Preface

Dynamical systems began with Poincaré's qualitative theory of differential equations near the end of the nineteenth century. Differentiable dynamical systems is the part that concerns structural stability, hyperbolicity, genericity, density, etc., developed since the 1960s. There is a survey article by Smale (1967, also 1980) on differentiable dynamical systems that is very enlightening.

This book is a graduate text in differentiable dynamical systems. It focuses on structural stability and the role of hyperbolicity, a topic that is central to the field. For the sake of simplicity we take the discrete setting, namely iterates of diffeomorphisms. It is well known that a periodic orbit is structurally stable if and only if it obeys so-called hyperbolicity (no eigenvalue of norm one). The case of a finite number of periodic orbits is treated similarly. However, it was long doubted whether a system of infinitely many periodic orbits could be structurally stable. A historic discovery from the early 1960s is the Smale horseshoe map. This is a structurally stable system that contains infinitely many periodic orbits. Together with the celebrated Anosov automorphism and the solenoid attractor found soon afterwards, they exhibit an amazing feature of the world: structural stability can be compatible to a high level of complexity (sometimes called "chaos"). The analytic condition that ensures such a chaotic set to be structurally stable is reasonably still called "hyperbolicity". This led to a new theory of *hyperbolic sets*, for which a hyperbolic periodic orbit serves as a special case. The Ω-stability theorem of Smale is then an early global result based on this theory. This book will develop along this line and consists of six chapters.

Chapter 1 introduces some basic concepts of dynamical systems such as limit set, nonwandering set, minimal set, transitive set, etc., as well as topological conjugacy and structural stability. As a comprehensive illustration of these concepts we give a short account of the classical theory of circle homeomorphisms. We also include in this chapter Conley's fundamental theorem of dynamical systems.

Chapter 2 is devoted to hyperbolicity, the main analytic concept of this book, for the case of a single fixed point. We study the stability of a hyperbolic fixed point, the persistence of hyperbolicity under perturbations, the Hartman-Grobman theorem, the stable manifold theorem, etc. While these subjects are classical, our treatment has kept in mind the need of the general case of a hyperbolic set of Chapter 4.

Chapter 3 presents three historic models, the Smale horseshoe, the Anosov toral automorphism, and the solenoid attractor, which led to the modern theory of differentiable dynamical systems.

Chapter 4 generalizes the concept of hyperbolicity from a fixed point to a general invariant set. We study the persistence of hyperbolicity, the stable manifold theorem, structural stability, the shadowing property, etc., for a hyperbolic set. This chapter constitutes the analytic foundation for the theory of structural stability and is technically the most difficult part of the book. Whenever some difficulty appears, the best way is to go back to check the corresponding part of Chapter 2, which is much more transparent.

Chapter 5 presents one direction of the theory: hyperbolicity implies (essentially) structural stability. A highlight is the Ω-stability theorem of Smale. We also include some equivalent descriptions by Newhouse and Franke-Selgrade.

Chapter 6 presents the theory of quasi-hyperbolicity and linear transversality. It provides alternate angles for looking at hyperbolicity. We also include a section that gives a glimpse of the stability conjectures.

There is some tough material in this book, notably the stable manifold theorem (Theorem 4.16) and the structural stability theorem (Theorem 4.21) for hyperbolic sets. In fact these two big theorems have caused an obstacle to teaching and learning the subject. A key objective of this book is to find some way to remove this obstacle. The strategy we found is to choose a suitable setting for proofs for a hyperbolic set so that they match the proofs for a hyperbolic fixed point, like a copy. A good example showing that this is possible is the proof of Lemma 4.5, which is almost a duplicate of the proof of Lemma 2.9. The reader might like to take one minute to compare the two proofs just formally. Indeed, with this strategy we have been able to give straightforward proofs for these two big theorems. The author believes that, unlike in art or in literature, in mathematics one does not have to avoid analogous presentations; rather, one uncovers the identical nature behind different exteriors. After all, the theory of hyperbolic sets is difficult to digest. We wish to make the presentation as plain as possible so the reader can quickly get through the obstacle to reach the heart of differentiable dynamical systems.

The prerequisites for reading this book are essentially undergraduate analysis, linear algebra, and basic topology. The framework of differentiable manifolds, in particular some basic concepts such as tangent bundles and tangent maps, submanifolds, Riemannian metric, the exponential map, are also important to the development of the text. When some less standard facts of basic topology are needed, definitions are inserted as a refresher. The book includes a number of figures to help the exposition.

There have been a number of books on dynamical systems in the literature. See the references at the end of this book, where the two big books Katok-Hasselblatt (1995) and Robinson (1995) give a panorama for modern dynamical systems. I have benefited from these nice books in many ways. I am especially thankful for the book by Zhang (1986), which I used as a text in the early years when I taught the course at Peking University.

This book is short. It might miss many original references for the results involved in the text. I am thankful for the book by Robinson (1995), which I found to be very helpful.

I have taught most parts of the first five chapters of the book as a one-semester course many times at Peking University. I also taught the course at Taiwan University (spring 2003) and Providence University of Taiwan (fall 2004). Part of the material was taught as short courses at Nankai University (1989), Sun Yat-Sen University (1990), Fuzhou University (1995), Nanjing University (1998), Center of Theoretical Sciences of Taiwan (1999), University of Science and Technology of China (2001), Jilin University (2007), Chiao Tung University of Taiwan (2011), and Chungnam University of Korea (2014). I wish to take this opportunity to thank the participants of all these courses. I particularly thank Shaobo Gan for collaborations over the years, countless discussions about the course, including a thorough proofreading of the entire manuscript. I also thank Xiao Wen and Dawei Yang for creating elegant figures and many of the exercises for this book and Xiao Wen for discussions on the C^k part of the stable manifold theorem. Finally, I wish to thank Wenxiang Sun and the people in our seminar for stimulating talks and discussions over the years.

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