Chapter 1

Set Theory and Metric Spaces

In this chapter we revisit basic notions of set theory and metric spaces. We consider sets in a naive fashion. As Cantor said, "A *set* is a collection into a whole of definite, distinct objects of our intuition or our thought." We say that the sets A and B are *equivalent* if there is a 1-1, onto function $f : A \to B$. A *finite* set is one that is empty, denoted $\emptyset$, or equivalent to $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$; any set that is not finite is called *infinite*. Infinite sets equivalent to $\mathbb{N}$ are called *countable*, all other infinite sets are *uncountable*. Often the term countable is applied to a set that is equivalent to any subset of $\mathbb{N}$.

Equivalent sets cannot essentially be told apart, which motivates the following informal definition. We associate with a set A its *cardinal number*, denoted $\text{card}(A)$ or a, with the property that any two equivalent sets have the same cardinality. 0 is the cardinal number of the class of sets equivalent to $\emptyset$, n that of $\{1, \ldots, n\}$, $\aleph_0$ that of $\mathbb{N}$, and c that of $[0, 1]$ or $\mathbb{R}$.

The inclusion relation for sets translates into a comparison relation for cardinal numbers. More precisely, given cardinals a, b, we say that $a \le b$ if there are sets A, B with $\text{card}(A) = a$ and $\text{card}(B) = b$ such that A is equivalent to a subset of B. The Cantor-Bernstein-Schröder theorem asserts that if there exist injective functions $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \to B$. Thus if $\text{card}(A) \le \text{card}(B)$ and $\text{card}(B) \le \text{card}(A)$, then $\text{card}(A) = \text{card}(B)$.

As for the arithmetic operations, given cardinals a, b and disjoint sets A, B with $\text{card}(A) = a$ and $\text{card}(B) = b$, $a + b$ is defined as the cardinal of

$A \cup B$; the product $a \cdot b$ is similarly defined as the cardinal of $A \times B$. And b^a is defined as the cardinal of B^A, the collection of maps from A into B.

We say that $(M, \prec)$ is an *ordered set* if the relation $\prec$ on $M \times M$ is a partial order on M, i.e., it satisfies the following three properties: (i) $m \prec m$ for every $m \in M$. (ii) If $m_1 \prec m_2$ and $m_2 \prec m_1$, then $m_1 = m_2$. (iii) If $m_1 \prec m_2$ and $m_2 \prec m_3$, then $m_1 \prec m_3$.

Given an ordered set $(M, \prec)$, we say that $m \in M$ is the *first element* of M if m precedes any other element of M. We say that an ordered set $(M, \prec)$ is *well-ordered* if it has a first element and any of its subsets ordered with the restriction order has a first element. Zermelo proved that every set can be well-ordered provided the axiom of choice is assumed. In one of its equivalent formulations the axiom of choice states that given an arbitrary family $\mathcal{A} = \{A_i : i \in I\}$ of nonempty sets indexed by a (nonempty) set I, there exists a function $f : I \to \bigcup_{i \in I} A_i$, called the choice or selection function, such that $f(i) \in A_i$ for each $i \in I$.

In particular, the axiom of choice is equivalent to Zorn's lemma — or Zorn's dilemma as Zorn used to say — which can be stated as follows. M is said to be *totally ordered* if for any $m \neq m' \in M$, $m \prec m'$ or $m' \prec m$. And $M' \subset M$ is said to have an *upper bound* $m \in M$ if $m' \prec m$ for all $m' \in M'$; note that m need not be an element of M'. An element $m \in M$ is said to be *maximal* if there is no $m' \in M$ so that $m \prec m'$. Finally, we say that $A \subset M$ is a *chain* in M if A, equipped with the induced order relation $\prec |_A$, is totally ordered. Zorn's lemma asserts that if every chain in a partially ordered set $(M, \prec)$ has an upper bound, then $(M, \prec)$ has a maximal element.

As an immediate application of Zorn's lemma it follows that a linear space over a field contains a maximal linearly independent set, i.e., a basis. A Hamel basis is a basis of $\mathbb{R}$ as a linear space over $\mathbb{Q}$.

Finally, we need *ordinals*, in particular one, Ω. By Zermelo's theorem there exist uncountable well-ordered sets and there is one with the property that all of its initial segments are countable. The ordinal of this set is denoted Ω.

Recall that a *metric space* (X, d) is a nonempty set X together with a nonnegative real-valued function d on $X \times X$, called a metric, such that for all $x, y, z \in X$ the following three properties hold: (i) $d(x, y) = 0$ iff $x = y$. (ii) $d(x, y) = d(y, x)$. (iii) $d(x, y) \le d(x, z) + d(z, y)$.

In a metric space the *balls* $B(x, r) = \{y \in X : d(x, y) < r\}$, $r > 0$, induce a natural topology on X where the *open sets* O are those sets such that if $x \in O$, there exists $B(y, r) \subset O$ with $x \in B(y, r)$; *closed sets* are the complements of open sets. We say that $x \in A \subset X$ is an *interior point* of

A if $x \in B(y,r)$ where $B(y,r) \subset A$; $\mathrm{int}(A)$, the *interior* of A, denotes the collection of interior points of A and is the largest open subset of A. The *closure* $\overline{A}$ of A is the smallest closed subset of X that contains A, i.e., the intersection of all closed sets containing A.

We say that a metric space (X,d) is *complete* if all Cauchy sequences of (X,d) converge. Cantor's nested theorem asserts that the intersection of any nested sequence of nonempty compact subsets of a metric space X is nonempty iff (X,d) is complete.

We say that G is a G_δ *set* in X if G is the intersection of a countable family of open sets in X; similarly, F is an F_σ *set* in X if F is the countable union of closed sets in X.

We say that $D \subset X$ is *dense* if $D \cap O \neq \emptyset$ for every open set O in X. A set in X is said to be *nowhere dense* if its closure has empty interior. The sets of *first category* in X are those that are countable unions of nowhere dense sets; all other sets are said to be of *second category* in X.

We say that a metric space (X,d) is a *Baire space* if every set of first category in X has empty interior. The Baire category theorem asserts that a complete metric space (X,d) is of second category in itself, i.e., X cannot be represented as a countable union of nowhere dense sets.

The problems in this chapter cover the various areas described above. They include observations dealing with the basic set-theoretical nature of sets, including the fact that the countable intersection of dense G_δ sets is a dense G_δ set, Problem 4, and the construction and properties of the Cantor set, or Cantor discontinuum, Problems 18–19, as well as the Cantor-Lebesgue function Problem 20. In the area of the Baire category problems include the existence (and abundance) of functions satisfying various properties, Problems 37–38, as well as the nature of the set of discontinuities of a continuous function, Problems 35–37. In the area of limits of continuous functions, we consider if $\chi_{\mathbb{Q}}$, the characteristic function of the rationals, is the limit of continuous functions, Problem 41, and whether pointwise convergence corresponds to metric convergence for an appropriate metric, Problem 42. The properties of the Baire class $\mathcal{B}_1$, i.e., those functions that are pointwise limits of continuous functions, are covered in Problems 43–51. Cardinality and cardinal arithmetic are discussed in Problems 54–70, and the Hamel basis is discussed in Problems 72–74.

The interested reader can further consult, for instance, K. Devlin, *The Joy of Sets: Fundamentals of Contemporary Set Theory*, Springer-Verlag, 2000; W. Brito, *El Teorema de Categoría de Baire. Sus Aplicaciones*, Editorial Académica Española, 2011; R.-L. Baire, *Sur les fonctions de variables réelles*, Annali di Mat. Ser. **3** (1899), no. 3, 1–123.

Problems

1. Let (X, d) be a metric space. Prove that the following statements are equivalent: (a) (X, d) is a Baire space. (b) The countable intersection of open dense sets is dense. (c) If A is of first category, A^c contains a dense G_δ subset.

2. Let (X, d) be a Baire space. Discuss the validity of the following statement: $A \subset X$ is nowhere dense iff $\overline{A^c} = X$.

3. Let (X, d) be a Baire space. Prove: (a) An open subset O of X is a Baire space in the induced metric. (b) If $\{F_n\}$ are closed subsets of X with $X = \bigcup_n F_n$, then $\bigcup_n \operatorname{int}(F_n)$ is dense in X.

4. Let (X, d) be a complete metric space. Prove: (a) The countable intersection of dense G_δ sets in X is a dense G_δ set in X. (b) If a set and its complement are dense subsets of X, at most one can be G_δ. (c) A countable dense subset of X cannot be G_δ.

5. Give an example of: (a) A sequence of open dense subsets of $[0, 1]$ whose intersection is a countable subset of $[0, 1]$. (b) A G_δ subset of $\mathbb{R}$ that is neither open nor closed. (c) A subset of $\mathbb{R}$ that is neither G_δ nor F_σ. (d) $A \subset \mathbb{R}$ such that $A \in F_{\sigma\delta} \setminus F_\sigma$ and $B \subset \mathbb{R}$ such that $B \in G_{\delta\sigma} \setminus G_\delta$.

6. Let (X, d) be a complete metric space. Prove: (a) A nonempty countable closed subset A of X has isolated points. (b) If X is perfect, X is uncountable.

7. We say that $A \subset \mathbb{R}$ has the Baire property if $A = G \Delta P$ with G open and P of first category. Prove that if A, B are sets of second category that have the Baire property, then $A + B$ and $A - B$ contain an interval.

8. Let (X, d) be a metric space, $\varphi : X \to X$ a homeomorphism, and $A \subset X$. Prove that A and $\varphi(A)$ are of the same category.

9. Let $d(x) = d(x, \mathbb{Z})$ denote the distance from $x \in \mathbb{R}$ to $\mathbb{Z}$. For $q \in \mathbb{N}$ and $\alpha > 0$, let $U_\alpha(q) = \{x \in \mathbb{R} : d(qx) < q^{-\alpha}\}$ and $Y_\alpha = \{x \in \mathbb{R} : x$ belongs to infinitely many $U_\alpha(q)\}$. Prove: (a) Y_α is a G_δ subset of $\mathbb{R}$ and $X = \bigcap_{\alpha \in \mathbb{R}^+} Y_\alpha$ is a dense G_δ subset of $\mathbb{R}$. (b) For $x \in \mathbb{R}$, $x \notin X$ iff there exists a polynomial P with real coefficients such that $P(n)d(nx) > 1$ for all $n \in \mathbb{N}$.

10. We say that a property in a metric space is generic if it holds except possibly in a set of first category. Prove that a generic point in $\mathbb{R}^2$ has both coordinates irrational.

11. We say that a real number x is Diophantine of exponent $\alpha > 0$ if there exists a constant $c > 0$ such that $|x - p/q| > cq^{-\alpha}$ for all rationals p/q. We denote by $\mathcal{D}(\alpha)$ the set of Diophantine numbers of exponent α and by $\mathcal{D} = \bigcup_\alpha \mathcal{D}(\alpha)$ the collection of Diophantine numbers. A real number x that is neither rational nor Diophantine is said to be a Liouville number; $\mathcal{L}$ denotes the collection of Liouville numbers. Prove: (a) If an irrational number x is algebraic of degree $d > 1$, $x \in \mathcal{D}(d)$. (b) $\mathcal{D}$ is of first category and, therefore, generic real numbers are Liouville.

12. Prove that if $A \subset \mathbb{R}$ is of first category, then $A^c - A^c = \mathbb{R}$.

13. Let $\mathcal{A} = \{x \in \mathbb{R} :$ the decimal expansion of x contains every possible finite pattern of digits$\}$. Prove that $\mathcal{A}$ is a dense G_δ subset of $\mathbb{R}$.

14. Let M be a closed subset of $\mathbb{R}$. Prove that M cannot be written as $M = \bigcup_n M_n$ with $M_n \subset \overline{M \setminus M_n}$ for all n.

15. Let ξ be an irrational number. Prove that $X = \{x \in \mathbb{R} : x = m + n\,\xi, m, n$ integers$\}$ is dense in $\mathbb{R}$.

16. Let $G \subset \mathbb{R}$ be an open set unbounded above and let $\{\lambda_n\}$ be such that $\lambda_n \to \infty$ and $d = \limsup_n(\lambda_{n+1} - \lambda_n) = 0$. Prove that every open interval of $\mathbb{R}$ contains a point x with the property that $x + \lambda_n \in G$ for infinitely many n.

17. Let G be an unbounded open subset of $(0, \infty)$ and $D = \{x \in (0, \infty) : nx \in G$ for infinitely many integers $n\}$. Prove that D is dense in $(0, \infty)$.

18. Let $\{a_n\}$ be a fixed sequence in $[0, 1]$ such that $a_0 = 1$ and $0 < 2a_n < a_{n-1}$ for $n \geq 1$. Let $P_0 = [0, 1]$ and let P_1 be the set obtained by removing the middle open interval of P_0 of length $a_0 - 2a_1$, i.e., $P_1 = [0, a_1] \cup [1 - a_1, 1]$; note that each interval of P_1 is closed and has length a_1. Next, having constructed P_n let P_{n+1} be the subset of P_n obtained by removing the middle open interval of length $a_n - 2a_{n+1}$ of each of the 2^n disjoint closed intervals, each of length a_n, that comprise P_n. Thus P_{n+1} consists of 2^{n+1} closed intervals each of length a_{n+1}. Finally, let $P = \bigcap_{n=0}^\infty P_n$.

(a) Give an explicit description of the P_n. (b) Let $S = \{x : x_n = 0$ or $x_n = 1\}$ be the space of sequences with terms 0 or 1. Set $r_n = a_{n-1} - a_n$ and let $\varphi : S \to [0, 1]$ be given by $\varphi(x) = \sum_n x_n r_n$. Prove that $\varphi : S \to P$ is 1-1 and onto. That is, P consists precisely of those numbers in $[0, 1]$ of the form $\sum_n x_n r_n$ with $x_n = 0$ or $= 1$.

Also, prove: (c) P has empty interior. (d) Every point in P is an accumulation point of P and, consequently, P is uncountable. (e) Each P_n is closed and has Lebesgue measure $|P_n| = 2^n a_n$. Thus P is a Borel set with Lebesgue measure $|P| = \lim_n 2^n a_n$. (f) If $0 \leq \beta < 1$, $\{a_n\}$ can be chosen so that P has Lebesgue measure $|P| = \beta$.

19. Let p be an integer greater than or equal to 3 and λ a real number such that $0 < \lambda \le p - 2$. Proceeding as in the construction of the P_n in Problem 18 remove open intervals of length λp^{-n} at the n-th stage and let $P_\lambda = \bigcap_n P_n$. (a) Find the Lebesgue measure of P_λ.

Now set $p = 3$ and $\lambda = 1$. The resulting set is called the Cantor set, or Cantor discontinuum, and is denoted C. Prove: (b) C is an uncountable perfect set that contains no intervals and has Lebesgue measure 0. Also, C consists of those $x \in [0,1]$ with ternary expansion $x = \sum_n x_n/3^n$ where $x_n \neq 1$ for all n. (c) C is symmetric about the point $1/2$. Moreover, if $x \in C$, $x/3 \in C$ and, if $x < 1/3$, $3x \in C$. (d) Each point of $[0,1]$ is the midpoint of not necessarily distinct points of C. (e) $C - C = [-1,1]$ and $C + C = [0,2]$. Also: (f) Give examples of rational and irrational $x \in [0,1]$ that are, and are not, in C. (g) Characterize the set of left endpoints of the removed open intervals in the construction of C.

20. The Cantor-Lebesgue function $f(x)$ is defined on $[0,1]$ in two steps, first on C and then on $[0,1] \setminus C$. If $x \in C$, let

$$f(x) = \frac{1}{2} \sum_n \frac{x_n}{2^n} \quad \text{where} \quad x = \sum_n \frac{x_n}{3^n}\,,\ x_n = 0, 2\,.$$

Prove: (a) $f : C \to [0,1]$ is onto. (b) If a, b are the endpoints of any of the open intervals removed in the construction of C, then $f(a) = f(b)$.

Next, if $x \in [0,1] \setminus C$, x is in exactly one of the intervals (a,b) removed from $[0,1]$. By (b), $f(a) = f(b)$ and we define $f(x) = f(a)$ for all $x \in (a,b)$. Prove: (c) f is monotone and continuous. (d) f satisfies $f(x) = 2f(x/3)$ for all $x \in [0,1]$. (e) Determine where f fails to be differentiable.

21. Discuss the validity of the following statement: There exist an interval $J = [a,b]$ and a strictly increasing function g on J such that $g'(x) = 0$ a.e. on J.

22. Let $X = \{\text{sequences } x : x_n = 0 \text{ or } = 1 \text{ for all } n\}$. Prove that equipped with the metric $d(x,y) = \sum_n 2^{-n} |\, x_n - y_n|$, (X,d) is homeomorphic to the Cantor discontinuum, i.e., there is a continuous bijection $\varphi : X \to C$ with φ^{-1} continuous.

23. Prove that $I = [0,1]$ is not the union of a countable family of: (a) Pairwise disjoint nonempty closed sets. (b) Cantor sets of positive Lebesgue measure.

24. Suppose that to every $x \in \mathbb{R}$ there corresponds a set $P(x) \subset \mathbb{R}$ such that $x \notin P(x)$ and x is not a limit point of $P(x)$. We say that two points $x, y \in \mathbb{R}$ are independent if $x \notin P(y)$ and $y \notin P(x)$; $A \subset \mathbb{R}$ is said to be independent if every pair of points of A is independent. Prove that there is an independent subset of $\mathbb{R}$ which is of second category in $\mathbb{R}$.

25. Discuss the validity of the following statements: There is an ordering $r_1, r_2, \ldots$ of the rationals in $[0,1]$ such that: (a) $\lim_n r_n$ exists. (b) $\sum_n r_n^k < \infty$ for some integer k. (c) $\sum_n r_n^n < \infty$.

26. For $f : [0,1] \to \mathbb{R}$, let

$$D^+ f(a) = \limsup_{x \to a^+} \frac{f(x) - f(a)}{x - a}\,.$$

Prove that for each $a \in [0,1)$, $G_a = \{f \in C(I) : D^+ f(a) = \infty\}$ is a dense G_δ subset of $C(I)$.

27. Let (X,d) be a complete metric space and $\mathcal{F} = \{f_\lambda\}_{\lambda \in \Lambda}$ a pointwise bounded family of continuous functions on X, i.e., for each $x \in X$ there is $M(x) < \infty$ such that $|f_\lambda(x)| \le M(x)$ for all $f_\lambda \in \mathcal{F}$. Prove that $\mathcal{F}$ is uniformly bounded in a ball B of X, i.e., there exist a constant $M > 0$ and a ball B of X such that $|f_\lambda(x)| \le M$ for all $f_\lambda \in \mathcal{F}$ and $x \in B$.

28. Let (X,d) be a Baire space and $f : X \to \mathbb{R}$ a lower semicontinuous function on X. Prove that f is bounded on a nonempty open subset O of X.

29. Let $\{f_n\}$ be continuously differentiable functions on $\mathbb{R}$. Prove: (a) If $|f_n(x)|, |f_n'(x)| \le M$ for each $x \in \mathbb{R}$ and all n, $\{f_n\}$ has a uniformly convergent subsequence. (b) If for each $x \in \mathbb{R}$ there are finite numbers $M(x), N(x)$ such that $|f_n(x)| \le M(x), |f_n'(x)| \le N(x)$ for all n, a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ converges uniformly in an interval $J \subset \mathbb{R}$.

30. Let $\{f_n\} \subset C(\mathbb{R})$ be such that for each $x \in \mathbb{R}$ there exists an integer n so that $f_n(x) = 0$. Let $O = \{x \in \mathbb{R}$: there exist an integer n and a neighborhood V_x of x such that $f_n|_{V_x} = 0\}$. Prove that O is an open dense subset of $\mathbb{R}$.

31. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Prove that f is a polynomial or, given a positive integer n and a closed disk $D_r(z_0)$, $f^{(n)}(z) \ne 0$ for some $z \in D_r(z_0)$.

32. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that for all $x \in \mathbb{R}$ there exists an integer n (which may depend on x) with $f^{(n)}(x) = 0$. Prove that f is a polynomial.

33. Let (X,d) be a metric space and $A \subset X$. Prove that χ_A is lower semicontinuous iff A is open.

34. Let (X,d) be a Baire space and O an open subset of X. Prove: (a) The set of points of discontinuity $D(\chi_O)$ of the characteristic function of O is a nowhere dense subset of X. (b) Given open sets $\{O_n\}$, there exists $x \in X$ such that χ_{O_n} is continuous at x for each n.

35. Let (X, d) be a Baire space and f a function on X. Prove: (a) $D(f) = \{x \in X : f \text{ is discontinuous at } x\}$ is an F_σ set. (b) f is continuous on a dense subset of X iff $D(f)$ is of first category in X.

36. Discuss the validity of the following statements: There exists a real-valued function f on $[0, 1]$ such that $C(f)$, the points of continuity of f, are: (a) The rationals in $[0, 1]$. (b) The numbers with a finite decimal expansion. (c) The algebraic numbers in $[0, 1]$.

37. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is left-continuous at $x \in \mathbb{R}$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for $x - \delta < t < x$; similarly for right-continuous. Discuss the validity of the following statement: There exists a function $f : \mathbb{R} \to \mathbb{R}$ that is left-continuous everywhere and right-continuous nowhere.

38. For f defined on an open interval $J \subset \mathbb{R}$, let $f'_g(x)$ and $f'_d(x)$ denote the left-hand side derivative and the right-hand side derivative of f at $x \in J$, respectively, whenever they exist. Prove that $A = \{x \in J : f'_g(x) \neq f'_d(x)\}$ is countable.

39. Let $A = \{a_n\}$ be a countable subset of $\mathbb{R}$. Construct a nondecreasing real-valued function f on $\mathbb{R}$ with $D(f) = A$.

40. Let (X, d) be a Baire space, (Y, d') a separable metric space, and $f : X \to Y$ with the property that the preimage of an open set in Y is F_σ in X. Prove that f is continuous in a dense G_δ subset of X.

41. Prove that $\chi_{\mathbb{Q}}$ is not the pointwise limit of a sequence of continuous functions. However, show that there exist continuous functions $\{f_{m,n}\}$ on $\mathbb{R}$ such that $\lim_n \lim_m f_{m,n}(x) = \chi_{\mathbb{Q}}(x)$ for all $x \in \mathbb{R}$.

42. Prove that $X = \{f : \mathbb{R} \to \mathbb{R}\}$ cannot be equipped with a metric d so that convergence in (X, d) is equivalent to pointwise convergence.

43. Let (X, d) be a metric space. We say that a function f on X is of Baire class 1, and this is denoted $f \in \mathcal{B}_1$, if f is the pointwise limit of a sequence of continuous functions on X. Prove that if X is a Baire space and $f \in \mathcal{B}_1$, the set $C(f)$ of points of continuity of f is a dense G_δ subset of X.

44. Prove that if f on $[0, 1]$ has a finite number of discontinuities, $f \in \mathcal{B}_1$. In particular, $\mathcal{B}_1$ contains the step functions.

45. Let $\{f_n\} \subset \mathcal{B}_1$ be defined on I and $\sum_n M_n$ a convergent series of positive real numbers. Prove that if $|f_n(x)| \leq M_n$ for $x \in I$ and all n, then $\sum_n f_n(x) = f(x) \in \mathcal{B}_1$.

46. Let $\{f_n\}$ be $\mathcal{B}_1$ functions on I. Discuss the validity of the following statement: If f_n converges uniformly to f in I, then $f \in \mathcal{B}_1$.

47. Let (X,d) be a metric space and f a semicontinuous function on X. Prove that $D(f)$ is of first category in X.

48. Prove that if a metric space (X,d) is of the first category in itself there exists an everywhere discontinuous bounded lower semicontinuous function on X.

49. Prove that if $f:[0,1]\to\mathbb{R}$ is semicontinuous, then $f\in\mathcal{B}_1$.

50. Let (X,d) be a metric space and A a first category F_σ subset of X. Construct $f\in\mathcal{B}_1$ with $D(f)=A$.

51. Let (X,d) be a complete metric space and $f\in\mathcal{B}_1$. Prove that if $O\subset\mathbb{R}$ is open, then $f^{-1}(O)$ is an F_σ subset of X.

52. Let $f:(0,\infty)\to\mathbb{R}$ be continuous and suppose that for $\lambda_n\to\infty$ with $d_n=\lambda_{n+1}-\lambda_n\to 0$, $\lim_n f(x+\lambda_n)$ exists for all x in an open interval J. Prove that $\lim_{x\to\infty} f(x)$ exists.

53. Let $f:(0,\infty)\to\mathbb{R}$ be a continuous function such that $\lim_n f(nx)=0$ for each $x>0$. Prove that $f(x)\to 0$ as $x\to\infty$.

54. Prove that the set of real numbers in $[0,1]$ which have two decimal expansions (one terminating in 9's and one in 0's) is countable.

55. Prove that $X=\{0,1\}^{\mathbb{N}}$ is uncountable.

56. Discuss the validity of the following statement: There is a set $\mathcal{F}\subset\mathcal{P}(\mathbb{N})$ that satisfies the following two properties: $\mathrm{card}(\mathcal{F})=c$ and, if $A,B\in\mathcal{F}$, $\mathrm{card}(A\cap B)<\infty$.

57. Let A be a countable subset of $\mathbb{R}$. Is it possible to translate A by a real number r into $r+A$ so that $A\cap(r+A)=\emptyset$?

58. Let $A\subset\mathbb{R}$ be uncountable. Prove that for some $t\in\mathbb{R}$, $A\cap(-\infty,t)$ and $A\cap(t,\infty)$ are both uncountable. Furthermore prove that there exist $a<b\in\mathbb{R}$ such that $A\cap(-\infty,a)$ and $A\cap(b,\infty)$ are both uncountable.

59. Let $\mathcal{A}$ be a set of real numbers with the property that $|a_1+\cdots+a_n|\le 1$ for every finite subset $a_1,\dots,a_n$ of $\mathcal{A}$. Prove that $\mathcal{A}$ is at most countable.

60. Construct $A\subset\mathbb{R}$ with $\mathrm{card}(A)=c$ such that A intersects every closed nowhere dense subset of $\mathbb{R}$ in a countable set.

61. Prove: (a) $\aleph_0+n=\aleph_0$ for all $n=1,2,\dots$. (b) $\aleph_0+\aleph_0=\aleph_0$.

62. Let A be an infinite set. Prove: (a) A contains a countable subset. (b) If $\mathrm{card}(A)=a$, then $a+\aleph_0=a$. (c) For infinite cardinals $a\le b$, $a+b=b$.

63. Let A be an infinite set. Prove: (a) A can be expressed as a pairwise disjoint union of countable subsets of A. (b) If A is uncountable, A can be expressed as $A=X\cup X^c$ where X and X^c are uncountable.

64. Prove that if A is an infinite set, then $A \times A \sim A$. Therefore, if $\text{card}(A) = a$, then $a \cdot a = a$.

65. Given an infinite cardinal b, let $\{a_d\}_{d \le b}$ be cardinals such that $a_d \le b$ for all d. Prove that $\sum_{d \le b} a_d \le b$. In particular, prove that the countable union of countable sets is countable.

66. Discuss the validity of the following statements: (a) There exists an uncountable pairwise disjoint collection of balls in $\mathbb{R}^n$. (b) There exists an uncountable pairwise disjoint collection of spheres in $\mathbb{R}^n$. (c) There exists an uncountable pairwise disjoint collection of figure eights in the plane.

67. We say that a sequence $\{a_n\}$ is eventually constant if there is an integer N such that $a_n = a_N$ for all $n \ge N$. What is the cardinality of the set of all eventually constant sequences of rational numbers? Real numbers?

68. Find the cardinality of $C(I)$ and $\mathcal{B}_1$.

69. Prove that $\text{card}(\mathcal{B}(\mathbb{R})) = c$.

70. Compute the cardinality of the family $\mathcal{C}$ of the compact subsets of $\mathbb{R}$.

71. Construct a set $B \subset \mathbb{R}$ such that both B and B^c intersect every uncountable compact subset of the line.

72. Prove that if H is a Hamel basis of $\mathbb{R}$, $\text{card}(H) = c$.

73. Prove that there are 2^c different Hamel bases of $\mathbb{R}$.

74. Prove that the Cantor discontinuum contains a Hamel basis.

75. Prove that there exist pairwise disjoint $\{A_n\} \subset \mathbb{R}$ such that $\mathbb{R} = \bigcup_n A_n$ and for any open interval $J \subset \mathbb{R}$, $A_n \cap J$ is of second category for all n.

Set Theory and Metric Spaces

Solutions

1. Recall the following relations in a metric space: $X \setminus \overline{A} = \text{int}(X \setminus A)$ and $X \setminus \text{int}(A) = \overline{X \setminus A}$. They imply that $\text{int}(\overline{X \setminus A}) = \text{int}(X \setminus \text{int}(A)) = X \setminus \overline{\text{int}(A)}$, and, consequently, A is an open dense subset of X iff its complement $X \setminus A$ is closed and nowhere dense.

(a) implies (b) Let $\{O_n\}$ be open dense sets in X and put $G = \bigcap_n O_n$. Then $G^c = \bigcup_n (X \setminus O_n)$ where each $X \setminus O_n$ is closed nowhere dense and so G^c is of first category and has empty interior. Now, by the second relation above with $A = G^c$ there, $\overline{G} = X \setminus \text{int}(G^c) = X \setminus \emptyset = X$ and G is dense in X.

(b) implies (c) Let $A = \bigcup_n A_n$ with A_n nowhere dense, all n. Then $\overline{A}_n$ is nowhere dense and $A \subset \bigcup_n \overline{A}_n$; hence $A^c \supset \bigcap_n (X \setminus \overline{A}_n) = G$. Note that since $(X \setminus \overline{A}_n)$ is open and dense for all n, $G \subset A^c$ is a dense G_δ subset in X.

(c) implies (a) If A is of first category, $X \setminus A$ contains a dense G_δ set and, consequently, $\overline{(X \setminus A)} = X$. Also, since $\text{int}(A) = X \setminus \overline{(X \setminus A)}$, $\text{int}(A) = \emptyset$.

3. (a) Let $\{O_n\}$ be open dense subsets of O in the induced metric and put $G = \bigcap_n O_n$; we claim that G is dense in O. Let $U = X \setminus \overline{O}$; since O, U are open in X, $U_n = O_n \cup U$ is open in X. We claim that the U_n are dense in X, i.e., given a nonempty open subset V of X, $V \cap U_n \neq \emptyset$. Now, this is clearly true if $V \subset U$. Otherwise $V \cap \overline{O} \neq \emptyset$, and, consequently, $V \cap O \neq \emptyset$. Thus, since $V \cap O$ is open in O, $V \cap O \cap O_n \neq \emptyset$, and, consequently, $V \cap U_n \neq \emptyset$. Therefore $\{U_n\}$ are open dense subsets of X and by Problem 1(b), $H = \bigcap_n U_n = G \cup U$ is dense in X. If V is a nonempty open subset of O, V is an open subset of X and, therefore, $V \cap H \neq \emptyset$. Hence $V \cap G = V \cap H \neq \emptyset$ and G is dense in O.

(b) Let V be a nonempty open subset of X. Then with $F'_n = F_n \cap V$, $V = \bigcup_n F'_n$ where the F'_n are closed in the relative topology of V. Now, by (a) V is a Baire space and, consequently, one of these closed sets, F'_m, say, has nonempty interior (in V). Let $O = \mathrm{int}_V(F'_m)$. Since V is open, O is open in X and since $O \subset F'_m$ it follows that $O \subset \mathrm{int}(F'_m)$. But $O \subset V$ and so $O \subset \mathrm{int}(F'_m) \cap V$. Thus $\bigcup_n \mathrm{int}(F_n) \cap V \neq \emptyset$.

4. (a) Let $\{G_n\}$ be dense G_δ subsets of X and consider $F_n = G_n^c$; each F_n is F_σ and nowhere dense. For the sake of argument suppose that $\bigcap_n G_n$ is not dense. Then its complement $\bigcup_n F_n$ contains an open ball and, hence, a smaller nonempty closed ball B, say. Now, since B is closed, B is a complete metric space in its own right and, since $B = \bigcup_n (B \cap F_n)$, B is the countable union of nowhere dense F_σ sets. By the Baire category theorem this cannot happen.

Note that this condition is equivalent to: If $\{C_n\}$ are closed sets, none of which contains a ball, then $\bigcup_n C_n \neq X$. This follows by complementation since $G_n = X \setminus C_n$ is open and dense for each n and, therefore, $\bigcap_n G_n \neq \emptyset$.

(b) For the sake of argument suppose that A and A^c are dense G_δ subsets of X. Then, by (a), $A \cap A^c = \emptyset$ is a dense G_δ subset of X, which is not the case.

(c) For the sake of argument suppose that A is a countable dense G_δ subset of X; then A^c is F_σ. Now, since A is countable, A is F_σ, and since A is dense, A^c is nowhere dense. Thus, contrary to the Baire category theorem, $X = A \cup A^c$ is the countable union of nowhere dense sets.

5. (a) You should not believe everything you read, no such example is possible: By Problem 1(b) the intersection is a dense G_δ set and by Problem 4(c) it cannot be countable.

7. Since the result for $A + B = A - (-B)$ follows from that for $A - B$ we prove the latter. Let $A = G_1 \Delta P_1$, $B = G_2 \Delta P_2$ where G_1, G_2 are nonempty open sets (for otherwise A, B would be of first category) and P_1, P_2 are of first category. First, observe that $G_1 - G_2$, which is a nonempty open set, is contained in $A - B$. Now, if $x \in G_1 - G_2$, $(x + B) \cap A \supset ((x + G_2) \cap G_1) - ((x + P_2) \cup P_1)$ where $(x + G_2) \cap G_1$ is a nonempty open set. Since $(x + P_2) \cup P_1$ is of first category and since a nonempty open set is not of first category, $((x + G_2) \cap G_1) - ((x + P_2) \cup P_1) \neq \emptyset$ and so $(x + B) \cap A \neq \emptyset$. Thus $x \in A - B$ and $G_1 - G_2 \subset A - B$.

9. (a) Since $Y_\alpha = \bigcap_n \bigcup_{q=n}^\infty U_\alpha(q)$, Y_α is a G_δ subset of $\mathbb{R}$. Moreover, for $\alpha \geq \beta$ and an integer q, we have $U_\alpha(q) \subset U_\beta(q)$ and so $Y_\alpha \subset Y_\beta$. Thus

$X = \bigcap_{\alpha\in\mathbb{R}^+} Y_\alpha = \bigcap_{n\in\mathbb{N}} Y_{n+1}$ is G_δ in $\mathbb{R}$. And since $\mathbb{Q} \subset X$, X is dense in $\mathbb{R}$.

(b) Let $x \in \mathbb{R}$. Suppose that $x \in X$, let P be a polynomial with real coefficients, and α strictly larger than the degree of P; note that $P(n) < n^\alpha$ for n sufficiently large. Now, since $\{n : d(nx) < n^{-\alpha}\}$ is infinite there exists an integer n such that $P(n) < n^\alpha$ and $d(nx) < n^{-\alpha}$, and, consequently, $d(nx)P(n) < 1$. On the other hand, if $x \notin X$, $x \notin Y_q$ for some integer $q > 0$; since $\{n \in \mathbb{N} : d(nx) < n^{-\alpha}\}$ is a finite set there exists $a > 0$ such that $ad(nx) > 1$ for all n in this set. Then the polynomial $P(t) = t^q + a$ satisfies $P(n)d(nx) > 1$ for all $n \in \mathbb{N}$.

10. We claim that the set of points with at least one of its coordinates rational is of first category in $\mathbb{R}^2$. If $\{r_n\}$ is an enumeration of the rationals in $\mathbb{R}$, let $X_n = \{(r_n, y) : y \in \mathbb{R}\}$ and $Y_n = \{(x, r_n) : x \in \mathbb{R}\}$; each X_n and Y_n is a line in $\mathbb{R}^2$ and is therefore closed and nowhere dense. Now, the complement of the set of points in $\mathbb{R}^2$ with both of their coordinates irrational is $(\bigcup_n X_n) \cup (\bigcup_n Y_n)$ and, therefore, of first category in $\mathbb{R}^2$.

11. (a) Let $P(t) = a_0 t^d + \cdots + a_d$ be an irreducible polynomial of degree $d > 1$ with integer coefficients satisfied by x; since $q^d P(p/q)$ is an integer, if not 0, one has $q^d|P(p/q)| \geq 1$. Now, since $P'(t)$ is bounded near x by M, say, it follows that $q^{-d} \leq |P(x) - P(p/q)| \leq M\,|x - p/q|$ and, consequently, $|x - p/q| \geq 1/(Mq^d)$.

(b) For an integer n let $A_n = \{x \in \mathbb{R} :$ there exists a rational $p/q, q > 1$, such that $|x - p/q| < 1/nq^\tau\}$; clearly A_n is open and since $\mathbb{Q} \subset A_n$, dense in $\mathbb{R}$. Hence, by Problem 1(b), $\mathcal{L} = \bigcap_n A_n$ is a dense G_δ subset of $\mathbb{R}$. Now, $\mathbb{R} \setminus \bigcap_n L_n = \mathcal{D}(\tau)$ is of first category and so is $\mathcal{D} = \bigcup_n \mathcal{D}(n)$. Finally, $\mathcal{L}$ is the complement of $\mathbb{Q}$ in the second category set $\mathbb{R} \setminus \mathcal{D}$ and since $\mathbb{Q}$ is of first category, $\mathcal{L}$ is of second category in $\mathbb{R}$.

12. For the sake of argument suppose that for some $r \in \mathbb{R}$, $r \neq x - y$ for all $x, y \in A^c$. Then $r + A^c \subset A$ and so $r + A^c$ is of first category and by translation so is A^c. Thus $\mathbb{R} = A \cup A^c$ is of first category, which is not the case.

13. Let $d = d_1 \ldots d_n$, $0 \leq d_k \leq 9$, denote a finite pattern of digits; the set $\{d\}$ of all possible finite patterns is then the countable union of finite sets and, hence, countable. Let X_d denote the set of real numbers whose decimal expansion does not contain the pattern d; we claim that X_d is nowhere dense. First, X_d is closed. Indeed, if $x \notin X_d$, the pattern d can be found in x and $x = x_0.x_1 \ldots x_k d_1 \ldots d_n x_{k+n+1} \ldots$, say. Let $\varepsilon = 10^{-(k+n+1)}$ and note that if $|y - x| < \varepsilon$, x and y do not differ in the first $n + k$ digits, y contains the pattern d, and so $y \in X_d^c$, which is therefore open. Next, X_d has empty interior. For the sake of argument suppose that there are

$x \in X_d$ and $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset X_d$; we may assume that $\varepsilon = 10^{-N}$ for some integer N. Then if the first N digits of y are the same as those of x, the distance between x and y is less than ε. Now consider a number $y = x_0.x_1 \ldots x_N d_1 \ldots d_n$ whose first N digits are those of x followed by the pattern d. Then on the one hand $y \in (x-\varepsilon, x+\varepsilon) \subset X_d$ while at the same time the decimal expansion of y contains the pattern d, which cannot happen. Thus X_d is nowhere dense and X_d^c is open dense. Therefore $\mathcal{A} = \bigcap_d X_d^c$ is the countable intersection of open dense subsets of $\mathbb{R}$ and by Problem 1(b), $\mathcal{A}$ is a dense G_δ set in $\mathbb{R}$.

14. For the sake of argument suppose that each $x \in M_n$ is the limit of a sequence $\{x_k\}$ with $x_k \notin M_n$ for all k. Let $x_0 \in M_1$ and I_0 an open interval containing x_0. Since x_0 is a limit point of $M \setminus M_1$ by the well-ordering of the integers there is a least integer $n_1 > 1$ such that I_0 contains a point $x_1 \in M_{n_1} \setminus M_1$. Let $I_1 \subset \overline{I}_1 \subset I_0$ be an open interval containing x_1 and select a point $x_2 \in M \setminus M_{n_1}$ such that $x_2 \in M_{n_2}$ and $n_2 > n_1$ is the least integer for which this happens; next let I_2 be an open interval containing x_2, $I_2 \subset \overline{I}_2 \subset I_1$ and I_2 contains no point of M_{n_1}. $\overline{I}_2$ contains no point of $M_1 \cup M_2$. Continuing this way one constructs a nested sequence $\{M \cap \overline{I}_k\}$ where each set is closed and bounded and so the intersection contains a point which, by the selection process, is not in M_n for any n. But, since $M = \bigcup_n M_n$, this cannot happen.

17. Let $G/n = \{y \in \mathbb{R} : ny \in G\}$; observe that $D = \limsup_n G/n = \bigcap_m \bigcup_{n=m}^\infty G/n$. Now, since G/n is open for every n, $\bigcup_{n=m}^\infty G/n$ is open for every m; we claim that it is also dense in $(0,\infty)$. For the sake of argument suppose that this is not the case and let $J = (a,b)$ be an open interval such that $J \cap \bigcup_{n=m}^\infty G/n = \emptyset$, i.e, $G \cap \bigcup_{n=m}^\infty (na, nb) = \emptyset$. Let M be the smallest integer such that $M > a/(b-a)$ and note that for $m \geq M$, $ma < (m+1)a < mb$ and, consequently, the above intervals overlap and $\bigcup_{n=m}^\infty (na,nb) = (ma,\infty)$. Therefore $G \cap (ma,\infty) = \emptyset$, which is not the case since G is unbounded. Finally, if $m < M$, $\bigcup_{n=m}^\infty G/n \supset (Ma,\infty)$ and $G \cap \bigcup_{n=m}^\infty (na,nb) \neq \emptyset$. Hence $\bigcup_{n=m}^\infty G/n$ is dense for every m and by Problem 1(b), $\bigcap_m \bigcup_{n=m}^\infty G/n = D$ is dense in $(0,\infty)$.

It is also true that if $\{G_k\}$ is a sequence of open sets unbounded above there is x_0 with the property that nx_0 is in each G_k infinitely often; indeed, it suffices to replace G in the above argument by G_1, G_1, G_2, G_1, G_2, G_3, G_1, G_2, G_3, $G_4, \ldots$.

The result also has the following interesting consequence: If $A \subset \mathbb{N}$ is an infinite set of positive integers, there exists $a > 1$ such that infinitely many $[a^k]$ (here $[x]$ denotes $x -$ fractional part of x) are in A; to obtain this let $G = \bigcup_n (\ln(n), \ln(n+1))$.

18. (a) With $P_0 = [0,1]$ the construction of the P_n proceeds by induction. First, some shorthand notation: let $d_n = a_{n-1} - 2a_n > 0$, all $n \ge 1$. Let $J_{n-1,k}$, $n \ge 1$, $1 \le k \le 2^{n-1}$, denote the 2^{n-1} pairwise disjoint closed intervals, each of length a_{n-1}, that comprise P_{n-1}, i.e., $P_{n-1} = \bigcup_{k=1}^{2^{n-1}} J_{n-1,k}$, and let $J_{n,2k-1}$ and $J_{n,2k}$ be the closed intervals of length a_n such that $J_{n,2k-1}$ has the same left endpoint as $J_{n-1,k}$ and $J_{n,2k}$ has the same right endpoint as $J_{n-1,k}$. Note that if $I_{n,k} = J_{n-1,k} \setminus (J_{n,2k-1} \cup J_{n,2k})$, $n > 0$, $1 \le k \le 2^{n-1}$, $I_{n,k}$ is the open interval of length d_n having the same midpoint as $J_{n-1,k}$. It is clear that $\{I_{n,k} : n \in \mathbb{N}, 1 \le k \le 2^{n-1}\}$ consists of pairwise disjoint open intervals. Also $[0,1] = P_0 \supset P_1 \supset \dots$, $P_{n-1} \setminus P_n = \bigcup_{k=1}^{2^n-1} I_{n,k}$, and $P_0 \setminus P = \bigcup_n (P_{n-1} \setminus P_n)$. In short, P_n is obtained from P_{n-1} by removing the center of $J_{n-1,k}$ for each k, i.e., the open interval $I_{n,k}$, $1 \le k \le 2^{n-1}$, and then P is obtained by removing from $[0,1]$ all the open intervals $I_{n,k}$, $n \ge 1, 1 \le k \le 2^{n-1}$. Specifically, $P_n = \bigcup_{k=1}^{2^n} J_{n,k}$ and $P = \bigcap_{n=0}^{\infty} P_n$.

For details of this construction consult K. Stromberg, *Introduction to classical real analysis*, Wadsworth International Mathematics Series, 1981.

19. (a) We need to define the sequence $\{a_n\}$ with $2a_n < a_{n-1}$ that corresponds to this construction. Let $a_0 = 1$ and for $n \ge 0$ define a_n by the relation $a_n - 2a_{n+1} = \lambda p^{-(n+1)}$. Note that $1 - 2a_1 = \lambda/p$ or $2a_1 = 1 - \lambda/p$; then $a_1 - 2a_2 = \lambda/p^2$ or $4a_2 = 1 - \lambda/p - (2/p)\lambda/p$ at the next step, and so on. More precisely, by induction it readily follows that

$$2^n a_n = \Big(1 - \frac{\lambda}{p}\Big) - \Big(\frac{2}{p} + \cdots + \Big(\frac{2}{p}\Big)^{n-1}\Big)\frac{\lambda}{p}$$

and, consequently,

$$|P_\lambda| = \lim_n 2^n a_n = \Big(1 - \frac{\lambda}{p}\Big) - \Big(\frac{2}{p-2}\Big)\frac{\lambda}{p} = 1 - \frac{\lambda}{p-2} \,.$$

(b) The construction of the P_n is simplest in this case. Since $a_n = 3^{-n}$ for all n,

$$r_n = \frac{1}{3^{n-1}} - \frac{1}{3^n} = 2\,\frac{1}{3^n}\,, \quad n = 1, 2, \dots,$$

and the Cantor discontinuum consists of those $x \in [0,1]$ with expansion $x = \sum_n x_n r_n = 2\sum_n 3^{-n} x_n$ where $x_n = 0$ or $= 1$.

(e) Note that $C - C = [-1,1]$ iff $C + C = [0,2]$. Indeed, if $C + C = [0,2]$ and $\lambda \in [-1,1]$, then $1 + \lambda \in [0,2]$ and so $1 + \lambda = c_1 + c_2$ for some $c_1, c_2 \in C$. Then $\lambda = c_1 - (1 - c_2) \in C - C$. Since these steps are reversible the statements are equivalent. Now, clearly $C + C \subset [0,2]$. And, given $x \in [0,2]$, $x/2 \in [0,1]$ and by (d), $x/2 = (y+z)/2$ with $y, z \in C$. Thus $x = y + z$ with y, z in C and so $[0,2] \subset C + C$.

(f) First, the rationals. For a positive integer p,

$$x = \frac{2}{3^p} + \frac{2}{3^{2p}} + \cdots = \frac{2}{3^p - 1}, \quad p = 1, 2, \ldots.$$

is in C; for instance, 1, 1/4, 1/13, are in C. Also $0 = 1 - 1$, $3/4 = 1 - 1/4$, $12/13 = 1 - 1/13$ are in C as are $1/39 = 1/(3 \cdot 13)$, and so on.

As for the irrational numbers, let $\{n_k\}$ be given by $n_1 = 1$ and $n_k = n_{k-1} + k$ for $k \geq 2$. Then $x = 2\sum_k 3^{-n_k} = 2\,(1/3 + 1/3^3 + 1/3^6 + \cdots)$ is in C. Another way to see this is to note that the ternary expansion of x is $.202002000200002\ldots$. Finally, $1/\sqrt{2}$, which satisfies $19/27 < 1/\sqrt{2} < 20/27$, has ternary decimal expansion beginning $.201002\ldots$, and $\pi/4$, whose ternary decimal expansion begins with $.210012\ldots$, are not in C.

(g) The left endpoints of the open intervals removed in the k-th step in the construction of the Cantor discontinuum are those points in C with ternary expansion ending in 1 at the k spot; they can also be thought of as those points with 0 in the k spot followed by a string of 2's. Moreover, since the right endpoints are obtained by adding $1/3^k$ to the left endpoints they are those points in the Cantor discontinuum with ternary decimal expansion with 2 in the k spot and ending in a string of 0's.

20. (a) Let $y = \sum_k a_k/2^k$, $a_k = 0, 1$ for all k, be the dyadic expansion of $y \in I$. Then $x = \sum_k (2a_k)/3^k$ is in C and since

$$f(x) = \frac{1}{2}\sum_k \frac{2a_k}{2^k} = \sum_k \frac{a_k}{2^k} = y\,,$$

f is onto.

(d) Let $x \in C$ have ternary expansion $x = \sum_n x_n/3^n$; thus $x/3 = \sum_n x_n/3^{n+1}$. Then by the definition of f,

$$f(x/3) = \frac{1}{2}\sum_n \frac{x_n}{2^{n+1}} = \frac{1}{2}\,\frac{1}{2}\sum_n \frac{x_n}{2^n} = \frac{1}{2}\,f(x)\,.$$

Next, if $x \in [0,1] \setminus C$, x and $x/3$ belong to different open intervals and the conclusion follows as before.

(e) Since f is constant in the open components that comprise the complement of the Cantor discontinuum, $f' = 0$ there. Next, let $x \in C$. For each integer n take $x_n = x \pm 2/3^n$ where the sign is chosen so that $x_n \in C$. Then

$$\frac{f(x_n) - f(x)}{x_n - x} = \frac{3^n}{2^{n+1}}$$

and since these quotients increase to ∞ with n, f is not differentiable at any $x \in C$.

21. The statement is true. Let f be the Cantor-Lebesgue function on I extended to be 0 for $x < 0$ and 1 for $x \geq 1$. Let $\{[a_n, b_n]\}$ be an enumeration of the closed subintervals of $[0,1]$ with rational endpoints $a_n \neq b_n$ and put

$$f_n(x) = f\Big(\frac{x-a_n}{b_n-a_n}\Big), \quad x \in [0,1].$$

f_n is a scaling of f on the subinterval $[a_n, b_n]$ and so is nondecreasing and $f_n' = 0$ a.e. Now let $g(x) = \sum_n 2^{-n} f_n(x)$, $x \in [0,1]$; g is well-defined since $f_n(x) \leq 1$ for all n. Moreover, by Fubini's lemma, $g'(x) = \sum_n f_n'(x) 2^{-n} = 0$ a.e. It only remains to prove that g is strictly increasing. Now, if $0 \leq x < y \leq 1$, pick $[a_n, b_n] \subset [x, y]$, $a_n \neq x, b_n \neq y$. Then

$$\frac{x-a_n}{b_n-a_n} < 0 < 1 < \frac{y-a_n}{b_n-a_n},$$

and so

$$\frac{1}{2^n} f_n(x) = \frac{1}{2^n} f\Big(\frac{x-a_n}{b_n-a_n}\Big) = 0 < \frac{1}{2^n} = \frac{1}{2^n} f\Big(\frac{y-a_n}{b_n-a_n}\Big) = \frac{1}{2^n} f_n(y),$$

which implies that $g(x) < g(y)$.

22. Let φ be given by $\varphi(x) = 2\sum_n x_n/3^n$. Clearly φ maps X onto C and is 1-1: If $x \neq y$ and if m is the first place where the ternary expansions of x and y differ, then

$$|\varphi(x) - \varphi(y)| \geq 2\Big(\frac{1}{3^m} - \sum_{n=m+1}^{\infty} \frac{1}{3^n}\Big) = \frac{1}{3^m} > 0.$$

And, since $|\varphi(y) - \varphi(x)| \leq 2\sum_n |y_n - x_n|/3^n \leq 2\big(\sum_n (2/3)^n\big)\, d(y,x)$, φ is continuous.

Next, we verify that $\{x^k\} \subset X$ converges to x in X iff $\lim_k x_n^k = x_n$ for $n = 1, 2, \ldots$. If $d(x^k, x) \to 0$, given $\varepsilon > 0$, there exists N such that $d(x^k, x) < \varepsilon/2^n$ for $k \geq N$. Then $|x_n^k - x_n|/2^n \leq d(x^k, x) < \varepsilon/2^n$, and, consequently, $|x_n^k - x_n| \leq \varepsilon$ for $k \geq N$. Conversely, if $\lim_k x_n^k = x_n$ for each n, given $\varepsilon > 0$, let N be such that $\sum_{n=N}^{\infty} 2^{-n} < \varepsilon/2$. Next pick k large enough so that $\sum_{n=0}^{N-1} |x_n^k - x_n|/2^n < \varepsilon/2$ and observe that $d(x^k, x) \leq \sum_{n=0}^{N-1} |x_n^k - x_n|/2^n + \sum_{n=N}^{\infty} 1/2^n < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for those k.

Finally, φ^{-1} is continuous. Suppose that $|\varphi(x^k) - \varphi(x)| \to 0$ as $k \to \infty$ and fix n. Then let $\varepsilon < 3^{-n}$ and pick N so that $|\varphi(x^k) - \varphi(x)| < \varepsilon$ for $k \geq N$. As we saw above, if m is the first index where $x_m^k \neq x_m$, $1/3^m \leq |\varphi(x^k) - \varphi(x)| < \varepsilon < 1/3^n$ for all $k \geq N$ and, therefore, such an index m verifies $m > n$ for that choice of ε, and $x_n^k = x_n$ for all $k \geq N$. In particular, this means that $\lim_k x_n^k = x_n$ and, since n is arbitrary, by the above observation $d(x^k, x) \to 0$ and φ^{-1} is continuous.

23. (a) For the sake of argument suppose that $[0,1] = \bigcup_n A_n$ where the A_n are pairwise disjoint nonempty closed sets and let $O_n = \operatorname{int}(A_n)$. Then $X = [0,1] \setminus \bigcup_n O_n = \bigcup_n (A_n \setminus O_n) \neq \emptyset$ is complete with the metric inherited from $[0,1]$ and, therefore, by the Baire category theorem there exist n_0 and an open interval $J = (a, b)$, say, such that $\emptyset \neq X \cap J \subset A_{n_0}$. We claim that $J \cap O_n = \emptyset$ for $n \neq n_0$. Let $x \in X \cap J$ and suppose that $y \in J \cap O_n$; with no loss of generality we may assume that $x < y$. Then there exists $z \in A_n \setminus O_N$ such that $a < x, z < y < b$. Now, this implies that $z \in X \cap J$, which in turn implies that $z \in A_{n_0}$ but, since A_n is disjoint from A_{n_0}, this cannot happen. Therefore $J \subset X \cup O_{n_0}$, $J = (J \cap X) \cup (J \cap O_{n_0}) \subset A_{n_0}$, and so $J \subset O_{n_0}$, which is not the case since $X \cap J \neq \emptyset$.

(b) By the Baire category theorem one of the sets must contain an interval of positive length and no Cantor set of positive Lebesgue measure does.

24. Since no point $x \in \mathbb{R}$ is a limit point of $P(x)$, to each $x \in \mathbb{R}$ we can associate an interval $J(x)$ with rational endpoints such that $x \in J(x)$ and $J(x) \cap P(x) = \emptyset$. Now, there are countably many intervals with rational endpoints, $J_1, J_2, \ldots$, say, and for every n let $A_n = \{x \in \mathbb{R} : J(x) = J_n\}$. Then $\mathbb{R} = \bigcup_n A_n$ and by the Baire category theorem one of these sets, A_N, say, is of second category. Hence, if x, y are two points in A_N, then $P(x) \cap A_N = P(y) \cap A_N = \emptyset$, x and y are independent, and, consequently, A_N is an independent set.

26. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{f \in C(I) : f(x) - f(a) > n(x - a)$ for some $x \in (a, a + 1/n) \cap [0, 1]\}$. Then $G_a = \bigcap_n \mathcal{V}_n$ and by Problem 1(b) it suffices to prove that each $\mathcal{V}_n$ is open and dense in $C(I)$.

First, $\mathcal{V}_n$ is open. Let $f \in \mathcal{V}_n$ and pick $x \in (a, a + 1/n) \cap [0, 1]$ such that $f(x) - f(a) > n(x - a)$. Then find $\varepsilon > 0$ such that $f(x) - f(a) > n(x - a) + 2\varepsilon$ and observe that if $g \in B(f, \varepsilon)$, $g(x) - g(a) \geq f(x) - f(a) - 2\varepsilon > n(x - a)$, which implies that $g \in \mathcal{V}_n$ and $\mathcal{V}_n$ is open.

Next, $\mathcal{V}_n$ is dense. Observe that $\mathcal{PL}(I) = \{g \in C(I) : g$ is piecewise linear$\}$ is dense in $C(I)$ (proof by pictures). Then let $g \in \mathcal{PL}(I)$ and pick $h \in \mathcal{PL}(I)$ such that $\|h\| < \varepsilon$ and $D^+h(a) > n - D^+g(a)$; h is easily constructed graphically, turning rapidly the slopes of the lines that define it, which is always possible since $D^+g(a) < \infty$ for g in $\mathcal{PL}(I)$. Then $D^+(g + h)(a) > n$ and since $g + h \in \mathcal{PL}(I)$, $g + h \in \mathcal{V}_n$. Now, since $\mathcal{PL}(I)$ is dense in $C(I)$ and $g + h \in B(g, \varepsilon)$, this guarantees that $\mathcal{V}_n$ is dense in $C(I)$.

27. For each k let $A_k = \bigcap_{\lambda \in \Lambda} \{x \in X : |f_\lambda(x)| \leq k\}$; since the f_λ are continuous, A_k is the intersection of closed sets and, hence, closed. Now, since each $x \in X$ belongs to A_k provided that $k \geq M(x)$, $X = \bigcup_k A_k$ and by the Baire category theorem A_n has nonempty interior for some n, i.e., a ball $B(x_0, \varepsilon) \subset A_n$ for some $x_0 \in A_n$ and $\varepsilon > 0$. Hence $|f_\lambda(x)| \leq n$ for all $\lambda \in \Lambda$ and $x \in B(x_0, \varepsilon)$ and the conclusion holds for $M = n$.

28. Recall that $f : X \to \mathbb{R}$ is lower semicontinuous if $O_\lambda = \{x \in X : f(x) > \lambda\}$ is open in X for all $\lambda \in \mathbb{R}$. For the sake of argument suppose that f is unbounded on every nonempty open subset on X; we claim that then O_λ is dense in X for all $\lambda \in \mathbb{R}$. Let O be a nonempty open subset of X. Since f is unbounded in O there exists $x \in O$ such that $f(x) > \lambda$, $x \in O \cap O_\lambda$, and O_λ is dense in X. Now, by Problem 1(b), $\bigcap_{n=-\infty}^{\infty} O_n$ is a dense G_δ subset of X, but this is impossible since, as is readily seen, $\bigcap_{n=-\infty}^{\infty} O_n = \emptyset$. Therefore f is bounded on a nonempty open subset O of X.

29. (a) Since $\{f_n\}$ is uniformly bounded to invoke Arzela-Ascoli it suffices to verify that the sequence is equicontinuous. Given $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then, for any $x, y \in \mathbb{R}$ with $|x - y| < \delta$, by the mean value theorem $|f_n(x) - f_n(y)| \le \varepsilon$ and the verification is complete.

(b) Let $A_{k,\ell} = \{x \in \mathbb{R} : M(x) \le k, N(x) \le \ell\}$; since the f_n and f_n' are continuous each $A_{k,\ell}$ is closed for all k, ℓ and $\mathbb{R} = \bigcup_{k,\ell} A_{k,\ell}$. Therefore by the Baire category theorem one of the sets, $A_{k,\ell}$, say, contains an open interval J and, in particular, both $\{f_n\}$ and $\{f_n'\}$ are uniformly bounded on any closed subinterval J_0 of J. Now, as in (a) it readily follows that such a sequence is equicontinuous in J_0 and, consequently, a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ converges uniformly in J_0.

32. By Problem 30 with $f_n = f^{(n)}$ there, f coincides with a piecewise polynomial continuous function in a dense open set $O \subset \mathbb{R}$. Let $F = \{x \in \mathbb{R}$: there is no neighborhood V_x of x such that $f|_{V_x}$ is a polynomial$\}$; F is the complement of O and, therefore, is closed and nowhere dense.

For the sake of argument suppose that $F \ne \emptyset$. First, note that F has no isolated points because if x_0 is an isolated point of F, f is a polynomial in $(x_0 - \eta, x_0)$ and in $(x_0, x_0 + \eta)$ for some $\eta > 0$ and by continuity also in $(x_0 - \eta, x_0 + \eta)$. Next, let $A_n = \{x \in \mathbb{R} : f^{(n)}(x) = 0\}$; A_n is closed and by assumption $\bigcup_n A_n = \mathbb{R}$. Applying the Baire category theorem to F, $A_n \cap F$ has nonempty interior in F for some n, i.e., there exists a nonempty interval $(x_0 - \delta, x_0 + \delta)$ such that $(x_0 - \delta, x_0 + \delta) \cap F \subset A_n \cap F$. Since $(x_0 - \delta, x_0 + \delta) = ((x_0 - \delta, x_0 + \delta) \cap F) \cup ((x_0 - \delta, x_0 + \delta) \cap O) = A \cup B$, say, the proof will be complete once we verify that f has vanishing derivatives of order $\ge n$ in both A and B for then f coincides with a polynomial in $(x_0 - \delta, x_0 + \delta)$, which is then contained in O, and this gives the desired contradiction.

First, if $x \in A$, $f^{(k)}(x) = 0$ for all $k \ge n$. Indeed, since F has no isolated points there exists a sequence $\{x_k\} \subset A$ with $\lim_k x_k = x$. Thus, since $A \subset A_n$, $f^{(n)}(x_k) = 0$ for all k and, therefore, by Rolle's theorem we can construct a sequence $\{x_k^1\}$ with x_k^1 between x_k and x_{k+1} such that $f^{(n+1)}(x_k^1) = 0$, all k. Since $\lim_k x_k^1 = x$ it follows that $f^{(n+1)}(x) = \lim_k f^{(n+1)}(x_k^1) = 0$. Repeating this argument with $\{x_k^1\}$ in place of $\{x_k\}$ we get that $f^{(n+2)}(x) = 0$, and so on.

Now, suppose that $x \in B$. Then f is a polynomial in some neighborhood of x and since $F \neq \emptyset$ this neighborhood is not the whole interval $(x_0-\delta, x_0+\delta)$. Let (a,b) be the maximal subinterval of $(x_0-\delta, x_0+\delta)$ containing x on which f is a polynomial. Then a or b, to fix ideas b, say, is in A and so by the above argument $f^{(k)}(b) = 0$ for all $k \geq n$. Now, if f is of degree d in (a,b), $f^{(d)}(x) \neq 0$ for $x \in (a,b)$ and taking limits, $f^{(d)}(a), f^{(b)}(b) \neq 0$. Therefore $d < n$ and, consequently, $f^{(k)}(x) = 0$ for all $k \geq n$ and all x in (a,b). This argument works for all $x \in B$ and combining with the first part we conclude that $f^{(k)}(x) = 0$ for all $k \geq n$ and all $x \in (x_0-\delta, x_0+\delta)$. Then, integrating n times and using the fundamental theorem of calculus we deduce that f is a polynomial of degree $\leq n$ in $(x_0-\delta, x_0+\delta)$, which is not the case. Therefore $F = \emptyset$.

34. (a) Since χ_O assumes only two values, $x \in D(\chi_O)$ iff every neighborhood of x contains points of O and O^c and so $D(\chi_O)$ is precisely the boundary of O. Thus $D(\chi_O)$ is closed and since as noted above its interior is empty, $D(\chi_O)$ is nowhere dense.

(b) Since by (a) the set of continuity of each χ_{O_n} is a dense open subset of X, by Problem 1(b) their intersection is a dense G_δ set in X and, in particular, not empty.

35. (a) Given an open ball $B(x,r) = \{y \in X : d(x,y) < r\}$, let $w(f; B(x,r))$ denote the oscillation of f in $B(x,r)$, i.e., $w(f;B(x,r)) = \sup\{|f(y)-f(z)| : y,z \in B(x,r)\}$. Now, with $\mathcal{B}_x = \{\text{open balls } B : x \in B\}$, the oscillation $w(f,x)$ of f at x is defined as $w(f,x) = \inf\{w(f;B) : B \in \mathcal{B}_x\}$; note that f is continuous at x iff $w(f,x) = 0$.

Now, $D(f) = \bigcup_n D_n(f)$ where $D_n(f) = \{x \in X : w(f,x) \geq 1/n\}$. Note that $D_n(f)$ is closed, all n. Indeed, if x is a limit point of $D_n(f)$, then every $B \in \mathcal{B}_x$ contains $y \in D_n(f)$, $x \neq y$, and so $w(f;B) \geq w(f,y) \geq 1/n$. Thus $w(f,x) \geq 1/n$, i.e., $x \in D_n(f)$, and $D_n(f)$ is closed.

(b) Necessity first. By (a) $D(f) = \bigcup_n D_n(f)$ where each $D_n(f)$ is closed. Moreover, since f is continuous on a dense subset of X, any open ball contained in $D(f)$ contains a point of continuity of f and, since $D_n(f) \subset D(f)$, any open ball contained in $D_n(f)$ is also contained in $D(f)$. But $D(f)$ cannot contain any such ball and so $D(f)$ is of first category.

As for sufficiency, by Problem 1(c) if $D(f)$ is of first category in X, its complement contains a dense G_δ subset of X.

36. (a) For the sake of argument suppose such an f exists. Then by Problem 35, $C(f)$ is a G_δ subset of $[0,1]$, and countable dense at that, but by Problem 4(c) no such set exists.

(b) and (c) The sets in question are countable and dense. Therefore as in (a) it readily follows that f does not exist.

37. The statement is false. Let f be left-continuous everywhere and $\varepsilon_k = 2^{-k}$, $k = 0, 1, \ldots$. Then for any $x_0 \in \mathbb{R}$, let δ_0 be such that $|f(x_0)-f(y)| < \varepsilon_0$ for $y \in J_0 = (x_0 - \delta_0, x_0)$. Next, for each $k \geq 1$ pick $x_k \in J_{k-1}$ and δ'_k such that $|f(x_k) - f(y)| < \varepsilon_k$ for $x_k - \delta'_k < y < x_k$. Now, since J_{k-1} is open, we may pick $\delta_k < \delta'_k$ so that, if $J_k = (x_k - \delta_k, x_k)$, $\overline{J}_k \subset J_{k-1}$; note that $|f(x_k) - f(y)| < \varepsilon_k$ for $y \in J_k$. Then $\{\overline{J}_k\}$ is a nested sequence of nonempty compact sets and by Cantor's nested property $\bigcap_k \overline{J}_k \neq \emptyset$. Let $x \in \bigcap_k \overline{J}_k$ and, given $\varepsilon > 0$, let k be such that $2\varepsilon_k < \varepsilon$. Since $x \in \overline{J}_{k+1} \subset J_k$ we can pick $r_\varepsilon > 0$ such that $(x, x + r_\varepsilon) \subset J_k$ and so, for any $y \in (x, x + r_\varepsilon)$, $|f(x) - f(y)| \leq |f(x) - f(x_k)| + |f(x_k) - f(y)| < 2\varepsilon_k < \varepsilon$ and f is right-continuous at x. Now, since in the above proof J_0 could be an arbitrary open subset of $\mathbb{R}$ it follows that the set of points where f is right-continuous is dense in $\mathbb{R}$.

38. Let $\{r_n\}$ be an enumeration of the rationals and $B = \{x \in A : f'_g(x) < f'_d(x)\}$; note that if $x \in B$ there is a smallest integer k, say, such that $f'_g(x) < r_k < f'_d(x)$. Let $\varphi(x, y) = (f(x)-f(y))/(x-y)$; since $\varphi(x, y) \to f'_g(x)$ as $y \to x^-$ it follows that $\varphi(x, y) < r_k$ for y sufficiently close to x. Thus there is a smallest integer m, say, such that $r_m < x$ and $\varphi(x, y) < r_k$ for all y such that $r_m < y < x$. Similarly, there is a smallest integer n, say, such that $r_n > x$ and $\varphi(x, y) < r_k$ for $x < y < r_n$. Then $f(y) - f(x) > r_k(y - x)$ for $r_m < y < r_n$. We claim that the mapping $x \to (k, m, n)$ is injective. Indeed, if $x \neq x_1$ corresponds to the same triplet, then $f(x_1) - f(x) > r_k(x_1 - x)$ and the opposite inequality must simultaneously hold, which is impossible. Thus B has at most cardinality $\aleph_0^3 = \aleph_0$.

39. Let $\varphi_n(x) = 2^{-n}\chi_{(a_n,\infty)}(x)$ and put $f(x) = \sum_n \varphi_n(x)$.

40. Since Y is separable, given an integer n, there exist open sets $\{O^n_m\} \subset Y$ such that $Y = \bigcup_m O^n_m$ and $\text{diam}(O^n_m) \leq 1/2n$ for all m. Now, $f^{-1}(O^n_m) = \bigcup_k Z^n_{k,m}$, say, where the $Z^n_{k,m}$ are closed in X and since the O^n_m cover Y for each n, $X = \bigcup_{k,m} Z^n_{k,m}$, all n.

Now, since by Problem 3(b) $\bigcup_{k,m} \text{int}(Z^n_{k,m})$ is dense in X, by Problem 1(b) it suffices to prove that f is continuous on $G = \bigcap_n \bigcup_{m,k} \text{int}(Z^n_{m,k})$, which is a dense G_δ subset in X. So, let $x \in G$ and note that for every n there exist m, k such that $x \in \text{int}(Z^n_{m,k})$. Given $\varepsilon > 0$, choose n such that $1/n < \varepsilon$. Then $x \in \text{int}(Z^n_{m,k})$ for some m, k, n, and, therefore, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $y \in Z^n_{m,k}$. Hence $f(x), f(y) \in U^n_m$ and, therefore, $d'(f(x), f(y)) < 1/n < \varepsilon$.

Finally, since it is readily seen that the preimage of an open set by a lower semicontinuous function $f : \mathbb{R} \to \mathbb{R}$ is an F_σ set in $\mathbb{R}$, it follows that such an f is continuous on a dense G_δ subset of $\mathbb{R}$.

41. For the sake of argument suppose that f_n are continuous functions such that $\lim_n f_n(x) = \chi_{\mathbb{Q}}(x)$ and let $A_k = \bigcap_{n\geq k} f_n^{-1}([-1/2, 1/2])$; the A_k are the intersection of closed sets and, hence, closed. Moreover, since for $x \in A_k$, $|f_n(x)| < 1/2$ for $n \geq k$, it follows that $\lim_n f_n(x) = \chi_{\mathbb{Q}}(x) = 0$ and A_k consists entirely of irrational numbers. Now, if x is irrational, $x \in A_k$ for sufficiently large k and, therefore, the irrationals in $\mathbb{R}$ can be written as the F_σ set $\bigcup_k A_k$; then the rationals are a G_δ set in $\mathbb{R}$ and by Problem 4(c) this is not the case.

Next, let $f_{m,n}(x) = \cos^{2m}(n!\pi x)$. Note that if x is rational with irreducible expression p/q and $q \leq n$, then $\lim_m \cos^{2m}(n!\pi x) = 1$. In all other cases, i.e., if x is irrational or q does not divide $n!$, $\lim_m \cos^{2m}(n!\pi x) = 0$. Thus with $f_n(x) = \lim_m \cos^{2m}(n!\pi x)$ it follows that $\lim_n f_n(x) = \chi_{\mathbb{Q}}(x)$ and $\chi_{\mathbb{Q}}$ is the limit of functions which are in turn limits of continuous functions.

42. For the sake of argument suppose that d is a metric in $X \times X$ such that convergence in (X, d) is equivalent to pointwise convergence. Now, if $\{f_{m,n}(x)\}$ is as in Problem 41, the pointwise limit $\lim_m f_{m,n}(x) = \varphi_n(x)$, say, exists for all n, and, consequently, $d(f_{m,n}, \varphi_n) \to 0$ as $m \to \infty$. Therefore by the triangle inequality $\lim_n d(\varphi_n, \chi_{\mathbb{Q}}) = 0$ and, using the Cantor diagonal process, it follows that for all m there exists $n(m)$ such that $d(f_{m,n(m)}, \chi_{\mathbb{Q}}) \to 0$ and so, since convergence in the metric is equivalent to pointwise convergence, $\chi_{\mathbb{Q}}$ is the limit of a sequence of continuous functions, which by Problem 41 is not the case.

43. Fix an integer n and for an integer m let $A_m^n = \bigcap_{k=m}^\infty \{x \in X : |f_m(x) - f_k(x)| \leq 1/n\}$; by the continuity of the f_k each A_m^n is closed. Now, since the numerical sequence $\{f_m(x)\}$ converges for all $x \in X$ it is Cauchy there and so there exists an integer m (depending on x) such that $|f_m(x) - f_k(x)| \leq 1/n$ for all $k \geq m$, that is to say, $x \in A_m^n$. Therefore $X = \bigcup_m A_m^n$ and if $O_{m,n} = \operatorname{int}(A_m^n)$, by Problem 3(b) $O_n = \bigcup_m O_{m,n}$ is an open dense subset of X.

We claim that f is continuous in $G = \bigcap_n O_n$, which by Problem 1(b) is a dense G_δ subset of X. To see this let $x \in G$ and given $\varepsilon > 0$, pick n such that $1/n < \varepsilon/3$. Now, since $x \in O_n$, $x \in O_{m,n}$ for some m. Moreover, since $O_{m,n} \subset A_m^n$ we have $|f_m(y) - f_k(y)| \leq 1/n$ for $y \in O_{m,n}$ and $k \geq m$. Thus letting $k \to \infty$ it follows that $|f_m(y) - f(y)| \leq 1/n$, an inequality that holds in particular for $y = x$. Next, since f_m is continuous at x there exists a neighborhood V of x, which we may assume is contained in $O_{m,n}$, such that $|f_m(y) - f_m(x)| < \varepsilon/3$ for all $y \in V$. Thus we have proved that for $y \in V$, $|f(y) - f(x)| \leq |f(y) - f_m(y)| + |f_m(y) - f_m(x)| + |f_m(x) - f(x)| < \varepsilon$, which gives the continuity of f at x. So f is at least continuous on the G_δ dense set G, which is of second category.

Note that the result applies to an everywhere differentiable function $f : \mathbb{R} \to \mathbb{R}$. Indeed, consider the sequence

$$f_n(x) = n\left(f\left(x + \frac{1}{n}\right) - f(x)\right), \quad n = 1, 2, \ldots.$$

Then $\{f_n\} \subset C(\mathbb{R})$ and, consequently, $f'(x) = \lim_n f_n(x)$ is continuous on a dense G_δ subset of $\mathbb{R}$.

46. The statement is true. By assumption there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $|f_{n_k}(x) - f(x)| \le 2^{-k}$ for all $x \in I$. Consider now $\{f_{n_{k+1}} - f_{n_k}\}$. By the triangle inequality,

$$\begin{aligned} |f_{n_{k+1}}(x) - f_{n_k}(x)| &\le |f_{n_{k+1}}(x) - f(x)| + |f(x) - f_{n_k}(x)| \\ &\le 2^{-(k+1)} + 2^{-k} = (3/2)\, 2^{-k} \end{aligned}$$

and, consequently, by Problem 45, $\sum_k (f_{n_{k+1}}(x) - f_{n_k}(x)) \in \mathcal{B}_1$. Thus $\sum_k (f_{n_{k+1}}(x) - f_{n_k}(x)) = \lim_N \sum_{k=1}^N (f_{n_{k+1}}(x) - f_{n_k}(x)) = f(x) - f_{n_1}(x) \in \mathcal{B}_1$ and since $f_{n_1} \in \mathcal{B}_1$, by Problem 45, $f = f - f_{n_1} + f_{n_1} \in \mathcal{B}_1$.

47. Suppose f is lower semicontinuous and for each $r \in \mathbb{Q}$ let $O_r = \{x \in X : f(x) > r\}$; since the O_r are open the sets $F_r = \overline{O_r} \setminus O_r$ are closed and nowhere dense. Thus it suffices to prove that $D(f) \subset \bigcup_{r \in \mathbb{Q}} F_r$. Let $x \in D(f)$. Then there exists $\varepsilon > 0$ such that for every neighborhood U of x and every $r \in (f(x), f(x) + \varepsilon)$, the set $U \cap O_r$ is nonempty. Therefore, if $r \in (f(x), f(x) + \varepsilon)$, $x \in F_r$. Compare with Problem 40.

48. Let $X = \bigcup_n X_n$ where the X_n are closed and nowhere dense and for $x \in X$ let $f(x) = \inf\{n : x \in X_n\}$ and $g = 1 - (1/f)$; g is clearly bounded on X, we claim that g is lower semicontinuous. For this it suffices to prove that f is lower semicontinuous which is obvious since $\{x \in X : f(x) \le k\} = \bigcup_{n=1}^k X_n$ for every $k = 1, \ldots$. Moreover, f is continuous at no point of X. Indeed, if O is a nonempty open subset of X, then no finite number of X_n cover O. Therefore f is not bounded on O. Clearly g is continuous at no point of X.

As an immediate consequence of this result it follows that a metric space (X, d) is of second category iff every semicontinuous function on X is continuous at some point of X.

49. We do the case when f is lower semicontinuous. For each integer n let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = \inf_{t \in [0,1]} (f(t) + n|t - x|)$. First, each f_n is continuous. Since a lower semicontinuous function achieves its minimum in $[0,1]$, f is bounded below and f_n takes finite values. Now, for $x, y \in [0,1]$, $f_n(x) = \inf_{t \in [0,1]} (f(t) + n|t - x|) \le f_n(y) + n|y - x|$ and exchanging x and y, also $f_n(y) \le f_n(x) + n|y - x|$. Therefore $|f_n(x) - f_n(y)| \le n|x - y|$ and so each f_n is continuous in $[0,1]$.

Next, $f(x) = \lim_n f_n(x)$ everywhere. Let $x \in [0,1]$; by definition $f_n(x) \le f(t) + n|t-x|$ for all $t \in [0,1]$ and, in particular, setting $t = x$, $f_n(x) \le f(x)$ and so $\limsup_n f_n(x) \le f(x)$. Next, suppose that $f(x) > r$, $r \in \mathbb{R}$. Since f is lower semicontinuous at x there exists $\delta > 0$ such that $f(t) > r$ for all $t \in (x-\delta, x+\delta) \cap [0,1] = J$, say. For these values of t, $\inf_{t\in J} (\, f(t)+n|t-x|\,) \ge r$ and, since $|t-x| \ge \delta$ for $t \in [0,1] \setminus J$, $\inf_{t\in[0,1]\setminus J}(\, f(t)+n|t-x|\,) \ge -M + n\delta$ where $-M$ is a lower bound for f. Thus $f_n(x) \ge r$ for n sufficiently large and so $\liminf_n f_n(x) \ge r$. Hence, since $f(x) > r$ is arbitrary, $\liminf_n f_n(x) \ge f(x)$ and finally, combining these inequalities, $\lim_n f_n(x) = f(x)$.

50. Let $A = \bigcup_n A_n$ where each A_n is closed and nowhere dense. Given $x \in A$, $m_x = \min\{n : x \in A_n\}$ is well-defined, and let $f : X \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1/m_x, & x \in A, \\ 0, & x \notin A. \end{cases}$$

We claim that $f \in \mathcal{B}_1$ and $D(f) = A$. First, $D(f) = A$. Let $x \in A$; since A contains no open ball, given $\varepsilon > 0$, there is $y \in B(x,\varepsilon) \setminus A$. Then $|f(y) - f(x)| = |0 - 1/m_x| = 1/m_x$ and f is not continuous at x. Conversely, let $x \in X \setminus A$, $\varepsilon > 0$, and pick N such that $1/N < \varepsilon$. Since $F = \bigcup_{n=1}^N A_n$ is closed there exists $\delta > 0$ such that $B(x,\delta) \cap F = \emptyset$. Then $|f(y) - f(x)| < 1/N < \varepsilon$ for all $y \in B(x,\delta)$ and f is continuous at x. Thus $D(f) = A$.

Next, we verify that f is upper semicontinuous. For $r \in \mathbb{R}$ let $B_r = \{x \in X : f(x) \ge r\}$. Then

$$B_r = \begin{cases} X, & r \le 0, \\ \bigcup_{n=1}^m A_n, & 1/(m+1) < r \le 1/(m), \\ \emptyset, & r > 1. \end{cases}$$

Thus B_r is closed in all cases, f is upper semicontinuous, and by Problem 49, $f \in \mathcal{B}_1$.

51. Since O can be written as a countable union of pairwise disjoint open intervals and since we can write an open interval $(a,b) = (-\infty,b) \cap (a,\infty)$, it suffices to prove that $f^{-1}((-\infty,q))$ and $f^{-1}((q,\infty))$ are F_σ in X for each rational q. Let $\{f_n\}$ be continuous functions on X such that $\lim_n f_n(x) = f(x)$ for each $x \in X$. Observe that with $\liminf_n\{f_n \le p\} = \bigcup_m \bigcap_{n=m}^\infty \{f_n \le p\}$, $f^{-1}((-\infty,q)) = \bigcup_{p\in\mathbb{Q},\, p<q} \liminf_n\{f_n \le p\}$. Now, the continuity of the f_n implies that the sets $\{f_n \le p\}$ are closed and, therefore, $\liminf_n\{f_n \le p\}$ is an F_σ subset of X, as is $f^{-1}(-\infty,q)$. Similarly, $\{f > q\} = \bigcup_{p\in\mathbb{Q},\, p>q} \liminf_n\{f_n \ge p\}$ is an F_σ subset of X.

The converse to the statement is also true and is due to Baire. Finally, observe that since $\chi_{\mathbb{Q}}^{-1}((-1/2,1/2)) = \mathbb{R} \setminus \mathbb{Q}$ is not F_σ, $\chi_{\mathbb{Q}} \notin \mathcal{B}_1$.

53. Given $\varepsilon > 0$, let $B_N = \bigcap_{n=N}^{\infty}\{x \in (0,\infty) : |f(nx)| \le \varepsilon\}$; since f is continuous, B_N is closed. Now, if $x \in (0,\infty)$, there is N_x such that $|f(nx)| \le \varepsilon$ for $n \ge N_x$ and so $x \in B_{N_x} \subset \bigcup_N B_N$. Thus $(0,\infty) = \bigcup_N B_N$ and by the Baire category theorem there are N_0 and an interval $J = (a,b)$ such that $|f(nx)| \le \varepsilon$ for $x \in J$ and all $n \ge N_0$, and, consequently, $|f(x)| \le \varepsilon$ for all $x \in V = \bigcup_{n \ge N_0}(na, nb)$. Observe that if N is an integer greater than $a/(b-a)$, then for $n > N$ we have $(n+1)a < nb$, and, consequently, $(Na,\infty) \subset V$. Hence, given $\varepsilon > 0$, there exists $A > 0$ such that $x \ge A$ implies $|f(x)| \le \varepsilon$, i.e., $\lim_{x\to\infty} f(x) = 0$.

54. Such a number x can be written with only a finite number of nonzero decimals, say n (depending on x). Then for some integer m, $x = m/10^n$, x is rational, and the set in question is countable.

55. For the sake of argument suppose that X is countable and let $f : \mathbb{N} \to X$ be a bijection from $\mathbb{N}$ into sequences of 0's and 1's. If $f(n) = (x_1^n, x_2^n, \ldots)$ is the sequence that corresponds to n let y be the sequence with terms $y_n = 1 - x_n^n$, all n. Then y is a sequence of 0's and 1's but since $y_n \ne x_n^n$ for all n, y cannot be any of the sequences in $f(\mathbb{N})$.

56. The statement is true. Since $\mathbb{N}$ and $\mathbb{Q} \cap [0,1]$ are equivalent we work with the latter set instead. $\mathcal{F}$ is then defined as the set of sequences of rationals in $[0,1]$ that converge to the irrationals there, one sequence per irrational. Since $\operatorname{card}([0,1] \setminus \mathbb{Q}) = c$, $\mathcal{F}$ has the right cardinality. Now, if $A, B \in \mathcal{F}$ have infinite intersection, they contain a common subsequence that must converge to the irrational number determined by both A and B. But since these numbers are different, the intersection is finite.

57. Yes. Enumerate $A = \{a_1, a_2, \ldots\}$ and observe that the set of all possible distances $|a_n - a_m|$ is countable. Now, since $\mathbb{R}$ is uncountable there is a real number r that is not equal to any of the distances and so $A \cap (r + A) = \emptyset$.

58. For the sake of argument suppose that $A \cap (-\infty, t)$ or $A \cap (t,\infty)$ is countable for all $t \in \mathbb{R}$; since A is uncountable at most one of these sets can be countable. Let $\{r_n\}$ be an enumeration of the rationals of $\mathbb{R}$ and for every n let A_n be $A \cap (-\infty, r_n)$ if $A \cap (-\infty, r_n)$ is countable and $A \cap (r_n, \infty)$ otherwise. Consider now $A \setminus \bigcup_n A_n$. If $A \setminus \bigcup_n A_n = \emptyset$, A is countable, which is not the case. If not, let $y \in A \setminus \bigcup_n A_n$ and write $A = (\bigcup_{r_n < y}(A \cap (-\infty, r_n))) \cup \{y\} \cup (\bigcup_{r_n > y}(A \cap (r_n, \infty)))$. Now, for all $r_n < y$, $A \cap (-\infty, r_n)$ is countable because $y \notin \bigcup_n A_n$; similarly, for all $r_n > y$, $A \cap (r_n, \infty)$ is countable. Thus $A \setminus \bigcup_n A_n = \{y\}$ and since $A \setminus \bigcup_n A_n$ is countable, A is countable, which is not the case. The second statement follows by applying the first statement twice.

60. Since there are c open dense sets in $\mathbb{R}$ and, consequently, c dense G_δ subsets of $\mathbb{R}$, there are c closed nowhere dense subsets of $\mathbb{R}$. Now, with Ω the first uncountable ordinal let $\{A_\alpha : \alpha < \Omega\}$ be the collection of all closed nowhere dense sets. Inductively choose $x_\alpha \in \mathbb{R}$ as follows: x_1 is any point in A_1 and, having chosen x_β for $\beta < \alpha < \Omega$, let x_α be a point not in $\{x_\beta : \beta < \alpha\} \cup \bigcup_{\beta<\alpha} A_\beta$; this choice is always possible by the Baire category theorem. Then $A = \{x_\alpha : \alpha < \Omega\}$ is the required set.

62. (a) A contains a finite subset A_n of cardinality n for each integer n and $\bigcup_n A_n \subset A$ is countable.

(b) Suppose that $A \cap \mathbb{N} = \emptyset$ and with $B = \{a_1, \ldots, a_n, \ldots\}$ a countable subset of A, let $\varphi : A \cup \mathbb{N} \to A$ be given by

$$\varphi(x) = \begin{cases} x, & x \in A \setminus B, \\ a_{2n-1}, & x = a_n \in B, \\ a_{2n}, & x = n \in \mathbb{N}. \end{cases}$$

Then φ is a bijection and $a + \aleph_0 = \operatorname{card}(A \cup \mathbb{N}) = \operatorname{card}(A) = a$.

(c) Let $\mathcal{F} = \{(X, f) : X \subset A \text{ and } f : X \to \{0,1\} \times X \text{ is a bijection}\}$; by Problem 61, $X \sim \{0,1\} \times X$ for $X \subset A$ countable and $\mathcal{F} \neq \emptyset$. Now, $\mathcal{F}$ is partially ordered by set inclusion and extension of functions, i.e., $(X, f) \prec (X', f')$ if $X \subset X'$ and $f'|_X = f$. Moreover, if $\mathcal{C} \subset \mathcal{F}$ is a chain, $(\widetilde{X}, \widetilde{f})$ defined by $\widetilde{X} = \bigcup_{(X,f)\in\mathcal{C}} X$ and $\widetilde{f}(x) = f(x)$ for $x \in X$, $(X, f) \in \mathcal{C}$, is an upper bound of $\mathcal{C}$. Note that $\widetilde{f}$ is well-defined since for $(X, f), (X', f') \in \mathcal{C}$, since $\mathcal{C}$ is a chain we may assume that $(X, f) \prec (X', f')$. So, if $x \in X$, $x \in X'$ and $f'(x) = f(x)$. Thus every chain in $\mathcal{F}$ has an upper bound and, consequently, by Zorn's lemma there is a maximal element (D, g) in $\mathcal{F}$ with $g : D \to \{0,1\} \times D$ a bijection. Clearly $D \subset A$ has infinite cardinality and we have $A = (A \setminus D) \cup D$. Now, if $A \setminus D$ is finite, by (a) $\operatorname{card}(A) = \operatorname{card}(D)$ and we are done. So, for the sake of argument suppose that $A \setminus D$ is infinite and let B be a countable subset of $A \setminus D$ and $f : B \to \{0,1\} \times B$ a bijection. Then $B \cap D = \emptyset$, $X = D \cup B \subset A$, and the mapping $h : X \to \{0,1\} \times X$ given by

$$h(x) = \begin{cases} g(x), & x \in D, \\ f(x), & x \in B, \end{cases}$$

is 1-1 and onto, $(h, X) \in \mathcal{F}$ and, since $h|_D = g$, (h, X) extends (g, D), contrary to its maximality. Therefore this case cannot occur and we have established that $a + a = a$. Finally, since $b + b = b$ and $b \le a + b \le b + b = b$, it readily follows that $a + b = b$.

63. (a) Let $\mathcal{A} = \{A_\alpha \subset A : \text{the } A_\alpha \text{ are countable and pairwise disjoint}\}$ and $\mathcal{F} = \{\mathcal{A}\}$; $\mathcal{F}$ consists of all pairwise disjoint collections of countable subsets of A and by Problem 62(a), $\mathcal{F} \neq \emptyset$. Now, $\mathcal{F}$ is partially ordered by set inclusion, i.e., given two collections $\mathcal{A}_1, \mathcal{A}_2$ in $\mathcal{F}$, we say that $\mathcal{A}_1 \prec \mathcal{A}_2$ if every countable subset of A in $\mathcal{A}_1$ is in $\mathcal{A}_2$. And, if $\mathcal{C} = \{P_\alpha\}_{\alpha\in I}$ is a chain in $\mathcal{F}$, the collection of all countable subsets of A which belong to P_α for some $\alpha \in I$ is an upper bound of $\mathcal{C}$. Hence by Zorn's lemma $\mathcal{F}$ has a maximal element $\mathcal{M} = \{M_\alpha\}$, say, comprised of countable subsets of A. Now, if $A \setminus \bigcup_\alpha M_\alpha \neq \emptyset$, $A \setminus \bigcup_\alpha M_\alpha$ is finite or by Problem 62(a) contains a countable set, which is not the case by the maximality of $\mathcal{M}$. Therefore $A = \bigcup_\alpha M_\alpha$ and we have finished.

(b) Let A be a set with at least two elements and consider $\mathcal{F} = \{(X, f) : X \subset A, f : X \to X^c \text{ is injective}\}$; $\mathcal{F}$ is partially ordered by set inclusion and extension of functions and every chain in $\mathcal{F}$ has an upper bound. Therefore by Zorn's lemma $\mathcal{F}$ has a maximal element $(\widetilde{X}, \widetilde{f})$, say. Observe that $\widetilde{X}^c \cap \widetilde{f}(\widetilde{X})^c$ contains at most one element. Indeed, if $x \neq x' \in \widetilde{X}^c \cap \widetilde{f}(\widetilde{X})^c$, then $(\bar{X}, \bar{f})$ defined by

$$\bar{X} = \widetilde{X} \cup \{x\}, \quad \bar{f}\big|_{\widetilde{X}} = \widetilde{f}, \quad \bar{f}(x) = x'$$

belongs to $\mathcal{F}$ and satisfies $(\widetilde{X}, \widetilde{f}) \prec (\bar{X}, \bar{f})$, which is not possible by the maximality of $(\widetilde{X}, \widetilde{f})$. Therefore A is the disjoint union of $\widetilde{X}, \widetilde{f}(\widetilde{X})$ and a set S of at most one element. Since $\widetilde{f}$ is 1-1, $\widetilde{X} \sim \widetilde{f}(\widetilde{X})$ and these sets have the same cardinality. Finally, since A is uncountable, $\widetilde{X}$ and $\widetilde{X}^c = \widetilde{f}(\widetilde{X}) \cup S$ are both uncountable.

64. First, if $A = \{a_1, \ldots, a_n, \ldots\}$ is countable, the mappings $f : A \times \mathbb{N} \to A$ given by $f(a_n, m) = a_{2^n 3^m}$ and $g : A \to A \times \mathbb{N}$ given by $g(a_n) = (a_n, 1)$ are injective and, consequently, by the Cantor-Bernstein-Schröder theorem, there exists a bijection $\phi : A \times \mathbb{N} \to A$. In particular, $\aleph_0 \cdot \aleph_0 = \aleph_0$.

Let $\mathcal{F} = \{(X, f) : X \subset A, f : X \times X \to X \text{ is a bijection}\}$; since for a countable subset X of A, $X \times X \sim X$, $\mathcal{F} \neq \emptyset$. Note that $\mathcal{F}$ is partially ordered by set inclusion and extension of functions and every chain in $\mathcal{F}$ has an upper bound. Therefore, by Zorn's lemma $\mathcal{F}$ has a maximal element (D, f), say. If $\operatorname{card}(D) = \operatorname{card}(A)$ we are done. Now, for the sake of argument suppose that $\operatorname{card}(D) < \operatorname{card}(A)$. Then $A \setminus D \neq \emptyset$ and there is an injection from $A \setminus D$ into D or an injection from D into $A \setminus D$. In the former case there is an injection of $A = (A \setminus D) \cup D$ into $\{0, 1\} \times D$ and, since $D \times D \sim D$, also an injection of A into D and, therefore, by the Cantor-Bernstein-Schröder theorem $A \sim D$, which is not the case.

In the latter case D is equivalent to a subset Y of $A \setminus D$ and put $Z = D \cup Y$; then

$$Z \times Z = (D \times D) \cup (D \times Y) \cup (Y \times D) \cup (Y \times Y)\,.$$

Now, since $D \sim Y$, $(D \times Y) \cup (Y \times D) \cup (Y \times Y) \sim \{0, 1, 2\} \times D$. Also, since $D \times D \sim D$ there is an injection from $\{0, 1, 2\} \times D$ into $D \times D$ and, therefore, an injection from $(D \times Y) \cup (Y \times D) \cup (Y \times Y)$ into $(Y \times Y)$. Hence, by the Cantor-Bernstein-Schröder theorem there is a bijection $f_1 : (D\times Y)\cup(Y\times D)\cup(Y\times Y) \to Y$. Define now $g : Z \to Z\times Z$ by $g|_D = f$ and $g|Y = f_1$. Then $(Z, g) \in \mathcal{F}$ and (D, f) precedes it, but this is not possible since (D, f) is a maximal element of $\mathcal{F}$. In other words, $\mathrm{card}(D) = \mathrm{card}(A)$ and we are done.

The reader will have no difficulty in verifying that given cardinals a, b, $a \cdot b = b \cdot a$, $a \cdot (b \cdot d) = (a \cdot b) \cdot d$ and $a \cdot (b + d) = a \cdot b + a \cdot d$. Also, if $a, b \ge \aleph_0$ and $a \le b$, then $a \cdot b = b$. The proof follows along similar lines to the ones discussed above; we will not deprive the reader the pleasure of carrying them out.

65. If $\{A_d\}$ are such that $\mathrm{card}(A_d) = a_d$, all $d \le b$, the sum $\sum_{d\le b} a_d$ is defined as $\mathrm{card}(\bigcup_{d\le b}(A_d \times \{d\}))$; manipulating bijections it readily follows that $\sum_{d\le b} a_d = \mathrm{card}(\bigcup_{d\le b} B_d)$ where $\{B_d\}_{d\le b}$ are pairwise disjoint sets with $\mathrm{card}(B_d) = a_d$. It is then clear that $\mathrm{card}(\bigcup_{d\le b} B_d) \le b \cdot b = b$.

In particular, with $b = \aleph_0$, if $\{A_n\}$ is any sequence of countable sets, $\mathrm{card}(\bigcup_n A_n) \le \aleph_0 \cdot \aleph_0 = \aleph_0$. Alternatively, since there exist 1-1 and onto mappings $f_n : A_n \to \mathbb{N}$ for every n, one can define $\phi : \bigcup_n A_n \to \mathbb{N}$ as follows: If $x \in \bigcup_n A_n$, let $x \in A_N$, say, and put $\phi(x) = 2^N 3^{f_N(x)}$; clearly ϕ is 1-1 and onto a subset on $\mathbb{N}$ and so $\bigcup_n A_n$ is at most countable.

66. (a) The statement is false. Let $\mathcal{B}$ be a collection of pairwise disjoint balls in $\mathbb{R}^n$. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, every ball in $\mathcal{B}$ contains a point in $\mathbb{Q}^n$, and different balls contain different points. Hence, since $\mathbb{Q}^n$ is countable, so is $\mathcal{B}$.

(b) The statement is true. If $S_r = \{x \in \mathbb{R}^n : |x| = r\}$, then $\mathcal{S} = \{S_r : r > 0\}$ is a collection of pairwise disjoint spheres in $\mathbb{R}^n$.

(c) The statement is false. Recall that a figure eight in the plane is a set of the form $D_1 \cup D_2$ where D_1 and D_2 are circles whose bounded disks intersect at exactly one point. Since the plane can be written as a countable union of an increasing sequence of bounded disks it suffices to prove that a bounded disk contains at most countably many pairwise disjoint eights. Then let D be a bounded disk, $\mathcal{C} = \{E_\alpha : E_\alpha$ is a figure eight totally contained in $D\}$, and, with $d(E_\alpha)$ denoting the diameter of E_α, $\eta = \{\sup_\alpha d(E_\alpha) : E_\alpha \in \mathcal{C}\}$; since the E_α are totally contained in C, $\eta < \infty$. Let $\mathcal{C}' = \{E_\alpha \in \mathcal{C} : d(E_\alpha) > \eta/2\}$; by the definition of η, $\mathcal{C}' \ne \emptyset$. Observe that no two distinct elements of $\mathcal{C}'$ can be contained in one another (for otherwise the diameter of the larger would exceed η) and that by area considerations $\mathcal{C}'$ contains finitely many E_α. Now let $\mathcal{C}_1 = \mathcal{C} \setminus \mathcal{C}'$; then $\eta_1 = \sup_\alpha\{d(E_\alpha) : E_\alpha \in \mathcal{C}_1\} \le \eta/2$ and repeating the argument for $\mathcal{C}_1, \eta_1$ in place of $\mathcal{C}, \eta$ we can extract a second pairwise disjoint finite family of E_α with $d(E_\alpha) > \eta_1/2$. The procedure is now clear: having picked $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}$ and $\eta_k = \{\sup_\alpha d(C_\alpha) : C_\alpha \in \mathcal{C}_k\}$ with $\eta_1 \geq 2\eta_2 \geq \ldots \geq 2^{n-1}\eta_{n-1}$ and $\eta_k \leq \eta/2^k$ for $1 \leq k \leq n$, pick $\mathcal{C}_n \subset \mathcal{C} \setminus \bigcup_{k=1}^{n-1} \mathcal{C}_k$ and $\eta_n = \sup_\alpha\{d(E_\alpha) : E_\alpha \in \mathcal{C}_n\} \leq \eta_{n-1}/2 \leq \eta/2^n$. Observe that $\mathcal{C}$ can be written as the countable union of the $\mathcal{C}_n$ because, since $\eta_n \to 0$ as $n \to \infty$, if $E_\alpha \in \mathcal{C}$, then $E_\alpha \in \mathcal{C}_n$ the first time that $d(E_\alpha) > \eta_n/2$. Thus $\mathcal{C}$ is the countable union of finite sets and is therefore countable.

67. We consider the rational sequences. Suppose $\{a_n\}$ is such that $a_n = a_N$ for all $n \geq N$. Then there are $\aleph_0^{N-1}$ possible choices for the first $N-1$ terms and $\aleph_0$ choices for a_N, giving a total of $\aleph_0^N = \aleph_0$ possible such sequences. So, since $N \in \mathbb{N}$, there are $\aleph_0^2 = \aleph_0$ such possible sequences.

68. Since continuous functions on I are determined by their values on $\mathbb{Q} \cap [0,1]$, there are $c^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = c$ of them. Now, for the limits of the sequences of continuous functions, there are $c^{\aleph_0} = c$ of them.

69. Let Ω denote the first uncountable ordinal. We first define classes $\mathcal{B}_\alpha$ of cardinality c for each ordinal $\alpha < \Omega$ and then prove that $\mathcal{B}(\mathbb{R}) = \bigcup_{\alpha<\Omega} \mathcal{B}_\alpha$; the cardinality assertion follows readily from this. Let $\mathcal{B}_0$ denote the collection of open intervals in $\mathbb{R}$. In order to define $\mathcal{B}_\lambda$ we need to consider two cases, namely, λ is a countable limit ordinal or $\lambda = \alpha + 1$ is a successor ordinal. In the former case let $\mathcal{B}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{B}_\alpha$ and in the latter let $\mathcal{B}_{\alpha+1}$ be the collection of all subsets of $\mathbb{R}$ that are a countable union of elements in $\mathcal{B}_\alpha$, or a countable intersection of elements in $\mathcal{B}_\alpha$, or differences of two elements in $\mathcal{B}_\alpha$. Clearly the families are increasing in the sense that $\mathcal{B}_\alpha \subset \mathcal{B}_\beta$ for $\alpha < \beta$.

Let $\mathcal{U} = \bigcup_{\alpha<\Omega} \mathcal{B}_\alpha$; we claim that $\mathcal{U} = \mathcal{B}(\mathbb{R})$. The inclusion $\mathcal{U} \subset \mathcal{B}(\mathbb{R})$ is immediate since $\mathcal{B}(\mathbb{R})$ is a σ-algebra and $\mathcal{B}_0 \subset \mathcal{B}(\mathbb{R})$. As for the opposite inclusion, since $\mathcal{B}_0 \subset \mathcal{U}$ it suffices to prove that $\mathcal{U}$ is a σ-algebra. Suppose that $U, V \in \mathcal{U}$. Then for some $\alpha, \beta < \Omega$, $U \in \mathcal{B}_\alpha$ and $V \in \mathcal{B}_\beta$ and so, if $\lambda = \max(\alpha, \beta)$, $U \setminus V \subset \mathcal{B}_{\lambda+1} \subset \mathcal{U}$. Next, suppose that $\{X_n\} \subset \mathcal{U}$ and for each n let $\alpha_n < \Omega$ be such that $X_n \in \mathcal{B}_{\alpha_n}$; note that α_n is countable for each n and that there exists $\gamma < \Omega$ such that $\alpha_n \leq \gamma$ for all n. Then $X_n \in \mathcal{B}_\gamma$ for all n and $\bigcup_n X_n, \bigcap_n X_n \in \mathcal{B}_{\gamma+1} \subset \mathcal{U}$.

Finally, we prove by induction that the cardinality of $\mathcal{B}_\lambda = c$ for all λ. The statement is true for $\mathcal{B}_0$. Also, if the assertion is true for $\mathcal{B}_\alpha$, since countable unions and intersections have cardinality $c^{\aleph_0} = c$, and so do the differences, the statement is also true for $\mathcal{B}_{\alpha+1}$. As for the limiting ordinals, their cardinality is $\aleph_0 \cdot c = c$ and we have finished.

70. Since the complement of a compact set is open there is an injective mapping from $\mathcal{C}$ into $\mathcal{O}$, the open sets in $\mathbb{R}$, and so $\operatorname{card}(\mathcal{C}) \leq \operatorname{card}(\mathcal{O})$. Now, since every open set in $\mathbb{R}$ can be written as at most a countable union of pairwise disjoint open intervals and each such interval contains a distinct rational number, $\text{card}(\mathcal{O}) \le \text{card}(\mathbb{Q}^{\mathbb{N}}) = \aleph_0^{\aleph_0} \le c^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = c$. Thus $\text{card}(\mathcal{C}) \le c$. On the other hand, for each $x > 0$, $[0, x]$ is compact and so $\text{card}(\mathcal{C}) \ge c$. Also, since each $[0, x]$, $x > 0$, is uncountable and has positive Lebesgue measure, the collection of uncountable compact subsets of $\mathbb{R}$ with positive Lebesgue measure has cardinality c.

71. By Problem 70 the class $\mathcal{C}$ of uncountable compact subsets of the line can be indexed by the ordinals less than Ω, the first uncountable ordinal, i.e., $\mathcal{C} = \{K_\alpha : \alpha < \Omega\}$. We may also assume that $\mathbb{R}$ and, consequently, also every K_α, has been well-ordered. Let x_1, y_1 be the first two elements of K_1. Next, if $1 < \alpha < \Omega$ and if x_β and y_β have been chosen for all $\beta < \alpha$, let x_α, y_α be the first two elements of $K_\alpha \setminus \bigcup_{\beta<\alpha}\{x_\beta, y_\beta\}$; such a choice is always possible since the set in question has cardinality c for each α. Now put $B = \{x_\alpha : \alpha < \Omega\}$. Since $y_\alpha \in B^c$, all $\alpha < \Omega$, the intersection property is readily verified.

72. Clearly $a = \text{card}(H) \le c$. Moreover, since the set of linear combinations of the elements of a finite set with coefficients in $\mathbb{Q}$ is countable but $\mathbb{R}$ is not, a is infinite. For each integer n let $A_n = \{x \in \mathbb{R} : x = \sum_{k=1}^n q_k\, h_k, q_k$ rational, $h_k \in H\}$; we claim that $\text{card}(A_n) = a$ for all n. Note that listing the rationals $\{r_1, r_2, \ldots\}$ we have $A_1 = \bigcup_n r_n H$, the union being disjoint, and since $\text{card}(r_n H) = \text{card}(H)$ for every $r_n \ne 0$, by Problem 64 it follows that $\text{card}(A_1) = \aleph_0 \cdot \text{card}(H) = \aleph_0 \cdot a = a$. We proceed now by induction; we just saw that the statement is true for $n = 1$. Now, since $A_n \subset A_{n+1} \subset A_n + A_1$, it readily follows that $\text{card}(A_n) \le \text{card}(A_{n+1}) \le \text{card}(A_n + A_1) \le \text{card}(A_n \times A_1) = \text{card}(A_n)$ and if $\text{card}(A_n) = a$, then $\text{card}(A_{n+1}) = a$. Finally, by the definition of Hamel basis, $\mathbb{R} = \bigcup_n A_n$ and $c \le \text{card}(\bigcup_n A_n) \le \aleph_0 \cdot a = a$. Therefore $a = c$.

73. By Zorn's lemma $\mathbb{R}$ has a Hamel basis H, say. Let $h_0 \in H$, for $\mathcal{I} \subset H \setminus \{h_0\}$ let $B_\mathcal{I} = \{h_0 + h : h \in \mathcal{I}\}$, and put $H_\mathcal{I} = B_\mathcal{I} \cup (H \setminus \mathcal{I})$. Then $H_\mathcal{I}$ is a Hamel basis of $\mathbb{R}$ and for any two different subsets $\mathcal{I}, \mathcal{I}'$ of $H \setminus \{h_0\}$, $H_\mathcal{I} \ne H_{\mathcal{I}'}$. Since $\text{card}\,(H) = c$ there are 2^c such subsets and, consequently, 2^c different Hamel bases of $\mathbb{R}$.

74. Since $C + C = [0, 2]$ every $x \in \mathbb{R}$ can be written as $x = n + x_1 + x_2$ where $n \in \mathbb{Z}$ and $x_1, x_2 \in C$. Thus C spans $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and, consequently, by Zorn's lemma C contains a minimal spanning set of $\mathbb{R}$, i.e., a Hamel basis.

75. Let $H = \{h_\alpha\}$ be a Hamel basis of $\mathbb{R}$, h_{α_0} the rational element of H, and $B = \text{sp}\{h_\alpha\}$, $\alpha \ne \alpha_0$. For the sake of argument suppose that for some $J = (a, b)$ and $r \in \mathbb{Q}$, $(a, b) \cap (r + B)$ is of first category and let $E = (a - r, b - r)$; $E \cap B$ is of first category and since $qB = B$ for each rational $q \neq 0$, $q(E \cap B) = (qE) \cap (qB) = (qE) \cap B$ is of first category. Then, with $\{q_n\}$ an enumeration of the rationals, $B = \bigcup_n q_n(E \cap B)$ is of first category and the same is true of the rational translations $A_n = (q_n + B)$, all n, of B. Therefore $\mathbb{R} = \bigcup_n A_n$ is of first category, which is not the case. Thus the sets A_n, all n, will do.