Preface to Third Edition: Part 1

Algebra is used by virtually all mathematicians, be they analysts, combinatorists, computer scientists, geometers, logicians, number theorists, or topologists. Nowadays, everyone agrees that some knowledge of linear algebra, group theory, and commutative algebra is necessary, and these topics are introduced in undergraduate courses. Since there are many versions of undergraduate algebra courses, I will often review definitions, examples, and theorems, sometimes sketching proofs and sometimes giving more details.[1] Part 1 of this third edition can be used as a text for the first year of graduate algebra, but it is much more than that. It and the forthcoming Part 2 can also serve more advanced graduate students wishing to learn topics on their own. While not reaching the frontiers, the books provide a sense of the successes and methods arising in an area. In addition, they comprise a reference containing many of the standard theorems and definitions that users of algebra need to know. Thus, these books are not merely an appetizer, they are a hearty meal as well.

When I was a student, Birkhoff–Mac Lane, *A Survey of Modern Algebra* [**8**], was the text for my first algebra course, and van der Waerden, *Modern Algebra* [**118**], was the text for my second course. Both are excellent books (I have called this book *Advanced Modern Algebra* in homage to them), but times have changed since their first publication: Birkhoff and Mac Lane's book appeared in 1941; van der Waerden's book appeared in 1930. There are today major directions that either did not exist 75 years ago, or were not then recognized as being so important, or were not so well developed. These new areas involve algebraic geometry, category

[1] It is most convenient for me, when reviewing earlier material, to refer to my own text FCAA: *A First Course in Abstract Algebra*, 3rd ed. [**94**], as well as to LMA, the book of A. Cuoco and myself [**23**], *Learning Modern Algebra from Early Attempts to Prove Fermat's Last Theorem.*

theory,[2] computer science, homological algebra, and representation theory. Each generation should survey algebra to make it serve the present time.

The passage from the second edition to this one involves some significant changes, the major change being organizational. This can be seen at once, for the elephantine 1000 page edition is now divided into two volumes. This change is not merely a result of the previous book being too large; instead, it reflects the structure of beginning graduate level algebra courses at the University of Illinois at Urbana–Champaign. This first volume consists of two basic courses: Course I (Galois theory) followed by Course II (module theory). These two courses serve as joint prerequisites for the forthcoming Part 2, which will present more advanced topics in ring theory, group theory, algebraic number theory, homological algebra, representation theory, and algebraic geometry.

In addition to the change in format, I have also rewritten much of the text. For example, noncommutative rings are treated earlier. Also, the section on algebraic geometry introduces regular functions and rational functions. Two proofs of the Nullstellensatz (which describes the maximal ideals in $k[x_1, \ldots, x_n]$ when k is an algebraically closed field) are given. The first proof, for $k = \mathbb{C}$ (which easily generalizes to uncountable k), is the same proof as in the previous edition. But the second proof I had written, which applies to countable algebraically closed fields as well, was my version of Kaplansky's account [**55**] of proofs of Goldman and of Krull. I should have known better! Kaplansky was a master of exposition, and this edition follows his proof more closely. The reader should look at Kaplansky's book, *Selected Papers and Writings* [**58**], to see wonderful mathematics beautifully expounded.

I have given up my attempted spelling reform, and I now denote the ring of integers mod m by $\mathbb{Z}_m$ instead of by $\mathbb{I}_m$. A star * before an exercise indicates that it will be cited elsewhere in the book, possibly in a proof.

The first part of this volume is called Course I; it follows a syllabus for an actual course of lectures. If I were king, this course would be a transcript of my lectures. But I am not king and, while users of this text may agree with my global organization, they may not agree with my local choices. Hence, there is too much material in the Galois theory course (and also in the module theory course), because there are many different ways an instructor may choose to present this material.

Having lured students into beautiful algebra, we present Course II: module theory; it not only answers some interesting questions (canonical forms of matrices, for example) but it also introduces important tools. The content of a sequel algebra course is not as standard as that for Galois theory. As a consequence, there is much more material here than in Course I, for there are many more reasonable choices of material to be presented in class.

To facilitate various choices, I have tried to make the text clear enough so that students can read many sections independently.

Here is a more detailed description of the two courses making up this volume.

---

[2] *A Survey of Modern Algebra* was rewritten in 1967, introducing categories, as Mac Lane–Birkhoff, *Algebra* [**73**].

Course I

After presenting the cubic and quartic formulas, we review some undergraduate number theory: division algorithm; Euclidian algorithms (finding $d = \gcd(a, b)$ and expressing it as a linear combination), and congruences. Chapter 3 begins with a review of commutative rings, but continues with maximal and prime ideals, finite fields, irreducibility criteria, and euclidean rings, PIDs, and UFD's. The next chapter, on groups, also begins with a review, but it continues with quotient groups and simple groups. Chapter 5 treats Galois theory. After introducing Galois groups of extension fields, we discuss solvability, proving the Jordan-Hölder Theorem and the Schreier Refinement Theorem, and we show that the general quintic is not solvable by radicals. The Fundamental Theorem of Galois Theory is proved, and applications of it are given; in particular, we prove the Fundamental Theorem of Algebra ($\mathbb{C}$ is algebraically closed). The chapter ends with computations of Galois groups of polynomials of small degree.

There are also two appendices: one on set theory and equivalence relations; the other on linear algebra, reviewing vector spaces, linear transformations, and matrices.

Course II

As I said earlier, there is no commonly accepted syllabus for a sequel course, and the text itself is a syllabus that is impossible to cover in one semester. However, much of what is here is standard, and I hope instructors can design a course from it that they think includes the most important topics needed for further study. Of course, students (and others) can also read chapters independently.

Chapter 1 (more precisely, Chapter B-1, for the chapters in Course I are labeled A-1, A-2, etc.) introduces modules over noncommutative rings. Chain conditions are treated, both for rings and for modules; in particular, the Hilbert Basis Theorem is proved. Also, exact sequences and commutative diagrams are discussed. Chapter 2 covers Zorn's Lemma and many applications of it: maximal ideals; bases of vector spaces; subgroups of free abelian groups; semisimple modules; existence and uniqueness of algebraic closures; transcendence degree (along with a proof of Lüroth's Theorem). The next chapter applies modules to linear algebra, proving the Fundamental Theorem of Finite Abelian Groups as well as discussing canonical forms for matrices (including the Smith normal form which enables computation of invariant factors and elementary divisors). Since we are investigating linear algebra, this chapter continues with bilinear forms and inner product spaces, along with the appropriate transformation groups: orthogonal, symplectic, and unitary. Chapter 4 introduces categories and functors, concentrating on module categories. We study projective and injective modules (paying attention to projective abelian groups, namely free abelian groups, and injective abelian groups, namely divisible abelian groups), tensor products of modules, adjoint isomorphisms, and flat modules (paying attention to flat abelian groups, namely torsion-free abelian groups). Chapter 5 discusses multilinear algebra, including algebras and graded algebras, tensor algebra, exterior algebra, Grassmann algebra, and determinants. The last chapter, Commutative Algebra II, has two main parts. The first part discusses "old-fashioned algebraic geometry," describing the relation between zero sets of polynomials (of several variables) and ideals (in contrast to modern algebraic geometry, which extends this discussion using sheaves and schemes). We prove the Nullstellensatz (twice!), and introduce the category of affine varieties. The second part discusses algorithms arising from the division algorithm for polynomials of several variables, and this leads to Gröbner bases of ideals.

There are again two appendices. One discusses categorical limits (inverse limits and direct limits), again concentrating on these constructions for modules. We also mention adjoint functors. The second appendix gives the elements of topological groups. These appendices are used earlier, in Chapter B-4, to extend the Fundamental Theorem of Galois Theory from finite separable field extensions to infinite separable algebraic extensions.

I hope that this new edition presents mathematics in a more natural way, making it simpler to digest and to use.

I have often been asked whether solutions to exercises are available. I believe it is a good idea to have some solutions available for undergraduate students, for they are learning new ways of thinking as well as new material. Not only do solutions illustrate new techniques, but comparing them to one's own solution also builds confidence. But I also believe that graduate students are already sufficiently confident as a result of their previous studies. As Charlie Brown in the comic strip *Peanuts* says,

> "In the book of life, the answers are not in the back."

Acknowledgments

The following mathematicians made comments and suggestions that greatly improved the first two editions: Vincenzo Acciaro, Robin Chapman, Daniel R. Grayson, Ilya Kapovich, T.-Y. Lam, David Leep, Nick Loehr, Randy McCarthy, Patrick Szuta, and Stephen Ullom. I thank them again for their help.

For the present edition, I thank T.-Y. Lam, Bruce Reznick, and Stephen Ullom, who educated me about several fine points, and who supplied me with needed references.

I give special thanks to Vincenzo Acciaro for his many comments, both mathematical and pedagogical, which are incorporated throughout the text. He carefully read the original manuscript of this text, apprising me of the gamut of my errors, from detecting mistakes, unclear passages, and gaps in proofs, to mere typos. I rewrote many pages in light of his expert advice. I am grateful for his invaluable help, and this book has benefited much from him.

Joseph Rotman

Urbana, IL, 2015

.