Chapter 3

Mean Curvature Equations

In this chapter, we discuss the equation of the prescribed mean curvature, or the mean curvature equation. We derive various a priori estimates for its solutions and solve the Dirichlet boundary-value problems.

In Section 3.1, we introduce principal curvatures of hypersurfaces in Euclidean spaces and discuss distance functions to hypersurfaces.

In Section 3.2, we derive global estimates up to first derivatives of solutions of the mean curvature equation. We prove the global estimates of L^∞-norms by Alexandrov's maximum principle and the boundary gradient estimates by constructing appropriate barrier functions, respectively. The global gradient estimates are based on Bernstein's method. We point out that the mean curvature equation is not uniformly elliptic. The structure of the equation plays an important role in the derivation.

In Section 3.3, we derive interior gradient estimates of solutions of the mean curvature equation. Again, we make use of the structure of the equation essentially. As an application of the interior gradient estimates, we prove a Liouville type theorem for the minimal surface equation.

In Section 3.4, we discuss the Dirichlet boundary-value problems for the mean curvature equation. First, we solve the Dirichlet problem by the method of continuity under a suitable condition on the boundary mean curvature. Then, we present a nonsolvability result if such a condition is not satisfied.

3.1. Principal Curvatures

In this section, we introduce principal curvatures of hypersurfaces in Euclidean spaces and discuss how they are related to distance functions to hypersurfaces.

We first introduce hypersurfaces in $\mathbb{R}^{n+1}$. A subset Σ in $\mathbb{R}^{n+1}$ is a C^l-*hypersurface* if, for any $p_0 \in \Sigma$, there exist a domain $\Omega \subset \mathbb{R}^n$, a domain $U \subset \mathbb{R}^{n+1}$ containing p_0, and a C^l-immersion $\mathbf{r} : \Omega \to U$ such that $\Sigma \cap U = \mathbf{r}(\Omega)$. Recall that a mapping $\mathbf{r}$ is an *immersion* if the matrix $(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ has a full rank. Here and hereafter, $\mathbf{r}_i = \mathbf{r}_{x_i}$ and $\mathbf{r}_{ij} = \mathbf{r}_{x_i x_j}$. The pair $(\Omega, \mathbf{r})$ is called a *local representation* of Σ. The *tangent space* $T_{p_0}\Sigma$ of Σ at p_0 is the subspace in $\mathbb{R}^{n+1}$ spanned by $\mathbf{r}_1, \ldots, \mathbf{r}_n$, which is n-dimensional due to the immersion of $\mathbf{r}$. A vector normal to $T_{p_0}\Sigma$ is called a *normal vector* of Σ at p_0.

Let Σ be a C^2-hypersurface in $\mathbb{R}^{n+1}$ and $(\Omega, \mathbf{r})$ be a local representation of Σ. We define, for $i, j = 1, \ldots, n$,

$$g_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j.$$

The matrix (g_{ij}) is positive definite under the assumption that $\mathbf{r}$ is an immersion. Its associated bilinear form is called the *first fundamental form* of Σ. Next, let ν be a unit normal vector to Σ. Define

$$h_{ij} = \mathbf{r}_{ij} \cdot \nu.$$

Its associated bilinear form is called the *second fundamental form* of Σ. We usually write the first and second fundamental forms as

$$I = g_{ij}dx_i dx_j, \quad II = h_{ij}dx_i dx_j.$$

We call g_{ij} and h_{ij} the coefficients of the first and second fundamental forms associated with the local representation $(\Omega, \mathbf{r})$, respectively.

Definition 3.1.1. Let Σ be a C^2-hypersurface in $\mathbb{R}^{n+1}$, $(\Omega, \mathbf{r})$ be a local representation of Σ, and $p \in \Sigma$ be a point with $p = \mathbf{r}(x)$. A number κ is a *principal curvature* of Σ at p with respect to a unit normal vector ν if κ is an eigenvalue of $(h_{ij}(x))$ with respect to $(g_{ij}(x))$; namely,

$$\det\big(h_{ij}(x) - \kappa g_{ij}(x)\big) = 0,$$

where g_{ij} and h_{ij} are the coefficients of the first and second fundamental forms associated with the local representation $(\Omega, \mathbf{r})$, respectively.

We point out that principal curvatures depend on the choice of normal vectors.

We now prove a simple result.

Lemma 3.1.2. *Principal curvatures are well-defined real numbers.*

Proof. Let Σ be a C^2-hypersurface in $\mathbb{R}^{n+1}$ and $p \in \Sigma$ be a fixed point. Assume that $(\Omega, \mathbf{r})$ and $(\widetilde{\Omega}, \widetilde{\mathbf{r}})$ are two local representations of Σ with $p = \mathbf{r}(x_0) = \widetilde{\mathbf{r}}(y_0)$, for some $x_0 \in \Omega$ and $y_0 \in \widetilde{\Omega}$. By shrinking Ω and $\widetilde{\Omega}$ appropriately, there is a C^2-diffeomorphism $x : \widetilde{\Omega} \to \Omega$ with $x(y_0) = x_0$ and $\widetilde{\mathbf{r}}(y) = \mathbf{r}(x(y))$. Then,

$$\begin{aligned}\widetilde{\mathbf{r}}_{y_i} &= \mathbf{r}_{x_k} \partial_{y_i} x_k, \\ \widetilde{\mathbf{r}}_{y_i y_j} &= \mathbf{r}_{x_k x_l} \partial_{y_i} x_k \partial_{y_j} x_l + \mathbf{r}_{x_k} \partial_{y_i y_j} x_k.\end{aligned}$$

Let ν be a unit normal vector and g_{ij}, h_{ij} and $\widetilde{g}_{ij}$, $\widetilde{h}_{ij}$ be the coefficients of the first and second fundamental forms associated with $(\Omega, \mathbf{r})$ and $(\widetilde{\Omega}, \widetilde{\mathbf{r}})$, respectively. Then,

$$\widetilde{g}_{ij} = \widetilde{\mathbf{r}}_{y_i} \cdot \widetilde{\mathbf{r}}_{y_j} = \mathbf{r}_{x_k} \cdot \mathbf{r}_{x_l} \partial_{y_i} x_k \partial_{y_j} x_l = g_{kl} \partial_{y_i} x_k \partial_{y_j} x_l,$$

and

$$\begin{aligned}\widetilde{h}_{ij} = \widetilde{\mathbf{r}}_{y_i y_j} \cdot \nu &= \big(\mathbf{r}_{x_k x_l} \partial_{y_i} x_k \partial_{y_j} x_l + \mathbf{r}_{x_k} \partial_{y_i y_j} x_k\big) \cdot \nu \\ &= \mathbf{r}_{x_k x_l} \cdot \nu \partial_{y_i} x_k \partial_{y_j} x_l = h_{kl} \partial_{y_i} x_k \partial_{y_j} x_l,\end{aligned}$$

where we used $\mathbf{r}_{x_k} \cdot \nu = 0$. Hence, it is easy to see that

$$\det\big((h_{ij} - \kappa g_{ij})\big) = 0$$

if and only if

$$\det\big((\widetilde{h}_{ij} - \kappa \widetilde{g}_{ij})\big) = 0.$$

Therefore, κ is independent of representations of Σ. Since both (g_{ij}) and (h_{ij}) are symmetric matrices, any such κ has to be real. □

Next, we introduce an important combination of principal curvatures.

Definition 3.1.3. Let Σ be a C^2-hypersurface in $\mathbb{R}^{n+1}$, with principal curvatures $\kappa_1, \ldots, \kappa_n$ corresponding to a unit normal vector ν. The *mean curvature* H of Σ corresponding to ν is the sum of the principal curvatures corresponding to ν; i.e.,

$$H = \sum_{i=1}^{n} \kappa_i.$$

There are other versions of curvatures. For example, the *Gauss curvature* K is the product of principal curvatures; i.e.,

$$K = \prod_{i=1}^{n} \kappa_i.$$

In this chapter, we discuss mean curvatures only.

We now express the mean curvature in terms of the first and the second fundamental forms.

Proposition 3.1.4. *Let Σ be a C^2-hypersurface in $\mathbb{R}^{n+1}$ and ν be its unit normal vector. Suppose that g_{ij} and h_{ij} are coefficients of the first fundamental form of Σ and the second fundamental form of Σ corresponding to ν, respectively. Then, the mean curvature H of Σ corresponding to ν is given by*

$$H = g^{ij} h_{ij},$$

where (g^{ij}) is the inverse matrix of (g_{ij}).

Proof. Set $G = (g_{ij})$ and $A = (h_{ij})$. Then, G is invertible. Let κ be a principal curvature of Σ. Then, $\det(A - \kappa G) = 0$, or equivalently

$$\det(AG^{-1} - \kappa I) = 0.$$

In other words, the principal curvature is an eigenvalue of AG^{-1}. The mean curvature H, being the sum of all principal curvatures, is given by

$$H = \mathrm{tr}(AG^{-1}).$$

This implies the desired result. □

Next, we calculate the mean curvature of graphs in $\mathbb{R}^{n+1}$.

Lemma 3.1.5. *Let Ω be a domain in $\mathbb{R}^n$ and Σ be a graph in $\mathbb{R}^{n+1}$ given by $x_{n+1} = u(x)$ for some $u \in C^2(\Omega)$. Then, the mean curvature $H(x)$ of Σ at $\big(x, u(x)\big)$ corresponding to the upward unit normal vector of Σ is given by*

$$H(x) = \frac{1}{\sqrt{1+|\nabla u(x)|^2}} \left(\Delta u(x) - \frac{u_i(x) u_j(x)}{1+|\nabla u(x)|^2} u_{ij}(x) \right).$$

Proof. We set

$$\mathbf{r}(x) = \big(x, u(x)\big).$$

Then, for $i, j = 1, \ldots, n$,

$$\mathbf{r}_i = \big(\mathbf{e}_i, u_i\big), \quad \mathbf{r}_{ij} = \big(0, u_{ij}\big),$$

where $\mathbf{e}_i$ is the unit vector along the x_i-axis. Now, we take

$$\nu = \frac{1}{\sqrt{1+|\nabla u|^2}} (-\nabla u, 1).$$

Then, ν is the upward unit normal vector of Σ, and

$$\begin{aligned} g_{ij} &= \mathbf{r}_i \cdot \mathbf{r}_j = \delta_{ij} + u_i u_j, \\ h_{ij} &= \mathbf{r}_{ij} \cdot \nu = \frac{u_{ij}}{\sqrt{1+|\nabla u|^2}}. \end{aligned}$$

A straightforward calculation yields that the component g^{ij} of the inverse matrix $(g_{ij})^{-1}$ is given by

$$g^{ij} = \delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}.$$

By Proposition 3.1.4, we have

$$H = g^{ij}h_{ij} = \frac{1}{\sqrt{1+|\nabla u|^2}}\left(\Delta u - \frac{u_i u_j}{1+|\nabla u|^2}u_{ij}\right).$$

This is the desired identity. □

Remark 3.1.6. The mean curvature can also be expressed by

$$H(x) = \operatorname{div}\left(\frac{\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}}\right).$$

Remark 3.1.7. We now examine a special case in the proof of Lemma 3.1.5. Let x_0 be a point in Ω. If $\nabla u(x_0) = 0$, then at $(x_0, u(x_0))$, $\nu = (0,1)$ and

$$g_{ij} = \delta_{ij}, \quad h_{ij} = u_{ij}.$$

As a consequence, the principal curvatures of Σ at $(x_0, u(x_0))$ is simply the eigenvalues of the Hessian matrix $\nabla^2 u(x_0)$.

Let Ω be a domain in $\mathbb{R}^n$ and $u \in C^2(\Omega)$. Suppose that the graph of u in $\mathbb{R}^{n+1}$ has a mean curvature $H(x)$ at the point $\big(x, u(x)\big)$, for $x \in \Omega$. Here, the mean curvature is calculated corresponding to the upward unit normal vector. Then, u satisfies

$$\Delta u - \frac{u_i u_j}{1+|\nabla u|^2}u_{ij} = H(x)\sqrt{1+|\nabla u|^2} \quad \text{in } \Omega.$$

This is the *equation of prescribed mean curvature*, or, simply, the *mean curvature equation*. If $H \equiv 0$, the mean curvature equation has the form

$$\Delta u - \frac{u_i u_j}{1+|\nabla u|^2}u_{ij} = 0 \quad \text{in } \Omega.$$

This is referred to as the *minimal surface equation*. Its solutions are the critical points of the area functional

$$\mathcal{A}(u) = \int_\Omega \sqrt{1+|\nabla u|^2}\,dx.$$

Their graphs are called *minimal surfaces*.

We now write the mean curvature equation as

$$a_{ij}(\nabla u)u_{ij} = H(x)\sqrt{1+|\nabla u|^2} \quad \text{in } \Omega,$$

where, for any $p \in \mathbb{R}^n$,

$$a_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2}.$$

It is easy to check that, for any $p \in \mathbb{R}^n$ and any $\xi \in \mathbb{R}^n$,

$$\frac{1}{1+|p|^2}|\xi|^2 \le a_{ij}(p)\xi_i\xi_j \le |\xi|^2.$$

For any $p \in \mathbb{R}^n \setminus \{0\}$, the left equality is attained if $\xi = p$ and the right equality is attained if $\xi \perp p$. The difficulty in discussions of the mean curvature equation arises from the lack of the uniform ellipticity. Although elliptic, the mean curvature equation has its ellipticity constants determined by the gradients of solutions. These ellipticity constants are controlled only after the C^1-norms of solutions are derived.

In the rest of this section, we discuss relations between the distance to a hypersurface and the mean curvature of this surface.

Let Ω be a bounded domain in $\mathbb{R}^n$. The *distance function* d to $\partial\Omega$ is defined by

$$d(x) = \text{dist}(x, \partial\Omega) \quad \text{for any } x \in \mathbb{R}^n.$$

It is easy to check that d is a Lipschitz function. In fact, for any $x, y \in \mathbb{R}^n$, we take $z \in \partial\Omega$ such that $d(y) = |y - z|$. Then,

$$d(x) \le |x - z| \le d(y) + |x - y|.$$

By interchanging x and y, we obtain

$$|d(x) - d(y)| \le |x - y|.$$

In general, distance functions are not smooth globally. For example, the distance function to a sphere is not C^1 at the center. In the next result, we will relate the regularity of the distance function to that of the boundary $\partial\Omega$, at least in a region close to the boundary.

In the following, we set, for some positive constant $\mu > 0$,

$$\Omega_\mu = \{x \in \Omega : d(x) < \mu\},$$

where d is the distance function to $\partial\Omega$.

Lemma 3.1.8. *Let Ω be a bounded domain with a C^k-boundary, for some $k \ge 2$. Then, there exists a positive constant μ, depending on Ω, such that $d \in C^k(\Omega_\mu)$.*

Proof. The assumption of the C^2-regularity of the boundary implies that Ω satisfies a *uniform interior sphere* condition; namely, at each $y_0 \in \partial\Omega$, there exists a ball B such that $\bar{B} \cap (\mathbb{R}^n \setminus \Omega) = \{y_0\}$ and the radius of the ball B is bounded from below by a positive constant independent of y_0, which we take to be μ. For each $x \in \Omega_\mu$, there exists a unique $y = y(x) \in \partial\Omega$ such that $|x - y| = d(x)$. In fact, we have

$$x = y + \nu(y)d, \tag{1}$$

where $\nu(y)$ is the inner unit normal vector at y.

The relation (1) determines y and d as functions of x. To discuss the regularity of these functions, we fix an $x_0 \in \Omega_\mu$ and assume $y_0 = y(x_0)$ is the origin such that $\mathbf{e_n}$ is the inner unit normal vector of $\partial\Omega$ at y_0. Then, $\mathbb{R}^{n-1} \times \{0\}$ is the tangent plane of $\partial\Omega$ at y_0. Locally, we express $\partial\Omega$ as a function $y_n = \rho(y')$ for y' in a ball $B'_r \subset \mathbb{R}^{n-1}$. Then, $\nabla\rho(0) = 0$. By an appropriate rotation in $\mathbb{R}^{n-1}$, we assume $\nabla^2\rho(0)$ is diagonal. This implies

$$\nabla^2\rho(0) = \mathrm{diag}(\kappa_1, \ldots, \kappa_{n-1}),$$

where $\kappa_1, \ldots, \kappa_{n-1}$ are principal curvatures of $\partial\Omega$ at y_0. Since μ is the radius of an interior tangent ball, it is straightforward to check that, for $i = 1, \ldots, n-1$,

$$\kappa_i \le \frac{1}{\mu}.$$

Next, for each $y' \in B'_r$, we view the inner unit normal vector $\nu(y)$ at $y = \big(y', \rho(y')\big)$ as $\bar\nu(y')$, a function of y'. Then, for $i = 1, \ldots, n-1$,

$$\bar\nu_i(y') = -\frac{\rho_i(y')}{\sqrt{1 + |\nabla\rho(y')|^2}},$$

and

$$\bar\nu_n(y') = \frac{1}{\sqrt{1 + |\nabla\rho(y')|^2}}.$$

Hence, for any $i, j = 1, \ldots, n-1$,

$$\partial_j\bar\nu_i(0) = -\kappa_i\delta_{ij}. \tag{2}$$

We now view the relation (1) as a map $x = x(y', d) : B'_r \times (0, \mu) \to \mathbb{R}^n$ for $y = \big(y', \rho(y')\big)$. Then, $x \in C^{k-1}\big(B'_r \times (0, \mu)\big)$ and its Jacobian matrix is given by

$$\frac{\partial x}{\partial(y', d)} = \begin{pmatrix} \delta_{ij} + \partial_{y_i}\nu_j d & \partial_{y_i}\rho + \partial_{y_i}\nu_n d \\ \nu_j & \nu_n \end{pmatrix}. \tag{3}$$

In particular,

$$\left.\frac{\partial x}{\partial(y', d)}\right|_{(0,d)} = \mathrm{diag}(1 - \kappa_1 d, \ldots, 1 - \kappa_{n-1}d, 1). \tag{4}$$

Hence, the Jacobian of x at $\big(0, d(x_0)\big)$ is given by

$$\det\left(\left.\frac{\partial x}{\partial(y', d)}\right|_{(0,d(x_0))}\right) = \big(1 - \kappa_1 d(x_0)\big)\cdots\big(1 - \kappa_{n-1}d(x_0)\big) > 0$$

since $d(x_0) < \mu$. By the inverse mapping theorem, the mapping y' is C^{k-1} in $B_s(x_0)$ for some $s > 0$. Note that $|\nu|^2 = 1$ and, for $i = 1, \ldots, n-1$,

$$\sum_{j=1}^{n-1}(\delta_{ij} + \partial_{y_i}\nu_j d)\nu_j + (\partial_{y_i}\rho + \partial_{y_i}\nu_n d)\nu_n = 0.$$

Hence, (3) implies

$$\nabla d(x) = \nu\big(y(x)\big) = \bar\nu\big(y'(x)\big). \tag{5}$$

Hence, ∇d is C^{k-1} in $B_s(x_0)$. Then, $d \in C^k(B_s(x_0))$, and thus $d \in C^k(\Omega_\mu)$. □

We now calculate derivatives of the distance functions and relate the distance function to the principal curvatures of the boundary.

Consider a bounded domain Ω in $\mathbb{R}^n$ with a C^2-boundary, which can be viewed as a C^2-hypersurface in $\mathbb{R}^n$. Then, we have the well-defined principal curvatures, and in particular the *mean curvature* $H_{\partial\Omega}$, of $\partial\Omega$ with respect to the inner unit normal vector.

Lemma 3.1.9. *Let Ω be a bounded domain with a C^2-boundary and μ be the positive constant as in Lemma* 3.1.8. *Then, for any $x \in \Omega_\mu$,*

$$\nabla d(x) = \nu(y),$$

and the eigenvalues of $\nabla^2 u(x)$ are given by

$$-\frac{\kappa_1(y)}{1-\kappa_1(y)d(x)}, \ldots, -\frac{\kappa_{n-1}(y)}{1-\kappa_{n-1}(y)d(x)}, 0,$$

where $y \in \partial\Omega$ is the unique point such that $d(x) = |x - y|$, ν is the inner unit normal vector to $\partial\Omega$, and $\kappa_1, \ldots, \kappa_{n-1}$ are the principal curvatures of $\partial\Omega$ with respect to ν. In particular,

$$\Delta d(x) \le -H_{\partial\Omega}(y),$$

where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ with respect to ν.

Proof. First, (5) in the proof of Lemma 3.1.8 is the identity concerning the gradient of d. Next, we adopt the setting in the proof of Lemma 3.1.8 to calculate the Hessian matrix $\nabla^2 d$.

Assume $\partial\Omega$ in a neighborhood of $0 \in \partial\Omega$ is given by a C^2-function $x_n = \rho(x')$ such that $\rho(0) = 0$, $\nabla\rho(0) = 0$, and $\nabla^2\rho(0)$ is diagonal. Then,

$$\nabla^2\rho(0) = (\kappa_1, \ldots, \kappa_{n-1}),$$

where $\kappa_1, \ldots, \kappa_{n-1}$ are the principal curvatures of $\partial\Omega$ at 0. Then, for any x along the x_n-axis with $d(x) < \mu$,

$$\nabla d(x) = (0, \ldots, 0, 1).$$

In the following, we fix such an x. Hence, $d_{in}(x) = 0$ for $i = 1, \ldots, n$. Next, for any $i, j = 1, \ldots, n-1$, we have

$$d_{ij}(x) = \partial_{x_j}\nu_i\big(0\big) = \partial_{y_k}\bar{\nu}_i(0)\partial_{x_j}y_k(x) = \frac{-\kappa_i}{1-\kappa_i d(x)}\delta_{ij},$$

by (2) and (4) in the proof of Lemma 3.1.8. Therefore, the eigenvalues of $\nabla^2 u(x)$ are given by

$$-\frac{\kappa_1}{1-\kappa_1 d(x)}, \ldots, -\frac{\kappa_{n-1}}{1-\kappa_{n-1}d(x)}, 0.$$

In particular,

$$\Delta d(x) = -\sum_{i=1}^{n-1} \frac{\kappa_i}{1 - \kappa_i d(x)} \le -\sum_{i=1}^{n-1} \kappa_i.$$

The expression in the right-hand side is the negative mean curvature $-H_{\partial\Omega}$ of $\partial\Omega$ at 0. □

If $H_{\partial\Omega} \ge 0$ on $\partial\Omega$, then $\Delta d \le 0$ in Ω_μ. Set

$$M(u) = (1 + |\nabla u|^2)\Delta u - u_i u_j u_{ij}.$$

This is called the *minimal surface operator*. Since $|\nabla d| = 1$, then $d_i d_{ij} = 0$, and hence

$$M(d) = 2\Delta d \le 0.$$

In other words, d is a supersolution of the minimal surface operator. This fact plays an important role in the derivation of the boundary gradient estimates for the mean curvature equation in the next section.

3.2. Global Estimates

Let Ω be a domain in $\mathbb{R}^n$. The mean curvature equation has the form

$$\Delta u - \frac{u_i u_j}{1 + |\nabla u|^2} u_{ij} = H\sqrt{1 + |\nabla u|^2} \quad \text{in } \Omega,$$

where H is a given function in Ω. This means that the hypersurface given by $x_{n+1} = u(x)$ in $\mathbb{R}^{n+1}$ has its mean curvature given by H. In this section, we derive estimates of solutions and their first-order derivatives.

We first derive a sup-norm estimate for solutions, due to Bakelman [**6**].

Theorem 3.2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $H \in C(\Omega)$ satisfy*

$$\|H\|_{L^n(\Omega)} < n \left(\int_{\mathbb{R}^n} (1 + |p|^2)^{-\frac{n+2}{2}} dp \right)^{\frac{1}{n}}.$$

Suppose that u is a $C(\bar{\Omega}) \cap C^2(\Omega)$-solution of

$$\Delta u - \frac{u_i u_j}{1 + |\nabla u|^2} u_{ij} = H\sqrt{1 + |\nabla u|^2} \quad \textit{in } \Omega.$$

Then,

$$\sup_\Omega |u| \le \max_{\partial\Omega} |u| + C\,\mathrm{diam}(\Omega),$$

where C is a positive constant depending only on n and $\|H\|_{L^n(\Omega)}$.

Proof. We set, for any $x \in \Omega$ and any $p \in \mathbb{R}^n$,

$$a_{ij}(p) = (1 + |p|^2)\delta_{ij} - p_i p_j$$

and

$$f(x, p) = H(x)(1 + |p|^2)^{\frac{3}{2}}.$$

Then, we write the mean curvature equation as

$$a_{ij}(\nabla u)u_{ij} = f(x, \nabla u) \quad \text{in } \Omega.$$

We first consider the special case $H \equiv 0$, in which the equation has the form

$$a_{ij}(\nabla u)u_{ij} = 0.$$

The maximum principle implies

$$\sup_{\Omega} |u| \le \max_{\partial\Omega} |u|.$$

This is the desired result.

Next, we discuss the general case. A simple calculation yields

$$D \equiv \det\big(a_{ij}(p)\big) = (1 + |p|^2)^{n-1}$$

and

$$D^* \equiv \sqrt[n]{D} = (1 + |p|^2)^{\frac{n-1}{n}}.$$

We then have

$$\frac{|f(x,p)|}{nD^*} = \frac{|H(x)|(1+|p|^2)^{\frac{3}{2}}}{n(1+|p|^2)^{\frac{n-1}{n}}} = \frac{1}{n}|H(x)|(1+|p|^2)^{\frac{n+2}{2n}}.$$

Now we set

$$h(p) = (1 + |p|^2)^{-\frac{n+2}{2n}}.$$

Then,

$$\int_{\mathbb{R}^n} h^n(p)dp = \int_{\mathbb{R}^n} (1 + |p|^2)^{-\frac{n+2}{2}} dp < \infty.$$

Suppose Γ^+ is the upper contact set of u, and set

$$M = \sup_{\Omega} u - \max_{\partial\Omega} u^+$$

and $d = \operatorname{diam}(\Omega)$. Note that $\nabla^2 u$ is nonpositive in Γ^+. Then, $-a_{ij}u_{ij} \ge 0$ in Γ^+, and hence $-f(x, \nabla u) \ge 0$ in Γ^+. Therefore,

$$\frac{f^-(x,\nabla u)}{nD^*} = \frac{|H(x)|}{nh(\nabla u)} \quad \text{in } \Gamma^+ \cap \Omega^+.$$

By applying Lemma 1.2.4 to h^n, we obtain

$$\begin{aligned}\int_{B_{M/d}} h^n dp &\le \int_{\Gamma^+\cap\Omega^+} h^n(\nabla u)\left(\frac{f^-}{nD^*}\right)^n dx \\ &= \int_{\Gamma^+\cap\Omega^+} \frac{|H(x)|^n}{n^n} dx \le \int_{\Omega} \frac{|H(x)|^n}{n^n} dx < \int_{\mathbb{R}^n} h^n dp.\end{aligned}$$

Therefore, $M/d \le C$ for some positive constant C, depending only on n and $\|H\|_{L^n(\Omega)}$. This implies

$$\sup_{\Omega} u \le \max_{\partial\Omega} u^+ + Cd.$$

A similar argument yields the lower bound of u. We then have the desired result. □

We now derive a boundary gradient estimate, due to Serrin [**134**].

Theorem 3.2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a C^2-boundary and $H \in C^1(\bar{\Omega})$ satisfy, for any $y \in \partial\Omega$,*

$$|H(y)| \leq H_{\partial\Omega}(y),$$

where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ corresponding to the inner unit normal vector to $\partial\Omega$. Suppose that u is a $C^1(\bar{\Omega}) \cap C^2(\Omega)$-solution of

$$\begin{aligned} \Delta u - \frac{u_i u_j}{1+|\nabla u|^2} u_{ij} &= H\sqrt{1+|\nabla u|^2} \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega, \end{aligned}$$

for some $\varphi \in C^2(\bar{\Omega})$. Then,

$$\max_{\partial\Omega} |\nabla u| \leq \max_{\partial\Omega} |\nabla \varphi| + \exp\left\{ C\big(1 + |\nabla\varphi|^3_{C^1(\bar{\Omega})}\big)\big(1 + |H|_{C^1(\bar{\Omega})}\big) \sup_{\Omega} |u| \right\},$$

where C is a positive constant depending only on n and Ω.

Proof. We set, for any $v \in C^2(\Omega)$,

$$Q(v) = (1+|\nabla v|^2)\Delta v - v_i v_j v_{ij} - H(1+|\nabla v|^2)^{\frac{3}{2}}.$$

Hence, $Q(u) = 0$ in Ω.

Let d be the distance function to $\partial\Omega$. Then d is C^2 in $\Omega_\mu = \{x \in \Omega : d(x) < \mu\}$, for some positive constant μ. Let ψ be a strictly increasing function defined in $[0, \mu)$, with

$$\psi(0) = 0. \tag{1}$$

Set

$$w = \psi(d) \quad \text{in } \Omega_\mu.$$

We now calculate $Q(\varphi + w)$. First,

$$w_i = \psi' d_i, \quad w_{ij} = \psi'' d_i d_j + \psi' d_{ij}.$$

Then, $|\nabla w| = \psi'$ and

$$\begin{aligned} Q(\varphi + w) = (1 &+ |\nabla\varphi + \psi'\nabla d|^2)(\Delta\varphi + \psi'' + \psi'\Delta d) \\ &- (\varphi_i + \psi' d_i)(\varphi_j + \psi' d_j)(\varphi_{ij} + \psi'' d_i d_j + \psi' d_{ij}) \\ &- H(1 + |\nabla\varphi + \psi'\nabla d|^2)^{\frac{3}{2}}. \end{aligned}$$

We write

$$Q(\varphi + w) = I + II + III,$$

where

$$I = \psi''(1 + |\nabla\varphi + \psi'\nabla d|^2) - \psi''(\varphi_i + \psi' d_i)(\varphi_j + \psi' d_j)d_i d_j,$$

$$II = (1 + |\nabla\varphi + \psi'\nabla d|^2)(\Delta\varphi + \psi'\Delta d)$$

$$- (\varphi_i + \psi' d_i)(\varphi_j + \psi' d_j)(\varphi_{ij} + \psi' d_{ij}),$$

$$III = -H(1 + |\nabla\varphi + \psi'\nabla d|^2)^{\frac{3}{2}}.$$

We now estimate I, II, and III. First, we write I as

$$I = \psi'' \left(1 + |\nabla\varphi + \psi'\nabla d|^2 - \left[(\nabla\varphi + \psi'\nabla d)\cdot\nabla d\right]^2\right).$$

The Cauchy inequality implies

$$\left[(\nabla\varphi + \psi'\nabla d)\cdot\nabla d\right]^2 \le |\nabla\varphi + \psi'\nabla d|^2$$

since $|\nabla d| = 1$. By requiring

$$\psi'' < 0, \tag{2}$$

we obtain

$$I \le \psi''.$$

For II, we have

$$II = (1 + |\nabla\varphi|^2 + 2\psi'\nabla d\cdot\nabla\varphi + \psi'^2)(\Delta\varphi + \psi'\Delta d)$$

$$- (\varphi_i\varphi_j + 2\psi'\varphi_i d_j + \psi'^2 d_i d_j)(\varphi_{ij} + \psi' d_{ij}).$$

Since $d_i d_{ij} = 0$, then

$$II = (1 + \psi'^2)\psi'\Delta d + (\Delta\varphi + 2\nabla d\cdot\nabla\varphi\Delta d - d_i d_j\varphi_{ij})\psi'^2$$

$$+ (|\nabla\varphi|^2\Delta d - \varphi_i\varphi_j d_{ij} + 2\nabla d\cdot\nabla\varphi\Delta\varphi - 2d_i\varphi_j\varphi_{ij})\psi'$$

$$+ (\Delta\varphi + |\nabla\varphi|^2\Delta\varphi - \varphi_i\varphi_j\varphi_{ij}).$$

In the following, we rewrite

$$II + III = \widetilde{II} + \widetilde{III},$$

where

$$\widetilde{II} = (\Delta\varphi + 2\nabla d\cdot\nabla\varphi\Delta d - d_i d_j\varphi_{ij})\psi'^2$$

$$+ (|\nabla\varphi|^2\Delta d - \varphi_i\varphi_j d_{ij} + 2\nabla d\cdot\nabla\varphi\Delta\varphi - 2d_i\varphi_j\varphi_{ij})\psi'$$

$$+ (\Delta\varphi + |\nabla\varphi|^2\Delta\varphi - \varphi_i\varphi_j\varphi_{ij})$$

and

$$\widetilde{III} = (1 + \psi'^2)\psi'\Delta d - H(1 + |\nabla\varphi + \psi'\nabla d|^2)^{\frac{3}{2}}.$$

It is easy to check that $\widetilde{II}$ satisfies

$$\widetilde{II} \le c_0(|\nabla\varphi| + |\nabla^2\varphi|)\psi'^2 + c_1|\nabla\varphi|(|\nabla\varphi| + |\nabla^2\varphi|)\psi' + c_2(1 + |\nabla\varphi|^2)|\nabla^2\varphi|,$$

where c_0, c_1, and c_2 are positive constants depending only on the C^2-norm of d in Ω_μ. Next, for any $x \in \Omega_\mu$, let $y \in \partial\Omega$ be the unique point such that $d(x) = |x - y|$. Lemma 3.1.9 implies

$$\Delta d(x) \le -H_{\partial\Omega}(y).$$

By the assumption on H, we have

$$\Delta d(x) \le H(y). \tag{3}$$

Then,

$$\Delta d(x) \le (H(y) - H(x)) + H(x) \le d|\nabla H|_{L^\infty} + H(x),$$

and hence

$$\widetilde{III} \le (1 + \psi'^2)d\psi'|\nabla H|_{L^\infty} + H\psi'(1 + \psi'^2) - H(1 + |\nabla\varphi + \psi'\nabla d|^2)^{\frac{3}{2}}.$$

For the last two terms, we note that

$$\begin{aligned}
&\psi'(1 + \psi'^2) - (1 + |\nabla\varphi + \psi'\nabla d|^2)^{\frac{3}{2}} \\
&\quad = (1 + \psi'^2)\left(\psi' - \sqrt{1 + |\nabla\varphi + \psi'\nabla d|^2}\right) \\
&\qquad + \left((1 + \psi'^2) - (1 + |\nabla\varphi + \psi'\nabla d|^2)\right)\sqrt{1 + |\nabla\varphi + \psi'\nabla d|^2} \\
&\quad = -\frac{1 + |\nabla\varphi|^2 + 2\psi'\nabla d \cdot \nabla\varphi}{\psi' + \sqrt{1 + |\nabla\varphi + \psi'\nabla d|^2}} \cdot (1 + \psi'^2) \\
&\qquad - (|\nabla\varphi|^2 + 2\psi'\nabla d \cdot \nabla\varphi)\sqrt{1 + |\nabla\varphi + \psi'\nabla d|^2}.
\end{aligned}$$

It is then straightforward to check that

$$|\psi'(1 + \psi'^2) - (1 + |\nabla\varphi + \psi'\nabla d|^2)^{\frac{3}{2}}| \le (1 + |\nabla\varphi|^3)(c_0\psi'^2 + c_1\psi' + c_2),$$

by adjusting c_0, c_1, and c_2. Therefore,

$$\widetilde{III} \le (1 + \psi'^2)d\psi'|\nabla H|_{L^\infty} + |H|_{L^\infty}(1 + |\nabla\varphi|^3)(c_0\psi'^2 + c_1\psi' + c_2).$$

Next, we require

$$\psi' \ge 1 \quad \text{and} \quad d\psi' \le 1. \tag{4}$$

By combining estimates of I, $\widetilde{II}$, and $\widetilde{III}$, we obtain

$$Q(\varphi + w) \le \psi'' + L\psi'^2,$$

where

$$L = C\left\{1 + |\nabla H|_{L^\infty} + (1 + |\nabla\varphi|^3_{L^\infty})|H|_{L^\infty} + (1 + |\nabla\varphi|^2_{L^\infty})|\nabla\varphi|_{C^1}\right\}$$

and C is a positive constant depending only on the C^2-norm of d. All norms in L are evaluated in Ω_μ. We point out that an extra 1 (the first term in the parenthesis) is inserted in the expression of L for a later purpose. In addition, we require

$$\psi'' + L\psi'^2 \le 0. \tag{5}$$

By examining the arguments above, we also have

$$Q(\varphi - w) \ge -(\psi'' + L\psi'^2).$$

In getting this, we employ instead of (3)

$$\Delta d(x) \le -H(y).$$

Now, we collect requirements (1), (2), (4), and (5) on ψ. In summary, we need to find a constant $d_0 \in (0, \mu]$ and a function ψ such that

$$\begin{aligned}
&\psi'' + L\psi'^2 \le 0 \quad \text{on } (0, d_0),\\
&\psi'' < 0,\ \psi' > 0,\ d\psi' \le 1 \quad \text{on } (0, d_0),\\
&\psi(0) = 0,\ \psi(d_0) \ge M,\ \psi'(d_0) \ge 1,
\end{aligned}$$

where $M = \sup_\Omega |u - \varphi|$. To this end, we first solve

$$\psi'' + L\psi'^2 = 0.$$

With $\psi'(0) = A/L$ for some positive constant A, we have

$$\psi'(d) = \frac{A}{L(1 + Ad)}.$$

Next, with $\psi(0) = 0$, we obtain

$$\psi(d) = \frac{1}{L} \log(1 + Ad).$$

Obviously, $\psi' > 0$, $\psi'' < 0$, and $d\psi' \le 1$ if $L \ge 1$. Note that $\psi(d_0) = M$ is equivalent to $\log(1 + Ad_0) = LM$, or

$$1 + Ad_0 = \exp\{LM\}.$$

Then, $\psi'(d_0) = 1$ is reduced to $A = L(1 + Ad_0)$, or

$$A = L \exp\{LM\}.$$

We take A as above and then

$$d_0 = \frac{\exp\{LM\} - 1}{L \exp\{LM\}}.$$

Hence,

$$d_0 \le \frac{1}{L} \le \frac{1}{C} \le \mu,$$

by taking C sufficiently large. Such a ψ satisfies all the requirements we imposed.

Now we set

$$\Omega_0 = \{x \in \Omega : d(x) < d_0\}.$$

We have constructed a function w in $\bar{\Omega}_0$ such that

$$\begin{aligned} Q(\varphi - w) \geq 0 \geq Q(\varphi + w) \quad &\text{in } \Omega_0, \\ w = 0 \quad &\text{on } \partial\Omega_0 \cap \partial\Omega, \\ w \geq \sup_{\Omega} |u - \varphi| \quad &\text{on } \partial\Omega_0 \cap \Omega. \end{aligned}$$

Therefore,

$$\begin{aligned} Q(\varphi - w) \geq Q(u) \geq Q(\varphi + w) \quad &\text{in } \Omega_0, \\ \varphi - w \leq u \leq \varphi + w \quad &\text{on } \partial\Omega_0. \end{aligned}$$

By the maximum principle, we obtain

$$\varphi - w \leq u \leq \varphi + w \quad \text{in } \Omega_0.$$

Since $\varphi - w = u = \varphi + w$ on $\partial\Omega$, we can take normal derivatives on $\partial\Omega$ and get

$$\frac{\partial \varphi}{\partial \nu} - \frac{\partial w}{\partial \nu} \leq \frac{\partial u}{\partial \nu} \leq \frac{\partial \varphi}{\partial \nu} + \frac{\partial w}{\partial \nu} \quad \text{on } \partial\Omega,$$

where ν is the inner unit normal vector to $\partial\Omega$. Hence,

$$\left|\frac{\partial u}{\partial \nu}\right| \leq \left|\frac{\partial \varphi}{\partial \nu}\right| + \frac{\partial w}{\partial \nu} \quad \text{on } \partial\Omega.$$

Note that

$$\left.\frac{\partial w}{\partial \nu}\right|_{\partial\Omega} = \psi'(0) = \frac{A}{L} = \exp\{LM\}.$$

We have the desired estimate by the definition of L. □

If boundary values are only continuous, we can estimate the modulus of continuity of solutions near the boundary by a similar method.

Theorem 3.2.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a C^2-boundary and $H \in C^1(\bar{\Omega})$ satisfy, for any $y \in \partial\Omega$,*

$$|H(y)| \leq H_{\partial\Omega}(y),$$

where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ with respect to the inner unit normal vector to $\partial\Omega$. Suppose that u is a $C(\bar{\Omega}) \cap C^2(\Omega)$-solution of

$$\begin{aligned} \Delta u - \frac{u_i u_j}{1 + |\nabla u|^2} u_{ij} = H\sqrt{1 + |\nabla u|^2} \quad &\text{in } \Omega, \\ u = \varphi \quad &\text{on } \partial\Omega, \end{aligned}$$

for some $\varphi \in C(\partial\Omega)$. Then, for any $x \in \Omega$ and any $x_0 \in \partial\Omega$,

$$|u(x) - u(x_0)| \leq \omega(|x - x_0|),$$

where ω is a nondecreasing function on $(0, \mathrm{diam}(\Omega))$, with $\lim_{r\to 0}\omega(r) = 0$, depending only on Ω, $\sup_\Omega |u|$, $\max_{\partial\Omega}|\varphi|$, $|H|_{C^1(\bar\Omega)}$, and the modulus of continuity of φ on $\partial\Omega$.

Proof. We fix an $x_0 \in \partial\Omega$. For any constant $\varepsilon > 0$, we take a constant $\delta > 0$ such that, for any $x \in \partial\Omega \cap B_\delta(x_0)$,

$$|\varphi(x) - \varphi(x_0)| < \varepsilon.$$

We note that δ can be chosen independent of x_0 by the uniform continuity of φ on $\partial\Omega$. We define

$$\varphi^\pm(x) = \varphi(x_0) \pm \left(\varepsilon + \frac{2}{\delta^2}\max_{\partial\Omega}|\varphi||x - x_0|^2\right).$$

Then, $\varphi^\pm \in C^2(\bar\Omega)$ and

$$\varphi^- \le \varphi \le \varphi^+ \quad \text{on } \partial\Omega.$$

Set

$$Q(v) = (1 + |\nabla v|^2)\Delta v - v_i v_j v_{ij} - H\sqrt{1 + |\nabla v|^2}.$$

As in the proof of Theorem 3.2.2, we can construct a function $w \in C(\bar\Omega_0) \cap C^2(\Omega_0)$ such that

$$Q(\varphi^- - w) \ge Q(u) \ge Q(\varphi^+ + w) \quad \text{in } \Omega_0,$$
$$\varphi^- - w \le u \le \varphi^+ + w \quad \text{on } \partial\Omega_0,$$

where $\Omega_0 = \{x \in \Omega : d(x) < d_0\}$ for some constant $d_0 > 0$. By the maximum principle, we obtain

$$\varphi^- - w \le u \le \varphi^+ + w \quad \text{in } \Omega_0.$$

Hence, for any $x \in \Omega_0$,

$$|u(x) - \varphi(x_0)| \le \varepsilon + w(x) + \frac{2}{\delta^2}\max_{\partial\Omega}|\varphi||x - x_0|^2.$$

Therefore, there exists a positive constant $\delta' \le \min\{d_0, \delta\}$ such that, for any $x \in \Omega \cap B_{\delta'}(x_0)$,

$$|u(x) - \varphi(x_0)| \le 2\varepsilon.$$

We note that δ' can be chosen independent of x_0. This implies the desired result. □

We now derive a global gradient estimate.

Theorem 3.2.4. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a C^1-boundary and $H \in C^1(\bar\Omega)$. Suppose that u is a $C^1(\bar\Omega) \cap C^3(\Omega)$-solution of*

$$\Delta u - \frac{u_i u_j}{1 + |\nabla u|^2}u_{ij} = H\sqrt{1 + |\nabla u|^2} \quad \text{in } \Omega.$$

Then,

$$\sup_{\Omega}|\nabla u| \le \Big(\max_{\partial\Omega}|\nabla u| + \sup_{\Omega}|H| + 2\Big) \cdot \exp\Big\{c_0\Big(\sup_{\Omega}|\nabla H|^{\frac{1}{2}} + 1\Big)\operatorname*{osc}_{\Omega} u\Big\},$$

where c_0 *is a universal positive constant.*

Proof. We set, for any $x \in \Omega$ and any $p \in \mathbb{R}^n$,

$$a_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2}$$

and

$$f(x,p) = H(x)\sqrt{1+|p|^2}.$$

Then, we write the mean curvature equation as

$$a_{ij}(\nabla u)u_{ij} = f(x, \nabla u) \quad \text{in } \Omega. \tag{1}$$

Without loss of generality, we assume $u \ge 0$; otherwise, we consider $u - \inf_\Omega u$ instead.

Let γ and ψ be two positive functions defined in $[0,\infty)$, with positive derivatives. These functions will be determined. Set

$$v = \gamma(u)\psi(|\nabla u|^2) \quad \text{in } \Omega.$$

We assume v attains its maximum at $x_0 \in \bar{\Omega}$. Then,

$$\gamma(u)\psi(|\nabla u|^2) \le \gamma\big(u(x_0)\big)\psi\big(|\nabla u(x_0)|^2\big) \quad \text{in } \Omega. \tag{2}$$

We will estimate $\nabla u(x_0)$.

If $x_0 \in \partial\Omega$, then

$$|\nabla u(x_0)| \le \max_{\partial\Omega}|\nabla u|. \tag{3}$$

Next, we consider the case $x_0 \in \Omega$. Then,

$$(\log v)_i(x_0) = 0, \quad \big((\log v)_{ij}(x_0)\big) \le 0.$$

Now, we evaluate $a_{ij}(\log v)_{ij}$ at x_0.

A simple differentiation yields

$$(\log v)_i = \frac{2\psi'}{\psi}u_k u_{ki} + \frac{\gamma'}{\gamma}u_i$$

and

$$(\log v)_{ij} = \frac{2\psi'}{\psi}(u_k u_{kij} + u_{ki}u_{kj}) + 4\Big(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\Big)u_k u_l u_{ki} u_{lj} + \frac{\gamma'}{\gamma}u_{ij} + \Big(\frac{\gamma''}{\gamma} - \frac{\gamma'^2}{\gamma^2}\Big)u_i u_j.$$

Hence,

$$
\begin{aligned}
a_{ij}(\log v)_{ij} = & \frac{2\psi'}{\psi}(u_k a_{ij} u_{kij} + a_{ij} u_{ki} u_{kj}) \\
& + 4\left(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\right) a_{ij} u_k u_l u_{ki} u_{lj} \\
& + \frac{\gamma'}{\gamma} a_{ij} u_{ij} + \left(\frac{\gamma''}{\gamma} - \frac{\gamma'^2}{\gamma^2}\right) a_{ij} u_i u_j.
\end{aligned}
$$

To eliminate the third derivatives of u in the expression above, we differentiate (1) and get

$$a_{ij} u_{kij} + a_{ij,p_l} u_{kl} u_{ij} = \partial_k f,$$

where

$$a_{ij,p_l} = -\frac{\delta_{li} p_j + \delta_{lj} p_i}{1 + |p|^2} + \frac{2 p_i p_j p_l}{(1 + |p|^2)^2}.$$

A simple substitution yields

$$a_{ij}(\log v)_{ij} = I + II + \left(\frac{\gamma''}{\gamma} - \frac{\gamma'^2}{\gamma^2}\right) a_{ij} u_i u_j, \tag{4}$$

where

$$I = \frac{2\psi'}{\psi}(-u_k a_{ij,p_l} u_{kl} u_{ij} + a_{ij} u_{ki} u_{kj}) + 4\left(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\right) a_{ij} u_k u_l u_{ki} u_{lj}$$

and

$$II = \frac{2\psi'}{\psi} u_k \partial_k f + \frac{\gamma'}{\gamma} f.$$

In the following, all calculations are made at x_0. Our goal is to eliminate second derivatives of u in I and II.

To simplify our calculation, we first choose an appropriate coordinate such that

$$\nabla u(x_0) = (u_1(x_0), 0, \dots, 0),$$

with $u_1(x_0) > 0$. In particular, for $i = 2, \dots, n$,

$$u_i(x_0) = 0.$$

Hence,

$$I = \frac{2\psi'}{\psi}(-u_1 a_{ij,p_l} u_{1l} u_{ij} + a_{ij} u_{ki} u_{kj}) + 4\left(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\right) a_{ij} u_1^2 u_{1i} u_{1j}.$$

Moreover,

$$\begin{aligned}
&a_{11}=\frac{1}{1+u_1^2},\quad a_{ii}=1\ (i\neq 1),\quad a_{ij}=0\ (i\neq j),\\
&a_{11,p_1}=-\frac{2u_1}{(1+u_1^2)^2},\quad a_{11,p_l}=0\ (l\neq 1),\\
&a_{1j,p_j}=-\frac{u_1}{1+u_1^2}\ (j\neq 1),\quad a_{1j,p_l}=0\ (j\neq 1,l\neq j),\\
&a_{ij,p_l}=0\ (i\neq 1,j\neq 1).
\end{aligned}$$

In fact, we can write

$$a_{11,p_1}=-\frac{2u_1}{1+u_1^2}a_{11}$$

and, for $i=2,\ldots,n$,

$$a_{1i,p_i}=-\frac{u_1}{1+u_1^2}a_{ii}.$$

Then,

$$\begin{aligned}
-u_1a_{ij,p_l}u_{1l}u_{ij}&=-u_1a_{11,p_1}u_{11}^2-2\sum_{j=2}^n u_1a_{1j,p_j}u_{1j}^2\\
&=\frac{2u_1^2}{1+u_1^2}a_{11}u_{11}^2+\sum_{j=2}^n\frac{2u_1^2}{1+u_1^2}a_{1j}u_{1j}^2=\frac{2u_1^2}{1+u_1^2}\sum_{i=1}^n a_{ii}u_{1i}^2.
\end{aligned}$$

Hence,

$$I=\frac{2\psi'}{\psi}\left(\frac{2u_1^2}{1+u_1^2}\sum_{i=1}^n a_{ii}u_{1i}^2+\sum_{i,k=1}^n a_{ii}u_{ki}^2\right)+4\left(\frac{\psi''}{\psi}-\frac{\psi'^2}{\psi^2}\right)\sum_{i=1}^n a_{ii}u_1^2u_{1i}^2.$$

By expanding the summation in terms of i and k and using the expressions of a_{ii}, we obtain

$$\begin{aligned}
I=&\left\{\frac{2\psi'}{\psi}\frac{1+3u_1^2}{1+u_1^2}+4\left(\frac{\psi''}{\psi}-\frac{\psi'^2}{\psi^2}\right)u_1^2\right\}\frac{u_{11}^2}{1+u_1^2}\\
&+\left\{\frac{2\psi'}{\psi}\frac{2+3u_1^2}{1+u_1^2}+4\left(\frac{\psi''}{\psi}-\frac{\psi'^2}{\psi^2}\right)u_1^2\right\}\sum_{i=2}^n u_{1i}^2+\frac{2\psi'}{\psi}\sum_{i,k=2}^n u_{ik}^2.
\end{aligned}\tag{5}$$

Concerning II, we have

$$II=\frac{2\psi'}{\psi}u_1\partial_1 f+\frac{\gamma'}{\gamma}f.$$

The expression of f implies

$$\begin{aligned}
\partial_1 f&=H_1\sqrt{1+u_1^2}+\frac{H}{\sqrt{1+u_1^2}}u_lu_{1l}\\
&=H_1\sqrt{1+u_1^2}+\frac{H}{\sqrt{1+u_1^2}}u_1u_{11}.
\end{aligned}$$

Then,

$$\text{(6)} \qquad II = \frac{H}{\sqrt{1+u_1^2}}\left(\frac{\gamma'}{\gamma}(1+u_1^2) + \frac{2\psi'}{\psi}u_1^2 u_{11}\right) + \frac{2\psi'}{\psi}u_1 H_1\sqrt{1+u_1^2}.$$

Next, we eliminate second derivatives of u from I and II.

The expression of $(\log v)_i$ and the condition $v_i(x_0) = 0$ imply

$$\frac{2\psi'}{\psi}u_1 u_{1i} = -\frac{\gamma'}{\gamma}u_i.$$

Hence,

$$\text{(7)} \qquad \begin{aligned} u_{11} &= -\frac{\gamma'\psi}{2\gamma\psi'}, \\ u_{1i} &= 0 \quad \text{for } i = 2, \ldots, n. \end{aligned}$$

By substituting (7) in (5) and (6), we have

$$I = \left\{\frac{2\psi'}{\psi}\frac{1+3u_1^2}{(1+u_1^2)^2} + \left(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\right)\frac{4u_1^2}{1+u_1^2}\right\}\frac{\psi^2}{\psi'^2}\frac{\gamma'^2}{4\gamma^2} + \frac{2\psi'}{\psi}\sum_{i,k=2}^{n} u_{ik}^2$$

and

$$II = \frac{\gamma'}{\gamma}\frac{H}{\sqrt{1+u_1^2}} + \frac{2\psi'}{\psi}u_1 H_1\sqrt{1+u_1^2}.$$

By substituting the expressions above in (4), we obtain, at x_0,

$$\text{(8)} \qquad \begin{aligned} a_{ij}(\log v)_{ij} \geq &\left\{\frac{2\psi'}{\psi}\frac{1+3u_1^2}{(1+u_1^2)^2} + \left(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\right)\frac{4u_1^2}{1+u_1^2}\right\}\frac{\psi^2}{\psi'^2}\frac{\gamma'^2}{4\gamma^2} \\ &+ \frac{\gamma'}{\gamma}\frac{H}{\sqrt{1+u_1^2}} + \frac{2\psi'}{\psi}u_1 H_1\sqrt{1+u_1^2} \\ &+ \left(\frac{\gamma''}{\gamma} - \frac{\gamma'^2}{\gamma^2}\right)\frac{u_1^2}{1+u_1^2}. \end{aligned}$$

The right-hand side depends only on u, u_1, H, and H_1 at x_0. We now choose γ and ψ. Set, for some α to be determined,

$$\text{(9)} \qquad \gamma(t) = e^{\alpha t}, \quad \psi(t) = t.$$

Then,

$$\gamma'(t) = \alpha e^{\alpha t}, \quad \gamma''(t) = \alpha^2 e^{\alpha t},$$

and

$$\psi'(t) = 1, \quad \psi''(t) = 0.$$

Hence,

$$\frac{\gamma'}{\gamma}(u) = \alpha, \quad \frac{\gamma''}{\gamma}(u) = \alpha^2,$$

and

$$\frac{\psi'}{\psi}(|\nabla u|^2) = \frac{1}{u_1^2}, \quad \frac{\psi''}{\psi}(|\nabla u|^2) = 0.$$

By substitutions in (8) and straightforward calculations, we have

$$a_{ij}(\log v)_{ij} \geq \frac{\alpha^2 u_1^2(u_1^2-1)}{2(u_1^2+1)^2} + \frac{2H_1}{u_1}\sqrt{1+u_1^2} + \frac{\alpha H}{\sqrt{1+u_1^2}}.$$

Since $a_{ij}(\log v)_{ij}(x_0) \leq 0$, we obtain

$$\frac{\alpha^2 u_1^2(u_1^2-1)}{2(u_1^2+1)^2} + \frac{2H_1}{u_1}\sqrt{1+u_1^2} + \frac{\alpha H}{\sqrt{1+u_1^2}} \leq 0,$$

or

$$\sqrt{1+u_1^2}\left(\frac{\alpha^2 u_1^2(u_1^2-1)}{2(u_1^2+1)^2} + \frac{2H_1}{u_1}\sqrt{1+u_1^2}\right) \leq -\alpha H.$$

If $u_1^2 \geq 3$, then

$$\frac{u_1^2(u_1^2-1)}{2(u_1^2+1)^2} = \frac{1}{2}\left(1 - \frac{1}{1+u_1^2}\right)\left(1 - \frac{2}{1+u_1^2}\right) \geq \frac{3}{16},$$

and

$$\frac{1+u_1^2}{u_1^2} = 1 + \frac{1}{u_1^2} \leq \frac{4}{3}.$$

This implies

$$u_1\left(\frac{3}{16}\alpha^2 - \frac{4}{\sqrt{3}}|\nabla H|\right) \leq \alpha|H|.$$

By taking

$$\alpha = c_0 + c_1 \sup_\Omega |\nabla H|^{\frac{1}{2}},$$

for some large universal constants c_0 and c_1, we have

$$\frac{3}{16}\alpha^2 - \frac{4}{\sqrt{3}}|\nabla H| \geq \alpha \geq 1.$$

In fact, it suffices to take $c_0 = 6$ and $c_1 = 4$. Hence, if $u_1(x_0) \geq \sqrt{3}$, then

$$u_1(x_0) \leq \sup_\Omega |H|.$$

Therefore,

$$|\nabla u(x_0)| \leq \sup_\Omega |H| + 2. \tag{10}$$

By combining (3) and (10), we obtain

$$|\nabla u(x_0)| \leq \max_{\partial\Omega} |\nabla u| + \sup_\Omega |H| + 2.$$

With the choice of γ and ψ in (9), (2) becomes

$$|\nabla u| \leq |\nabla u(x_0)| \exp\left\{\frac{1}{2}\alpha \sup_\Omega u\right\} \quad \text{in } \Omega.$$

We then have the desired result. □

We now make a comment on the proof. Besides γ and the known quantities such that H and H_1, the expression in the right-hand side of (8) is a function of u_1 since $\psi = \psi(u_1^2)$. Our goal is to choose γ and ψ appropriately so that (8) yields an upper bound of u_1. For this goal, the choice of γ and ψ in (9) is by no means unique. In the next section, we will modify the arguments above and derive interior gradient estimates by choosing different γ and ψ.

3.3. Interior Gradient Estimates

In this section, we discuss the interior gradient estimates of solutions of the mean curvature equation. As we pointed out, the mean curvature equation is uniformly elliptic only after the gradient estimate is established. In general, gradient estimates are difficult to prove for nonuniformly elliptic quasilinear equations. The structure of equations plays an important role.

Theorem 3.3.1. *Suppose that u is an $L^\infty(B_R) \cap C^3(B_R)$-solution of*

$$\Delta u - \frac{u_i u_j}{1+|\nabla u|^2} u_{ij} = H\sqrt{1+|\nabla u|^2} \quad \text{in } B_R,$$

for some $H \in C^1(B_R)$ with $|H|_{C^1(B_R)} < \infty$. Then,

$$|\nabla u(0)| \le \exp\left\{C\left(1+\frac{\omega^2}{R^2}\right)+C\left(\omega+\frac{\omega^2}{R}\right)|H|_{L^\infty(B_R)}+C\omega^2|\nabla H|_{L^\infty(B_R)}\right\},$$

where $\omega = \sup_{B_R} u - \inf_{B_R} u$ and C is a positive constant depending only on n.

Proof. We set, for any $x \in B_R$ and any $p \in \mathbb{R}^n$,

$$a_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2}$$

and

$$f(x,p) = H(x)\sqrt{1+|p|^2}.$$

Then, we write the mean curvature equation as

$$a_{ij}(\nabla u)u_{ij} = f(x, \nabla u) \quad \text{in } B_R.$$

Without loss of generality, we assume $u \ge 0$; otherwise, we consider $u - \inf_{B_R} u$ instead.

Let γ and ψ be two positive functions defined in $[0,\infty)$, with positive derivatives. These functions will be determined. Let η be a nonnegative function in B_R with $\eta = 0$ on ∂B_R. Set

$$v = \eta\gamma(u)\psi(|\nabla u|^2) \quad \text{in } B_R.$$

We assume v attains its maximum at $x_0 \in B_R$. Then,

$$\eta\gamma(u)\psi(|\nabla u|^2) \le \eta(x_0)\gamma\big(u(x_0)\big)\psi\big(|\nabla u(x_0)|^2\big) \quad \text{in } B_R.$$

In the following, we assume $\eta(x_0) \neq 0$. Then,

$$(\log v)_i(x_0) = 0, \quad \big((\log v)_{ij}(x_0)\big) \le 0.$$

Now we evaluate $a_{ij}(\log v)_{ij}$ at x_0.

A simple differentiation yields

$$(\log v)_i = \frac{2\psi'}{\psi}u_k u_{ki} + \frac{\gamma'}{\gamma}u_i + \frac{\eta_i}{\eta}$$

and

$$\begin{aligned}(\log v)_{ij} = &\frac{2\psi'}{\psi}(u_k u_{kij} + u_{ki}u_{kj}) + 4\bigg(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\bigg)u_k u_l u_{ki} u_{lj}\\ &+ \frac{\gamma'}{\gamma}u_{ij} + \bigg(\frac{\gamma''}{\gamma} - \frac{\gamma'^2}{\gamma^2}\bigg)u_i u_j + \bigg(\frac{\eta_{ij}}{\eta} - \frac{\eta_i\eta_j}{\eta^2}\bigg).\end{aligned}$$

Comparing with similar expressions in the proof of Theorem 3.2.4, we note that the only extra terms here are those involving η. We now proceed as in the proof of Theorem 3.2.4 and calculate at x_0.

We first choose an appropriate coordinate such that

$$\nabla u(x_0) = (u_1(x_0), 0, \dots, 0),$$

with $u_1(x_0) > 0$. In particular, for $i = 2, \dots, n$,

$$u_i(x_0) = 0.$$

As in the proof of Theorem 3.2.4, we obtain

$$(1) \qquad a_{ij}(\log v)_{ij} = I + II + \bigg(\frac{\gamma''}{\gamma} - \frac{\gamma'^2}{\gamma^2}\bigg)a_{ij}u_i u_j + a_{ij}\bigg(\frac{\eta_{ij}}{\eta} - \frac{\eta_i\eta_j}{\eta^2}\bigg),$$

where

$$(2) \qquad \begin{aligned} I = &\bigg\{\frac{2\psi'}{\psi}\frac{1+3u_1^2}{1+u_1^2} + 4\bigg(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\bigg)u_1^2\bigg\}\frac{u_{11}^2}{1+u_1^2}\\ &+ \bigg\{\frac{2\psi'}{\psi}\frac{2+3u_1^2}{1+u_1^2} + 4\bigg(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\bigg)u_1^2\bigg\}\sum_{i=2}^n u_{1i}^2 + \frac{2\psi'}{\psi}\sum_{i,k=2}^n u_{ik}^2\end{aligned}$$

and

$$(3) \qquad II = \frac{H}{\sqrt{1+u_1^2}}\bigg(\frac{\gamma'}{\gamma}(1+u_1^2) + \frac{2\psi'}{\psi}u_1^2 u_{11}\bigg) + \frac{2\psi'}{\psi}u_1 H_1\sqrt{1+u_1^2}.$$

Next, we eliminate second derivatives from I and II.

The expression of $(\log v)_i$ and the condition $v_i(x_0) = 0$ imply

$$\frac{2\psi'}{\psi}u_1 u_{1i} = -\bigg(\frac{\gamma'}{\gamma}u_i + \frac{\eta_i}{\eta}\bigg).$$

Hence,

$$
\begin{aligned}
u_{11} &= -\frac{\psi}{2\psi'}\left(\frac{\gamma'}{\gamma} + \frac{\eta_1}{u_1\eta}\right),\\
u_{1i} &= -\frac{\psi}{2\psi'}\frac{\eta_i}{u_1\eta} \quad \text{for } i = 2, \ldots, n.
\end{aligned} \tag{4}
$$

By substituting (4) in (2) and (3), we have

$$
\begin{aligned}
I = &\left\{\frac{2\psi'}{\psi}\frac{1+3u_1^2}{1+u_1^2} + 4\left(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\right)u_1^2\right\}\frac{\psi^2}{4(1+u_1^2)\psi'^2}\left(\frac{\gamma'}{\gamma} + \frac{\eta_1}{u_1\eta}\right)^2\\
&+ \left\{\frac{2\psi'}{\psi}\frac{2+3u_1^2}{1+u_1^2} + 4\left(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\right)u_1^2\right\}\frac{\psi^2}{4u_1^2\psi'^2}\sum_{i=2}^{n}\frac{\eta_i^2}{\eta^2} + \frac{2\psi'}{\psi}\sum_{i,k=2}^{n}u_{ik}^2
\end{aligned} \tag{5}
$$

and

$$
II = \frac{H}{\sqrt{1+u_1^2}}\left(\frac{\gamma'}{\gamma} - \frac{\eta_1}{\eta}u_1\right) + \frac{2\psi'}{\psi}u_1 H_1\sqrt{1+u_1^2}. \tag{6}
$$

We now choose γ and ψ. Set $M = \sup_{B_R} u$ and

$$
\gamma(t) = 1 + \frac{t}{M}, \quad \psi(t) = \log t.
$$

Then,

$$
\gamma'(t) = \frac{1}{M}, \quad \gamma''(t) = 0,
$$

and

$$
\psi'(t) = \frac{1}{t}, \quad \psi''(t) = -\frac{1}{t^2}.
$$

Hence, $1 \le \gamma(u) \le 2$ since $0 \le u \le M$, and

$$
\frac{\psi'}{\psi}(|\nabla u|^2) = \frac{1}{2u_1^2 \log u_1}, \quad \frac{\psi''}{\psi}(|\nabla u|^2) = -\frac{1}{2u_1^4 \log u_1}.
$$

A straightforward calculation yields

$$
\begin{aligned}
&\frac{2\psi'}{\psi}\frac{1+3u_1^2}{1+u_1^2} + 4\left(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\right)u_1^2\\
&\qquad = \frac{1}{u_1^2 \log u_1}\left(1 - \frac{2}{1+u_1^2} - \frac{1}{\log u_1}\right).
\end{aligned}
$$

Hence, if $u_1 \ge e^3$, then

$$
\frac{2\psi'}{\psi}\frac{1+3u_1^2}{1+u_1^2} + 4\left(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\right)u_1^2 \ge \frac{1}{2u_1^2 \log u_1} > 0.
$$

In particular, the summation corresponding to i from 2 to n in (5) has a positive coefficient. By keeping only the first term in the right-hand side of (5) and using $1+u_1^2 \le 2u_1^2$, we obtain

$$I \ge \frac{1}{2u_1^2 \log u_1} \frac{\psi^2}{8u_1^2 \psi'^2} \left(\frac{\gamma'}{\gamma} + \frac{\eta_1}{u_1 \eta} \right)^2 \ge \frac{1}{4} \log u_1 \left(\frac{\gamma'}{\gamma} + \frac{\eta_1}{u_1 \eta} \right)^2.$$

Concerning II in (6), we have

$$\begin{aligned} II &= \frac{H}{\sqrt{1+u_1^2}} \left(\frac{\gamma'}{\gamma} - \frac{\eta_1}{\eta} u_1 \right) + H_1 \frac{\sqrt{1+u_1^2}}{u_1 \log u_1} \\ &\ge -|H| \left(\frac{1}{M} + \frac{|\eta_1|}{\eta} \right) - |H_1| \end{aligned}$$

if $u_1 > e^2$. We also note, with the expression of a_{11},

$$\left| \left(\frac{\gamma''}{\gamma} - \frac{\gamma'^2}{\gamma^2} \right) a_{ij} u_i u_j \right| = \frac{\gamma'^2}{\gamma^2} a_{11} u_1^2 \le \frac{1}{M^2}.$$

By substitutions in (1) and the condition $a_{ij}(\log v)_{ij}(x_0) \le 0$, we obtain

$$\frac{1}{4} \log u_1 \left(\frac{\gamma'}{\gamma} + \frac{\eta_1}{u_1 \eta} \right)^2 \le a_{ij} \left(\frac{\eta_i \eta_j}{\eta^2} - \frac{\eta_{ij}}{\eta} \right) + \frac{1}{M^2} + |H| \left(\frac{1}{M} + \frac{|\eta_1|}{\eta} \right) + |\nabla H|.$$

We now choose η such that

$$|\nabla^2 \eta| + \frac{|\nabla \eta|^2}{\eta} \le \frac{c}{R^2} \quad \text{in } B_R,$$

for some positive constant c. For example, we can take

$$\eta(x) = \left(1 - \frac{|x|^2}{R^2} \right)^2.$$

Then,

$$\log u_1 \left(\frac{\gamma'}{\gamma} + \frac{\eta_1}{u_1 \eta} \right)^2 \le C \left\{ \frac{1}{R^2 \eta} + \frac{1}{M^2} + \left(\frac{1}{M} + \frac{1}{R\sqrt{\eta}} \right) |H| + |\nabla H| \right\},$$

where C is a positive constant depending only on n. For simplicity, we set

$$\widetilde{M} = \frac{M^2}{R^2} + 1 + M \left(1 + \frac{M}{R} \right) |H|_{L^\infty} + M^2 |\nabla H|_{L^\infty}.$$

Then, if $u_1 > e^3$,

$$\log u_1 \left(\frac{\gamma'}{\gamma} + \frac{\eta_1}{u_1 \eta} \right)^2 \le \frac{C \widetilde{M}}{M^2 \eta}.$$

We first consider the case

$$\left| \frac{\eta_1}{u_1 \eta} \right| \le \frac{\gamma'}{2\gamma}.$$

Then,

$$\frac{\gamma'^2}{4\gamma^2}\log u_1 \le \frac{C\widetilde{M}}{M^2\eta}.$$

Since $1 \le \gamma(u) \le 2$ in B_R, we get

$$\eta\gamma\log u_1 \le C\widetilde{M}.$$

Now we consider the case

$$\frac{\gamma'}{2\gamma} \le \left|\frac{\eta_1}{u_1\eta}\right|.$$

Then,

$$\eta\gamma u_1 \le C\frac{M}{R},$$

and hence, if $u_1 > e$,

$$\eta\gamma\log u_1 \le C\frac{M}{R}.$$

In conclusion, we have, at x_0,

$$\eta\gamma\log u_1 \le C\left\{1 + \frac{M^2}{R^2} + \frac{M}{R} + M\left(1 + \frac{M}{R}\right)|H|_{L^\infty} + M^2|\nabla H|_{L^\infty}\right\}.$$

This implies the desired result since $u_1 = |\nabla u|$ at x_0. □

Here, we follow Wang [**162**] for the proof of Theorem 3.3.1. Refer to [**95**] for an alternative proof. We will prove an improved interior gradient estimate in Theorem 4.3.2.

As an application, we prove a Liouville type result for the minimal surface equation.

Theorem 3.3.2. *Suppose that u is a $C^3(\mathbb{R}^n)$-solution of*

$$\Delta u - \frac{u_i u_j}{1+|\nabla u|^2}u_{ij} = 0 \quad \text{in } \mathbb{R}^n.$$

If u is bounded, then u is constant. If u has at most a linear growth, then u is an affine function.

Proof. By setting, for any $p \in \mathbb{R}^n$,

$$a_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2},$$

we write the minimal surface equation as

$$a_{ij}(\nabla u)u_{ij} = 0 \quad \text{in } \mathbb{R}^n. \tag{1}$$

It is easy to check that

$$\sup_{p\in\mathbb{R}^n}\big(|p||\nabla a_{ij}(p)|\big) \le 2.$$

Since u has at most a linear growth, then for any $R > 0$,

$$\sup_{B_R} |u| \le C_0(1+R),$$

for some positive constant C_0. By Theorem 3.3.1, we have

$$|\nabla u| \le K_0 \quad \text{in } \mathbb{R}^n,$$

where K_0 is a positive constant depending only on n and C_0 above. Therefore, the equation (1) is now uniformly elliptic. If u is bounded, Theorem 2.1.3 implies that u is constant. If u has at most a linear growth, Theorem 2.4.2 implies that u is an affine function. □

3.4. Dirichlet Problems

In this section, we employ the method of continuity to solve the Dirichlet problem for the mean curvature equations. We proceed similarly as in Section 2.6.

Theorem 3.4.1. *Let $\alpha \in (0,1)$ be a constant and $\Omega \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$ with a $C^{3,\alpha}$-boundary. Suppose that $H \in C^{1,\alpha}(\bar{\Omega})$ satisfies*

$$\|H\|_{L^n(\Omega)} < n \left(\int_{\mathbb{R}^n} (1+|p|^2)^{-\frac{n+2}{2}} dp \right)^{\frac{1}{n}}$$

and, for any $y \in \partial\Omega$,

$$|H(y)| \le H_{\partial\Omega}(y),$$

where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ corresponding to the inner unit normal vector to $\partial\Omega$. Then for any $\varphi \in C^{3,\alpha}(\bar{\Omega})$, there exists a unique solution $u \in C^{3,\alpha}(\bar{\Omega})$ of

$$\begin{aligned} \Delta u - \frac{u_i u_j}{1+|\nabla u|^2} u_{ij} &= H\sqrt{1+|\nabla u|^2} \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Proof. We need only prove the existence of a $C^{2,\alpha}(\bar{\Omega})$-solution of (1). We note that the uniqueness follows from Corollary 2.1.2 and the higher regularity $u \in C^{3,\alpha}(\bar{\Omega})$ from Proposition 2.1.5. With $u \in C^3(\bar{\Omega})$, we can apply Theorems 3.2.1, 3.2.2, and 3.2.4 and obtain

$$|u|_{C^1(\bar{\Omega})} \le C_1,$$

where C_1 is a positive constant depending only on n, $\|H\|_{L^n(\Omega)}$, $|H|_{C^1(\bar{\Omega})}$, $|\varphi|_{C^2(\bar{\Omega})}$, and Ω. Then, by applying Theorems 2.5.1 and 2.5.2, we conclude

$$|u|_{C^{2,\alpha}(\bar{\Omega})} \le C_*, \tag{2}$$

where C_* is a positive constant depending only on n, α, $\|H\|_{L^n(\Omega)}$, $|H|_{C^1(\bar{\Omega})}$, $|\varphi|_{C^{2,\alpha}(\bar{\Omega})}$, and Ω.

In the following, we set, for any $p \in \mathbb{R}^n$,

$$a_{ij}(p) = \frac{1}{\sqrt{1+|p|^2}}\left(\delta_{ij} - \frac{p_i p_j}{1+|p|^2}\right),$$

and write (1) as

$$\begin{aligned} a_{ij}(\nabla u)u_{ij} &= H \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega. \end{aligned}$$

For each $t \in [0,1]$, consider a family of Dirichlet problems

$$\begin{aligned} a_{ij}(\nabla u)u_{ij} &= tH \quad \text{in } \Omega, \\ u &= t\varphi \quad \text{on } \partial\Omega. \end{aligned} \tag{3}$$

For $t = 0$, (3) corresponds to a Dirichlet problem whose unique solution is given by $u = 0$. Any $C^{2,\alpha}$-solutions of (3) are $C^3(\bar{\Omega})$ and satisfy the estimate (2).

Letting $v = u - t\varphi$, (3) is equivalent to

$$\begin{aligned} a_{ij}(\nabla v + t\nabla\varphi)v_{ij} + ta_{ij}(\nabla v + t\nabla\varphi)\varphi_{ij} &= tH \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4}$$

Moreover, any solutions of (4) satisfy the estimate (2).

Next, we set

$$\mathcal{X} = \{v \in C^{2,\alpha}(\bar{\Omega}) : \; v = 0 \text{ on } \partial\Omega\}$$

and

$$Q(v,t) = a_{ij}(\nabla v + t\nabla\varphi)v_{ij} + ta_{ij}(\nabla v + t\nabla\varphi)\varphi_{ij} - tH.$$

Solving (4) is equivalent to finding a function $v \in \mathcal{X}$ such that $Q(v,t) = 0$ in Ω.

Set

$$I = \{t \in [0,1] : \text{there exists a } v \in \mathcal{X} \text{ such that } Q(v,t) = 0\}.$$

We note that $0 \in I$. To prove $1 \in I$, we need to prove that I is both open and closed in $[0,1]$. For the openness, we note that $Q : \mathcal{X} \times [0,1] \to C^\alpha(\bar{\Omega})$ is of class C^1 and its Frèchet derivative with respect to $v \in \mathcal{X}$ is given by

$$Q_v(v,t)w = a_{ij}(\nabla v + t\nabla\varphi)w_{ij} + (v_{ij} + t\varphi_{ij})a_{ij,p_k}w_k.$$

For any fixed $v \in \mathcal{X}$ and $t \in [0,1]$, $Q_v(v,t)$ is a uniformly elliptic linear operator with C^α-coefficients. By the Schauder theory, $Q_v(v,t)$ is an invertible operator from $\mathcal{X}$ to $C^\alpha(\bar{\Omega})$. Suppose $t_0 \in I$; i.e., $Q(v^{t_0}, t_0) = 0$ for some $v^{t_0} \in \mathcal{X}$. By the implicit function theorem, for any t close to t_0, there is a unique $v^t \in \mathcal{X}$, close to v^{t_0} in the $C^{2,\alpha}$-norm, satisfying $Q(v^t,t) = 0$. Hence $t \in I$ for all such t, and therefore I is open. For the closedness, we note by (2) that any solution $v \in \mathcal{X}$ of $Q(v,t) = 0$ in Ω satisfies a uniform $C^{2,\alpha}(\bar{\Omega})$-estimate, independent of t; i.e.,

$$|v^t|_{C^{2,\alpha}(\bar{\Omega})} \le C_*, \text{ independent of } t.$$

Hence, the closedness of I follows from the compactness in $C^2(\bar{\Omega})$ of bounded sets in $C^{2,\alpha}(\bar{\Omega})$, a consequence of the Arzela-Ascoli theorem. Therefore, I is the whole unit interval. The function v^1 is then our desired solution of (4) corresponding to $t = 1$. □

In Theorem 3.4.1, we proved the existence of $C^{3,\alpha}$-solutions in $\bar{\Omega}$ if boundary values are $C^{3,\alpha}$ on $\partial\Omega$. By an approximation, we can conclude the existence of solutions under weaker conditions. In particular, when boundary values are only continuous, we have solutions which are continuous up to the boundary.

Theorem 3.4.2. *Let $\alpha \in (0,1)$ be a constant and $\Omega \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$ with a $C^{3,\alpha}$-boundary. Suppose that $H \in C^{1,\alpha}(\bar{\Omega})$ satisfies*

$$\|H\|_{L^n(\Omega)} < n\left(\int_{\mathbb{R}^n} (1+|p|^2)^{-\frac{n+2}{2}}dp\right)^{\frac{1}{n}}$$

and, for any $y \in \partial\Omega$,

$$|H(y)| \le H_{\partial\Omega}(y),$$

where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ corresponding to the inner unit normal vector to $\partial\Omega$. Then for any $\varphi \in C(\partial\Omega)$, there exists a unique solution $u \in C(\bar{\Omega}) \cap C^{2,\alpha}(\Omega)$ of

$$\begin{aligned} \Delta u - \frac{u_i u_j}{1+|\nabla u|^2}u_{ij} &= H\sqrt{1+|\nabla u|^2} \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Proof. Without loss of generality, we assume $\varphi \in C(\bar{\Omega})$. Let $\{\varphi_m\}$ be a sequence of $C^{3,\alpha}(\bar{\Omega})$-functions such that

$$\varphi_m \to \varphi \quad \text{uniformly in } \bar{\Omega} \text{ as } m \to \infty.$$

Now, we consider the Dirichlet problem

$$\begin{aligned} \Delta u - \frac{u_i u_j}{1+|\nabla u|^2}u_{ij} &= H\sqrt{1+|\nabla u|^2} \quad \text{in } \Omega, \\ u &= \varphi_m \quad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

By Theorem 3.4.1, there exists a unique solution $u_m \in C^{3,\alpha}(\bar{\Omega})$ of (2).

We fix an m and proceed to derive an estimate of u_m. By Theorem 3.2.1, we have

$$\sup_{\Omega} |u_m| \le \max_{\partial\Omega} |u_m| + C\text{diam}(\Omega), \tag{3}$$

where C is a positive constant depending only on n and $\|H\|_{L^n(\Omega)}$. It is obvious that the right-hand side of (3) can be made independent of m by the uniform convergence of φ_m. Moreover, by Theorem 3.2.3, we have, for any $x \in \Omega$ and any $x_0 \in \partial\Omega$,

$$|u_m(x) - u_m(x_0)| \leq \omega(|x - x_0|), \tag{4}$$

where ω is a nondecreasing function on $(0, \text{diam}(\Omega))$, with $\lim_{r\to 0} \omega(r) = 0$, depending only on n, Ω, $|H|_{C^1(\bar{\Omega})}$, $|u_m|_{L^\infty(\Omega)}$, $|\varphi_m|_{L^\infty(\Omega)}$, and the modulus of continuity of φ_m on $\partial\Omega$. Again, ω can be made independent of m.

Next, we consider an arbitrary subdomain $\Omega' \Subset \Omega$. By using Theorem 3.3.1, Theorem 2.4.1, and Theorem 2.4.3 successively, we obtain

$$|u_m|_{C^{2,\alpha}(\Omega')} \leq C_*, \tag{5}$$

where C_* is a positive constant depending only on n, $|u_m|_{L^\infty(\Omega)}$, $|H|_{C^1(\bar{\Omega})}$, Ω', and Ω. Again, C_* can be made independent of m. By (3), (4), (5), and the Arzela-Ascoli theorem, there exists a subsequence of $\{u_m\}$ convergent uniformly in $\bar{\Omega}$ and also in C^2 in any subdomain $\Omega' \Subset \Omega$ to a function $u \in C(\bar{\Omega}) \cap C^{2,\alpha}(\Omega)$. Hence by passing the limit, u satisfies (1). □

In the following, we present a result of the nonsolvability of the Dirichlet problem for the mean curvature equation if the condition in Theorem 3.4.1 concerning the boundary mean curvature is not satisfied. We note that such a condition is used only in the establishment of the boundary gradient estimates, as in Theorem 3.2.2. In this sense, the nonsolvability is due to the lack of the boundary gradient estimates.

We first prove a comparison principle. For a fixed $H \in C(\bar{\Omega})$, set, for any $v \in C^2(\Omega)$,

$$Q(v) = (1 + |\nabla v|^2)\Delta v - v_i v_j v_{ij} - H(1 + |\nabla v|^2)^{\frac{3}{2}}.$$

Lemma 3.4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and Σ a relatively open C^1-portion of $\partial\Omega$. Suppose that $u \in C(\bar{\Omega}) \cap C^2(\Omega \cup \Sigma)$ and $v \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy $Q(u) \geq Q(v)$ in Ω, $u \leq v$ on $\partial\Omega \setminus \Sigma$, $\partial v/\partial\nu = -\infty$ on Σ, where ν is the inner unit normal vector to $\partial\Omega$. Then, $u \leq v$ in Ω.*

Proof. Set $w = u - v$. Then, w satisfies $Lw \geq 0$ in Ω, for a linear elliptic operator L of the form $L = a_{ij}\partial_{ij} + b_i\partial_i$. By the maximum principle, we have

$$\sup_{\Omega} w \leq \sup_{\partial\Omega} w.$$

Since $\partial w/\partial\nu = \infty$ on Σ, the function w cannot achieve a maximum on Σ. Hence, $w \leq 0$ in Ω. □

With Lemma 3.4.3, we establish an estimate at boundary points where the condition on the boundary mean curvature in Theorem 3.4.1 is not satisfied.

Lemma 3.4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^2-boundary. Suppose that $H \in C(\bar{\Omega})$ is nonnegative in $\bar{\Omega}$ and, for some $y_0 \in \partial\Omega$,*

$$H_{\partial\Omega}(y_0) < H(y_0),$$

where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ corresponding to the inner unit normal vector to $\partial\Omega$. Assume that $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ is a solution of

$$\Delta u - \frac{u_i u_j}{1+|\nabla u|^2} u_{ij} = H\sqrt{1+|\nabla u|^2} \quad \text{in } \Omega.$$

Then, for any $a < a_0$,

$$u(y_0) \le \max_{\partial\Omega\setminus B_a(y_0)} u + \eta(a),$$

where $a_0 \in (0, \mathrm{diam}(\Omega))$ is a constant and η is a positive continuous function on $(0, \mathrm{diam}(\Omega))$, depending only on n, $\mathrm{diam}(\Omega)$, $H(y_0) - H_{\partial\Omega}(y_0)$, the modulus of continuity of H at y_0, and the C^2-property of $\partial\Omega$ at y_0, with

$$\lim_{a\to 0} \eta(a) = 0.$$

Proof. Without loss of generality, we assume $y_0 = 0$. Set $D = \mathrm{diam}(\Omega)$ and, for any $v \in C^2(\Omega)$,

$$Q(v) = (1+|\nabla v|^2)\Delta v - v_i v_j v_{ij} - H(1+|\nabla v|^2)^{\frac{3}{2}}.$$

Take a constant $a \in (0, D)$ to be fixed. We now divide the proof into two steps.

Step 1. Set $r = |x|$ and consider

$$w(x) = c + \psi(r) \quad \text{in } \Omega \setminus B_a,$$

where c is a constant and ψ is a C^2-function in (a, D) such that

$$\psi(D) = 0, \quad \psi' \le 0, \quad \psi'(a) = -\infty. \tag{1}$$

A straightforward calculation yields

$$Q(w) = \psi'' + \frac{n-1}{r}\psi'(1+\psi'^2) - H(1+\psi'^2)^{\frac{3}{2}}.$$

By $H \ge 0$ in Ω and $\psi' \le 0$ in (a, D), we have

$$Q(w) \le \psi'' + \frac{n-1}{r}\psi'^3 \quad \text{in } \Omega \setminus B_a.$$

We now solve
$$\psi'' + \frac{n-1}{r}\psi'^3 = 0 \quad \text{in } (a, D),$$
with the condition (1). This yields
$$\psi(r) = \frac{1}{\sqrt{2(n-1)}} \int_r^D \left(\log \frac{t}{a}\right)^{-\frac{1}{2}} dt \quad \text{in } (a, D).$$
With such a ψ, we have $Q(w) \leq 0$ in $\Omega \setminus B_a$. Next, we take
$$c = \max_{\partial\Omega \setminus B_a} u.$$
Then,
$$u \leq w \text{ on } \partial\Omega \setminus B_a, \quad \frac{\partial w}{\partial \nu} = -\infty \text{ on } \Omega \cap \partial B_a,$$
where ν is the inner unit normal vector to $\partial(\Omega \setminus B_a)$. By Lemma 3.4.3, we obtain $u \leq w$ in $\Omega \setminus B_a$, and hence

$$u \leq \max_{\partial\Omega \setminus B_a} u + \psi(a) \quad \text{in } \Omega \setminus B_a. \tag{2}$$

A simple change of variables yields
$$\psi(a) = \frac{a}{\sqrt{2(n-1)}} \int_1^{\frac{D}{a}} \frac{1}{\sqrt{\log s}}\, ds.$$
We conclude easily $\psi(a) \to 0$ as $a \to 0$.

Step 2. Without loss of generality, we assume that e_n is the inner unit normal vector to $\partial\Omega$ at $y_0 = 0$ and that $\partial\Omega$ in a neighborhood of the origin is given by
$$\rho(x') = \frac{1}{2}\sum_{i=1}^{n-1} \kappa_i x_i^2 + o(|x'|^2),$$
where $\kappa_1, \ldots, \kappa_{n-1}$ are the principal curvatures of $\partial\Omega$ at 0 with respect to the inner unit normal vector to $\partial\Omega$. Consider the paraboloid $\mathcal{S}$ given by
$$\widetilde{\rho}(x') = \frac{1}{2}\sum_{i=1}^{n-1} \widetilde{\kappa}_i x_i^2,$$
for constants $\widetilde{\kappa}_i > \kappa_i$, for $i = 1, \ldots, n-1$. In the following, we will take $\widetilde{\kappa}_i$ sufficiently close to κ_i, for $i = 1, \ldots, n-1$, and then take a sufficiently small such that
$$\widetilde{\rho}(x') \geq \rho(x') \quad \text{for any } |x'| < a.$$
Take any $\varepsilon \in (0, a)$. Set $d(x) = \text{dist}(x, \mathcal{S})$ and
$$\widetilde{\Omega}_\varepsilon = \{x \in \Omega : x \in B_a,\, x_n > \widetilde{\rho}(x'),\, d(x) > \varepsilon\},$$
$$\widetilde{\Omega} = \{x \in \Omega : x \in B_a,\, x_n > \widetilde{\rho}(x')\}.$$

Then, $d(x) < 2a$ for any $x \in \widetilde{\Omega}_\varepsilon$. Consider

$$\widetilde{w}(x) = \widetilde{c} + \widetilde{\psi}(d) \quad \text{in } \widetilde{\Omega}_\varepsilon,$$

where $\widetilde{c}$ is a constant and $\widetilde{\psi}$ is a C^2-function in $(\varepsilon, 2a)$ such that

$$\widetilde{\psi}(2a) = 0, \quad \widetilde{\psi}' \le 0, \quad \widetilde{\psi}'(\varepsilon) = -\infty. \tag{3}$$

A straightforward calculation yields

$$Q(\widetilde{w}) = \widetilde{\psi}'' + \widetilde{\psi}'(1 + \widetilde{\psi}'^2)\Delta d - H(1 + \widetilde{\psi}'^2)^{\frac{3}{2}}.$$

By $H \ge 0$ in Ω and $\widetilde{\psi}' \le 0$ in $(\varepsilon, 2a)$, we have

$$Q(\widetilde{w}) \le \widetilde{\psi}'' + \widetilde{\psi}'(1 + \widetilde{\psi}'^2)(\Delta d + H).$$

Since $\mathcal{S}$ is given by a quadratic polynomial, its principal curvatures are constants. By Lemma 3.1.9, we have

$$\Delta d = -\sum_{i=1}^{n-1} \frac{\widetilde{\kappa}_i}{1 - \widetilde{\kappa}_i d} = -\sum_{i=1}^{n-1} \frac{\widetilde{\kappa}_i^2 d}{1 - \widetilde{\kappa}_i d} - \sum_{i=1}^{n-1} \widetilde{\kappa}_i,$$

and hence

$$-(\Delta d + H) = \sum_{i=1}^{n-1} \frac{\widetilde{\kappa}_i^2 d}{1 - \widetilde{\kappa}_i d} + \sum_{i=1}^{n-1} (\widetilde{\kappa}_i - \kappa_i) + (H(0) - H) - (H(0) - H_{\partial\Omega}(0)).$$

By $H_{\partial\Omega}(0) < H(0)$ and the continuity of H at $y_0 = 0$, we can choose $\widetilde{\kappa}_i > \kappa_i$ sufficiently close to κ_i, for $i = 1, \dots, n-1$, and choose a sufficiently small, such that

$$-(\Delta d + H) \le -\delta \quad \text{in } \widetilde{\Omega}_\varepsilon,$$

for some positive constant δ. In fact, we can take $\delta = (H(0) - H_{\partial\Omega}(0))/2$. We point out that δ is a fixed constant, independent of ε. By $\widetilde{\psi}' \le 0$ in $(\varepsilon, 2a)$ again, we have

$$Q(\widetilde{w}) \le \widetilde{\psi}'' + \delta\widetilde{\psi}'(1 + \widetilde{\psi}'^2) \le \widetilde{\psi}'' + \delta\widetilde{\psi}'^3 \quad \text{in } \widetilde{\Omega}_\varepsilon.$$

We now solve

$$\widetilde{\psi}'' + \delta\widetilde{\psi}'^3 = 0 \quad \text{in } (\varepsilon, 2a),$$

with the condition (3). This yields

$$\widetilde{\psi}(d) = K\left[\sqrt{2a - \varepsilon} - \sqrt{d - \varepsilon}\right] \quad \text{in } (\varepsilon, 2a),$$

for $K = \sqrt{2/\delta}$. With such a $\widetilde{\psi}$, we have $Q(\widetilde{w}) \le 0$ in $\widetilde{\Omega}_\varepsilon$. Next, we take

$$\widetilde{c} = \sup_{\Omega \cap \partial B_a} u.$$

Then,

$$u \le \widetilde{w} \text{ on } \partial\widetilde{\Omega}_\varepsilon \cap \partial B_a, \quad \frac{\partial w}{\partial \nu} = -\infty \text{ on } \partial\widetilde{\Omega}_\varepsilon \setminus \partial B_a,$$

where ν is the inner unit normal vector to $\partial\widetilde{\Omega}_\varepsilon$. By Lemma 3.4.3, we obtain $u \le \widetilde{w}$ in $\widetilde{\Omega}_\varepsilon$, and hence

$$u \le \sup_{\Omega\cap\partial B_a} u + K\sqrt{2a-\varepsilon} \quad \text{in } \widetilde{\Omega}_\varepsilon.$$

Letting $\varepsilon \to 0$, we have

$$u \le \sup_{\Omega\cap\partial B_a} u + K\sqrt{2a} \quad \text{in } \widetilde{\Omega}$$

and, in particular,

$$u(0) \le \sup_{\Omega\cap\partial B_a} u + K\sqrt{2a}. \tag{4}$$

We point out that $\widetilde{\Omega}$ may be a proper subset of $\Omega \cap B_a$.

By combining (2) and (4), we obtain

$$u(0) \le \max_{\partial\Omega\setminus B_a} + \psi(a) + K\sqrt{2a}.$$

This is the desired result. □

Lemma 3.4.4 demonstrates that the boundary values of the solutions of the mean curvature equation satisfy a constraint and hence cannot be arbitrarily prescribed. This leads to the following nonexistence result.

Theorem 3.4.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^2-boundary. Suppose that $H \in C(\bar{\Omega})$ is either nonnegative or nonpositive in $\bar{\Omega}$ and, for some $y_0 \in \partial\Omega$,*

$$H_{\partial\Omega}(y_0) < |H(y_0)|,$$

where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ corresponding to the inner unit normal vector to $\partial\Omega$. Then, for any constant $\varepsilon > 0$, there exists a $\varphi \in C(\partial\Omega)$ with $|\varphi| \le \varepsilon$ on $\partial\Omega$, such that there exist no $C(\bar{\Omega}) \cap C^2(\Omega)$-solutions of

$$\Delta u - \frac{u_i u_j}{1+|\nabla u|^2} u_{ij} = H\sqrt{1+|\nabla u|^2} \quad \text{in } \Omega,$$
$$u = \varphi \quad \text{on } \partial\Omega.$$

Proof. We first assume $H \ge 0$ in $\bar{\Omega}$. For any given constant $\varepsilon > 0$, we take $a > 0$ such that $\eta(a) < \varepsilon$, where η is the function in Lemma 3.4.4. We construct a function $\varphi \in C(\partial\Omega)$ such that $\varphi(y_0) = \varepsilon$, $0 \le \varphi \le \varepsilon$ on $\partial\Omega$, and $\varphi = 0$ on $\partial\Omega \setminus B_a(y_0)$. Then, such a φ violates the conclusion of Lemma 3.4.4 and hence cannot be the boundary value of the solutions of the mean curvature equation. If $H \le 0$ in $\bar{\Omega}$, we consider the equation satisfied by $-u$ and repeat the argument. □

To end this section, we formulate results in this section for the minimal surface equation. For the existence of solutions of the Dirichlet problem, we have the following results.

Theorem 3.4.6. *Let $\alpha \in (0,1)$ be a constant and $\Omega \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$ with a $C^{3,\alpha}$-boundary satisfying*

$$H_{\partial\Omega} \geq 0 \quad \text{on } \partial\Omega,$$

where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ corresponding to the inner unit normal vector to $\partial\Omega$. Then for any $\varphi \in C^{3,\alpha}(\bar{\Omega})$, there exists a unique solution $u \in C^{3,\alpha}(\bar{\Omega}) \cap C^\infty(\Omega)$ of

$$\Delta u - \frac{u_i u_j}{1+|\nabla u|^2} u_{ij} = 0 \quad \text{in } \Omega,$$
$$u = \varphi \quad \text{on } \partial\Omega.$$

Theorem 3.4.7. *Let $\alpha \in (0,1)$ be a constant and $\Omega \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$ with a $C^{3,\alpha}$-boundary satisfying*

$$H_{\partial\Omega} \geq 0 \quad \text{on } \partial\Omega,$$

where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ corresponding to the inner unit normal vector to $\partial\Omega$. Then for any $\varphi \in C(\partial\Omega)$, there exists a unique solution $u \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ of

$$\Delta u - \frac{u_i u_j}{1+|\nabla u|^2} u_{ij} = 0 \quad \text{in } \Omega,$$
$$u = \varphi \quad \text{on } \partial\Omega.$$

For the nonexistence of solutions of the Dirichlet problem, we have the following result.

Theorem 3.4.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^2-boundary such that for some $y_0 \in \partial\Omega$,*

$$H_{\partial\Omega}(y_0) < 0,$$

where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ corresponding to the inner unit normal vector to $\partial\Omega$. Then, for any constant $\varepsilon > 0$, there exists a $\varphi \in C(\partial\Omega)$ with $|\varphi| \leq \varepsilon$ on $\partial\Omega$, such that there exist no $C(\bar{\Omega}) \cap C^2(\Omega)$-solutions of

$$\Delta u - \frac{u_i u_j}{1+|\nabla u|^2} u_{ij} = 0 \quad \text{in } \Omega,$$
$$u = \varphi \quad \text{on } \partial\Omega.$$