]. Later work with Neumann [**BNS87**] extended their construction to a wider class of discrete groups. In what follows we restrict ourselves to metabelian groups whose corresponding module is cyclic. This special case suffices in order to explain the general idea and to shed light on the mystery of why Theorem 1.6.1 might be true.

We begin with some commutative algebra definitions. Consider the Laurent polynomial ring $S = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ over the integers $\mathbb{Z}$. The units in S are the *monomials* $\pm x^{\mathbf{a}} = \pm x_1^{a_1} \cdots x_n^{a_n}$ where $\mathbf{a} = (a_1, \ldots, a_n)$ runs over $\mathbb{Z}^n$. For $f \in S$ and $\mathbf{w} \in \mathbb{R}^n$, the *initial form* $\mathrm{in}_{\mathbf{w}}(f)$ is the sum of all terms in f whose $\mathbf{w}$-weight is minimal. If I is a proper ideal in S, then the *initial ideal* $\mathrm{in}_{\mathbf{w}}(I)$ is the ideal generated by all initial forms $\mathrm{in}_{\mathbf{w}}(f)$ where f runs over I. Computing $\mathrm{in}_{\mathbf{w}}(I)$ from a generating set of I requires *Gröbner bases over the integers*. The relevant algorithm for computing $\mathrm{in}_{\mathbf{w}}(I)$ from I is implemented in computer algebra systems such as `Macaulay2` and `Magma`.

The *tropical variety* (over $\mathbb{Z}$) of the ideal I is the following subset of $\mathbb{R}^n$:

$$\mathrm{trop}_{\mathbb{Z}}(I) \;=\; \bigl\{\, \mathbf{w} \in \mathbb{R}^n \,:\, \mathrm{in}_{\mathbf{w}}(I) \neq S \,\bigr\}.$$

This tropical variety contains the tropical variety over the field $\mathbb{Q}$ as a subset:

$$\mathrm{trop}_{\mathbb{Z}}(I) \;\supseteq\; \mathrm{trop}_{\mathbb{Q}}(I).$$

This containment is strict in general. For example, if $n = 2$ and $I = \langle x_1 + x_2 + 3 \rangle$, then $\mathrm{trop}_{\mathbb{Q}}(I)$ is the tropical line (1.3.1), which has three rays. However, $\operatorname{trop}_{\mathbb{Z}}(I)$ also contains the positive quadrant because 3 is not a unit in $\mathbb{Z}$.

We write $R = S/I$ for the quotient $\mathbb{Z}$-algebra, and, by mild abuse of notation, we write R^* for the multiplicative group generated by the images of the monomials. It follows from the results to be proved later in Chapter 3 that the complex variety of the ideal I is finite if and only if $\operatorname{trop}_{\mathbb{Q}}(I) = \{\mathbf{0}\}$. Here we state the analogous result for tropical varieties over the integers.

Theorem 1.6.2 (Bieri and Strebel)**.** *The $\mathbb{Z}$-algebra $R = S/I$ is finitely generated as a $\mathbb{Z}$-module if and only if*

$$\operatorname{trop}_{\mathbb{Z}}(I) = \{\mathbf{0}\}. \tag{1.6.4}$$

Proof. See [**BS80**, Theorem 2.4]. □

This raises the questions of how to test this criterion in practice and, if (1.6.4) holds, how to determine a finite set of monomials $\mathcal{U} \subset R^*$ that generate the abelian group R^*. It turns out that this can be done in `Macaulay2`.

Example 1.6.3. Fix integers m and n where $|m| > 1$. Consider the ideal

$$J = \langle\, ms^{-1}t^{-1} + s^{-1} + t^{-1} + n + st,\; mst + s + t + n + s^{-1}t^{-1} \,\rangle \subset \mathbb{Z}[s^{\pm 1}, t^{\pm 1}].$$

This ideal is a variation on Example 43 in Strebel's exposition [**Str84**]. The condition (1.6.4) is satisfied. To find a generating set $\mathcal{U}$, we can run the following four lines of `Macaulay2` code for various fixed values of m and n:

```
R = ZZ[s,t,S,T];            m = 7; n = 13;
J = ideal(m*S*T+S+T+n+s*t,m*s*t+s+t+n+S*T,s*S-1,t*T-1);
toString leadTerm J
toString basis(R/J)
```

The output of this script is the same for all m and n, namely,

$$\mathcal{U} \;=\; \left\{1, s, st^{-1}, t, s^{-1}, s^{-1}t^{-1}, t^{-1}, t^{-2}\right\}. \tag{1.6.5}$$

For a proof that $\mathbb{Z}\mathcal{U} = R/J$, it suffices to show that the initial ideal of J with respect to the reverse lexicographic term order is generated by

```
(m^2-1)*S*T,t*T,m*s*T,S^2,t*S,s*S,t^2,s*t,s^2,T^3,S*T^2,s*T^2.
```

This proof amounts to computing a Gröbner basis over the integers $\mathbb{Z}$. ♢

The integral tropical variety $\operatorname{trop}_{\mathbb{Z}}(I)$ is of interest even in the case $n = 1$.

Example 1.6.4. Suppose that ξ is an algebraic number over $\mathbb{Q}$ and I is the prime ideal of all Laurent polynomials $f(x)$ in $\mathbb{Z}[x, x^{-1}]$ such that $f(\xi) = 0$.

There are four possible cases of what the integral tropical variety can be:

- If ξ and $1/\xi$ are both algebraic integers, then $\mathrm{trop}_{\mathbb{Z}}(I) = \{0\}$.
- If ξ is an algebraic integer but $1/\xi$ is not, then $\mathrm{trop}_{\mathbb{Z}}(I) = \mathbb{R}_{\geq 0}$.
- If $1/\xi$ is an algebraic integer but ξ is not, then $\mathrm{trop}_{\mathbb{Z}}(I) = \mathbb{R}_{\leq 0}$.
- If neither ξ nor $1/\xi$ are algebraic integers, then $\mathrm{trop}_{\mathbb{Z}}(I) = \mathbb{R}$.

Examples of numbers for the first, third, and last cases are $\xi = \sqrt{2} + \sqrt{3}$, $\xi = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}$, and $\xi = \sqrt{2} + \frac{1}{\sqrt{3}}$, respectively. In particular, we see from Theorem 1.6.1 that G_ξ is finitely presented if and only if $\mathrm{trop}_{\mathbb{Z}}(I) \neq \mathbb{R}$. $\diamondsuit$

We now come to the punchline of this section, namely, the extension of Example 1.6.4 to $n \geq 2$ variables. Let I be any ideal in $S = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and let $R = S/I$. We associate with I the following group of 2×2-matrices:

$$G_I = \begin{pmatrix} 1 & R \\ 0 & R^* \end{pmatrix}.$$

This is a *metabelian group*, which means that the commutator subgroup of G_I is abelian. The elements of G_I are $\left(\begin{smallmatrix} 1 & f \\ 0 & m \end{smallmatrix}\right)$, where f is a Laurent polynomial and m is a Laurent monomial, but both are considered modulo I. The following result generalizes Theorem 1.6.1 to higher dimensions:

Theorem 1.6.5 (Bieri and Strebel)**.** *The metabelian group G_I is finitely presented if and only if the integer tropical variety $\mathrm{trop}_{\mathbb{Z}}(I)$ contains no line.*

This was the main theorem in the remarkable 1980 paper by Bieri and Strebel [**BS80**, Theorem A]. It predates the 1984 paper by Bieri and Groves [**BG84**], which has been cited by tropical geometers for its resolution of problems left open in Bergman's 1971 paper [**Ber71**] on the logarithmic limit set.

In what follows we aim to shed some light on the presentation of the metabelian group G_I. We begin with the observation that G_I is always finitely generated, namely, by a natural set of $n+1$ matrices over $R = S/I$:

Lemma 1.6.6. *The metabelian group G_I is generated by the $n+1$ matrices*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad X_i = \begin{pmatrix} 1 & 0 \\ 0 & x_i \end{pmatrix} \quad \textit{for } i = 1, 2, \ldots, n.$$

For $n = 1$, these are the two generators in (1.6.1), and $G_I = G_\xi$ if $I = \langle f(x) \rangle$ is the principal ideal generated by the minimal polynomial of ξ.

We now examine the relations among the $n+1$ generators in Lemma 1.6.6. Let us first assume that $I = \langle 0 \rangle$ is the zero ideal, so that $R = S$. The matrices X_i and X_j commute, so the commutator $[X_i, X_j] = X_i X_j X_i^{-1} X_j^{-1}$

is the 2×2-identity matrix. Next we consider the action of the group R^* on G_I by conjugation. For any monomial $m = x^{\mathbf{u}}$ we have $X^{\mathbf{u}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & x^{\mathbf{u}} \end{smallmatrix}\right)$, and the product $A^m = X^{-\mathbf{u}} A X^{\mathbf{u}}$ is equal to $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$. Likewise, we have $A^{-m} = X^{-\mathbf{u}} A^{-1} X^{\mathbf{u}} = \left(\begin{smallmatrix} 1 & -m \\ 0 & 1 \end{smallmatrix}\right)$, so the same identity holds for monomials whose coefficient is -1. In particular, for any monomial m in S, the two matrices A and A^m commute. Hence, in the group $G_{\langle 0 \rangle}$ we have

$$[X_i, X_j] = [A, A^m] = 1 \text{ for } 1 \le i < j \le n \text{ and monomials } m \in S^*. \tag{1.6.6}$$

Lemma 1.6.7. *The relations* (1.6.6) *define a presentation of the group* $G_{\langle 0 \rangle}$.

We next extend this to all ideals I. For any $f \in S$, consider the matrix

$$A^f = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}.$$

Matrix products such as (1.6.2) show that A^f lies in $G_{\langle 0 \rangle}$.

Proposition 1.6.8. *For any ideal I in S, the group G_I has the presentation*

$$[X_i, X_j] = [A, A^m] = A^f = 1, \tag{1.6.7}$$

where m runs over monomials, f runs over I, and $1 \le i < j \le n$.

This presentation is infinite. We wish to know whether (1.6.7) can be replaced by a finite subset. Is the group G_I finitely presented? To answer this, we first note that the conjugation action satisfies the relations

$$A^f A^g = A^g A^f = A^{f+g} \quad \text{and} \quad (A^f)^g = (A^g)^f = A^{fg} \text{ for } f, g \in S.$$

This shows that it suffices to take f from any finite generating set of the ideal I. So, the question is whether there exists a finite subset $\mathcal{U} \subset \mathbb{Z}^n$ such that the monomials $m = \pm x^{\mathbf{u}}$ with $\mathbf{u} \in \mathcal{U}$ suffice in the presentation (1.6.7).

Theorem 1.6.5 offers a criterion for testing whether such a finite set $\mathcal{U}$ exists. For instances in which the answer is affirmative, we can use the techniques in [**BS80**, §3] to construct an explicit generating set $\mathcal{U}$. These techniques are quite delicate and have not yet been developed into an actual algorithm. In what follows we outline a proposal for how to approach this.

The first step is to compute the integral tropical variety $\operatorname{trop}_{\mathbb{Z}}(I)$ from the given generators of I. We would replace I by its homogenization and compute the Gröbner fan. If K is a field, then the Gröbner fan of $I \subset K[x_0, x_1, \ldots, x_n]$ is a polyhedral fan in $\mathbb{R}^{n+1}$ such that the initial ideal $\operatorname{in}_{\mathbf{w}}(I)$ is constant as $\mathbf{w}$ ranges over the relative interior of any cone. See Corollary 2.5.12. However, here we need an extension to $K = \mathbb{Z}$, and this theory has yet to be developed. Gröbner fans over $\mathbb{Z}$ will be finer than those over $K = \mathbb{Q}$. For example, if $I = \langle 2x_1, x_1x_2 - x_1x_3 \rangle$, then the Gröbner fan over $\mathbb{Q}$ consists of a single cone, while the Gröbner fan over $\mathbb{Z}$ has a wall on the plane $\{w_2 = w_3\}$.

In the course of computing the Gröbner fan of I, we would obtain generators for every initial ideal $\mathrm{in}_{\mathbf{w}}(I)$. From these we would derive a finite generating set $\mathcal{B}$ of I with the property that, for every $\mathbf{w} \in \mathbb{R}^n$, either $\mathrm{in}_{\mathbf{w}}(I)$ is a proper ideal in S or the finite set $\{\mathrm{in}_{\mathbf{w}}(f) : f \in \mathcal{B}\}$ contains a unit. A subset $\mathcal{B}$ of the ideal I that enjoys this property is a *tropical basis* over $\mathbb{Z}$. Every Laurent polynomial in a tropical basis $\mathcal{B}$ can be scaled by a unit, so we can always assume that the relevant leading monomial is the constant 1.

Suppose now that I is an ideal in S which satisfies the condition of Theorem 1.6.5 and that we have computed a tropical basis $\mathcal{B}$ for I. Then

(1.6.8) For all $\mathbf{w} \in \mathbb{R}^n$ there is $f \in \mathcal{B}$ with $\mathrm{in}_{\mathbf{w}}(f) = 1$ or $\mathrm{in}_{-\mathbf{w}}(f) = 1$.

For each Laurent polynomial f in the tropical basis $\mathcal{B}$, let support(f) denote the set of all vectors $\mathbf{a} \in \mathbb{Z}^n$ such that the monomial $x^{\mathbf{a}}$ appears with nonzero coefficient in f. We define the Newton polytope of the tropical basis $\mathcal{B}$ as the convex hull of the union of these support sets for all f in $\mathcal{B}$:

$$\mathrm{Newt}(\mathcal{B}) \quad := \quad \mathrm{conv}\Big(\bigcup_{f \in \mathcal{B}} \mathrm{support}(f)\Big).$$

By examining the proof technique used in [**BS80**, §3.5], one can derive the following explicit version of the "if" direction in the Bieri–Strebel Theorem:

Theorem 1.6.9. *Fix a tropical basis $\mathcal{B}$ satisfying* (1.6.8) *for the ideal I. Then the metabelian group G_I is presented by the relations* (1.6.6)*, where f runs over the elements in the tropical basis $\mathcal{B}$ and $m = x^{\mathbf{u}}$ runs over the set* $\mathrm{Newt}(\mathcal{B}) \cap \mathbb{Z}^n$ *of lattice points $\mathbf{u}$ in the Newton polytope the tropical basis.*

Example 1.6.10. Let $n = 1$, and let I be the prime ideal of $\xi = \sqrt{2} + \sqrt{3}$. The singleton $\mathcal{B} = \{x^4 - 10x^2 + 1\}$ is a tropical basis of I satisfying (1.6.8). Then the group $G_\xi = G_I$ is presented by five relations. The first relation is the word in (1.6.3), and the other four required relations are the words

$$[A, A^{x^i}] \; = \; AX^{-i}AX^iA^{-1}X^{-i}A^{-1}X^i \quad \text{for } i = 1, 2, 3, 4. \qquad \diamondsuit$$

Example 1.6.11. Consider the group in [**Str84**, Example 43]. Let $S = \mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$ with $I = \langle\, f \,\rangle$ generated by the polynomial in Example 1.6.3:

$$f(s,t) \;\; = \;\; ms^{-1}t^{-1} + s^{-1} + t^{-1} + n + st.$$

The tropical variety $\mathrm{trop}_{\mathbb{Z}}(I)$ contains no line. A minimal tropical basis satisfying the condition (1.6.8) consists of three Laurent polynomials:

$$\mathcal{B} \;=\; \big\{\, s^{-1}t^{-1}f(s,t)\,,\; sf(s,t),\; tf(s,t) \,\big\}.$$

The corresponding polytope $\mathrm{Newt}(\mathcal{B})$ is a planar convex 7-gon that has 14 lattice points, corresponding to the 14 Laurent monomials:

$$m \;\;=\;\; s^2t,\; st^2,\; st,\; s,\; s/t,\; t,\; 1,\; 1/t,\; t/s,\; 1/s,\; 1/st,\; 1/st^2,\; 1/s^2t,\; 1/s^2t^2.$$

The metabelian group G_I has three generators A, X_1, X_2. The description in Theorem 1.6.9 gives a presentation with $17 = 3 + 14$ relations. ♢

In this section we saw a connection between Gröbner bases over $\mathbb{Z}$ and tropical geometry. The beautiful group theory results by Bieri, Groves, Neumann, and Strebel suggest that further research on this topic is desirable.

1.7. Curve Counting

The breakthrough that brought tropical methods to the attention of geometers was the work of Mikhalkin [**Mik05**] on Gromov–Witten invariants of the plane. These invariants count the number of complex algebraic curves of a given degree and genus passing through a given number of points. Mikhalkin proved that complex curves can be replaced by tropical curves, and he then derived a combinatorial formula for the count in the tropical case. We already saw the first case of this result in Section 1.3: there is a unique tropical line (degree 1 curve) through two general points in $\mathbb{R}^2$. The objective of this section is to present the basic ideas and the main result.

We begin by reviewing some classical facts about curves in the complex projective plane $\mathbb{P}^2$. If C is a smooth curve of degree d in $\mathbb{P}^2$, then its *genus* is the number of handles of the Riemann surface of C. That genus equals

$$g(C) \;=\; \frac{1}{2}(d-1)(d-2).$$

Moreover, that same number counts the lattice points in the interior of the Newton polygon of the general curve of degree d. That Newton polygon is the triangle with vertices $(d, 0, 0)$, $(0, d, 0)$, and $(0, 0, d)$. In symbols,

$$g(C) \;=\; \#\bigl(\mathrm{int}(\mathrm{Newt}(C)) \cap \mathbb{Z}^3\bigr).$$

The set of all curves of degree d forms a projective space of dimension

$$\binom{d+2}{2} - 1 \;=\; \frac{1}{2}(d-1)(d-2) \,+\, 3d-1. \tag{1.7.1}$$

As the $\binom{d+2}{2}$ coefficients of its defining polynomial vary, the curve C may acquire one or more singular points. The simplest type of singularity is a *node*. Each time the curve acquires a node, the genus drops by one. Thus for a singular curve C_{sing} with ν nodes and no other singularities, the genus is

$$g(C_{\mathrm{sing}}) \;=\; \frac{1}{2}(d-1)(d-2) - \nu. \tag{1.7.2}$$

We are interested in the following problem of enumerative geometry: *What is the number $N_{g,d}$ of irreducible curves of genus g and degree d that pass through $g + 3d - 1$ general points in the complex projective plane $\mathbb{P}^2$?*

This question makes sense because the moduli space of curves of degree d and genus g is expected to have dimension $g+3d-1$, by (1.7.1) and (1.7.2), since acquiring a node poses a codimension-1 condition on the curve. Thus we expect the number $N_{g,d}$ of curves satisfying all constraints to be finite. Gromov–Witten theory offers the tools for proving this finiteness result.

The numbers $N_{g,d}$ are called *Gromov–Witten invariants* of the plane $\mathbb{P}^2$. Their study has been a topic of considerable interest among geometers.

Example 1.7.1. The simplest Gromov–Witten invariants are $N_{0,1} = 1$ and $N_{0,2} = 1$. This translates into saying that a unique line passes through two points and that a unique quadric passes through five points. We also have $N_{1,3} = 1$, which says that a unique cubic passes through nine points. ♢

Example 1.7.2. The first nontrivial number is $N_{0,3} = 12$, and we wish to explain this in some detail. It concerns curves defined by cubic polynomials

$$f = c_0x^3+c_1x^2y+c_2x^2z+c_3xy^2+c_4xyz+c_5xz^2+c_6y^3+c_7y^2z+c_8yz^2+c_9z^3.$$

For general coefficients $c_0, \ldots, c_9$, the curve $\{f = 0\}$ is smooth of genus $g = 1$. The curve becomes rational, i.e., the genus drops to $g = 0$, precisely when it has a singular point. This happens if and only if the *discriminant* of f vanishes. The discriminant $\Delta(f)$ is a homogeneous polynomial of degree 12 in the ten unknown coefficients $c_0, c_1 \ldots, c_9$. It is a sum of 2040 monomials:

$$\Delta(f) = 19683c_0^4c_6^4c_9^4 - 26244c_0^4c_6^3c_7c_8c_9^3 + \cdots - c_2^2c_3c_4^4c_5^3c_6^2. \tag{1.7.3}$$

The study of discriminants and resultants is the topic of the book by Gel'fand, Kapranov, and Zelevinsky [**GKZ08**], which contains many formulas for computing them. Here is a simple determinantal formula for (1.7.3). The Hessian H of the quadrics $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ is a polynomial of degree 3. Form the 6×6-matrix $M(f)$ whose entries are the coefficients of the six quadrics $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$, $\frac{\partial H}{\partial x}$, $\frac{\partial H}{\partial y}$, and $\frac{\partial H}{\partial z}$. Then the discriminant (1.7.3) equals $\Delta(f) = \det(M(f))$.

Now, suppose the cubic $\{f = 0\}$ is required to pass through eight given points in $\mathbb{P}^2$. This translates into eight linear equations in $c_0, c_1, \ldots, c_9$. Combining the eight linear equations with the degree 12 equation $\Delta(f) = 0$, we obtain a system of equations that has 12 solutions in $\mathbb{P}^9$. These solutions are the coefficient vectors of the $N_{0,3} = 12$ rational cubics that we seek. ♢

Example 1.7.3. Quartic curves in the plane $\mathbb{P}^2$ can have genus 3, 2, 1, or 0. The Gromov–Witten numbers corresponding to these four cases are

$$N_{3,4} = 1, \; N_{2,4} = 27, \; N_{1,4} = 225, \text{ and } N_{0,4} = 620.$$

Here 27 is the degree of the discriminant of a ternary quartic. The last entry means that there are 620 rational quartics through 11 general points. ♢

The result of Mikhalkin [**Mik05**] can be stated informally as follows:

Theorem 1.7.4. *The Gromov–Witten numbers $N_{g,d}$ can be found tropically.*

The following discussion is aimed at making precise what this means. We consider tropical curves of degree d in $\mathbb{R}^2$. Each such curve C is the planar dual graph to a regular subdivision of the triangle with vertices $(0,0)$, $(0,d)$, and $(d,0)$. We say that the curve C is *smooth* if this subdivision consists of d^2 triangles each having unit area $1/2$. Equivalently, the tropical curve C is smooth if it has d^2 vertices. These vertices are necessarily trivalent.

We already encountered smoothness of tropical curves in Section 1.3. Proposition 4.5.1 explains this for hypersurfaces in arbitrary dimensions. The next property for plane tropical curves is more inclusive than "smooth".

A tropical curve C is called *simple* if each vertex is either trivalent or is locally the intersection of two line segments. Equivalently, C is simple if the corresponding subdivision consists only of triangles and parallelograms. Here the triangles are allowed to have large area. Let $t(C)$ be the number of trivalent vertices, and let $r(C)$ be the number of unbounded edges of C.

We define the *genus* of a simple tropical curve C by the formula

$$g(C) \;=\; \frac{1}{2}t(C) - \frac{1}{2}r(C) + 1. \qquad (1.7.4)$$

It is instructive to check that this definition makes sense for smooth tropical curves. Indeed, if C is smooth, then $t(C) = d^2$ and $r(C) = 3d$, and we recover the formula for the genus of a smooth classical complex curve:

$$g(C) \;=\; \frac{1}{2}d^2 - \frac{1}{2}3d + 1 \;=\; \frac{1}{2}(d-1)(d-2).$$

We finally define the *contribution* of a simple curve C as the product of the normalized areas of all triangles in the corresponding subdivision. Thus, in computing the contribution of C, we disregard the "nodal singularities", which correspond to 4-valent crossings. We just multiply positive integers attached to the trivalent vertices. The contribution of a trivalent vertex equals $w_1 w_2 |\det(\mathbf{u}_1, \mathbf{u}_2)|$, where w_1, w_2, w_3 are the weights of the adjacent edges and $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are their primitive edge directions. That formula is independent of the choice made because of the balancing condition $w_1\mathbf{u}_1 + w_2\mathbf{u}_2 + w_3\mathbf{u}_3 = 0$. If the curve is smooth, then its contribution equals 1.

Here now is the precise statement of what was meant in Theorem 1.7.4:

Theorem 1.7.5 (Mikhalkin's Correspondence Principle)**.** *The number of simple tropical curves of degree d and genus g that pass through $g + 3d - 1$ general points in $\mathbb{R}^2$, where each curve is counted with its contribution, equals the Gromov–Witten number $N_{g,d}$ of the complex projective plane $\mathbb{P}^2$.*

The proof of Theorem 1.7.5 given by Mikhalkin in [**Mik05**] uses methods from complex geometry, specifically, deformations of J-holomorphic curves. Subsequently, Gathmann and Markwig [**GM07a, GM07b**] developed an algebraic approach. See also the work of Tyomkin [**Tyo12**]. Mikhalkin's Correspondence Principle led to the systematic development of tropical moduli spaces and tropical intersection theory on such spaces.

We close with one more example of what can be done with tropical curves in enumerative geometry. The Gromov–Witten invariants $N_{0,d}$ for rational curves (genus $g = 0$) satisfy the following remarkable recursion:

$$(1.7.5) \qquad N_{0,d} \;=\; \sum_{\substack{d_1+d_2=d\\ d_1,d_2>0}} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} \;-\; d_1^3 d_2 \binom{3d-4}{3d_1-1} \right) N_{0,d_1} N_{0,d_2}.$$

This equation is due to Kontsevich, who derived it from the WDVV equations, named after the theoretical physicists Witten, Dijkgraaf, Verlinde, and Verlinde, which express the associativity of quantum cohomology of $\mathbb{P}^2$.

Using Mikhalkin's Correspondence Principle, Gathmann and Markwig [**GM08**] gave a proof of this formula using tropical methods. Namely, they establish the combinatorial result that simple tropical curves of degree d and genus 0 passing through $3d-1$ points satisfy the Kontsevich relations (1.7.5).

Students wishing to learn the foundations of tropical geometry as it pertains to the topic of this section are referred to the text by Mikhalkin and Rau [**MR**]. The present book does not contain a proof of Theorem 1.7.5. We do not focus on metric graphs, curves, and their moduli. Instead we study embedded tropical varieties that are derived from polynomial ideals.

1.8. Compactifications

Many of the advanced tools of algebraic geometry, such as intersection theory, are custom tailored for varieties that are compact, such as complex projective varieties. Yet, in concrete problems, the given spaces are often not compact. In such a case one first needs to replace the given variety X by a nice compact variety $\overline{X}$ that contains X as dense subset. Here the emphasis lies on the adjective "nice" because the advanced tools will not work or will give incorrect answers if the boundary $\overline{X}\backslash X$ is not good enough.

We begin by considering a nonsingular curve X in the n-dimensional complex torus $(\mathbb{C}^*)^n$. The curve X is not compact, and we wish to add a finite set of points to X so as to get a smooth compactification $\overline{X}$ of X.

From a geometric point of view, it is clear what must be done. Identifying the complex plane $\mathbb{C}$ with $\mathbb{R}^2$, the curve X becomes a surface. More precisely, X is a noncompact Riemann surface. It is an orientable smooth compact surface of some genus g with a certain number m of points removed.

The problem is to identify the m missing points and to fill them back in. What is the algebraic procedure that accomplishes this geometric process?

To illustrate the algebraic complications, we begin with a plane curve

$$X \;=\; \{\,(x,y)\in(\mathbb{C}^*)^2 \,:\, f(x,y)=0\,\}.$$

Our smoothness hypothesis says that the Laurent polynomial equations

$$(1.8.1)\qquad f(x,y) \;=\; \frac{\partial f}{\partial x}(x,y) \;=\; \frac{\partial f}{\partial y}(x,y) \;=\; 0$$

have no common solutions (x,y) in the algebraic torus $(\mathbb{C}^*)^2$. A first attempt at compactifying X is to replace $f(x,y)$ with the homogeneous polynomial

$$f^{\mathrm{hom}}(x,y,z) \;=\; z^N\cdot f\bigl(\frac{x}{z},\frac{y}{z}\bigr).$$

Here N is the smallest integer such that this expression is a polynomial. This homogeneous polynomial defines a curve in the complex projective plane $\mathbb{P}^2$:

$$X^{\mathrm{hom}} \;=\; \bigl\{\,(x:y:z)\in\mathbb{P}^2 \,:\, f^{\mathrm{hom}}(x,y,z)=0\,\bigr\}.$$

This curve is a compactification of X but it is usually not what we want.

Example 1.8.1. Let X be the curve in $(\mathbb{C}^*)^2$ defined by the polynomial

$$(1.8.2)\qquad f(x,y) \;=\; c_1 + c_2xy + c_3x^2y + c_4x^3y + c_5x^3y^2.$$

Here c_1,c_2,c_3,c_4,c_5 are any complex numbers that satisfy

$$(1.8.3)\quad c_2c_3^4 - 8c_2^2c_3^2c_4 + 16c_2^3c_4^2 - c_1c_3^3c_5 + 36c_1c_2c_3c_4c_5 - 27c_1^2c_4c_5^2 \;\neq\; 0.$$

This condition ensures that the given noncompact curve X is smooth. The discriminant polynomial in (1.8.3) is computed by eliminating x and y from (1.8.1). The homogenization of the polynomial $f(x,y)$ equals

$$f^{\mathrm{hom}}(x,y,z) \;=\; c_1z^5 + c_2xyz^3 + c_3x^2yz^2 + c_4x^3yz + c_5x^3y^2.$$

The corresponding projective curve X^{hom} in $\mathbb{P}^2$ is compact but it is not smooth. The boundary we have added to compactify consists of two points

$$X^{\mathrm{hom}}\backslash X \;=\; \bigl\{\,(1:0:0),\,(0:1:0)\,\bigr\}.$$

Both of these points are singular on the compact curve X^{hom}. Their respective multiplicities are 2 and 3. In this context, *multiplicities* refers to the lowest degrees seen in $f^{\mathrm{hom}}(1,y,z)$ and $f^{\mathrm{hom}}(x,1,z)$, respectively.

Another thing one might try is the closure of our curve $X\subset(\mathbb{C}^*)^2$ in the product of two projective lines $\mathbb{P}^1\times\mathbb{P}^1$. Then the ambient coordinates are $\bigl((x_0:x_1),(y_0:y_1)\bigr)$, and our polynomial is replaced by its *bihomogenization*

$$x_0^3y_0^2f\bigl(\frac{x_1}{x_0},\frac{y_1}{y_0}\bigr) \;=\; c_1x_0^3y_0^2 + c_2x_1y_1x_0^2y_0 + c_3x_1^2y_1x_0y_0 + c_4x_1^3y_1y_0 + c_5x_1^3y_1^2.$$

The compactification X^{bihom} of X is the zero set of this polynomial in $\mathbb{P}^1 \times \mathbb{P}^1$. Now, the boundary we have added to compactify consists of three points

$$X^{\text{bihom}} \backslash X \;=\; \big\{ \big((1:0),(0:1)\big),\; \big((0:1),(1:0)\big),\; \big((0:1),(c_5,-c_4)\big) \big\}.$$

The compactification X^{bihom} is better than X^{hom} but is still mildly singular. The first point above is singular, of multiplicity 2, but the last two points are smooth on X^{bihom}. They correctly fill in two of the holes in X. $\Diamond$

The solution to our problem offered by tropical geometry is to replace a given noncompact variety $X \subset (\mathbb{C}^*)^n$ by a *tropical compactification* X^{trop}. Each such compactification of X is characterized by a polyhedral fan in $\mathbb{R}^n$ whose support is the tropical variety corresponding to X. When the dimension or codimension is small, there is a unique coarsest fan structure. This includes all curves and all hypersurfaces. In these cases we obtain a canonical tropical compactification. However, in general, picking a tropical compactification requires making choices, and X^{trop} will depend on these choices. See Example 3.5.4 for a concrete illustration.

Tropical compactifications were introduced by Jenia Tevelev in [**Tev07**]. The geometric foundation for his construction is the theory of *toric varieties*. In Chapter 6, we shall explain the relationship between toric varieties and tropical geometry. In Section 6.4 we shall see the precise definition of tropical compactifications X^{trop} of a variety $X \subset (\mathbb{C}^*)^n$, and we shall prove its key geometric properties. In what follows, we keep the discussion informal and entirely elementary, and we simply go over a few examples.

Example 1.8.2. Let X be the plane complex curve in (1.8.2). Its tropical compactification X^{trop} is a smooth elliptic curve, i.e., it is a Riemann surface of genus $g = 1$. The boundary $X^{\text{trop}} \backslash X$ consists of $m = 4$ points. Unlike the extra points in the bad compactifications X^{hom} and X^{bihom} in Example 1.8.1, these four new points are smooth on X^{trop}. This confirms that the complex curve X is a real torus with $m = 4$ points removed.

The tropical compactification of a plane curve is derived from its Newton polygon, here the quadrilateral $\text{Newt}(f) = \text{conv}\big\{(0,0),(1,1),(3,2),(3,1)\big\}$. The genus g of X is the number of interior lattice points of $\text{Newt}(f)$.

The tropical curve is the union of the inner normal rays to the four edges of this quadrilateral. In other words, $\text{trop}(X)$ consists of the four rays spanned by $(1,-1)$, $(1,-2)$, $(-1,0)$, and $(-1,3)$. Each ray has multiplicity one because the edges of $\text{Newt}(f)$ have lattice length 1. This shows that $m = 4$ points need to be added to X to get X^{trop}. The directions of the rays specify how these new points should be glued into X in order to make them smooth in X^{trop}. Algebraically, this process can be described by replacing the given polynomial f by a certain homogeneous polynomial f^{trop}, but the homogenization process is now more tricky. One uses *homogeneous* *coordinates*, on the toric surface given by Newt(f). These generate the Cox homogeneous coordinate ring, to be defined in Section 6.1. Here, it suffices to think of the homogeneous coordinates we know for $\mathbb{P}^2$ and for $\mathbb{P}^1 \times \mathbb{P}^1$. $\Diamond$

The example of plane curves has two natural generalizations in $(\mathbb{C}^*)^n$, $n \geq 3$, namely curves and hypersurfaces. We briefly discuss both of these.

If X is a curve in $(\mathbb{C}^*)^n$, then the geometry is still easy. All we will do is fill in m missing points in a punctured Riemann surface of genus g. However, the algebra is more complicated than in Example 1.8.2. The curve X is given by an ideal $I \subset \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Our primary challenge is to determine the number m from I. The number m is the sum of the multiplicities of the rays in the tropicalization of X. The tropical curve trop(X) is a finite union of rays in $\mathbb{R}^n$ but it is generally impossible to find these rays from (the Newton polytopes of) the given generators of I. To understand how trop(X) arises from I, one needs the concepts pertaining to Gröbner bases and initial ideals, which will be introduced in Chapters 2 and 3. In practice, the software `Gfan`, due to Anders Jensen [**Jen**], can be used to compute the tropical curve trop(X) and the multiplicity of each of its rays.

If X is a hypersurface in $(\mathbb{C}^*)^n$, then the roles are reversed. The algebra is still easy but the geometry is more complicated now than in Example 1.8.2. Let $f = f(x_1, \ldots, x_n)$ be the polynomial that defines X. We compute its Newton polytope Newt(f) $\subset \mathbb{R}^n$, as introduced in Definition 2.3.4.

The tropical compactification X^{trop} has one boundary divisor for each facet of Newt(f). These boundary divisors are varieties of dimension $n-2$. They get glued to the $(n-1)$-dimensional variety X in order to create the compact $(n-1)$-dimensional variety X^{trop}. The precise nature of this gluing is determined by the ray normal to the facet. What is different from the curve case is that the boundary divisors are themselves nontrivial hypersurfaces in X, and they are no longer pairwise disjoint. In fact, describing their intersection pattern in $X^{\text{trop}} \backslash X$ is an essential part of the construction. The relevant combinatorics is encoded in the facial structure of the polytope Newt(f), and we record this data in the tropical hypersurface.

Tropical geometry furnishes such a compactification for any subvariety X of the algebraic torus $(\mathbb{C}^*)^n$. Starting from an ideal $I \subset \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with $X = V(I)$, we can compute the tropical variety trop(X). For small examples this can be done by hand, but for larger examples we use software such as `Gfan` for that computation. The output is a polyhedral fan Δ in $\mathbb{R}^n$ whose support $|\Delta|$ equals trop(X). That fan determines a tropical compactification $X^{\text{trop}}(\Delta)$ of the variety X. Now, this compactification may not be quite nice enough, so one sometimes has to replace the fan Δ by a refinement Δ'. This induces a map $X^{\text{trop}}(\Delta') \to X^{\text{trop}}(\Delta)$. For example, Δ may not be a simplicial fan, and we could take Δ' to be a smooth fan that triangulates Δ. Further, we may want to require the flatness condition in Definition 6.4.13.

Let us consider the case when X is an irreducible surface in $(\mathbb{C}^*)^n$. In any compactification $\overline{X}$ of X, the boundary $\overline{X}\backslash X$ is a finite union of irreducible curves. What is desired is that these curves are smooth and that they intersect each other transversally. If this holds, then the boundary $\overline{X}\backslash X$ has *normal crossings*. The tropical compactifications of a surface X usually have the normal crossing property. Here the tropical variety $\mathrm{trop}(X)$ supports a two-dimensional fan in $\mathbb{R}^n$. Such a fan has a unique coarsest fan structure. We identify the tropical surface $\mathrm{trop}(X)$ with that coarsest fan Δ, and we abbreviate $X^{\mathrm{trop}} = X^{\mathrm{trop}}(\Delta)$. The rays in the fan $\mathrm{trop}(X)$ correspond to the irreducible curves in $\overline{X}\backslash X$, and two such curves intersect if and only if the corresponding rays span a two-dimensional cone. Since the fan $\mathrm{trop}(X)$ is two dimensional, it has no cones of dimension ≥ 3. Hence, the intersection of any three of the irreducible curves in $\overline{X}\backslash X$ is empty.

Example 1.8.3. Let I be the ideal minimally generated by three linear polynomials $a_1x_1+a_2x_2+a_3x_3+a_4x_4+a_5x_5+a_6$ in $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}, x_5^{\pm 1}]$. Its variety X is a noncompact surface in $(\mathbb{C}^*)^5$. If we took the variety of I in affine space $\mathbb{C}^5$, then this would simply be an affine plane $\mathbb{C}^2$. But the torus $(\mathbb{C}^*)^5$ is obtained from $\mathbb{C}^5$ by removing the hyperplanes $\{x_i = 0\}$. Hence our noncompact surface X equals the affine plane $\mathbb{C}^2$ with five lines removed. Equivalently, X is the complex projective plane $\mathbb{P}^2$ with six lines removed.

If the three linear generators of I have random coefficients, then the six lines form a normal crossing arrangement in $\mathbb{P}^2$ and the tropical compactification simply fills the six lines back in, so that $X^{\mathrm{trop}} = \mathbb{P}^2$. Here, $\mathrm{trop}(X)$ consists of six rays and the 15 two-dimensional cones spanned by any two of the rays. Five of the rays are spanned by the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5$ of $\mathbb{R}^5$, and the sixth ray is spanned by $-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3-\mathbf{e}_4-\mathbf{e}_5$.

The situation is more interesting if the generators of I are special, e.g.,

$$I \quad = \quad \langle\, x_1 + x_2 - 1,\; x_3 + x_4 - 1,\; x_1 + x_3 + x_5 - 1 \,\rangle. \tag{1.8.4}$$

For this particular ideal, the configuration of six lines in $\mathbb{P}^2$ has four triples of lines that meet in one point. Two of these special intersection points are

$$\{x_1 = x_4 = x_5 = 0,\; x_2 = x_3 = 1\} \text{ and } \{x_2 = x_3 = x_5 = 0,\; x_1 = x_4 = 1\}.$$

The other two points lie on the line at infinity, where they are determined by $\{x_1 = x_2 = 0\}$ and $\{x_3 = x_4 = 0\}$, respectively. The tropical compactification is constructed by blowing up these four special points. This process replaces each triple intersection point with a new line that meets the three old lines transversally at three distinct points. Thus X^{trop} is a compact surface whose boundary $X^{\mathrm{trop}}\backslash X$ consists of ten lines, namely, the six old lines that had been removed from $\mathbb{P}^2$ plus the four new lines from blowing up. Now, no three lines intersect, so the boundary $X^{\text{trop}}\backslash X$ is normal crossing. There are 15 pairwise intersection points, three on each of the four new lines, and three old intersection points. The latter are determined by $\{x_1 = x_3 = 0\}$, $\{x_2 = x_4 = 0\}$ and by intersecting $\{x_5 = 0\}$ with the line at infinity.

The combinatorics of this situation is encoded in the tropical plane $\text{trop}(X)$. It consists of 15 two-dimensional cones which are spanned by ten rays. The rays correspond to the ten lines. The rays are spanned by

$$\mathbf{e}_1\,,\ \mathbf{e}_2\,,\ \mathbf{e}_3\,,\ \mathbf{e}_4\,,\ \mathbf{e}_5\,,\ -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5,$$
$$\mathbf{e}_1 + \mathbf{e}_4 + \mathbf{e}_5,\ \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_5,\ -\mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5,\ -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_5.$$

The tropical plane $\text{trop}(X)$ is the cone over the *Petersen graph*, shown in Example 4.1.12. The ten vertices of the Petersen graph correspond to the ten lines in $X^{\text{trop}}\backslash X$, and the 15 edges of the Petersen graph correspond to the pairs of lines that intersect on the tropical compactification X^{trop}. ♢

The previous example shows that tropical compactifications are nontrivial and interesting even for linear ideals I. Since linear ideals cut out linear spaces, we refer to $\text{trop}(X)$ as a *tropical linear space*. The combinatorics of tropical linear spaces is governed by the theory of *matroids*. This will be explained in Chapter 4. In the linear case, the open variety $X \subset (\mathbb{C}^*)^n$ is the complement of an arrangement of $n+1$ hyperplanes in a projective space, and the tropical compactification X^{trop} was already known before the advent of tropical geometry. It is essentially equivalent to the *wonderful compactifications* of a hyperplane arrangement complement due to De Concini and Procesi. This was shown in [**FS05**, Theorem 6.1].

1.9. Exercises

(1) Consider the 2×2-matrices $A = \left(\begin{smallmatrix} 2 & 3 \\ 5 & 9 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 9 & 5 \\ 3 & 2 \end{smallmatrix}\right)$. Compute $A \odot B$ and $A \oplus B$ tropically. Also compute $A \oplus A^2 \oplus \cdots \oplus A^{1000}$.

(2) Formulate and prove the Fundamental Theorem of Algebra in the tropical setting. Why is the tropical semiring "algebraically closed"?

(3) Find all roots of the quintic $x^5 \oplus 1{\odot}x^4 \oplus 3{\odot}x^3 \oplus 6{\odot}x^2 \oplus 10{\odot}x \oplus 15$.

(4) Let $p : \mathbb{R}^n \to \mathbb{R}$ be a function that is continuous, concave, and piecewise linear with finitely many pieces that are linear functions with integer coefficients. Show that p can be represented by a tropical polynomial in $x_1, \ldots, x_n$.

(5) Prove the following generalization of Proposition 1.2.1. Let $B \in (\mathbb{R} \cup \{\infty\})^{n \times n}$ be a matrix, and let G be the associated weighted directed graph as in Section 1.2. We now allow negative edge weights,

and G may have loops. Assume that G has no negative cost circuit, so there is no path from a vertex to itself in G for which the sum of the edge weights is negative. Consider the matrix

$$B^+ \;=\; B \oplus B^2 \oplus B^3 \oplus \cdots \oplus B^n.$$

Show that B^+_{ij} is the length of the shortest path from i to j. What goes wrong if G has a negative cost circuit?

(6) Prove Proposition 1.2.3. This concerns the tropical interpretation of the dynamic programming method for integer programming.

(7) Let $D = (d_{ij})$ be a symmetric $n{\times}n$-matrix with zeros on the diagonal and positive off-diagonal entries. We say that D represents a *metric space* if the triangle inequalities $d_{ik} \leq d_{ij} + d_{jk}$ hold for all indices i, j, k. Show that D represents a metric space if and only if the matrix equation $D \odot D = D$ holds.

(8) The tropical $3{\times}3$-determinant is a piecewise-linear real-valued function $\mathbb{R}^{3\times 3} \to \mathbb{R}$ on the nine-dimensional space of 3×3-matrices. Describe all the regions of linearity of this function and their boundaries. What does it mean for a matrix to be tropically singular?

(9) How many combinatorial types of quadratic curves are there?

(10) Prove that the stable self-intersection of a plane curve equals its set of vertices. What does this mean for classical algebraic geometry?

(11) Given five general points in $\mathbb{R}^2$, there exists a unique tropical quadric passing through these points. Compute and draw the quadratic curve through the points $(0,5)$, $(1,0)$, $(4,2)$, $(7,3)$, $(9,4)$.

(12) For any multiset of five points in the plane there is a unique tropical quadric passing through them. Argue how stable intersections can be used to get uniqueness for configurations in special position.

(13) A tropical cubic curve in $\mathbb{R}^2$ is *smooth* if it has precisely nine nodes. Prove that every smooth cubic curve has a unique bounded region, and that this region can have either three, four, five, six, seven, eight, or nine edges. Draw examples for all seven cases.

(14) Install Anders Jensen's software `Gfan` [**Jen**] on your computer. Download the manual and try running a few examples.

(15) Find explicit tropical biquadratic polynomials whose curves look like those shown in Figure 1.3.4.

(16) The amoeba of a curve of degree 4 in the plane $\mathbb{C}^2$ can have either 0, 1, 2, or 3 bounded convex regions in its complement. Construct explicit examples for all four cases.

(17) Determine the logarithmic limit set $\mathcal{A}_\infty(I)$ for the line given by the ideal $I = \langle x_1 + x_2 + 1\rangle$ in $S = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$. Verify Theorem 1.4.2 for this example. How would the picture look for a line in 3-space?

(18) Consider the plane curve given by the parameterization

$$x \;=\; (t-1)^{13}t^{19}(t+1)^{29} \quad \text{and} \quad y = (t-1)^{31}t^{23}(t+1)^{17}.$$

Find the Newton polygon of its implicit equation $f(x, y) = 0$. How many terms do you expect the polynomial $f(x, y)$ to have?

(19) Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in $\mathbb{Z}^n$ that sum to zero: $\mathbf{v}_1 + \cdots + \mathbf{v}_m = 0$. Show that there exists an algebraic curve in $(\mathbb{C}^*)^n$ whose tropical curve in $\mathbb{R}^n$ consists of the rays spanned by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$.

(20) Construct a rational parameterization (1.5.1) of a plane curve whose Newton polygon is an octagon. Give $\phi_1(t)$ and $\phi_2(t)$ explicitly.

(21) Let G_ξ be the group generated by the matrices A and X in (1.6.1) for $\xi = \frac{1}{4}(1 + \sqrt{33})$. Can you construct a finite presentation of G_ξ?

(22) Find a nonzero ideal I in $\mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}]$ with $\mathrm{trop}_{\mathbb{Z}}(I) = \mathbb{R}^n$.

(23) What can the integral tropical variety $\mathrm{trop}_{\mathbb{Z}}(I)$ look like for an ideal I generated by two linear forms in $\mathbb{Z}[x, y, z]$? List all possibilities.

(24) Given 14 general points in the plane $\mathbb{C}^2$, what is the number of rational curves of degree 5 that pass through these 14 points?

(25) The two curves in Figure 1.3.4 are simple. For each of them, compute the genus using the formula in (1.7.4).

(26) Consider a curve X in $(\mathbb{C}^*)^3$ cut out by two general polynomials of degree 2. What is the genus g and the number m of punctures of this Riemann surface? Describe its tropical compactification X^{trop}.

(27) The set of singular 3×3-matrices with nonzero complex entries is a hypersurface X in the torus $(\mathbb{C}^*)^{3\times 3}$. Describe its tropical compactification X^{trop}. How many irreducible components does the boundary $X^{\mathrm{trop}}\backslash X$ have? How do these components intersect?

(28) Prove the tropical Bézout Theorem 1.3.2.

(29) For which values of x are the following matrices tropically singular?

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & x \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & x \end{pmatrix}, \quad \begin{pmatrix} x & 2 & 3 \\ 2 & x & 6 \\ 3 & 6 & x \end{pmatrix}, \quad \begin{pmatrix} x & 2 & 3 & 4 \\ 2 & x & 6 & 8 \\ 3 & 6 & x & 12 \\ 4 & 8 & 12 & x \end{pmatrix}$$

(30) The variety $X \subset (\mathbb{C}^*)^5$ defined by the ideal in (1.8.4) is the complement of an arrangement of six lines in the projective plane $\mathbb{P}^2$. Draw those six lines, and describe X^{trop} in terms of your arrangement.