Chapter 1

What is a singular perturbation?

Mathematical equations arising from physical sciences contain parameters. Perturbation theory examines parameter dependence of solutions *locally.* To present basic ideas simply, consider a one-parameter family of functions: For each x in a set R and real parameter ϵ in a punctured neighborhood of $\epsilon = 0$, the values of the functions $f(x, \epsilon)$ are in a metric space. The range is a metric space so that convergence of functions f as $\epsilon \to 0$ can be discussed. $f(x, \epsilon)$ is to be regarded as a solution of some set of equations containing ϵ as a parameter.

The equations are called a *regularly perturbed* problem if all solutions $f(x, \epsilon)$ converge *uniformly* on R as $\epsilon \to 0$. If there is a solution which does not converge uniformly, the problem is called *singularly perturbed.* Notice that the category, regular or singular, is formulated in terms of the solutions and *not* the equations.

This abstract definition of singular perturbation is very broad. But practical problems draw attention to a few dominant categories of singular behavior. What follows is a mini-survey of examples.

Prototypical examples.

Singularly perturbed polynomial equations

For $\epsilon > 0$, the polynomial equation

$$\epsilon z^8 - z^3 - 1 = 0 \tag{1.1}$$

has 8 complex roots. In the language of the preceding general discussion, the set R is the integers $1, 2, \ldots, 8$ labeling the roots, and the metric space of the roots $z = f(k, \epsilon)$, $k = 1, \dots, 8$, is the complex numbers. Figure 1.1 displays numerical approximations of the 8 roots for the sequence of ϵ's,

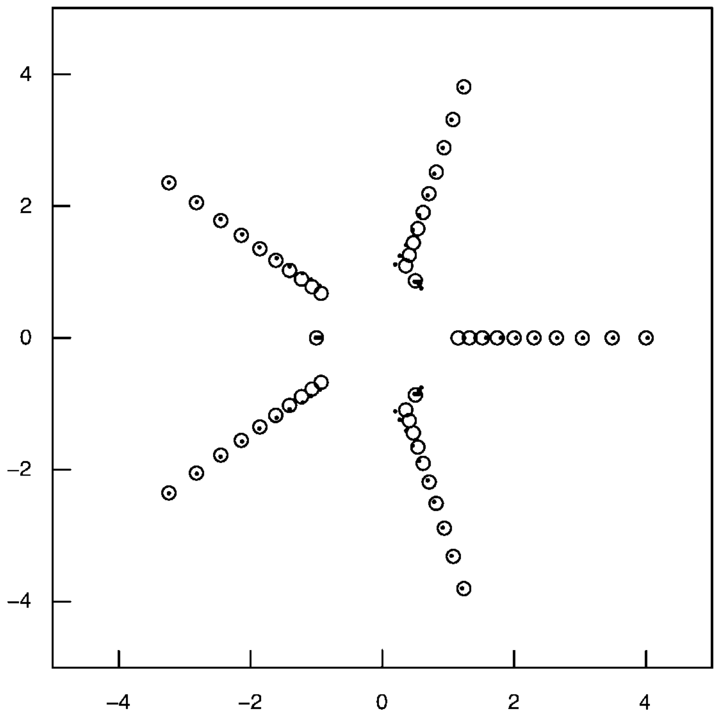

Figure 1.1. The roots (dots) and the approximated roots (circles)

$\epsilon = 2^{-n}$, $n = 1, \dots, 10$. Three of the roots appear to be converging to the cube roots of -1,

$$(-1)^{\frac{1}{3}} = e^{i\frac{\pi}{3}},\ e^{i\pi},\ e^{i\frac{5\pi}{3}},$$

as $\epsilon \to 0$. This is easy to see: Setting $\epsilon = 0$ in (1.1) gives the *reduced equation*

$$z^3 + 1 = 0. \tag{1.2}$$

The remaining 5 roots are diverging: Figure 1.2 is a log-log plot of the positive, real root as a function of ϵ. It appears that the diverging roots scale with ϵ like $\epsilon^{-\frac{1}{5}}$. If the polynomial equation (1.1) were regularly perturbed, all 8 roots would converge as $\epsilon \to 0$. Hence, it is singularly perturbed.

If it is *assumed* that the roots of (1.1) exhibit algebraic scalings with ϵ as $\epsilon \to 0^+$, direct constructive approximation is easy. Roots that scale like ϵ^{-p} as $\epsilon \to 0$ can be represented as

$$z(\epsilon) = \epsilon^{-p} Z(\epsilon) \tag{1.3}$$

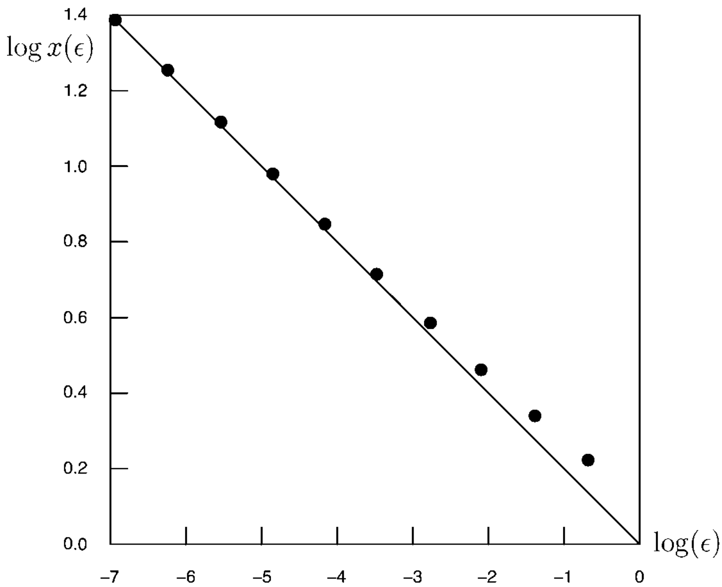

Figure 1.2

where it is assumed that $Z(\epsilon)$ converges to a non-zero value as $\epsilon \to 0$. The scaled version of (1.1) in terms of Z is

$$\epsilon^{1-8p} Z^8 - \epsilon^{-3p} Z^3 - 1 = 0. \tag{1.4}$$

The limit $\epsilon \to 0^+$ is examined. At this point, the exponent p is undetermined, so the limit equation depends on p, as shown in the table below:

		limit equation
	$p < 0$	$-1 = 0$
*	$p = 0$	$Z^3 + 1 = 0$
	$0 < p < \frac{1}{5}$	$Z^3 = 0$
*	$p = \frac{1}{5}$	$Z^8 - Z^3 = 0$
	$p > \frac{1}{5}$	$Z^8 = 0$

The $p < 0$ limit equation, $-1 = 0$, is absurd, so the $p < 0$ scalings are excluded. The limits for $0 < p < \frac{1}{5}$ and $p > \frac{1}{5}$ give only $Z = 0$, which is their way of saying that they don't resolve any roots: For instance, if the root $z(\epsilon)$ is bounded as $\epsilon \to 0$, the corresponding $Z(\epsilon) = \epsilon^p z(\epsilon)$ converges to zero for any $p > 0$. Hence, scalings with $p > 0$ are too coarse to resolve it. The limit equation $Z^8 = 0$ for $p > \frac{1}{5}$ indicates that scalings with $p > \frac{1}{5}$ are too coarse to resolve *any* roots at all. This leaves the so-called *distinguished limits* with $p = 0$ or $p = \frac{1}{5}$. The $p = 0$ limit equation is the same as (1.2), which is obtained by setting $\epsilon = 0$ in (1.1). This nails the three roots which converge to the cube roots of -1. The $p = \frac{1}{5}$ scaling nails the 5 diverging roots: The limit equation $Z^8 - Z^3 = 0$ has zero as a triple root. This corresponds to the three bounded roots which appear to vanish in this scaling. The remaining 5 roots are the 5-th roots of 1,

$$Z = e^{i\frac{2\pi}{5}k}, \ k = 0, 1, 2, 3, 4.$$

The corresponding approximations to the roots $z(\epsilon)$ of (1.1) are

$$z(\epsilon) \simeq \epsilon^{-\frac{1}{5}} e^{i\frac{2\pi k}{5}}. \tag{1.5}$$

The circles in Figure 1.1 mark the approximations (1.5) for the same sequence of ϵ's, $\epsilon = 2^{-k}$, $k = 1, \ldots, 10$, while the dots are the actual solutions.

Singularly perturbed polynomial equations often appear as the characteristic equations of linear, constant coefficient ODE's. Initial and boundary value problems involving such ODE's are singularly perturbed as well. The famous paradoxes of how charged particles interact with their own radiation can be discussed in this context.

Radiation reaction

The power radiated into the electromagnetic field by a particle of charge q moving along a non-relativistic trajectory $\mathbf{x} = \mathbf{x}(t)$ is given by a well-known formula due to Larmor:

$$p = \frac{2}{3}\frac{q^2}{c^3}|\ddot{\mathbf{x}}|^2. \tag{1.6}$$

Here, c is the speed of light. The heart of the calculation leading to (1.6) is the retarded solution of the wave equation with a point source moving along $\mathbf{x} = \mathbf{x}(t)$ and evaluation of the electromagnetic energy flux through a small sphere about $\mathbf{x} = \mathbf{x}(t)$.

How does the outgoing radiation affect the particle motion? The simplest classical model proposes a *radiation reaction force* $\mathbf{f}(t)$ so the rate of work $\mathbf{f}(t) \cdot \dot{\mathbf{x}}$ done by this force on the particle is negative and balances the dissipation rate (1.6). Specifically, we seek $\mathbf{f}(t)$ so

$$\mathbf{f}(t) \cdot \dot{\mathbf{x}}(t) = -p = -\frac{2}{3}\frac{q^2}{c^3}|\ddot{\mathbf{x}}(t)|^2. \tag{1.7}$$

It can't really be done: In the right-hand side,

$$|\ddot{\mathbf{x}}|^2 = -\dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{1}{2}(|\dot{\mathbf{x}}|^2)^{\cdot\cdot},$$

so the proposed balance (1.7) becomes

$$\mathbf{f} \cdot \dot{\mathbf{x}} = \frac{2}{3}\frac{q^2}{c^3}\dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} - \frac{1}{3}\frac{q^2}{c^3}(|\dot{\mathbf{x}}|^2)^{\cdot\cdot}. \tag{1.8}$$

The first term on the right-hand side of (1.8) suggests the identification

$$\mathbf{f} = \frac{2}{3}\frac{q^2}{c^3}\dddot{\mathbf{x}}, \tag{1.9}$$

but the second term on the right-hand side remains unaccounted for.

The usual argument is to restrict the discussion to *periodic* motions $\mathbf{x}(t)$ and look at the *average* of the energy balance (1.8) over a period. This conveniently removes the second term in (1.8). In any case, (1.9) summarizes the standard model in which the particle's own outgoing radiation is supposed to induce a radiation force upon itself.

As a specific example, consider the radiation damping of a particle of mass m and charge q in a uniform magnetic field $\mathbf{B}$. The equation of motion based on the model (1.9) of radiation reaction is

$$m\ddot{\mathbf{x}} = -\frac{q}{c}\dot{\mathbf{x}} \times \mathbf{B} + \frac{2}{3}\frac{q^2}{c^3}\dddot{\mathbf{x}}. \tag{1.10}$$

The dimensionless ODE using $\frac{mC}{qB}$ as the unit of time is

$$\ddot{\mathbf{x}} + \dot{\mathbf{x}} \times \hat{\mathbf{B}} - \epsilon\dddot{\mathbf{x}} = 0, \tag{1.11}$$

where $\hat{\mathbf{B}}$ is the unit vector in the $\mathbf{B}$ direction and ϵ is the dimensionless parameter

$$\epsilon := \frac{2}{3}\frac{q^3 B}{m^2 c^4}. \tag{1.12}$$

For an electron in a lab-strength field $B = 5000$ gauss, ϵ is tiny, $\epsilon = 10^{-12}$. Motion in a plane perpendicular to $\hat{\mathbf{B}}$ is possible. Adopt cartesian coordinates (x_1, x_2, x_3) so $\hat{\mathbf{B}} = (0, 0, 1)$. Let $v(t)$ be the complex function of time whose real and imaginary parts are the x_1 and x_2 components of velocity. It follows from (1.11) that

$$\dot{v} + iv - \epsilon\ddot{v} = 0. \tag{1.13}$$

For $\epsilon = 0$, the solutions of (1.13) are proportional to e^{-it}, corresponding to the well-known circular motion of a particle in a uniform magnetic field and no radiation damping. Now turn on the radiation damping, $0 < \epsilon \ll 1$. There are complex exponential solutions of (1.13),

$$v = e^{zt},$$

where z satisfies the characteristic equation

$$\epsilon z^2 - z - i = 0. \tag{1.14}$$

This polynomial equation is singularly perturbed in the sense of the preceding example. The roots and their expansions in powers of ϵ are

$$z = \frac{1-\sqrt{1+4i\epsilon}}{2\epsilon} = -i - \epsilon + \dots,$$
$$z' = \frac{1+\sqrt{1+4i\epsilon}}{2\epsilon} = \frac{1}{\epsilon} + i \dots . \tag{1.15}$$

As $\epsilon \to 0$, z converges to the root $-i$ of the reduced $\epsilon = 0$ equation, and z' diverges like $\frac{1}{\epsilon}$. The solution $v = e^{zt}$ is approximated by

$$v(t) \simeq e^{-\epsilon t} e^{-it}. \tag{1.16}$$

This represents an orbit that slowly spirals to the origin. That is what the theory of radiation force is supposed to describe. But the solution

$$v(t) = e^{z' t} \simeq e^{\frac{t}{\epsilon}} e^{it} \tag{1.17}$$

with violent exponential growth as $\epsilon \to 0$ is a disaster. Can one adopt a "don't ask, don't tell" policy and simply exclude the growing solutions from analysis of physical problems? As a simple example, suppose there is no magnetic field for $t < 0$, during which time the particle is at rest. For times $t > 0$, an incident electromagnetic wave propagating in the x_1 direction induces a uniform magnetic field in the x_3 direction and a uniform electric field in the x_2 direction, in a neighborhood of the particle. For $t > 0$, the equation of motion is

$$\dot{v} + iv - i - \epsilon \ddot{v} = 0.$$

(The electric field is represented by the constant term $-i$.) The solution in $t > 0$ with $z(0)$ and $\dot{z}(0)$ both zero (z and $\dot{z}$ are continuous across $t = 0$) is

$$v(t) = 1 - \frac{z' e^{zt} - z e^{z' t}}{z' - z}. \tag{1.18}$$

The growing term proportional to $e^{z' t}$ can be approximated by

$$-i\epsilon e^{\frac{t}{\epsilon}} e^{it}.$$

As $\epsilon \to 0$, the coefficient $i\epsilon \to 0$, but clearly growth is only delayed. And not by much: The growing term dominates the solution when

$$\epsilon e^{\frac{t}{\epsilon}} \gg 1 \quad \text{or} \quad t \gg \epsilon \log \frac{1}{\epsilon}.$$

The right-hand side vanishes as $\epsilon \to 0$, so disaster is almost instantaneous.

The breakdown of the standard model of radiation reaction has been examined intermittently throughout the twentieth century. In 1938, Dirac [**10**] attempted to reexamine carefully the energy-momentum balance between a charged point particle and its retarded field. He was ambushed by the non-integrable energy-momentum singularity along the particle world line. He found that the infinity could be formally canceled, leaving the radiation force (1.9) as a residual if the particle interacted with a combination of its advanced and retarded fields. So now the electron according to Dirac has memories of the future! He proposed that this future memory could be used to suppress violent self-acceleration of charged particles in ordinary macroscopic reality. In the preceding example with electric and magnetic fields turned on at $t = 0$, Dirac's procedure is to allow only the damped solution in $t > 0$ and uphold continuity of v and $\dot{v}$ at $t = 0$ with a small component of growing solution in $t < 0$. This leads to

$$v(t) = \begin{cases} -\dfrac{\epsilon z}{1-\epsilon z} e^{\frac{t}{\epsilon}} & t < 0, \\ 1 - \dfrac{1}{1-\epsilon z} e^{zt} & t \geq 0. \end{cases} \tag{1.19}$$

Notice that this solution asymptotes to uniform motion $v = 1$ as $t \to \infty$. A simple approximation to (1.19) based on the approximation $z \simeq -i - \epsilon$ is

$$v(t) = \begin{cases} i\epsilon e^{\frac{t}{\epsilon}} & t < 0, \\ 1 - (1 - i\epsilon) e^{-\epsilon t} e^{-it} & t \geq 0. \end{cases} \tag{1.20}$$

Figure 1.3 depicts the trajectory of $v(t)$ based on (1.20). The only hint of that notion, "memories of the future", is the tiny kink in the trajectory near the starting point $v = 0$.

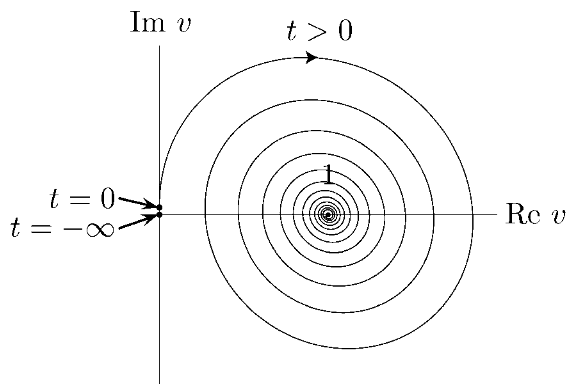

Figure 1.3. The trajectory of $v(t)$ based on (1.20)

Problems and Solutions

Problem 1.1 (Bad truncations). Let $x(t)$ be displacement of a particle in the $\hat{\mathbf{x}}$ direction subject to $\hat{\mathbf{x}}$-external force $f(t)$. The usual equation of

Newtonian mechanics is

$$\ddot{x}(t) = f(t). \tag{1.1.1}$$

Here, we examine a little model in which the particle feels its past history, so (1.1.1) is replaced by

$$\int_0^\infty e^{-s}\ddot{x}(t-\epsilon s)ds = f(t). \tag{1.1.2}$$

We assume that $f(t)$ is non-zero only in $0 < t < 1$ and $x(t) \equiv 0$ for $t < 0$.

a) Show that for $t > 1$, $\dot{x}(t) \equiv I := \int_0^1 f(t)dt$. This is the same prediction as in Newtonian mechanics, with I the total impulse delivered by the external force. (**Hint:** Introduce

$$y(t) := \int_0^\infty e^{-s}\ddot{x}(t-\epsilon s)ds \tag{1.1.3}$$

as a second dependent variable. $y(t)$ satisfies a first-order ODE with $\ddot{x}(t)$ as forcing.)

b) We formulate an ODE approximation to the initial value problem of part a): Instead of (1.1.2), we have the ODE, obtained by replacing $\ddot{x}(t-\epsilon s)$ with its two-term Taylor polynomial $\ddot{x}(t) - \epsilon\dddot{x}(t)s$. Determine $\ddot{x}(t)$ for $t > 1$. What do you observe?

c) Offer a critique of the truncated initial value problem that goes beyond "it does not work." (Given $\ddot{x}(t)$ as it actually is, how does the truncation error $\ddot{x}(t-\epsilon s) - \ddot{x}(t) + \epsilon\dddot{x}(t)s$ behave?)

Solution.

a) We claim that $y(t)$ satisfies ODE

$$\epsilon\dot{y} + y = \ddot{x}. \tag{1.1.4}$$

The solution with $y(t) \equiv 0$ for $t < 0$ is precisely $y(t)$ in (1.1.3). Setting $y(t) = f(t)$ as dictated by (1.1.2), (1.1.4) becomes

$$\ddot{x}(t) = \epsilon\dot{f}(t) + f(t),$$

and integration from $t = -\infty$ to $t > 1$ gives

$$\dot{x}(t) = \epsilon(f(t) - f(-\infty)) + \int_{-\infty}^t f(s)ds = \int_0^1 f(s)ds = I.$$

b) The proposed truncation of (1.1.2) is

$$\ddot{x}(t) - \epsilon\dddot{x}(t) = f(t),$$

which we can write as

$$(\ddot{x})^{\cdot} - \frac{\ddot{x}}{\epsilon} = -\frac{f}{\epsilon}.$$

The solution of this first-order ODE for $\ddot{x}$ with $\ddot{x} \equiv 0$ for $t < 0$ is

$$\ddot{x}(t) = -\frac{1}{\epsilon}\int_0^t e^{\frac{t-s}{\epsilon}} f(s)ds,$$

in $t > 0$. In $t > 1$, it reduces to

$$\ddot{x}(t) = -\frac{e^{\frac{t}{\epsilon}}}{\epsilon}\int_0^1 e^{-\frac{s}{\epsilon}} f(s)ds.$$

We see that $\ddot{x}(t)$ undergoes exponential growth with characteristic time ϵ in $t > 1$. This is of course *not* the behavior of the exact solution of (1.1.2), which has $\ddot{x}(t) \equiv 0$ for $t > 1$.

c) Take $\ddot{x}(t) = ae^{\frac{t}{\epsilon}}$, where a is a constant. We have

$$\ddot{x}(t - \epsilon s) - \ddot{x}(t) + \epsilon\dddot{x}(t)s = \frac{a}{\epsilon^2}e^{\frac{t}{\epsilon}}(e^{-s} - 1 + s),$$

and for s bounded away from zero this truncation error is *huge*.

Problem 1.2 (Harmonic oscillator with memory, and even worse truncations). $x(t)$ satisfies

(1.2.1) $$\int_0^\infty e^{-s}\ddot{x}(t - \epsilon s)ds + x(t) = 0.$$

The inertia term has the same kind of memory effects as in Problem 1.1.

a) Determine the characteristic equation for z so $x(t) = e^{zt}$ is a solution of (1.2.1), and find its roots. Show that for $\epsilon > 0$, these exponential solutions decay to zero as $t \to \infty$.

b) We get an ODE approximation to (1.2.1) of order $n+2$ by replacing $\ddot{x}(t - \epsilon s)$ with its n-th degree Taylor polynomial in ϵ. Determine the characteristic polynomial equation for z so e^{zt} solves the ODE. Carry out an analysis of its roots as $\epsilon \to 0$ by examining distinguished limits as in the text example.

Solution.

a) Substituting $x = e^{zt}$ into (1.2.1) gives

$$z^2\int_0^\infty e^{-(1+\epsilon z)s}ds + 1 = 0.$$

Assuming $\mathrm{Re}(1+\epsilon z) > 0$, we evaluate the integral to get

$$\frac{z^2}{1+\epsilon z} + 1 = 0 \tag{1.2.2}$$

or

$$z^2 + \epsilon z + 1 = 0. \tag{1.2.3}$$

The roots are

$$z = -\frac{\epsilon}{2} \pm \sqrt{\left(\frac{\epsilon}{2}\right)^2 - 1} \tag{1.2.4}$$

or, equivalently,

$$z = -\frac{\epsilon}{2} \pm i\sqrt{1 - \left(\frac{\epsilon}{2}\right)^2}. \tag{1.2.5}$$

If $\epsilon > 2$, (1.2.4) is the most appropriate representation of the roots, and we see that both are real and negative, signifying exponential decay. If $0 < \epsilon < 2$, we have $\mathrm{Re}\, z = -\frac{\epsilon}{2} < 0$ for *both* roots, so again exponential decay.

b) The proposed ODE is

$$(x - \epsilon\dot{x} + \epsilon^2\ddot{x} - \ldots (-1)^n \epsilon^n x^{(n)})^{\cdot\cdot} + x = 0,$$

and the characteristic equation in place of (1.2.2) is

$$z^2(1 - \epsilon z + (\epsilon z)^2 - \ldots (-1)^n(\epsilon z)^n) + 1 = 0. \tag{1.2.6}$$

We recognize that (1.2.6) results by replacing $\frac{1}{1+\epsilon z}$ in (1.2.2) by its n-th degree Taylor polynomial in ϵ. We represent roots of (1.2.6) by $z = \epsilon^{-p}Z(\epsilon)$ and seek exponents p corresponding to distinguished limits. (1.2.6) in terms of Z is

$$Z^2\{1 - (\epsilon^{1-p}Z) + \ldots (-1)^n(\epsilon^{1-p}Z)^n\} + \epsilon^{2p} = 0. \tag{1.2.7}$$

As in the text example, we take formal $\epsilon \to 0$ limits of (1.2.7), assuming $Z(0) \neq 0$:

	p-values	limit equation
	$p < 0$	$1 = 0$
*	$p = 0$	$Z^2 + 1 = 0$
	$0 < p < 1$	$Z^2 = 0$
*	$p = 1$	$Z^2\{1 - Z + \ldots (-1)^n Z^n\} = 0$
	$p > 1$	$Z^{n+2} = 0$

The distinguished limit equation for $p = 0$ has roots $Z = \pm i$, and we surmise two roots z of (1.2.6) converge to $\pm i$ as $\epsilon \to 0$. The distinguished limit $p = 1$ corresponds to roots which diverge like $\frac{1}{\epsilon}$ as $\epsilon \to 0$. The $p = 1$ limit equation has $Z = 0$ as a double root, corresponding to the two roots for z near $\pm i$ which aren't resolved in the $\frac{1}{\epsilon}$ scaling. The remaining non-zero roots satisfy

$$1 - Z + \dots (-1)^n Z^n = 0. \tag{1.2.8}$$

For $Z \neq -1$, the left-hand side is

$$\frac{1 - (-Z)^{n+1}}{1 + Z}.$$

Observe that $Z = -1$ is *never* a root of (1.2.8), so it reduces to $(-Z)^{n+1} = 1$ or $Z^{n+1} = (-1)^{n+1}$. If n is even, then $Z^{n+1} = -1$, and we get Z equal to one of the $n+1$-th roots of -1, *except* $Z = -1$. If n is odd, then $Z^{n+1} = 1$, and Z is one of the $n+1$-th roots of unity, *except* $Z = -1$.

Convection-diffusion boundary layer

The function $c(x, \epsilon)$ defined in $x \geq 0$ satisfies the boundary value problem

$$\epsilon c_x + \phi'(x) c = f \quad \text{in} \quad x > 0, \tag{1.21}$$

$$c = 0 \quad \text{at} \quad x = 0. \tag{1.22}$$

Here, $\epsilon > 0$ and $f > 0$ are given constants, and $\phi(x)$ is a given function. The solution $c(x, \epsilon)$ is examined in the limit $\epsilon \to 0$. A physical interpretation is helpful in understanding it: Think of c as the concentration of particles in a solution. $\phi(x)$ is the potential energy of a particle, so $-\phi'(x)$ is the force on it. The force causes a particle to drift with velocity proportional to $-\phi'(x)$. If this is the only way the particles move, the rate at which they cross into $(0, x)$ would be minus the product of their velocity and density at x, proportional to $\phi'(x)c(x, \epsilon)$. Transport of particles by a given velocity field is called *convection.* In addition there is a *diffusion* of particles induced by random molecular motions at finite temperature. The rate at which particles enter $(0, x)$ due to diffusion is proportional to $c_x(x, \epsilon)$. The net rate or *influx* of particles entering $(0, x)$ is the sum of convective and diffusive components. The quantity

$$\epsilon c_x + \phi' c$$

which appears in the ODE (1.21) is a non-dimensionalized influx, and ϵ is recognized as a dimensionless diffusion coefficient. Under steady conditions, the rate at which particles cross into $(0, x)$ must have a uniform value f independent of x. This is what the ODE (1.21) says. Otherwise, there would be subintervals of $x > 0$ in which particles would accumulate over time, and the density couldn't be time independent. The zero boundary condition at $x = 0$ represents absorption of particles once they cross $x = 0$.

The solution of the boundary value problem (1.21), (1.22) is

$$c(x,\epsilon) = \frac{f}{\epsilon}\int_0^x \exp\left(\frac{\phi(s)-\phi(x)}{\epsilon}\right) ds. \tag{1.23}$$

We examine its $\epsilon \to 0$ behavior.

The most straightforward limit process is $\epsilon \to 0$ with $x > 0$ fixed. Specifically, define

$$c^0(x) := \lim_{\epsilon\to 0} c(x,\epsilon). \tag{1.24}$$

It seems reasonable that $c^0(x)$ should satisfy the *reduced equation*

$$\phi'(x)c^0(x) = f,$$

obtained by setting $\epsilon = 0$ in (1.21). This would give

$$c^0(x) = \frac{f}{\phi'(x)}\,. \tag{1.25}$$

Assume for now that $\phi'(x)$ is uniformly positive in $x > 0$, so $c^0(x)$ is well defined in $x > 0$. The result (1.25) in fact follows from the integral representation (1.23). The systematic approximation of integrals like (1.23) as $\epsilon \to 0$ will be examined in more detail later. But the essential idea is captured by a formal calculation: Figure 1.4 is a qualitative graph of the exponential

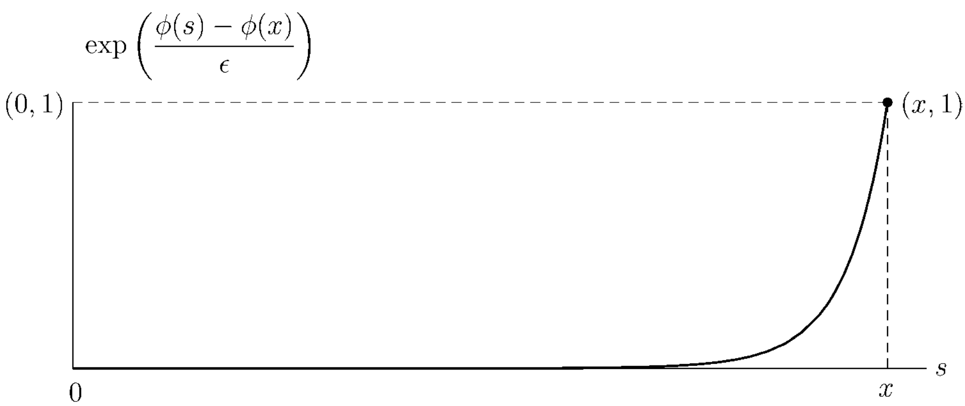

Figure 1.4. Qualitative graph of the exponential in (1.23) as a function of s

in (1.23) as a function of the integration variable s. For $0 < \epsilon \ll 1$, the exponential is much less than 1 over most of the interval $0 < s < x$ and rises steeply to value 1 as $s \to x^-$. Notice that

$$\frac{\partial}{\partial s}\left(\exp\left(\frac{\phi(s)-\phi(x)}{\epsilon}\right)\right)\Bigg|_{s=x} = \frac{1}{\epsilon}\phi'(x),$$

so it appears that most of this rise happens in an interval of thickness ϵ. This motivates a change of variable in the integral (1.23). The new variable in place of s is

$$S := \frac{x-s}{\epsilon}.$$

The integral (1.23) is rewritten as

$$c = f \int_0^{\frac{x}{\epsilon}} \exp\left(\frac{\phi(x-\epsilon S) - \phi(x)}{\epsilon} \right) dS,$$

and the formal $\epsilon \to 0$ limit is

$$f \int_0^\infty \exp(-\phi'(x)S) dS = \frac{f}{\phi'(x)} = c^0(x).$$

In general, $c^0(x)$ does not satisfy the zero boundary condition at $x = 0$:

$$c^0(0) = \frac{f}{u} \neq 0, \quad u := \phi'(0). \tag{1.26}$$

This is an indication that the convergence of $c(x, \epsilon)$ to $c^0(x)$ as $\epsilon \to 0$ is non-uniform at $x = 0$. Figure 1.5 depicts a numerical approximation of $c(x, \epsilon)$

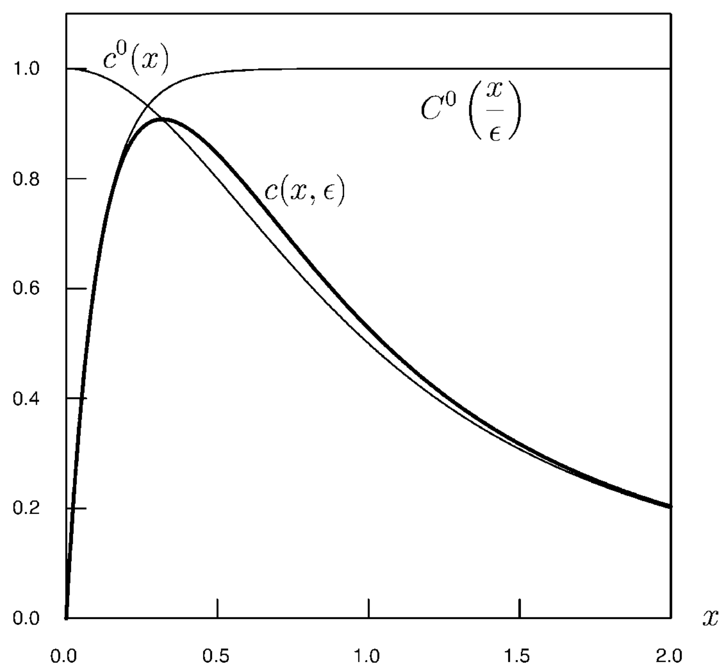

Figure 1.5. Numerical solution $c(x, \epsilon)$ (dark line) and inner and outer solutions

for specific choices

$$\phi(x) = x + \frac{x^3}{3}, \quad f = 1, \quad \epsilon = 0.1.$$

A graph of $c^0(x)$ is included for comparison. It is seen that $c(x,\epsilon)$ undergoes a rapid increase from zero in a thin *boundary layer* to the right of $x=0$.

The width of the boundary layer goes to zero as $\epsilon\to 0$. How does the width scale with ϵ? How does the magnitude of c in the boundary layer scale with ϵ? The *mathematical* approach to these questions is to introduce a representation of $c(x,\epsilon)$ in the boundary layer that embodies scalings of c and x with respect to ϵ:

$$c(x,\epsilon)=\epsilon^{-q}C(X:=\epsilon^{-p}x,\epsilon). \tag{1.27}$$

The exponents q and p are yet to be determined. It is assumed that $C(X,\epsilon)$ converges to a limit function $C^0(X)$ as $\epsilon\to 0$ with X fixed. The ideas embodied in (1.27) are that the boundary layer thickness is proportional to ϵ^p and the magnitude of c in the boundary layer is proportional to ϵ^{-q}. Substitution of (1.27) into the ODE (1.22) gives

$$\epsilon^{1-q-p}C_X+\epsilon^{-q}\phi'(\epsilon^pX)C=f. \tag{1.28}$$

For each choice of q, p one gets a *reduced equation* by taking the limit of (1.28) as $\epsilon\to 0$. The process is analogous to the analysis of the polynomial equation (1.1). In particular, there are *distinguished limits* characterized by balances between two or more terms of (1.28) in powers of ϵ. For instance the reduced equation $\phi'(x)c^0(x)=f$ that characterizes the solution in the limit $\epsilon\to 0$ with $x>0$ fixed is obtained with $q=0$, $p=0$. There is another distinguished limit that balances all three terms of (1.28). If $u:=\phi'(0)>0$, the limit equation with $q=0$, $p=1$ is

$$C^0_X+uC^0=f. \tag{1.29}$$

It is plausible that $q=0$, $p=1$ is the boundary layer limit. $p=1$ means that the boundary layer thickness is proportional to ϵ. One might have guessed $q=0$ at the outset, since the numerical solution in Figure 1.5 suggests that the net rise of $c(x,\epsilon)$ in the boundary layer appears to converge to the positive value $c^0(0)=\frac{f}{u}$ as $\epsilon\to 0$.

The conclusion that boundary layer thickness is proportional to ϵ can be argued by traditional dimensional analysis. Think of (1.21) as an equation in dimensional quantities. In particular, ϵ is a diffusion coefficient with units of $(\text{length})^2\div$ time, and the drift velocity at $x=0$, $u=\phi'(0)$ has units of length $\div$ time. The unique length formed from ϵ and u is $\frac{\epsilon}{u}$. Notice that c has units of $1\div$ length, but its magnitude does not enter into the balance of convection and diffusion, which are both linear in c.

Having established, by fair means or foul, that boundary layer thickness is proportional to ϵ, the solution (1.23) in the boundary layer can be examined rigorously: Introduce the scaled displacement,

$$X := \frac{x}{\epsilon},$$

and define

$$C(X,\epsilon) := c(\epsilon X,\epsilon) = f\int_0^X \exp\left(\frac{\phi(\epsilon X-\epsilon S)-\phi(\epsilon X)}{\epsilon}\right) dS. \tag{1.30}$$

$C(X,\epsilon)$ is the exact solution (1.23) expressed as a function of X and ϵ. The integral representation (1.30) is obtained from (1.23) by substituting $x = \epsilon X$ and changing the integration variable from s to $S := X - \frac{s}{\epsilon}$. Since x has magnitude ϵ in the boundary layer, the appropriate limit process for the boundary layer is $\epsilon \to 0$ with $X := \frac{x}{\epsilon}$ fixed. Under this limit process, the exponent in (1.30) has limit

$$\lim_{\epsilon\to 0} \frac{\phi(\epsilon X-\epsilon S)-\phi(\epsilon X)}{\epsilon} = -uS$$

and

$$C^0(X) := \lim_{\epsilon\to 0} C(X,\epsilon) = f\int_0^X e^{-uS} dS = \frac{f}{u}(1-e^{-uX}). \tag{1.31}$$

$C^0(X)$ is displayed as a function of x in Figure 1.5. Notice that $C^0(X)$ satisfies the boundary condition $C^0(0) = 0$ but generally does not converge to $c^0(x)$ in the limit $\epsilon \to 0$, $x > 0$ fixed.

In summary, two complementary limits of the exact solution $c(x,\epsilon)$ have been found. One is $c^0(x)$, in the limit $\epsilon \to 0$ and $x > 0$ fixed. The other is $C^0(X)$, in the limit $\epsilon \to 0$, $X := \frac{x}{\epsilon}$ is fixed. The limits describe the solution in the complementary intervals, outside and inside the boundary layer. Traditionally, $c^0(x)$ is called an *outer* approximation, and $C^0(X)$ an *inner* approximation. How can the outer and inner approximations be joined to form an approximation to $c(x,\epsilon)$ uniformly valid in $x > 0$? First, observe a certain compatibility between $c^0(x)$ and $C^0(X)$ called *matching*. From (1.26) and (1.31) it follows that

$$\lim_{X\to\infty} C^0(X) = \frac{f}{u} = \lim_{x\to 0} c^0(x). \tag{1.32}$$

A traditional verbal characterization of the matching is: "Outer limit of inner approximation = Inner limit of outer approximation." The common value $\frac{f}{u}$ of the limits in (1.32) is called the "common part of the inner and outer approximations." The matching (1.32) informs the sought-for joining of inner and outer approximations. Define

$$c^u(x,\epsilon) := c^0(x) + C^0(X) - \frac{f}{u} = \frac{f}{u}\left\{\frac{u}{\phi'(x)} - e^{-\frac{ux}{\epsilon}}\right\}. \tag{1.33}$$

This sum of inner and outer approximations minus the common part is the candidate for a uniform approximation to $c(x, \epsilon)$ in $0 < x < 1$. Figure 1.6 compares $c^u(x, \epsilon)$ to the numerical solution.

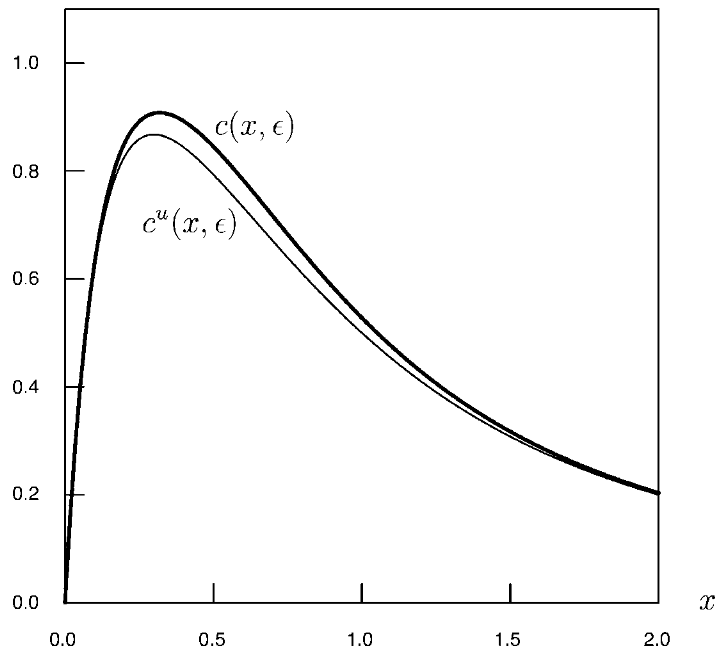

Figure 1.6. Numerical solution (dark line) compared with the uniform approximation c^u

Problems and Solutions

Problem 1.3 (A simple boundary layer). $c(x)$ satisfies the ODE

$$\epsilon c'' - f(x)c + 1 = 0 \tag{1.3.1}$$

in $x > 0$, with $f(x)$ a given uniformly positive function and ϵ a positive gauge parameter. Think of $c(x)$ as a concentration of particles which diffuse with diffusion coefficient ϵ, which are introduced with a uniform, constant source strength (the $+1$ on the left-hand side) and are absorbed with local rate constant $f(x)$. The concentration $c(x)$ is bounded as $x \to +\infty$, and particles are absorbed at $x = 0$, so there is the boundary condition

$$c(0) = 0. \tag{1.3.2}$$

a) Construct a formal outer approximation $c^0(x)$ based on the limit process $\epsilon \to 0$ with $x > 0$ fixed. What is the inner limit $\lim_{x\to 0} c^0(x)$?

b) Due to the absorbing boundary condition (1.3.2), there is a boundary layer at $x = 0$. Determine the exponent p in the inner representation of c,

$$c = C\left(X := \frac{x}{\epsilon^p}, \epsilon\right). \tag{1.3.3}$$

Determine the ODE for $C^0(x) := C(X, \epsilon = 0)$ and find the solutions with $C^0(0) = 0$. Select the solution for $C^0(X)$ which asymptotically matches the outer solution. Construct a uniformly valid approximation to c based on $c^0(x)$ and $C^0(X)$.

Solution.

a) Setting $\epsilon = 0$ in (1.3.1), we have $c^0(x) = \frac{1}{f(x)}$ and $\lim_{x\to 0} c^0(x) = \frac{1}{f(0)}$.

b) Substituting (1.3.3) into ODE (1.3.1), we have

$$\epsilon^{1-2p} C_{XX}(X, \epsilon) - f(\epsilon^p X) C(X, \epsilon) + 1 = 0.$$

The distinguished limit for the boundary layer clearly has $p = \frac{1}{2}$. In this case $C^0(X) := C(X, \epsilon = 0)$ satisfies the reduced ODE

$$(C^0)'' - f(0)C^0 + 1 = 0.$$

The solution with $C^0(0) = 0$ and $C^0(\infty) = \frac{1}{f(0)} = \lim_{x\to 0} c^0(x)$ is

$$C^0(X) = \frac{1}{f(0)}(1 - e^{-\sqrt{f(0)}X}).$$

The uniformly valid approximation is

$$c^u(x, \epsilon) = c^0(x) + C^0\left(\frac{x}{\sqrt{\epsilon}}\right) - \frac{1}{f(0)} = \frac{1}{f(x)} - \frac{1}{f(0)} e^{-\sqrt{f(0)}\frac{x}{\sqrt{\epsilon}}}.$$

Problem 1.4 (Pileup near $x = 0$). In the convection-diffusion boundary value problem (1.20), (1.21), assume that $\varphi'(x)$ is positive in $x > 0$ but approaches zero as $x \to 0^+$, with $\varphi'(0) = 0$, $\varphi''(0) > 0$. The outer solution $\frac{f}{\varphi'(x)}$ in (1.24) blows up like $\frac{1}{x}$ as $x \to 0$, so clearly we need to rethink the exponents p and q in the boundary layer representation

$$c(x, \epsilon) = \epsilon^{-q} C(X := \epsilon^{-p} x, \epsilon)$$

in (1.26).

a) Determine the exponents p and q for a distinguished limit which balances the convection and diffusion terms of the flux in (1.21).

b) Substitute the boundary layer representation (1.27) (with your exponents p and q from part a)) into the first-order ODE $\epsilon c_x + \varphi'(x)c = f$ in (1.21) and determine $C^0(X) := \lim_{\epsilon\to 0} C(X,\epsilon)$. Show that the leading order boundary layer solution exhibits asymptotic matching with the outer solution in $\epsilon^p \ll x \ll 1$. Write a uniformly valid approximation to $c(x,\epsilon)$ synthesized from the outer and boundary layer solutions.

Solution.

a) The diffusive flux in terms of $C(X,\epsilon)$ is

$$\epsilon c_x = \epsilon^{1-q-p} C_X(X,\epsilon),$$

and the convective flux is

$$\varphi' c = \varphi'(\epsilon^p X)\epsilon^{-q} C(X,\epsilon) \simeq \epsilon^{p-q}\varphi''(0) X C(X,\epsilon)$$

assuming $p > 0$. These balance each other and the right-hand side of (1.21) if $1-q-p=0$ and $p-q=0$, so $p=q=\frac{1}{2}$.

b) $\epsilon c_x + \varphi'(x)c = C_X + \epsilon^{-\frac{1}{2}}\varphi'(\epsilon^{\frac{1}{2}}X)C \to C^0_X + \varphi''(0)XC^0$ as $\epsilon \to 0$. The ODE for $C^0(X)$ is

$$C^0_X + \varphi''(0)XC^0 = f,$$

and the solution with $C^0(0) = 0$ is

(1.4.1) $$C^0(X) = f\int_0^X e^{\frac{\varphi''(0)}{2}(S^2 - X^2)} dS.$$

To check matching with the outer solution, we need the limiting behavior of $C^0(X)$ as $X \to \infty$. The latter is singled out with the change of integration variable from S to σ defined by $S = X - \frac{\sigma}{X}$. (1.4.1) expressed as a σ-integral is

$$C^0(X) = \frac{f}{X}\int_0^{X^2} e^{-\varphi''(0)\sigma + \frac{\varphi''(0)\sigma^2}{2X^2}} d\sigma.$$

In the limit $X \to \infty$,

$$\int_0^{X^2} e^{-\varphi''(0)\sigma + \frac{\varphi''(0)\sigma^2}{2X^2}} d\sigma \to \int_0^{\infty} e^{-\varphi''(0)\sigma} d\sigma = \frac{1}{\varphi''(0)},$$

so the outer $(X \to \infty)$ limit of the inner solution $\epsilon^{-\frac{1}{2}}C^0(X)$ is

(1.4.2) $$\frac{f}{\varphi''(0)\epsilon^{\frac{1}{2}}X} = \frac{f}{\varphi''(0)x}.$$

Using $\varphi'(x) \simeq \varphi''(0)x$ for $x \ll 1$, the inner $(x \to 0)$ limit of the outer solution $\frac{f}{\varphi'(x)}$ is $\frac{f}{\varphi''(0)x}$, in agreement with the outer limit (1.4.2) of the inner solution.

The uniformly valid approximation to $c(x,\epsilon)$ is given formally by sums of inner and outer solutions minus "common part" $\frac{f}{\varphi''(0)x}$:

$$c^u(x,\epsilon) = \frac{1}{\sqrt{\epsilon}} C^0\left(\frac{x}{\sqrt{\epsilon}}\right) + \frac{f}{\varphi'(x)} - \frac{f}{\varphi''(0)x}\,. \tag{1.4.3}$$

This approximation to $c(x,\epsilon)$ has $c^u(0,\epsilon) = -\frac{1}{2}\frac{\varphi'''(0)}{(\varphi''(0))^2}$, which is generally non-zero. But in the boundary layer where x has magnitude $\sqrt{\epsilon}$, the inner solution has nominal magnitude $\frac{1}{\sqrt{\epsilon}}$, and in this sense the error at $x = 0$ is small.

Modulated oscillations

The function $x = x(t,\epsilon)$ defined in $t \geq 0$ satisfies the initial value problem

$$\ddot{x} + \frac{x}{(1+\epsilon t)^2} = 0, \quad t > 0, \tag{1.34}$$

$$x(0) = 0, \quad \dot{x}(0) = 1, \tag{1.35}$$

where $\epsilon > 0$ is a given positive constant. Physically, (1.34) represents a harmonic oscillator whose natural frequency ω depends on time,

$$\omega = \frac{1}{1+\epsilon t}\,. \tag{1.36}$$

The exact solution is

$$x(t,\epsilon) = a \sin \xi, \tag{1.37}$$

where the *amplitude* a and *phase* ξ are given by

$$a = a(t,\epsilon) = \sqrt{\frac{1+\epsilon t}{1-\frac{\epsilon^2}{4}}}\,, \tag{1.38}$$

$$\xi = \xi(t,\epsilon) = \sqrt{1-\frac{\epsilon^2}{4}}\,\frac{\log(1+\epsilon t)}{\epsilon}\,. \tag{1.39}$$

Figure 1.7a is the graph of $x(t,\epsilon)$ for $\epsilon = 0.01$ in $0 < t < 1500$. The convergence of $x(t,\epsilon)$ as $\epsilon \to 0$ is non-uniform in $0 < t < \infty$: The limit of (1.37) as $\epsilon \to 0$ with $t > 0$ fixed is $x^0(t) = \sin t$. The graph of the residual $x(t,\epsilon) - x^0(t)$ in Figure 1.7b displays oscillations that grow with increasing t. The initial value problem (1.34), (1.35) is singularly perturbed.

(a)

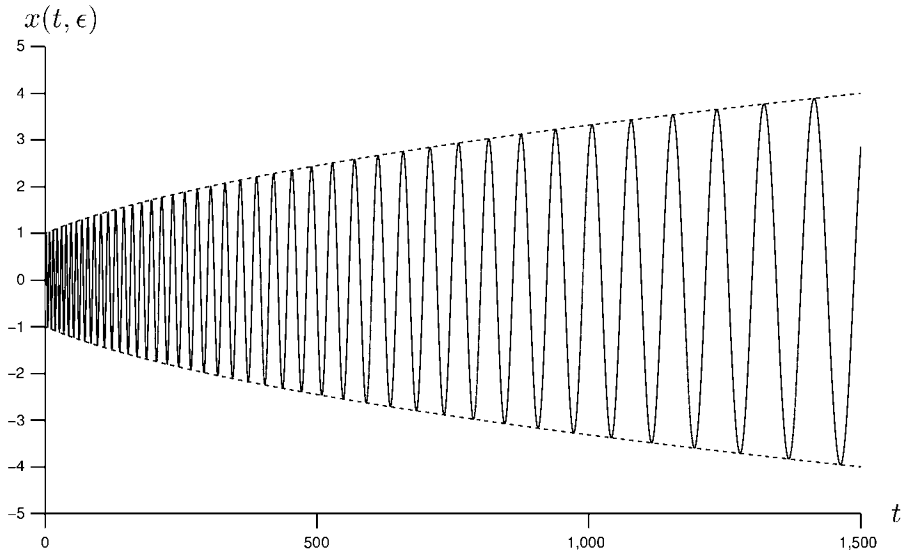

Figure 1.7a

(b)

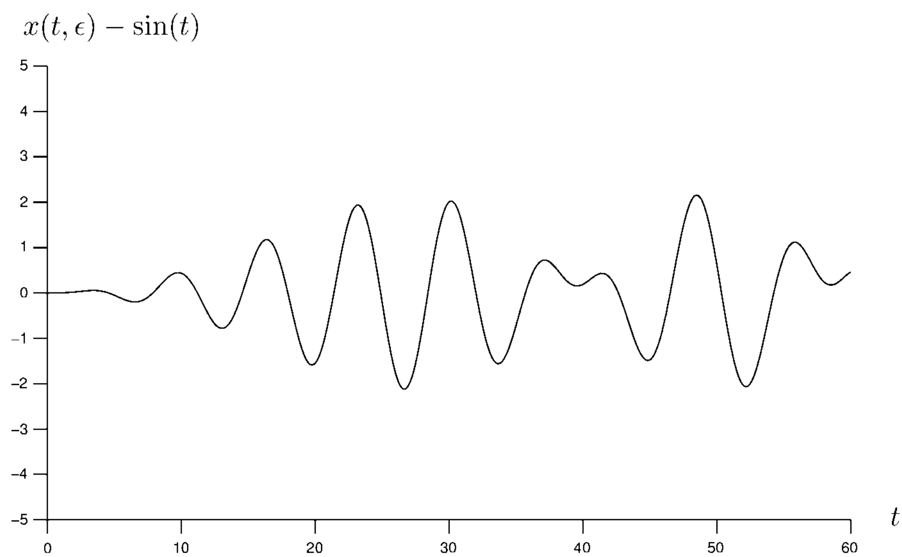

Figure 1.7b

Heuristically, it is clear what goes on: In the limit $\epsilon \to 0$, $x(t, \epsilon)$ converges uniformly to a sinusoidal function of t over any subinterval of $0 < t < \infty$ that contains a fixed number of oscillations, independent of ϵ. But after a large number of oscillations, on the order of $\frac{1}{\epsilon}$, the amplitude and frequency of the oscillations undergo large relative changes. This kind of non-uniform convergence is called *modulated oscillation*.

We carry out an $\epsilon \to 0$ limit process on the solution that displays both the local periodicity over a few cycles and the changes in the periodic structure after many cycles, on the order of $\frac{1}{\epsilon}$. In the expressions (1.38), (1.39) for amplitude and phase, time appears scaled by ϵ, so it is natural to examine a limit with the *slow time* ϵt near a fixed value T. To focus on a few cycles of oscillation about $t = \frac{T}{\epsilon}$, we introduce time with origin shifted to $\frac{T}{\epsilon}$. That is,

$$\tau := t - \frac{T}{\epsilon}. \tag{1.40}$$

We express the amplitude and phase (1.38), (1.39) in terms of τ:

$$a = \sqrt{\frac{1 + T + \epsilon\tau}{1 - \frac{\epsilon^2}{4}}},$$

$$\xi = \sqrt{1 - \frac{\epsilon^2}{4}} \frac{\log(1 + T + \epsilon\tau)}{\epsilon}.$$

In the limit $\epsilon \to 0$ with τ fixed,

$$a \to \sqrt{1 + T}, \quad \xi - \frac{\log(1 + T)}{\epsilon} \to \frac{\tau}{1 + T}. \tag{1.41}$$

The limit amplitude is independent of the shifted time τ, so in an interval of time about $\frac{T}{\epsilon}$ with duration independent of ϵ, the amplitude of oscillations is uniformly approximated by a constant (which happens to be $\sqrt{1 + T}$). So we say, "the *local amplitude* at slow time T is $\sqrt{1 + T}$." If we consider another interval with a different slow time T, we will obtain a different amplitude. In the second of equations (1.41), $\frac{\log(1+T)}{\epsilon}$ approximates the phase at time $\frac{T}{\epsilon}$. What's really important is the *phase difference* $\xi - \frac{\log(1+T)}{\epsilon}$: In the limit $\epsilon \to 0$, τ fixed, we see that its time rate of change is the constant $\frac{1}{1+T}$. So we say: "The *local frequency* at slow time T is $\frac{1}{1+T}$." That's what a physicist expects by simple inspection of the original ODE (1.34). Again: different values of T, different frequencies.

Problems whose solutions exhibit modulated oscillations are sometimes tractable by constructive approximation methods. Like the inner and outer approximations that we discovered in the boundary layer example, they acknowledge the two or more characteristic times that live in these problems. But unlike boundary layer problems, the two or more characteristic times are everywhere intermingled, so the constructive methods have names like *averaging* or *multiple scales*.

Here we present a constructive physicist approximation to the amplitude a in (1.37). Let's generalize the ODE (1.34) to

$$\ddot{x} + \Omega^2(\epsilon t)x = 0, \tag{1.42}$$

where the natural frequency is a given function $\Omega(T)$ of the slow time $T := \epsilon t$. For a few cycles of oscillation about time $\frac{T}{\epsilon}$, we expect a nearly uniform sinusoidal oscillation. In terms of the shifted time τ in (1.40), we'd say

$$x \to a(T)\cos\Omega(T)\tau \tag{1.43}$$

as $\epsilon \to 0$ with τ fixed. We dropped an additive constant from the phase: it can be absorbed by shifting the origin of τ. We simply insert the expected local frequency $\Omega(T)$, which is a given function of slow time T. The analysis we're about to do determines the slow time dependence of amplitude, $a = a(T)$.

The slow change of natural frequency over long time does cumulative work on the oscillator, and there is slow cumulative change in the *energy*

$$e := \frac{1}{2}\dot{x}^2 + \frac{1}{2}\Omega^2(\epsilon t)x^2. \tag{1.44}$$

We compute

$$\dot{e} = \epsilon\Omega(\epsilon t)\Omega'(\epsilon t)x^2. \tag{1.45}$$

Terms $\dot{x}\ddot{x}$ and $\Omega^2 x\dot{x}$ cancel by ODE (1.42), and indeed we see that change in energy is induced specifically by slow time dependence of $\Omega(T := \epsilon t)$. Introducing the shifted time τ in place of t and representing $e = e(\tau)$, (1.45) becomes

$$\frac{1}{\epsilon}\frac{de}{d\tau} = \Omega(T + \epsilon\tau)\Omega'(T + \epsilon\tau)x^2.$$

In the limit $\epsilon \to 0$ with τ fixed, we have

$$\frac{1}{\epsilon}\frac{de}{d\tau} \to \Omega(T)\Omega'(T)a^2(T)\cos^2(\Omega(T)\tau). \tag{1.46}$$

Here, we used the limit of x in (1.43). Let δe denote the change in energy over one cycle of oscillation with $0 < \tau < \frac{2\pi}{\Omega(T)}$. Integrating (1.46) gives

$$\frac{1}{\epsilon}\delta e \to (\Omega\Omega' a^2)(T)\int_0^{\frac{2\pi}{\Omega(T)}} \cos^2(\Omega(T)\tau)d\tau = \pi\Omega'(T)a^2(T) \tag{1.47}$$

as $\epsilon \to 0$. We propose that the leading approximation to energy over times comparable to $\frac{1}{\epsilon}$ is some function $e = e(T)$ of slow time T. In this case, we expect that δe in (1.47) is approximated by $\delta e \simeq e'(T)\epsilon\frac{2\pi}{\Omega(T)}$. Here $\epsilon\frac{2\pi}{\Omega(T)}$ is the elapsed slow time in one period of oscillation. Hence we expect that

$$\frac{1}{\epsilon}\delta e \to \frac{2\pi e'(T)}{\Omega(T)} \tag{1.48}$$

as $\epsilon \to 0$. Comparing (1.47), (1.48) we see that

$$e'(T) = \frac{1}{2}(\Omega\Omega' a^2)(T). \tag{1.49}$$

In (1.49) we can express $a(T)$ in terms of $e(T)$. Substituting (1.43) for x (with T fixed) into (1.44) we find

$$e(T) \to \frac{1}{2}(\Omega^2 a^2)(T) \tag{1.50}$$

as $\epsilon \to 0$. Hence, (1.49) becomes

$$e'(T) = \frac{\Omega'(T)}{\Omega(T)} e(T)$$

or

$$\frac{e'(T)}{e(T)} = \frac{\Omega'(T)}{\Omega(T)}. \tag{1.51}$$

We see that $e(T)$ is proportional to $\Omega(T)$. We recover $a(T)$ from (1.50) with e proportional to Ω, giving $a(T)$ proportional to $\frac{1}{\sqrt{\Omega(T)}}$. For our original example (1.34) with $\Omega = \frac{1}{1+T}$, this gives a proportional to $\sqrt{1+T}$, as we've already seen from the exact solution.

This physicist analysis has a superficial appearance of being elementary, but on closer examination, there are subtle insights and guesses.

Problems and Solutions

Problem 1.5 (Secular terms). The solution of the initial value problem (1.34), (1.35) has a power series expansion

$$x(t,\epsilon) = x^0(t) + \epsilon x^1(t) + \epsilon^2 x^2(t) + \dots .$$

Determine $x^0(t)$ and $x^1(t)$ by (i) expansion of the exact solution and (ii) by substituting the power series into the initial value problem, formulating and solving reduced initial value problems for $x^0(t)$ and $x^1(t)$. Show that the first-order term $\epsilon x^1(t)$ has the same order of magnitude as $x^0(t)$ when t is sufficiently large. This indicates the non-uniform validity of the power series expansion for all $t > 0$. In general, terms $\epsilon^k x^k(t)$, which become larger than ϵ^k in magnitude for t sufficiently large, are called *secular terms*.

Solution. The first two terms in the power series expansion of the amplitude (1.38) are

$$a(t,\epsilon) = \sqrt{\frac{1+\epsilon t}{1-\frac{\epsilon^2}{4}}} = 1 + \frac{\epsilon}{2}t + \dots .$$

For the phase (1.39) we have

$$\xi(t,\epsilon) = \sqrt{1-\frac{\epsilon^2}{4}}\,\frac{\log(1+\epsilon t)}{\epsilon} = t - \frac{\epsilon t^2}{2} + \dots .$$

Hence the expansion of $x(t,\epsilon)$ is

$$\begin{aligned} x(t,\epsilon) &= a(t,\epsilon)\sin\xi(t,\epsilon) \\ &= \left(1+\frac{\epsilon}{2}t+\dots\right)\sin\left(t-\frac{\epsilon t^2}{2}+\dots\right) \\ &= \sin t + \epsilon\left\{\frac{t}{2}\sin t - \frac{t^2}{2}\cos t\right\} + \dots , \end{aligned}$$

and we see that

$$x^0(t) = \sin t, \quad x^1(t) = \frac{t}{2}\sin t - \frac{t^2}{2}\cos t.$$

Next we derive reduced initial value problems by substituting the power series of $x(t,\epsilon)$ into (1.34), (1.35). Since we want to include terms up to first order in ϵ, we expand $\frac{1}{(1+\epsilon t)^2}$ in ODE (1.34) as $1 - 2\epsilon t + \dots$, and so

(1.5.1) $$\ddot{x} + (1 - 2\epsilon t + \dots)x = 0.$$

Substituting $x = x^0 + \epsilon x^1 + \dots$ into (1.5.1) and balancing zero and first powers of ϵ, we find reduced ODE's for x^0, x^1:

$$\begin{aligned} \ddot{x}^0 + x^0 &= 0, \\ \ddot{x}^1 + x^1 &= 2tx^0, \end{aligned}$$

in $t > 0$. Substituting the power series for x into the initial conditions and balancing zero and first powers of ϵ gives reduced initial conditions

$$\begin{aligned} x^0(0) = 0, \dot{x}^0(0) = 1, \\ x^1(0) = 0, \dot{x}^1(0) = 0. \end{aligned}$$

The solution for $x^0(t)$ is obviously $x^0(t) = \sin t$, and then the reduced ODE for $x^1(t)$ reads

(1.5.2) $$\ddot{x}^1 + x^1 = 2t\sin t.$$

We require the solution whose value and derivative vanish at $t = 0$. To find a particular solution of (1.5.2), we take the imaginary part of a particular solution to the complex ODE

$$\ddot{z} + z = 2te^{it}.$$

Substituting $z = \left(at + \frac{b}{2}t^2\right)e^{it}$, we deduce $a = \frac{1}{2}$, $b = -i$, so $z = \left(\frac{t}{2} - \frac{it^2}{2}\right)e^{it}$, and the imaginary part is $\frac{t}{2}\sin t - \frac{t^2}{2}\cos t$. This particular solution already has zero value and derivative at $t = 0$, so

$$x^1(t) = \frac{t}{2}\sin t - \frac{t^2}{2}\cos t.$$

In the two-term expansion

$$x(t,\epsilon) \sim \sin t + \epsilon\left(\frac{t}{2}\sin t - \frac{t^2}{2}\cos t\right),$$

the ϵ term has the same order of magnitude as the leading term when ϵt^2 is comparable to one or t is comparable to $\epsilon^{-\frac{1}{2}}$.

Problem 1.6 (Approach to limit cycle). $x(t,\epsilon)$ satisfies the van der Pol ODE

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0. \tag{1.6.1}$$

Here, ϵ is a positive constant and we examine the limit $\epsilon \to 0^+$.

a) For $\epsilon = 0$, show that the parametric curve $(x(t), \dot{x}(t))$ in $\mathbb{R}^2$ is a uniform circular motion about the origin.
b) For $0 < \epsilon \ll 1$, the orbits in the $(x, \dot{x})$ plane don't quite close due to the nonlinear damping in (1.6.1). Figure 1.6.1 (based on a numerical solution of (1.6.1) with $\epsilon = 0.1$) depicts one cycle in the $(x, \dot{x})$ plane with $R^* < R$. It could happen that $R^* > R$. In any case, compute $\lim_{\epsilon\to 0} \frac{R^* - R}{\epsilon}$.

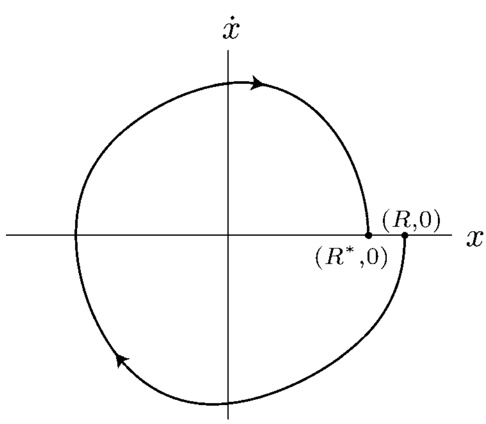

Figure 1.6.1

c) After many cycles (on the order of $\frac{1}{\epsilon}$) we expect that the radius R of the nearly circular orbit seen during one cycle undergoes significant change. We propose to approximate the slow variation of radius as a function of slow time $T := \epsilon t$, $R = R(T)$. Propose an ODE for $R(T)$ and give its general solution. Given $R(0) > 0$, what happens to $R(T)$ as $T \to \infty$?

d) Sketch representative curves $(x(t), \dot{x}(t))$ in $\mathbb{R}^2$ for $0 < \epsilon \ll 1$. Indicate the orientations of increasing t by arrows, as in Figure 1.6.1.

Solution.

a) The general solution of $\ddot{x} + x = 0$ is $x = R\cos(t + \theta)$, where $R > 0$ and θ are constants. Hence, $(x, \dot{x}) = R(\cos(t + \theta), -\sin(t + \theta))$, which is a parametric representation of uniform circular motion with radius R and frequency 1 (period 2π).

b) Look at energy $e := \frac{1}{2}(\dot{x}^2 + x^2)$. We have $\dot{e} = \dot{x}(\ddot{x} + x) = \epsilon\dot{x}^2(1 - x^2)$. The last equality follows from the ODE (1.6.1). Assume $(x, \dot{x}) = (R, 0)$ at $t = 0$. Then $x(t, \epsilon) \to x^0(t) := R\cos t$ as $\epsilon \to 0$ with t fixed. Hence the leading approximation to change δe in energy as we go from $(x, \dot{x}) = (R, 0)$ to $(R^*, 0)$ is

$$\begin{aligned}\epsilon\int_0^{2\pi}(\dot{x}^0)^2(1 - (x^0)^2)dt &= \epsilon\int_0^{2\pi} R^2\sin^2 t(1 - R^2\cos^2 t)dt \\ &= \epsilon\pi R^2\left(1 - \frac{R^2}{4}\right).\end{aligned}$$

We surmise

$$\frac{\delta e}{\epsilon} \to \pi R^2\left(1 - \frac{R^2}{4}\right) \tag{1.6.2}$$

as $\epsilon \to 0$. Energies corresponding to $(x, \dot{x}) = (R, 0)$, $(R^*, 0)$ are $\frac{1}{2}R^2$ and $\frac{1}{2}R^{*2}$ so

$$\delta e = \frac{1}{2}(R^{*2} - R^2) = \frac{1}{2}(R^* - R)(R^* + R),$$

so we also have

$$\frac{\delta e}{\epsilon} \to R\lim_{\epsilon\to 0}\frac{R^* - R}{\epsilon}. \tag{1.6.3}$$

Comparing (1.6.2), (1.6.3) we get

$$\lim_{\epsilon\to 0}\frac{R^* - R}{\epsilon} = \pi R\left(1 - \frac{R^2}{4}\right). \tag{1.6.4}$$

c) In Figure 1.6.1, the elapsed time δt between $(R, 0)$ and $(R^*, 0)$ is nearly 2π, and the corresponding increment δT of slow time is nearly $2\pi\epsilon$. Taking $R = R(T)$, we anticipate

$$\lim_{\epsilon\to 0} \frac{R^* - R}{\epsilon} = \lim_{\epsilon\to 0} \frac{R(T + 2\pi\epsilon) - R(T)}{\epsilon} = 2\pi R'(T).$$

Comparing with (1.6.4), we propose the ODE

$$2R' = R\left(1 - \frac{R^2}{4}\right).$$

This is equivalent to a linear ODE for $\frac{1}{R^2}$,

$$\left(\frac{1}{R^2}\right)' = \frac{1}{4} - \frac{1}{R^2},$$

and the solution with given $R(0) > 0$ is

$$\frac{1}{R^2(T)} = \frac{1}{4} + \left(\frac{1}{R^2(0)} - \frac{1}{4}\right) e^{-t}.$$

We see that $R(\infty) = 2$.

d) The phase plane trajectories $(x(t), x'(t))$ asymptote to a closed curve called a *limit* cycle as $t \to +\infty$. Figure 1.6.2 shows representative trajectories based on numerical solutions of the van der Pol ODE for $\epsilon = 0.1$ and $\epsilon = 1$. As the calculation in part c) suggests, the $\epsilon = 0.1$ limit cycle is close to the circle $x^2 + \dot{x}^2 = 2$.

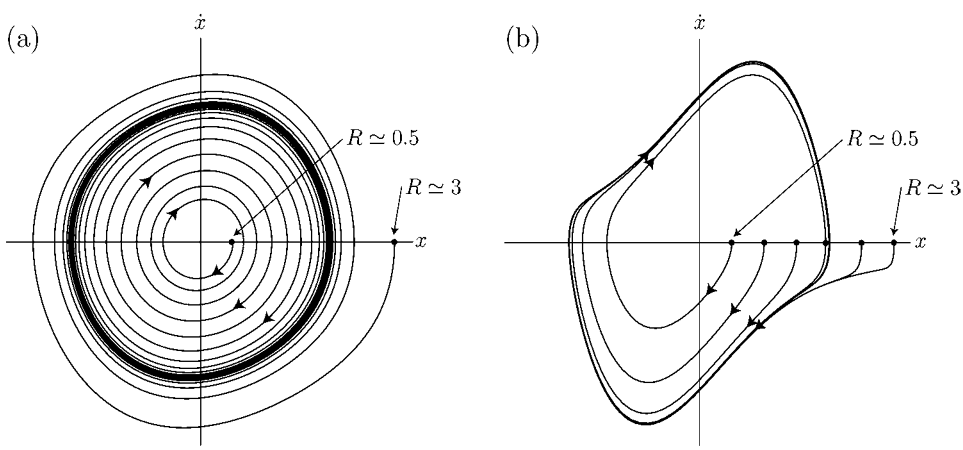

Figure 1.6.2

Problem 1.7 (Adiabatic invariant for particle in a box). A particle is confined between a fixed wall ($x = 0$) and a slowly moving wall ($x = L(T := \epsilon t)$, $0 < \epsilon \ll 1$). We look at the x-component of motion, $x = x(t)$. Between the walls, the particle is free, so $\ddot{x} = 0$ in $0 < x < L(T)$. Rebounds off the walls are elastic: In the instantaneous rest frame of a wall, the incoming and outgoing x-velocities are opposite and equal.

a) Consider a particle bouncing off the right wall $x = L(T)$ with incoming velocity $+v$. What is its rebound velocity?

b) The speed $v := |\dot{x}|$ of the particle as a function of time is piecewise constant, as depicted in Figure 1.7.1b. The small jump discontinuities correspond to rebounds off the right wall $x = L(T)$, according to your results of part a). We propose that the staircase graph in Figure 1.7.1b can be approximated by a smooth function of slow time, $v = v(T)$. (In Figure 1.7.1a, the graph of $v(T)$ interpolates the midpoints of the horizontal segments, corresponding to rebounds off the $x = 0$ wall, where there is no change of v). Propose an approximate ODE for $v(T)$, and show that within this approximation, $v(T)L(T)$ is a constant independent of T.

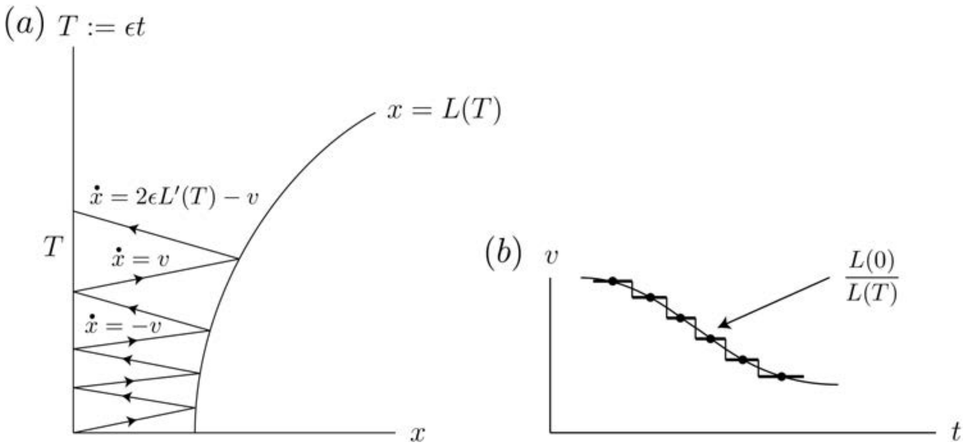

Figure 1.7.1

Solution.

a) The relative incoming velocity is $v - \epsilon L'(T)$ and relative outgoing velocity is $\epsilon L'(T) - v$ and rebound velocity (in frame fixed to left wall) is $2\epsilon L'(T) - v$.

b) Look at one cycle between two bounces from right wall: An incoming particle with velocity $+v$ rebounds from right wall with velocity

$2\epsilon L'(T) - v$. It flies to left wall, rebounds from it with *no* change in speed, and returns to right wall with velocity $v - 2\epsilon L'(T)$ in time approximately equal to $\frac{2L(T)}{v}$. The corresponding increment of slow time is $\frac{2\epsilon L(T)}{v}$. Hence we propose an approximate ODE for $v = v(T)$,

$$v'(T) = \frac{-2\epsilon L'(T)}{\left(\frac{2\epsilon L(T)}{v(T)}\right)} \quad \text{or} \quad \frac{v'(T)}{v(T)} = -\frac{L'(T)}{L(T)},$$

which implies $v(T)L(T) =$ constant independent of T.

Guide to bibliography.

Many of the references, [**5**], [**11**], [**14**]–[**16**], [**18**]–[**20**], [**24**]–[**26**], [**31**], [**33**], present concrete and elementary examples of singular perturbation. One gets the sense that concrete analysis of prototypical singular perturbation problems is more important than conceptualized definitions. The introductory chapters of O'Malley's texts [**24**], [**25**] especially emphasize clear examples of boundary and internal layers. In the books of Van Dyke [**32**], Kaplun [**17**], and Eckhaus [**11**], we see the historical lineage of singular perturbation ideas from deep problems associated with the Navier–Stokes equations. The text of Bender and Orszag [**5**] has a larger scope than singular perturbation, and introductory examples of boundary layers and modulated oscillations are found in Chapters 9 and 11, which are devoted to these subjects.

A note about the definition of singular perturbation: The usual characterization as "non-uniform convergence of a solution as $\epsilon \to 0$" prevails generally, but there are variations, and these are not necessarily equivalent to each other. For instance, on p. 324 of the text [**5**] of Bender and Orszag, it says: "We define a singular perturbation problem as one whose perturbation series does not take the form of a power series, or if it does, the power series has a vanishing radius of consequence." Problem 2.3 of this text presents a simple boundary value problem whose formal power series solution has zero radius of convergence in ϵ. Nevertheless, from an exact integral representation, it is easy to demonstrate the uniform convergence of the solution as $\epsilon \to 0$.