Step 3. $f : U \to E$ C^1, $0 \in E$ a hyperbolic fixed point.

Write $f = A + \phi_f$, extend ϕ_f to $\overline{\phi}_f$, then apply Step 2 to $A + \overline{\phi}_f$ to get a C^1 map $\sigma : E^s \to E^u$ such that $W^s(0, A + \overline{\phi}_f) = \mathrm{gr}(\sigma)$. Then we cut off the graph to get $W^s_r(0, f)$.

Step 4. $f : U \to E$ C^k, $k \geq 2$, $0 \in E$ a hyperbolic fixed point.

Consider the "tangent map" $F(x, v) = (fx,\ Df(x)v)$. It is C^{k-1} if f is C^k. We verify that $W^s_r((0,0), F)$ is (the r-neighborhood of the 0-set of) the tangent bundle of $W^s_r(0, f)$. Then the proof goes by induction.

4.4. Stable manifolds of hyperbolic sets

Now we come back to our compact manifold M. Taking closure if necessary, we assume that all hyperbolic sets are compact.

In this section we study stable manifolds for hyperbolic sets, a fundamental topic to differentiable dynamical systems. As Smale (1980) commented, the global stable manifolds "lie close to the heart of the subject". The basic references for the stable manifolds theory are Hirsch-Pugh (1970) and Hirsch-Pugh-Shub (1977).

As indicated in the preface, our strategy is to choose a suitable setting so that the proof for hyperbolic sets will match that for hyperbolic fixed points. This is the setting of "fiber-stable manifolds" defined below. First we make some technical preparations.

Recall that a basic approach in Chapter 2 is to take the difference

$$\phi = f - Df(0)$$

and try to get $\mathrm{Lip}\,\phi$ small. However, on a manifold, $f - Tf$ does not make sense in general. What we do is to use the exponential map to "lift" f locally to the tangent bundle TM so that a subtraction will make sense.

Let $x \in M$. Recall that the *exponential map*

$$exp_x : T_xM \to M$$

at $x \in M$ is defined to be

$$exp_x(v) = \sigma_v(1),$$

where $\sigma_v(t)$ is the geodesic determined by the Riemannian metric of M, through x at $t = 0$ with velocity v. (Here M is compact; hence exp_x can be defined on the whole T_xM.) See Figure 4.3.

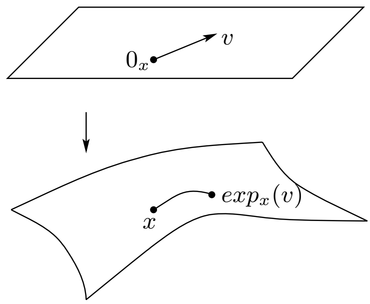

Figure 4.3. The exponential map.

Recall our notation

$$T_xM(\rho) = \{v \in T_xM \mid |v| \leq \rho\}.$$

The following theorem can be found in a general textbook on differential geometry, for instance Boothby (1975). We omit the proof.

Theorem 4.9. (1) *$exp_x(0_x) = x$, where 0_x is the origin of T_xM.*

(2) *$D(exp_x)(0_x) : T_xM \to T_xM$ is the identity.*

(3) *There is $\rho > 0$ such that, for any $x \in M$, $exp_x : T_xM(\rho) \to M$ is a C^∞ embedding. Moreover, $d(x, exp_x(v)) = |v|$, $\forall\ v \in T_xM(\rho)$, where d and $|\cdot|$ are both induced by the given Riemannian metric of M.*

(4) *The map $exp : TM \to M$, $exp(v) = exp_{\pi v}(v)$, is C^∞, where $\pi : TM \to M$ is the bundle projection.*

Thus, taking x as the base, any nearby point $y \in B(x, \rho)$ determines a vector $exp_x^{-1}y \in T_xM$ of length $|exp_x^{-1}y| = d(x, y)$. In a Euclidean space it is just the vector $y - x$ from x to y.

Through the exponential map, f is locally lifted to the tangent bundle to a fiber-preserving map over f, which is not linear on fibers. Precisely, fix $0 < r_\rho < \rho$ such that, for any two points $x, y \in M$, if $d(x, y) < r_\rho$, then $d(fx, fy) < \rho$. Recall by definition

$$TM(r_\rho) = \{v \in TM \mid |v| \leq r_\rho\}.$$

Define the *self-lifting*

$$F_f : TM(r_\rho) \to TM$$

of f to be

$$F_f(v) = exp_{f(x)}^{-1}\ f\ exp_x(v), \quad x = \pi v;$$

see Figure 4.4.

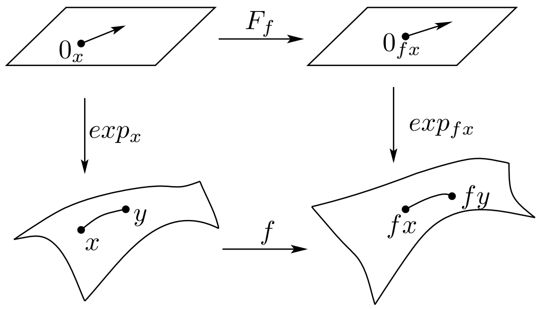

Figure 4.4. The self-lifting F_f of f.

Clearly, F_f is fiber-preserving over f. Since f is C^1, so is F_f. Briefly, thinking of points of an orbit of f as "moving origins", F_f exhibits the behavior of f near these origins. Clearly,

$$F_f(exp_x^{-1}y) = exp_{fx}^{-1}(fy).$$

Thus F_f takes the vector "from x to y" to the vector "from fx to fy". Inductively,

$$F_f^n(exp_x^{-1}y) = exp_{f^n x}^{-1}(f^n y).$$

In particular,

$$|F_f^n(exp_x^{-1}y)| = d(f^n x, f^n y).$$

In this way the distance $d(f^n x, f^n y)$ of two points on M is converted to the length $|exp_{f^n x}^{-1}(f^n y)|$ of a vector on TM. Of course, for these iterates $F_f^n(v)$ to make sense, where $v = exp_x^{-1}y$, we have to assume

$$d(f^i x, f^i y) < r_\rho, \ \forall\ 0 \le i \le n-1.$$

For a vector $v \in TM(r_\rho)$, the iterate $F_f^n(v)$ may not be well defined for all $n \ge 0$. This is as in Chapter 2, where a vector $v \in E(r)$ may not have the iterates $(A+\phi)^n(v)$ contained in $E(r)$ for all $n \ge 0$. The box $E(r)$ was like a window of size r with center $0 \in E$, where we looked at the dynamics of $A + \phi$ for $n \ge 0$. We only considered vectors v whose positive iterates $(A+\phi)^n v$ all remain in the window $E(r)$. Now for every $x \in M$ we have in the tangent plane of x a window $T_xM(r_\rho)$ of center $0_x \in T_xM$ and of size r_ρ, where we look at the dynamics of F_f for $n \ge 0$. The windows are by definition mutually disjoint. We only consider vectors v whose positive iterates $F_f^n(v)$ all remain in the union

$$TM(r_\rho) = \bigcup_{x \in M} T_xM(r_\rho)$$

of the windows. The only difference is that, when we look at the dynamics, we are moving from one window to another.

A fiber-preserving map $F : TM(r) \to TM$ over f, when restricted to every fiber, becomes a map between two Euclidean spaces:

$$F|_{T_xM(r)} : T_xM(r) \to T_{fx}M.$$

Define the *fiber-derivative* of F at $v \in TM(r)$ to be

$$D_2F(v) = D(F|_{T_xM(r)})(v) : T_xM(r) \to T_{fx}M,$$

where $x = \pi v$. In local coordinates, $TM(r)$ is represented as $B_1 \times B_2$, where B_1 and B_2 are balls of $\mathbb{R}^d$, $d = \dim M$, with B_1 representing the base part, and B_2 the fiber part. Thus D_2F is just the partial derivative of F with respect to the second variable. The *higher-order fiber-derivatives* are defined likewise to be the higher-order derivatives of the restricted map.

Lemma 4.10. *Let $f : M \to M$ be a C^1 diffeomorphism.*

(1) $F_f(0_x) = 0_{fx}, \ \ \forall\, x \in M.$

(2) $D_2(F_f)(0_x) = T_xf, \ \ \forall\, x \in M.$

(3) $D_2(F_f)$ *is continuous on* $TM(r_\rho)$.

Remark. In local coordinates, item (3) says that the partial derivative of F_f with respect to the second variable is continuous on $B_1 \times B_2$ (with respect to both variables).

Proof.

$$F_f(0_x) = exp_{fx}^{-1}\ f\ exp_x(0_x) = exp_{fx}^{-1}(fx) = 0_{fx}.$$

$$\begin{aligned} D_2(F_f)(0_x) &= D(exp_{fx}^{-1}\ f\ exp_x)(0_x) \\ &= id|_{T_{fx}M} \circ Df(x) \circ id|_{T_xM} = T_xf. \end{aligned}$$

Let $x = \pi v$. Then

$$\begin{aligned} D_2(F_f)(v) &= D(exp_{fx}^{-1}\ f\ exp_x)(v) \\ &= D(exp_{fx}^{-1})(f(exp_x v)) \circ Df(exp_x v) \circ D(exp_x)(v). \end{aligned}$$

Since f is C^1, $D_2(F_f)$ is continuous on $TM(r_\rho)$. □

Let

$$\phi_f = F_f - Tf : TM(r_\rho) \to TM.$$

Then ϕ_f is fiber-preserving for f and, by Lemma 4.10,

$$\phi_f(0_x) = 0_{fx}, \quad D_2\phi_f(0_x) = 0, \quad \forall\, x \in M.$$

Note that ϕ_f is C^0 only (since Tf is C^0 only), though F_f is C^1. However ϕ_f restricted to every fiber $T_xM(r_\rho)$ is C^1. In fact we have more: $D_2\phi_f$ is continuous on $TM(r_\rho)$. In terms of local coordinates, the former means that the partial derivative of ϕ_f with respect to the second variable is continuous with respect to the second variable, and the latter means that the partial derivative of ϕ_f with respect to the second variable is continuous with respect to both variables.

For a continuous fiber-preserving map $F : TM(r) \to TM$ that is Lipschitz on every fiber, define the *fiber-Lipschitz constant* of F to be

$$\mathrm{Lip}_2 F = \sup_{x\in M} \mathrm{Lip}(F|_{T_xM(r)}).$$

This will be the only form of Lipschitz constant below we will consider for a fiber-preserving map.

Lemma 4.11. *Let $f : M \to M$ be a diffeomorphism. Denote $\phi_g = F_g - Tg$. Then for any $\epsilon > 0$, there are a C^1 neighborhood $\mathcal{U}$ of f and a number $r > 0$ such that, for any $g \in \mathcal{U}$, $\mathrm{Lip}_2\phi_g < \epsilon$ on $TM(r)$.*

Proof. Since $D_2\phi_f$ is continuous on $TM(r_\rho)$ and since $D_2\phi_f(0_x) = 0$ and M is compact, for any $\epsilon > 0$, there is $r > 0$ such that for any $v \in TM(r)$ one has $|D_2\phi_f(v)| < \epsilon$. But $TM(r)$ is compact; hence there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ and any $v \in TM(r)$ one has $|D_2\phi_g(v)| < \epsilon$. Applying the generalized mean value theorem to fibers, we get $\mathrm{Lip}_2\phi_g < \epsilon$ on $TM(r)$. □

Now we study stable manifolds for hyperbolic sets. Since the global stable manifolds are obtained by iterating the local stable manifolds, we start with the local ones.

Recall that, in Chapter 2, the basic setting for the problem of stable manifold is $A + \phi$, where A is a hyperbolic linear isomorphism of a finite-dimensional normed vector space E and ϕ is Lipschitz with Lipschitz constant small. Thus, here we start with maps of the form

$$Tf + \phi$$

on $T_\Lambda M$, where ϕ is continuous, fiber-preserving over f, fiber-Lipschitz with $\mathrm{Lip}_2\phi$ small. It could be referred to as a *fiber-Lipschitz perturbation of a hyperbolic bundle isomorphism.*

Let

$$\phi : TM(r) \to TM,\ 0 < r \le \infty,$$

be fiber-preserving over f such that

$$\phi(0_x) = 0_{fx},\ \forall\, x \in M.$$

Let $|\cdot|$ be a norm on TM. Define, respectively, the *local fiber-stable manifold* and the *local fiber-unstable manifold* of 0_x of size r with respect to $Tf+\phi$ to be

$$W^s_r(0_x, Tf+\phi) = \{v \in T_xM \mid |(Tf+\phi)^n v| \le r \ \forall n \ge 0, \text{ and } \lim_{n\to+\infty} |(Tf+\phi)^n v| = 0\},$$

$$W^u_r(0_x, Tf+\phi) = \{v \in T_xM \mid |(Tf+\phi)^{-n} v| \le r \ \forall n \ge 0, \text{ and } \lim_{n\to+\infty} |(Tf+\phi)^{-n} v| = 0\}.$$

Here we have used the term "fiber-stable" because v is required to be in the same fiber of 0_x. This will be the only form of stable and unstable manifolds below that we will consider for a fiber-preserving map.

Our setting will be fiber-stable manifolds of a fiber-Lipschitz perturbation of Tf. Within this setting, the proof of the stable manifold theorem of a hyperbolic set will match that of a hyperbolic fixed point, like a copy.

Now we "repeat" what we did in Section 2.5. The next lemma corresponds to Lemma 2.14.

Lemma 4.12 (Characterization of W^s_r on fibers)**.** *Let $\Lambda \subset M$ be a hyperbolic set of f with splitting $T_\Lambda M = E^s \oplus E^u$ of skewness $0 < \tau < 1$ with respect to a C^0 norm $|\cdot|$ of $T_\Lambda M$ that is adapted to and of box type to $E^s \oplus E^u$. Let $r > 0$. Let $\phi : T_\Lambda M(r) \to T_\Lambda M$ be continuous, fiber-preserving over f, and fiber-Lipschitz such that*

$$\mathrm{Lip}_2\phi < 1-\tau, \quad \phi(0_x) = 0_{fx} \quad \forall x \in \Lambda.$$

Then for any $x \in \Lambda$,

$$\begin{aligned} W^s_r(0_x, Tf+\phi) &= \{v \in T_xM(r) \mid |(Tf+\phi)^n v| \le r, \ \forall n \ge 0\} \\ &= \{v \in T_xM(r) \mid (Tf+\phi)^n v \in T_{f^n x}M(r) \cap C_1(E^s(f^n x)), \ \forall n \ge 0\} \\ &= \{v \in T_xM(r) \mid |(Tf+\phi)^n v| \le (\tau + \mathrm{Lip}_2\phi)^n |v|, \ \forall n \ge 0\}. \end{aligned}$$

Likewise for W^u_r.

Proof. Here $T_\Lambda M(r)$, Tf, ϕ, and $\mathrm{Lip}_2\phi$ correspond to $E(r)$, A, ϕ, and $\mathrm{Lip}\,\phi$ of Lemma 2.14. We write the first part of the proof only.

Claim 1. *If $x \in \Lambda$ and $v, v' \in T_xM(r)$, then*

$$|(Tf+\phi)_s(v) - (Tf+\phi)_s(v')| \le (\tau + \mathrm{Lip}_2\phi)|v - v'|.$$

In fact,

$$\begin{aligned} |(Tf+\phi)_s(v) - (Tf+\phi)_s(v')| &= |(Tf)_{ss}(v_s - v'_s) + \phi_s(v) - \phi_s(v')| \\ &\le (\tau + \mathrm{Lip}_2\phi)|v - v'|. \end{aligned}$$

Claim 2. *If $v, v' \in T_xM(r)$ and $v - v' \notin C_1(E^s(x))$, then*

$$(Tf + \phi)v - (Tf + \phi)v' \notin C_1(E^s(fx)),$$

and

$$|(Tf + \phi)_u v - (Tf + \phi)_u v'| \geq (\tau^{-1} - \mathrm{Lip}_2\phi)|v - v'|.$$

In fact,

$$\begin{aligned}|(Tf + \phi)_u v - (Tf + \phi)_u v'| &= |(Tf)_{uu}(v_u - v'_u) + \phi_u(v) - \phi_u(v')| \\ &\geq \tau^{-1}|v_u - v'_u| - \mathrm{Lip}_2\phi|v - v'|.\end{aligned}$$

But $v - v' \notin C_1(E^s(x))$; hence $|v_u - v'_u| = |v - v'|$. Then

$$|(Tf + \phi)_u v - (Tf + \phi)_u v'| \geq (\tau^{-1} - \mathrm{Lip}_2\phi)|v - v'|.$$

Now $v - v' \notin C_1(E^s(x))$; hence $v - v' \neq 0$. Combined with Claim 1 we get

$$|(Tf + \phi)_u v - (Tf + \phi)_u v'| > |(Tf + \phi)_s v - (Tf + \phi)_s v'|.$$

Thus $(Tf + \phi)v - (Tf + \phi)v' \notin C_1(E^s(fx))$. This proves Claim 2.

Clearly, the proof is a duplicate of that of Lemma 2.14; hence we stop here. □

Remark. As remarked after Lemma 2.14, here $T_\Lambda M(r)$ could be the whole $T_\Lambda M$.

Now we pass to our manifold M. For any $x \in M$ and $r > 0$, define the *local stable manifold* and *local unstable manifold* of x of size r with respect to f to be, respectively,

$$\begin{aligned}&W^s_r(x, f) \\ &\quad = \{y \in M \mid d(f^n y, f^n x) \leq r \ \forall\, n \geq 0, \text{ and } \lim_{n\to+\infty} d(f^n y, f^n x) = 0\},\end{aligned}$$

$$\begin{aligned}&W^u_r(x, f) \\ &\quad = \{y \in M \mid d(f^{-n} y, f^{-n} x) \leq r, \ \forall\, n \geq 0, \text{ and } \lim_{n\to+\infty} d(f^{-n} y, f^{-n} x) = 0\}.\end{aligned}$$

Clearly, for any $x \in M$,

$$f(W^s_r(x)) \subset W^s_r(fx), \quad f(W^u_r(x)) \supset W^u_r(fx).$$

The next theorem corresponds to Theorem 2.15.

Theorem 4.13 (Characterization of W^s_r on manifold)**.** *Let $\Lambda \subset M$ be a hyperbolic set of f. There are $r > 0$, $C \geq 1$, and $0 < \lambda < 1$ such that for any $x \in \Lambda$,*

$$\begin{aligned}W^s_r(x, f) &= \{y \in M \mid d(f^n y, f^n x) \leq r, \ \forall\, n \geq 0\} \\ &= \{y \in M \mid d(f^n y, f^n x) \leq r \text{ and } d(f^n y, f^n x) \leq C\lambda^n d(y, x), \ \forall\, n \geq 0\}.\end{aligned}$$

Likewise,

$$W^u_r(x,f) = \{y \in M \mid d(f^{-n}y, f^{-n}x) \le r,\ \forall\ n \ge 0\}$$
$$= \{y \in M \mid d(f^{-n}y, f^{-n}x) \le r \text{ and } d(f^{-n}y, f^{-n}x) \le C\lambda^n d(y,x),\ \forall\ n \ge 0\}.$$

Proof. We give a proof for W^s_r only. We may assume the Riemannian norm $|\cdot|$ of M is adapted to Λ. It suffices to prove there are $r > 0$, $C \ge 1$, and $0 < \lambda < 1$ such that the second set is contained in the third.

Given $x \in \Lambda$, for $y \in M$ close to x, let

$$v = exp_x^{-1}(y).$$

Then

$$d(f^n y, f^n x) = |F^n_f(v)| = |(Tf + \phi_f)^n v|,$$

where

$$\phi_f = F_f - Tf,$$

as long as the iterates make sense. It then reduces to proving there are $r > 0$, $C \ge 1$, and $0 < \lambda < 1$ such that

$$\{v \in T_xM(r) \mid |(Tf + \phi_f)^n v| \le r,\ \forall\ n \ge 0\}$$
$$\subset \{v \in T_xM(r) \mid |(Tf + \phi_f)^n v| \le C\lambda^n |v|,\ \forall\ n \ge 0\}.$$

It suffices to prove there are $r > 0$, $C \ge 1$, and $0 < \lambda < 1$ such that this inclusion holds replacing the norm $|\cdot|$ by the box-adjusted norm $|\cdot|_\Lambda$ of $|\cdot|$ with respect to $T_\Lambda M = E^s \oplus E^u$.

Let $0 < \tau < 1$ be the skewness of Λ with respect to $|\cdot|$. Assume $|\cdot|_\Lambda$ is equivalent to $|\cdot|$ by a constant $K \ge 1$. Let $C = 1$, and fix

$$\tau < \lambda < 1.$$

By Lemma 4.11, there is $r > 0$ sufficiently small such that

$$\mathrm{Lip}_{2,|\cdot|}\phi_f \le K^{-2}(\lambda - \tau)$$

on $TM(Kr, |\cdot|)$. Hence

$$\mathrm{Lip}_{2,|\cdot|_\Lambda}\phi_f \le \lambda - \tau$$

on $T_\Lambda M(r, |\cdot|_\Lambda)$. (Changing the norm of a Euclidean space yields a multiplier of the relative constant for the length of vectors, but a multiplier of the square of the relative constant for the Lipschitz constants of the maps.) Then the conclusion follows directly from Lemma 4.12. This proves Theorem 4.13. □

Let X be a compact metric space. A homeomorphism $f : X \to X$ is called *expansive* if there is a constant $r > 0$ such that, for every pair of different points $x \ne y$ in X, there is an integer m such that $\mathrm{d}(f^m(x), f^m(y)) \ge r$. The number $r > 0$ is called an *expansive constant* of f.

The next theorem corresponds to Theorem 2.16.

Theorem 4.14 (Uniform expansivity of hyperbolic sets)**.** *Let $\Lambda \subset M$ be a hyperbolic set of f. Then $f|_\Lambda$ is expansive. In fact, there are a C^1 neighborhood $\mathcal{U}_0$ of f and two numbers $a_0 > 0$ and $r_0 > 0$ such that every compact invariant set $\Delta \subset B(\Lambda, a_0)$ of every $g \in \mathcal{U}_0$ is r_0-expansive.*

Proof. We may assume the Riemannian norm $|\cdot|$ of M is adapted to Λ. By definition, $g|_\Delta$ is r_0-expansive means that if $x, y \in \Delta$ satisfy

$$d(g^n x, g^n y) \le r_0 \quad \forall n \in \mathbb{Z},$$

then $x = y$. Let $v = exp_x^{-1}(y)$. Since

$$d(g^n x, g^n y) = |F_g^n(v)| = |(Tg + \phi)^n v|,$$

where

$$\phi = \phi_g = F_g - Tg,$$

it reduces to proving there is $r_0 > 0$ such that if a vector $v \in T_x M$ satisfies

$$|(Tg + \phi)^n v| \le r_0 \quad \forall n \in \mathbb{Z},$$

then $v = 0$.

Let $0 < \tau(\Lambda) < 1$ be the skewness of Λ of Tf with respect to $|\cdot|$. Fix

$$\tau(\Lambda) < \tau_0 < \lambda < 1.$$

Let

$$\phi' = (F_g)^{-1} - (Tg)^{-1}.$$

Hence

$$(Tg)^{-1} + \phi' = (T_g + \phi)^{-1}.$$

Note that $g \to f$ implies $g^{-1} \to f^{-1}$.

By Lemma 4.8, there are a C^1 neighborhood $\mathcal{U}_0$ of f and two numbers $a_0 > 0$ and $K \ge 1$ such that any compact invariant set $\Delta \subset B(\Lambda, a_0)$ of any $g \in \mathcal{U}_0$ is hyperbolic with skewness

$$\tau(\Delta) \le \tau_0$$

with respect to $|\cdot|$, and the box-adjusted norm $|\cdot|_\Delta$ of $|\cdot|$ with respect to the hyperbolic splitting of Δ is equivalent to $|\cdot|$ with constant K. By Lemma 4.11, there are a C^1 neighborhood of f, still denoted $\mathcal{U}_0$, and a number $r_0 > 0$ such that, for any $g \in \mathcal{U}_0$,

$$\mathrm{Lip}_{2,|\cdot|}\phi \le K^{-2}(\lambda - \tau_0), \qquad \mathrm{Lip}_{2,|\cdot|}\phi' \le K^{-2}(\lambda - \tau_0)$$

on $TM(K^2 r_0; |\cdot|)$. Hence

$$\mathrm{Lip}_{2,|\cdot|_\Delta}\phi \le \lambda - \tau_0, \qquad \mathrm{Lip}_{2,|\cdot|_\Delta}\phi' \le \lambda - \tau_0$$

on $T_\Delta M(K r_0; |\cdot|_\Delta)$.

Assume a vector $v \in T_xM$, $x \in \Delta$, satisfies

$$|(Tg + \phi)^n v| \le r_0 \quad \forall n \in \mathbb{Z}.$$

Then

$$|(Tg + \phi)^n v|_\Delta \le K r_0 \quad \forall n \in \mathbb{Z}.$$

By Lemma 4.12,

$$\begin{aligned}|v|_\Delta &= |(Tg^{-1} + \phi')(T_g + \phi)(v)|_\Delta \\ &\le (\tau + \mathrm{Lip}_2\phi')|(Tg + \phi)(v)|_\Delta \le (\tau + \mathrm{Lip}_2\phi')(\tau + \mathrm{Lip}_2\phi)|v|_\Delta \le \lambda^2 |v|_\Delta.\end{aligned}$$

Here the first "$\le$" holds because, with respect to $|\cdot|_\Delta$, $(Tg+\phi)(v)$ has length $\le Kr_0$ for all negative iterates. Likewise, the second "$\le$" holds because v has length $\le Kr_0$ for all positive iterates. Since $\lambda < 1$, we get $v = 0$. This proves Theorem 4.14. □

Remark. The strong uniformness that appears in the statement of Theorem 4.14 is actually common for results about hyperbolic sets. Nevertheless for simplicity we will not state every theorem in this uniform way but only Theorems 4.14 and 4.20, just for the use of Theorem 4.23 below.

For a fiber-preserving map $\sigma : E^u \to E^s$, define the *fiber-derivative* of σ at $v \in E^u$ to be

$$D_2\sigma(v) = D(\sigma|_{E^u(x)})(v),$$

where $x = \pi v$. Also, define the *fiber-Lipschitz constant* of σ to be

$$\mathrm{Lip}_2\sigma = \sup_{x\in\Lambda}\mathrm{Lip}(\sigma|_{E^u(x)}).$$

We explained right after Lemma 4.10 that saying that $D_2\phi$ is continuous on $T_\Lambda M$ is more than saying that ϕ restricted to every fiber T_xM is C^1. Here, likewise, saying that $D_2\sigma$ is continuous on E^u is more than saying that σ restricted to every fiber $E^u(x)$ is C^1.

Now we proceed to the main part of the stable manifolds theorem for a hyperbolic set. We first consider an ideal setting $Tf + \phi : T_\Lambda M \to T_\Lambda M$, where $\phi : T_\Lambda M \to T_\Lambda M$ is continuous and fiber-preserving over f, $\phi(0_x) = 0_{fx}$, and $\mathrm{Lip}_2\phi$ is small on the whole $T_\Lambda M$. Define the (global) *fiber-unstable manifold* of $0_x \in T_xM$ to be

$$W^u(0_x, Tf + \phi) = \left\{ v \in T_xM \mid \lim_{n\to+\infty} |(Tf + \phi)^{-n} v| = 0 \right\}.$$

We remark that here the "global" fiber-unstable manifold of 0_x is on fibers, which is not the global unstable manifold $W^u(x)$ we will eventually have on the manifold M. It is just an intermediate object through which we will obtain the local unstable manifold $W^u_r(x)$ on M.

The next lemma corresponds to Lemma 2.17.

Lemma 4.15. *Let $\Lambda \subset M$ be a hyperbolic set of f with splitting $T_\Lambda M = E^u \oplus E^s$, and let $|\cdot|$ be a C^0 norm of $T_\Lambda M$ that is adapted to and of box type to $E^u \oplus E^s$. Then there is $\delta > 0$ such that:*

(1) *If $\phi : T_\Lambda M \to T_\Lambda M$ is continuous, fiber-preserving over f, and fiber-Lipschitz such that*

$$\mathrm{Lip}_2 \phi < \delta, \quad \phi(0_x) = 0_{fx} \ \forall \ x \in \Lambda,$$

then there is a continuous fiber-preserving fiber-Lipschitz map $\sigma : E^u \to E^s$ over id, $\sigma(0_x) = 0_x$, $\mathrm{Lip}_2 \sigma \leq 1$, such that for any $x \in \Lambda$, $W^u(0_x, Tf + \phi)$ is exactly the graph of $\sigma_x : E^u(x) \to E^s(x)$, where $\sigma_x = \sigma|_{E^u(x)}$.

(2) *If $\phi : T_\Lambda M \to T_\Lambda M$ is continuous, fiber-preserving over f, and C^1 restricted to every fiber such that*

$$\mathrm{Lip}_2 \phi < \delta, \quad \phi(0_x) = 0_{fx} \ \forall \ x \in \Lambda,$$

then for any $x \in \Lambda$, the map $\sigma_x = \sigma|_{E^u(x)}$ guaranteed by item (1) is C^1, and the C^1 submanifold $W^u(0_x, Tf + \phi)$ is tangent at 0_x to the unstable subspace $G^u(x)$ at x of the hyperbolic bundle isomorphism $\{T_x f + D_2\phi(0_x) \mid x \in \Lambda\}$. Moreover, if $D_2\phi$ is continuous on $T_\Lambda M$, then $D_2\sigma$ is continuous on E^u.

Proof. We first prove item (1). Let

$$\Sigma(E^u, E^s; 0) = \{\sigma : E^u \to E^s \ |\sigma \text{ is continuous, fiber-preserving over } id, \ \sigma(0_x) = 0_x, \ |\sigma|_* < \infty\},$$

where

$$|\sigma|_* = \sup_{x \in \Lambda} |\sigma_x|_*$$

and $|\sigma_x|_*$ is defined as in the proof of Lemma 2.17. With this norm $\Sigma(E^u, E^s; 0)$ forms a Banach space, and

$$\Sigma(E^u, E^s; 0)[1] = \{\sigma \in \Sigma(E^u, E^s; 0) \mid \sigma \text{ is fiber-Lipschitz, } \mathrm{Lip}_2 \sigma \leq 1\}$$

is a closed subset of $\Sigma(E^u, E^s; 0)$.

Let $0 < \tau < 1$ be the skewness of Λ with respect to $|\cdot|$. Let

$$\delta = \min\left\{\frac{1-\tau}{2}, \ m(Tf, \ \Lambda)\right\},$$

where

$$m(Tf, \Lambda) = \inf\{m(T_x f) \mid x \in \Lambda\}.$$

This is a positive number since Λ is compact. (Later we will reduce δ further.)

Let $\phi : T_\Lambda M \to T_\Lambda M$ be continuous, fiber-preserving over f, and fiber-Lipschitz such that

$$\mathrm{Lip}_2 \phi < \delta, \quad \phi(0_x) = 0_{fx} \ \forall \ x \in \Lambda.$$

We prove there is $\sigma \in \Sigma(E^u, E^s; 0)[1]$ such that its graph

$$\mathrm{gr}(\sigma) = \bigcup_{x\in\Lambda} \mathrm{gr}(\sigma_x)$$

is invariant under $Tf + \phi$; that is, for every $x \in \Lambda$,

$$(Tf + \phi)\mathrm{gr}(\sigma_x) = \mathrm{gr}(\sigma_{fx}).$$

Then we prove $\mathrm{gr}(\sigma_x)$ is exactly $W^u(0_x, Tf + \phi)$. Below, till the end of the proof of Lemma 4.15, we abbreviate

$$Tf = A.$$

The proof will match that of Lemma 2.17, like a copy.

The invariance condition

$$(A + \phi)\mathrm{gr}(\sigma_x) \subset \mathrm{gr}(\sigma_{fx})$$

is equivalent to

$$\sigma_{fx}((A + \phi)_u(v + \sigma_x v)) = (A + \phi)_s(v + \sigma_x v)$$

for every $v \in E^u(x)$. Since $A_u(\sigma_x v) = 0$ and $A_s v = 0$, this reduces to

$$\sigma_{fx}((A_{uu})_x v + \phi_u(v + \sigma_x v)) = A_{ss}(\sigma_x v) + \phi_s(v + \sigma_x v).$$

That is,

$$\sigma_{fx}((A_{uu})_x + \phi_u(I_{u,x} + \sigma_x)) = A_{ss}(\sigma_x) + \phi_s(I_{u,x} + \sigma_x).$$

Since

$$m((A_{uu})_x) \geq \tau^{-1}, \quad \mathrm{Lip}(\phi_u(I_{u,x} + \sigma_x)) \leq 2\mathrm{Lip}_2\phi < 2\delta = 1 - \tau,$$

by Theorem 2.7, $(A_{uu})_x + \phi_u(I_{u,x} + \sigma_x)$ is invertible. Hence

$$\sigma_{fx} = (A_{ss}\sigma_x + \phi_s(I_{u,x} + \sigma_x))((A_{uu})_x + \phi_u(I_{u,x} + \sigma_x))^{-1}.$$

This suggests a map

$$T = T_\phi : \Sigma(E^u, E^s; 0)[1] \to \Sigma(E^u, E^s; 0)$$

$$(T(\sigma))_{fx} = (A_{ss}\sigma_x + \phi_s(I_{u,x} + \sigma_x))((A_{uu})_x + \phi_u(I_{u,x} + \sigma_x))^{-1}, \ \forall\, x \in \Lambda,$$

called the *graph transform* induced by $A + \phi$. Finding σ with

$$(A + \phi)\mathrm{gr}(\sigma) \subset \mathrm{gr}(\sigma)$$

then reduces to finding a fixed point of T.

We verify that T maps $\Sigma(E^u, E^s; 0)[1]$ into itself. Since $\sigma_x \in \Sigma(E^u(x), E^s(x); 0_x)[1]$, it is easy to see that $(T\sigma)_{fx}(0_{fx}) = 0_{fx}$ and $(T\sigma)_{fx}$ is Lipschitz with

$$\mathrm{Lip}((T\sigma)_{fx}) \leq \frac{\tau + 2\mathrm{Lip}_2\phi}{\tau^{-1} - 2\mathrm{Lip}_2\phi} < 1.$$

Hence T maps $\Sigma(E^u, E^s; 0)[1]$ into itself.

Next we verify that T is a contraction with respect to the norm $|\cdot|_*$. For any $\sigma, \sigma' \in \Sigma(E^u, E^s; 0)[1]$, abbreviate

$$F_x = (A_{uu})_x + \phi_u(I_{u,x} + \sigma_x) : E^u(x) \to E^u(fx),$$

$$F'_x = (A_{uu})_x + \phi_u(I_{u,x} + \sigma'_x) : E^u(x) \to E^u(fx);$$

that is,

$$(T(\sigma))_{fx} F_x = A_{ss}\sigma_x + \phi_s(I_{u,x} + \sigma_x),$$
$$(T(\sigma'))_{fx} F'_x = A_{ss}\sigma'_x + \phi_s(I_{u,x} + \sigma'_x).$$

Since $F_x : E^u(x) \to E^u(fx)$ is a homeomorphism that fixes the origin, when v runs through $E^u(x) - \{0_x\}$,

$$\begin{aligned}|T(\sigma) - T(\sigma')|_* &= \sup_{x\in\Lambda} \sup_{v\neq 0_x} \frac{|(T\sigma)_x(v) - (T\sigma')_x(v)|}{|v|} \\ &= \sup_{x\in\Lambda} \sup_{v\neq 0_x} \frac{|(T\sigma)_{fx}(F_x v) - (T\sigma')_{fx}(F_x v)|}{|F_x v|}.\end{aligned}$$

On one hand,

$$\begin{aligned}&|(T\sigma)_{fx}(F_x v) - (T\sigma')_{fx}(F_x v)| \\ &\quad\le |(T\sigma)_{fx}(F_x v) - (T\sigma')_{fx}(F'_x v)| + |(T\sigma')_{fx}(F'_x v) - (T\sigma')_{fx}(F_x v)| \\ &\quad\le |A_{ss}(\sigma_x(v) - \sigma'_x(v))| + |\phi_s(v + \sigma_x(v)) - \phi_s(v + \sigma'_x(v))| \\ &\quad\quad + \mathrm{Lip}((T\sigma')_{fx})|F'_x v - F_x v| \\ &\quad\le \tau|\sigma_x(v) - \sigma'_x(v)| + \mathrm{Lip}\,\phi_x|\sigma_x(v) - \sigma'_x(v)| + \mathrm{Lip}\,\phi_x|\sigma_x(v) - \sigma'_x(v)| \\ &\quad\le (\tau + 2\mathrm{Lip}\,\phi_x)|\sigma_x(v) - \sigma'_x(v)|.\end{aligned}$$

On the other hand, since $\phi_x(0_x) = 0_{fx}$ and $\sigma_x(0_x) = 0_x$,

$$\begin{aligned}|F_x v| &= |(A_{uu})_x v + \phi_u(v + \sigma_x v) - \phi_u(0_x + \sigma(0_x))| \\ &\ge \tau^{-1}|v| - \mathrm{Lip}\,\phi_x(|v| + \mathrm{Lip}\,\sigma_x|v|) \\ &\ge (\tau^{-1} - 2\mathrm{Lip}\,\phi_x)|v|.\end{aligned}$$

Thus

$$\begin{aligned}|T(\sigma) - T(\sigma')|_* &\le \frac{\tau + 2\mathrm{Lip}_2\phi}{\tau^{-1} - 2\mathrm{Lip}_2\phi} \sup_{x\in\Lambda} \sup_{v\neq 0_x} \frac{|\sigma_x(v) - \sigma'_x(v)|}{|v|} \\ &= \frac{\tau + 2\mathrm{Lip}_2\phi}{\tau^{-1} - 2\mathrm{Lip}_2\phi}|\sigma - \sigma'|_*.\end{aligned}$$

Since

$$\frac{\tau + 2\mathrm{Lip}_2\phi}{\tau^{-1} - 2\mathrm{Lip}_2\phi} < 1,$$

$T = T_\phi$ is a contraction with respect to the norm $|\cdot|_*$. By the contraction mapping principle, T has a unique fixed point $\sigma = \sigma_\phi \in \Sigma(E^u, E^s; 0)[1]$ such that

$$(A + \phi)\mathrm{gr}(\sigma) \subset \mathrm{gr}(\sigma).$$

Up to this point we have done the main part of the proof of item (1), which matches Lemma 2.17, like a copy. As proved before, we actually have

$$(A + \phi)\mathrm{gr}(\sigma) = \mathrm{gr}(\sigma).$$

Moreover, reducing δ further if necessary, for every $x \in \Lambda$,

$$\mathrm{gr}(\sigma_x) = W^u(0_x, A + \phi) = W^u(0_x, Tf + \phi).$$

This proves item (1).

Before proving item (2) we insert a definition. Let $\{H_m\}_{m\in\mathbb{Z}}$ be a sequence of d-dimensional Euclidean spaces. Denote

$$H = \bigsqcup_{m\in\mathbb{Z}} H_m,$$

where $\bigsqcup$ means *discrete union*; namely, the H_m are regarded as mutually isolated so that H is a d-dimensional manifold. Let

$$A : H \to H$$

be a map such that

$$A|_{H_m} : H_m \to H_{m+1}$$

is a linear isomorphism. We call A a *hyperbolic sequence* if $\{|A_m|,\ |A_m^{-1}|\}_{m\in\mathbb{Z}}$ is bounded and, for every $m \in \mathbb{Z}$, there are a direct sum

$$H_m = E^s_m \oplus E^u_m,$$

$$A(E^s_m) = E^s_{m+1},\ A(E^u_m) = E^u_{m+1},$$

and two constants $C \geq 1$ and $0 < \lambda < 1$ such that

$$|A^n(v)| \leq C\lambda^n|v|, \quad \forall v \in E^s_m,\ m \in \mathbb{Z},\ n \geq 0,$$

$$|A^{-n}(v)| \leq C\lambda^n|v|, \quad \forall v \in E^u_m,\ m \in \mathbb{Z},\ n \geq 0.$$

Now we prove item (2). Assume that ϕ restricted to every fiber T_xM of $T_\Lambda M$ is C^1. We prove that σ_x is C^1 and that the C^1 submanifold $W^u(0_x, Tf+\phi)$ of T_xM is tangent at 0_x to $G^u(x)$, where $G^u(x)$ is the unstable subspace at x for the hyperbolic bundle isomorphism $\{T_xf + D_2\phi(0_x) \mid x \in \Lambda\}$. Fix $x \in \Lambda$. We put the discrete topology on $\mathrm{Orb}(x)$ and treat

$$T_{f^m x}M = H_m, \quad Tf = A.$$

In other words, we single out Tf on $T_{\mathrm{Orb}(x)}M$ to form a hyperbolic sequence. We take this point of view for the proof of item (2) because we will take derivatives of $Tf + \phi$ inside a fiber, a process that would be unnecessarily confusing if the metric of the base set is involved. The reason we can take such a point of view is that the issue of smoothness of σ_x is intrinsically inside the fiber T_xM anyway. In this setting, since the H_m are separated from each other, we may write $\mathrm{Lip}_2\phi$ or $D_2\phi$ simply as $\mathrm{Lip}\,\phi$ or $D\phi$.

Abbreviate

$$g = (Tf + \phi)|_H : H \to H.$$

Thus g is just $Tf+\phi$ expressed in the discrete setting. We call g the *discrete version* of $Tf + \phi$. By Theorem 2.7, g is a diffeomorphism of H (mapping H_m onto H_{m+1}) if $\mathrm{Lip}\,\phi$ is small. See Figure 4.5. An important observation is that g *will be Anosov if* $\mathrm{Lip}\,\phi$ *is sufficiently small.*

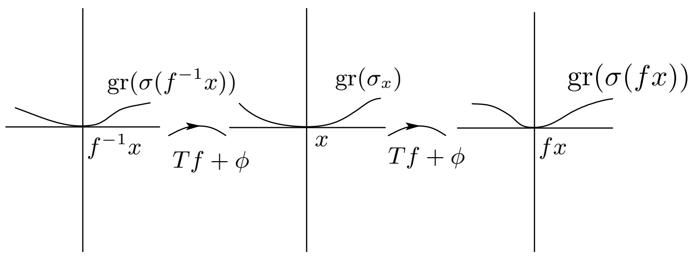

Figure 4.5. The discrete union H of linear spaces.

We verify this. The proof is the same as the proof of Lemma 2.17. For every $y \in H_m$, define

$$E^u(y) = \{y\} \times E^u(f^m x), \quad E^s(y) = \{y\} \times E^s(f^m x).$$

Then

$$T_yH = E^u(y) \oplus E^s(y), \quad y \in H.$$

This is a hyperbolic splitting for A; namely A is Anosov. If $\mathrm{Lip}\,\phi$ is sufficiently small, the same proof as in Section 4.3 will show that H is a hyperbolic set of g; namely g is Anosov. Let

$$T_yH = G^u(y) \oplus G^s(y), \quad y \in H,$$

be the hyperbolic splitting of Tg. Note that

$$\dim G^u(y) = \dim(E^u(y)).$$

Now we prove σ_x is C^1. Take any $v \in E^u(x)$. First we prove σ_x is differentiable at v. Denote $z = (v, \sigma_x(v)) \in \mathrm{gr}(\sigma_x)$ for short. By the criterion of Katok-Hasselblatt (1995) stated at the end of Chapter 2, we need to prove that the tangent set $T_z\mathrm{gr}(\sigma_x)$ is contained in a linear subspace of T_zH of dimension $\dim E^u(x)$. In fact, we prove

$$T_z\mathrm{gr}(\sigma_x) \subset G^u(z).$$

Since $\mathrm{Lip}(\sigma) \leq 1$ (here σ is the above fiber-preserving map over *id*; since we are in the discrete setting, we simply write $\mathrm{Lip}_2\sigma$ to be $\mathrm{Lip}\,\sigma$), for every $y \in \mathrm{gr}(\sigma)$, every generalized tangent line of $T_y\mathrm{gr}(\sigma)$ lies in the 1-cone $C_1(E^u(y))$ with respect to the direct sum $E^u(y) \oplus E^s(y)$. As long as $\mathrm{Lip}\,\phi$ is small enough, $G^u(y)$ and $G^s(y)$ will be close to $E^u(y)$ and $E^s(y)$, respectively; hence the generalized tangent lines of $T_y\mathrm{gr}(\sigma)$ will be contained in the 2-cone $C_2(G^u(y))$ with respect to the direct sum $G^u(y) \oplus G^s(y)$. Since $\mathrm{gr}(\sigma)$ is invariant under g, Tg maps generalized tangent lines of $\mathrm{gr}(\sigma)$ into generalized tangent lines of $\mathrm{gr}(\sigma)$. In particular, for every generalized tangent line l of $T_z\mathrm{gr}(\sigma_x)$,

$$Tg^{-n}(l) \subset C_2(G^u(g^{-n}z)),\ \forall n \geq 0.$$

By Theorem 4.2,

$$l \subset G^u(z).$$

This proves $T_z\mathrm{gr}(\sigma_x) \subset G^u(z)$.

By the criterion, σ_x is differentiable at v, and $\mathrm{gr}(\sigma_x)$ is tangent at z to the unstable subspace $G^u(z)$ of Tg. In particular, at the origin 0_x, $\mathrm{gr}(\sigma_x)$ is tangent to the unstable subspace $G^u(x)$ of $A + D_2\phi(0_x)$. By Theorem 4.3, $G^u(z)$ varies continuously in $z \in \mathrm{gr}(\sigma_x)$. Thus σ_x is C^1.

Finally, let $D_2\phi$ be continuous on $T_\Lambda M$. We prove $D_2\sigma$ is continuous on E^u. To this end we have to leave the discrete setting and go back to our tangent bundle $T_\Lambda M$. Note that the discrete setting is just a way of presentation. What we have proved are properties in the fibers of the tangent bundle. For instance, the direct sums $G^u(y) \oplus G^s(y)$, $y \in T_\Lambda M$, form a hyperbolic splitting for $\{T_{\pi y} f + D_2\phi(y) \mid y \in T_\Lambda M\}$. Now $D_2\phi$ is continuous on $T_\Lambda M$; hence the idea of the proof of Theorem 4.3 (hyperbolic estimates pass to the closure) shows that $G^u(y)$ varies continuously in $y \in T_\Lambda M$. Writing $z = (v, \sigma v)$ with $v \in E^u$, then $G^u(z)$ varies continuously in $v \in E^u$. Namely, $D_2\sigma(v)$ varies continuously in $v \in E^u$, proving Lemma 4.15. □

A family of embedded submanifolds $\{D_i\}_{i\in I}$ is called *self-coherent* if, for any $i, j \in I$, $\mathrm{int}\,D_i \cap \mathrm{int}\,D_j$ is open in both D_i and D_j. In particular, $\mathrm{int}\,D_i$ and $\mathrm{int}\,D_j$ do not cross each other. For instance, local solution curves of an autonomous C^1 ordinary differential equation form a family of self-coherent 1-dimensional embedded submanifolds.

The next theorem corresponds to Theorem 2.18.

Theorem 4.16 (Stable manifolds theorem for a hyperbolic set)**.** *Let $f : M \to M$ be a C^k diffeomorphism, $k \geq 1$, and let $\Lambda \subset M$ be a hyperbolic set of f with splitting $T_\Lambda M = E^s \oplus E^u$. Then there is $r > 0$ such that, for every $x \in \Lambda$:*

(1) *$W^s_r(x)$ is a C^k embedded submanifold of M of dimension* $\dim E^s(x)$ *tangent at x to E^s_x, and $W^s_r(x)$ varies continuously in $x \in \Lambda$ with respect to the C^k topology. Precisely, there are a neighborhood V of 0_Λ in E^s and a*

continuous fiber-preserving map $\sigma : V \to E^u$ *over id whose fiber-derivatives up to order* k *are continuous on* V, *and* $\sigma(0_x) = 0_x$, $D_2\sigma(0_x) = 0$, *such that* $W^s_r(x) = exp_x \mathrm{gr}(\sigma|_{V\cap E^s(x)})$.

(2) *The family* $\{W^s_r(x)\}_{x\in\Lambda}$ *is self-coherent.*

(3) *The global stable manifold* $W^s(x)$ *is an immersed* C^k *submanifold of* M *of dimension* $\dim E^s(x)$.

Remark. Here the global stable manifold is defined by $W^s(x) = \{y \in X \mid \lim_{n\to+\infty} d(f^n y, f^n x) = 0\}$ (see the paragraph before Theorem 3.8). Theorems 4.16 and 4.13 are often combined together and referred to as the stable manifold theorem.

Let us call the map $\sigma : V \to E^u$ in item (1) the *generating map* of the family of stable manifolds $\{W^s_r(x)\}_{x\in\Lambda}$. For every $x \in \Lambda$, through the graph and the exponential map, σ C^k embeds $V \cap E^s(x)$, a neighborhood of 0_x in $E^s(x)$, right onto $W^s_r(x)$.

Proof. First we prove item (1). We first prove there is $r > 0$ such that, for every $x \in \Lambda$, $W^s_r(x)$ is a C^k submanifold of M, tangent at x to E^s_x. Since

$$W^s_r(x, f) = \exp_x(W^s_r(0_x, F_f)),$$

it suffices to prove that the fiber-stable manifold $W^s_r(0_x, F_f)$ is a C^k submanifold of T_xM, tangent at 0_x to E^s_x. We will work with $W^s_r(0_x, F_f)$ instead of $W^s_r(x, f)$ till the end of the proof of item (1).

First we prove the case $k = 1$. Let f be C^1. We first work with a norm $|\cdot|$ of $T_\Lambda M$ that is adapted to and of box type to $E^s \oplus E^u$. We extend the locally defined F_f to the whole $T_\Lambda M$ and apply item (2) of Lemma 4.15 to obtain (global) stable manifolds, then cut them off to get local stable manifolds, and finally get back to the original Riemannian norm. The proof will match that of Theorem 2.18, like a copy.

Fix a C^∞ bump function $\alpha : \mathbb{R} \to \mathbb{R}$ with $0 \le \alpha \le 1$ such that $\alpha(t) = 1$ for $|t| \le 1/3$, $\alpha(t) = 0$ for $|t| \ge 2/3$. Write

$$\phi_f = F_f - Tf : TM(r_\rho) \to TM.$$

Then ϕ_f is C^0, and C^1 restricted to fibers, and

$$\phi_f(0_x) = 0_{fx}, \quad D_2\phi_f(0_x) = 0, \quad \forall\, x \in M.$$

(Here ϕ_f has an additional property of "tangency" compared to the map ϕ of Lemma 4.15.) Define

$$\overline{\phi}_f : TM \to TM$$

$$\overline{\phi}_f(v) = \alpha\left(\frac{|v|}{3r}\right)\phi_f(v),$$

where $0 < r < r_\rho/3$ will be shrunk shortly. Then $\overline{\phi}_f$ is C^0, and C^1 restricted to fibers, and $\overline{\phi}_f = \phi_f$ on $TM(r)$. By the claim in the proof of Theorem 2.13, if $r > 0$ is sufficiently small, then $\mathrm{Lip}_2\overline{\phi}_f$ on the whole TM, with respect to $|\cdot|$, will be small enough to satisfy Lemmas 4.12 and 4.15. By Lemma 4.15, there are a continuous fiber-preserving map

$$\sigma : E^s \to E^u$$

over id, C^1 restricted to fibers, and $\sigma(0_x) = 0_x$, $\mathrm{Lip}_2\sigma \leq 1$, such that for any $x \in \Lambda$,

$$W^s(0_x, Tf + \overline{\phi}_f) = \mathrm{gr}(\sigma|_{E^s(x)}).$$

Moreover, the C^1 submanifold $W^s(0_x, Tf + \overline{\phi}_f) \subset T_xM$ is tangent at 0_x to the stable subspace of the hyperbolic linear isomorphism $Tf + D_2\overline{\phi}_f(0_x)$. Since

$$D_2\phi_f(0_x) = 0,$$

this stable subspace is just E^s_x. This means

$$D_2\sigma(0_x) = 0.$$

Since the norm $|\cdot|$ is of box type, for any $x \in \Lambda$,

$$T_xM(r) = E^s_x(r) \times E^u_x(r).$$

Denote

$$i : E^s \to T_\Lambda M$$

to be

$$i(v) = (v, \sigma(v)).$$

Then for every $x \in \Lambda$, i is a C^1 embedding that takes E^s_x onto the C^1 submanifold $\mathrm{gr}(\sigma|_{E^s(x)})$ of T_xM, tangent at 0_x to E^s_x. Since $\mathrm{Lip}\,\sigma_x \leq 1$, we have

$$i(E^s_x(r)) = W^s(0_x, Tf + \overline{\phi}_f) \cap T_xM(r).$$

We prove

$$W^s_r(0_x, F_f) = W^s(0_x, Tf + \overline{\phi}_f) \cap T_xM(r).$$

Since $\overline{\phi}_f = \phi_f$ on $T_\Lambda M(r)$, the "$\subset$" part is obvious. We prove "$\supset$". Let $v \in W^s(0_x, Tf + \overline{\phi}_f) \cap T_xM(r)$. It suffices to prove $(Tf + \overline{\phi}_f)^n v \in T_{f^n x}M(r)$ for all $n \geq 1$. But this is obvious because, by Lemma 4.12 (for the case of the whole subbundle $T_\Lambda M$),

$$W^s(0_x, Tf + \overline{\phi}_f) = \{v \in T_xM \mid |(Tf + \overline{\phi}_f)^n v| \leq (\tau + \mathrm{Lip}_2\overline{\phi}_f)^n |v|, \ \forall n \geq 0\}.$$

This proves "$\supset$". Thus

$$W^s_r(0_x, F_f) = i(E^s_x(r))$$

is a C^1 submanifold of T_xM.

So far we have worked with a norm $|\cdot|$ of $T_\Lambda M$ that is adapted to and of box type to $E^s \oplus E^u$. Now we get back to the original Riemannian norm $\|\cdot\|$ of M. Take $0 < a < r$ such that for every $x \in \Lambda$,

$$W^s_a(0_x, F_f; \|\cdot\|) \subset W^s_r(0_x, F_f; |\cdot|).$$

Then $W^s_a(0_x, F_f; \|\cdot\|)$ is a C^1 submanifold of $T_x M$. Let

$$V_x = i^{-1}(W^s_a(0_x, F_f; \|\cdot\|)).$$

Then V_x is a neighborhood of 0_x in E^s_x that is mapped onto $W^s_a(0_x, F_f; \|\cdot\|)$ by the graph of σ_x. Let

$$V = \bigcup_{x\in\Lambda} V_x.$$

Then V is a neighborhood of 0_Λ in E^s satisfying the requirement of the theorem. This proves the case $k = 1$.

Now we prove the case $k \geq 2$. Since the tangent planes have been determined in the case $k = 1$, we prove the smoothness of $W^s_r(0_x, F_f)$ only. The proof is hinted at by Robinson (1995) for a fixed point (see the proof of Theorem 2.18).

Let f be C^k. Fix $x \in \Lambda$. We prove $W^s_r(0_x, F_f)$ is a C^k submanifold of $T_x M$. Since we will take derivatives inside the fibers $T_{f^m x} M$, for clearness of presentation we take the discrete setting of the hyperbolic sequence, namely letting

$$H_m = T_{f^m x} M, \;\; H = \bigsqcup_{m\in\mathbb{Z}} H_m, \;\; U_m = T_{f^m x} M(r_\rho), \;\; U = \bigsqcup_{m\in\mathbb{Z}} U_m, \;\; 0_m = 0_{f^m x}.$$

We only consider maps $g : U \to H$ with $g : U_m \to H_{m+1}$ (fiber-preserving over the shift map $m \to m+1$). We say a C^1 map $g : U \to H$ is *tangent at the origins to a hyperbolic sequence* if

$$g(0_m) = 0_{m+1}$$

and

$$Dg(0_m) : H_m \to H_{m+1}$$

is a hyperbolic sequence. Since the discrete version of F_f is such a map, it suffices to prove inductively that *if $g : U \to H$ is C^k and is tangent at the origins to a hyperbolic sequence, then there is $r > 0$ such that $W^s_r(0_0, g)$ is a C^k submanifold of H_0.* Note that $0_0 = 0_x$.

The case $k = 1$ is just proved (by applying the result of Lemma 4.15, hence without using the discrete setting of the hyperbolic sequence). Assume that the case of $k - 1$ is proved; we prove the case of k. Let g be C^k. The proof will be like a copy of that of Theorem 2.18.

Define a map G such that, for every $m \in \mathbb{Z}$,

$$G : U_m \times H_m \to H_{m+1} \times H_{m+1}$$
$$G(y, v) = (g(y),\ Dg(y)v).$$

Here the product spaces take the usual max metric. Then G is C^{k-1}. Clearly,

$$G^n(y, v) = (g^n(y),\ Dg^n(y)v).$$

Also,

$$G(0_m, 0_m) = (0_{m+1}, 0_{m+1}),$$

and

$$DG(0_m, 0_m) = \begin{pmatrix} Dg(0_m) & 0 \\ 0 & Dg(0_m) \end{pmatrix}.$$

Hence $\{DG(0_m, 0_m)\}_{m\in\mathbb{Z}}$ is a hyperbolic sequence. By induction, there is $r > 0$ such that $W_r^s((0_0, 0_0), G)$ is a C^{k-1} submanifold of $U_0 \times H_0$.

On the other hand, by Theorem 4.13, we may assume $r > 0$ has been chosen such that

$$W_r^s((0_0, 0_0),\ G) = \{(y, v) \in U_0 \times H_0 \mid |g^n y| \le r,\ |Dg^n(y)v| \le r,\ \forall n \ge 0\},$$

where $Dg^n(y)v$ is just $Tg^n(v)$. As proved before, g always agrees with an Anosov diffeomorphism $\overline{g}$ of H on a neighborhood of the set of origins 0_m. Let

$$T_y H = G^u(y) \oplus G^s(y), \quad y \in H,$$

be the hyperbolic splitting of $\overline{g}$. If $r > 0$ is sufficiently small, then $v \in T_y H$ satisfies

$$|Tg^n(v)| \le r,\ \forall n \ge 0$$

if and only if

$$v \in G^s(y)(r).$$

Hence the above equation can be rewritten as

$$W_r^s((0_0, 0_0), G) = \{(y, v) \in U_0 \times H_0 \mid y \in W_r^s(0_0, g),\ v \in G^s(y)(r)\}.$$

As proved before,

$$G^s(y) = T_y(W^s(0_0, g)).$$

Hence $W_r^s((0_0, 0_0), G)$ is just (the r-neighborhood of the 0-set of) the tangent bundle of $W_r^s(0_0, g)$. Since the degrees of smoothness of a manifold and its tangent bundle differ by 1 and since $W_r^s((0_0, 0_0), G)$ is C^{k-1}, so $W_r^s(0_0, g)$ is C^k. This proves the case of k.

So far we have proved that if f is C^k, then $W_r^s(0_x, F_f)$ is a C^k submanifold of $T_x M$. There is one more step to go, that is, to prove that the fiber-derivatives of σ of Theorem 4.16 up to order k are continuous on $E^s(r)$.

Since the problem restricted to fibers is just solved, it remains to prove that these fiber-derivatives are continuous with respect to the base points $x \in \Lambda$. To this end we have to leave the discrete setting and go back to our tangent bundle. But before that let us use the discrete setting to define a map Σ. Let g and G be as above. For simplicity we use the same notation σ to denote its discrete version. Thus $\sigma : E^s_m(r) \to E^u_m(r)$ is the map with $\mathrm{gr}(\sigma) = W^s_r(0_m, g)$, where $E^s_m \oplus E^u_m$ is the hyperbolic splitting of $Dg(0_m)$ and $r > 0$ is the small number just determined. Define

$$\Sigma : E^s_m(r) \times E^s_m(r) \to E^u_m \times E^u_m$$
$$\Sigma(a, b) = (\sigma(a),\ D\sigma(a)b).$$

Claim. Σ is the *generating map* of G, meaning that for every $m \in \mathbb{Z}$, $\mathrm{gr}(\Sigma_m) = W^s_r((0_m, 0_m), G)$, where $\Sigma_m = \Sigma|_{E^s_m(r)\times E^s_m(r)}$.

In fact, by definition, $(y, v) \in \mathrm{gr}(\Sigma_m)$ if and only if

$$y^u = \sigma(y^s),\quad y^s \in E^s_m(r) \ \text{ and }\ v^u = D\sigma(y^s)v^s,\quad v^s \in E^s_m(r).$$

That is,

$$y \in \mathrm{gr}(\sigma) = W^s_r(0_m, g) \ \text{ and }\ v \in \mathrm{gr}(D\sigma(y^s)|_{E^s_m(r)}) = G^s(y)(r).$$

This means

$$(y, v) \in W^s_r((0_m, 0_m), G),$$

proving the claim.

While a fiber-preserving map on the bundle has its discrete version, conversely, the map G defined in the discrete setting gives a map

$$G_\Lambda : T_xM(\rho) \times T_xM \to T_{fx}M \times T_{fx}M,\ \forall\ x \in \Lambda,$$

such that G is the discrete version of G_Λ. Likewise, the map Σ gives a map Σ_Λ in the bundle such that Σ is the discrete version of Σ_Λ.

Now we go back to our tangent bundle $T_\Lambda M$. Let f be C^k. We prove that the fiber-derivatives of σ up to order k are continuous in $x \in \Lambda$. We sketch the proof. Since F_f has fiber-derivatives up to order k continuous in $x \in \Lambda$ and since its discrete version is tangent at the origins to a uniform family of hyperbolic sequences, it suffices to prove inductively that *if a fiber-preserving map F of a bundle over $f : \Lambda \to \Lambda$ defined near the origins has fiber-derivatives up to order k continuous in $x \in \Lambda$ and if the discrete version of F is tangent at the origins to a uniform family of hyperbolic sequences, then there is $r > 0$ such that the generating map σ of F of size r has fiber-derivatives up to order k continuous in $x \in \Lambda$.* Note that we have not specified the bundle for F because the induction will concern a collection of bundles over the same base Λ, such as the one where the map G_Λ lives.

The case $k = 1$ is guaranteed by item (2) of Lemma 4.15. Assume that the case of $k - 1$ is proved. We prove the case of k. Thus, assume that F has fiber-derivatives up to order k continuous in $x \in \Lambda$ and that its discrete version g is tangent at the origins to a uniform family of hyperbolic sequences. Let G be the map induced by g as above. Since G is defined using g and Dg, which are just F and D_2F in the bundle, the fiber-derivatives of G_Λ up to order $k-1$ are made up of fiber-derivatives of F up to order k and hence are continuous in $x \in \Lambda$. As proved above, G is tangent at the origins to a uniform family of hyperbolic sequences and, by the claim, Σ_Λ is the generating map of G_Λ. Thus, by induction, there is $r > 0$ such that the generating map Σ_Λ of size r has fiber-derivatives up to order $k-1$ continuous in $x \in \Lambda$. By the definition of Σ_Λ, σ has fiber-derivatives up to order k continuous in $x \in \Lambda$. This proves item (1) of Theorem 4.16.

Now we prove item (3). It is easy to see that, for any $r > 0$,

$$W^s(x) = \bigcup_{n \geq 0} f^{-n} W_r^s(f^n x), \qquad W^u(x) = \bigcup_{n \geq 0} f^n W_r^u(f^{-n} x).$$

Now Λ is a hyperbolic set and $x \in \Lambda$. By item (1), $W_r^s(x)$ is a C^k embedded submanifold of M. Hence $f^{-n}W_r^s(x)$ is a C^k embedded submanifold for every $n \geq 0$. As a monotone union of a sequence of embedded submanifolds, $W^s(x)$ is a C^k immersed submanifold of M. Likewise for $W^u(x)$.

Finally we prove item (2); that is, the family of embedded submanifolds $\{W_r^s(x)\}_{x\in\Lambda}$ is self-coherent. Note that while $W_r^s(0_x) \subset T_xM$ and $W_r^s(0_y) \subset T_yM$ are by definition disjoint for $x \neq y$, $W_r^s(x) = exp_x(W_r^s(0_x))$ and $W_r^s(y) = exp_y(W_r^s(0_y))$ in M do intersect if $d(x, y)$ is small. Thus there is indeed a problem of coherence of intersections.

Let $W_r^s(x) \cap W_r^s(y) \neq \emptyset$, where $x, y \in \Lambda$. Then both $W_r^s(x)$ and $W_r^s(y)$ are contained in the global stable manifold $W^s(x)$. Since the three submanifolds have the same dimension, the coherence of the family $W_r^s(x)$, $x \in \Lambda$, follows immediately. This proves Theorem 4.16. □

Remark. The proof of the stable manifolds theorem for a hyperbolic set is long, going through Lemma 4.15 and Theorem 4.16. It follows the four steps of the proof for a hyperbolic fixed point, summarized after the proof of Theorem 2.18. There are some additional issues involved here, such as transferring the map f on the manifold to the self-lifting F_f on the tangent bundle, establishing a discrete setting to investigate the smoothness of the fiber-stable manifolds $W_r^s(0_x, F_f)$, comparing the tangent bundle and the discrete setting to investigate the continuous dependence on base points x for the fiber-derivatives of the generating map of the family $\{W_r^s(0_x, F_f)\}_{x\in\Lambda}$, exploring the self-coherence of the family $\{W_r^s(x)\}_{x\in\Lambda}$, etc. Nevertheless the main part of the proof still matches the proof for a hyperbolic fixed point, like a copy.

Theorem 4.17. *Let $\Lambda \subset M$ be a hyperbolic set of f. There are $r > 0$ and $\delta > 0$ such that for any $x, y \in \Lambda$, if $d(x, y) \le \delta$, then $W_r^s(x) \cap W_r^u(y) \ne \emptyset$ transversely.*

Proof. Let $\Lambda_i = \{x \in \Lambda : \dim E^s(x) = i\}$. Then $\Lambda_0, \dots, \Lambda_{\dim M}$ are finitely many disjoint compact invariant sets. By Theorem 4.1, Λ_0 and $\Lambda_{\dim M}$ are of finite many points. Thus we may assume all points of Λ have the same index $1 \le i \le \dim M - 1$.

By Theorem 4.16, there is $r > 0$ such that, for every $x \in \Lambda$, $W_r^s(x)$ and $W_r^u(x)$ are C^1 submanifolds of M and vary continuously in x with respect to the C^1 topology. Since $W_r^s(x)$ and $W_r^u(x)$ intersect transversely at x, there is $\delta(x) > 0$ such that if $y \in \Lambda$ satisfies $d(x, y) \le \delta(x)$, then $W_r^s(x) \cap W_r^u(y) \ne \emptyset$ transversely. Since Λ is compact, $\delta(x)$ can be chosen independent of $x \in \Lambda$. This proves Theorem 4.17. □

There is a nice geometrical explanation for the expansiveness of a hyperbolic set using local stable manifolds. Let Λ be a hyperbolic set of f. Since $W_r^s(x)$ and $W_r^u(x)$ intersect transversely at x, if $r > 0$ is small enough, then $W_r^s(x) \cap W_r^u(x) = \{x\}$ for any $x \in \Lambda$. If $d(f^n y, f^n x) \le r$ for all $n \in \mathbb{Z}$, then $y \in W_r^s(x) \cap W_r^u(x)$ by Theorem 4.13. Hence $y = x$, proving that f is r-expansive on Λ.

The best picture for the family of global stable manifolds $\{W^s(x, f)\}$ is presented when f is Anosov. In that case Theorem 4.16 applies to every point and hence yields the local stable manifold $W_r^s(x)$ for every $x \in M$. The self-coherence of $\{W_r^s(x)\}$ then guarantees that the family of global stable manifolds $W^s(x)$ forms a C^0 foliation of the whole manifold M with C^k leaves, a beautiful global picture.

(By definition, for $m < n$, a decomposition of an n-dimensional manifold M into a disjoint union of m-dimensional C^k immersed submanifolds of M is called an m-dimensional C^0 *foliation* with C^k *leaves* if, in C^0 local charts, the union is like $\mathbb{R}^n = \bigcup(\mathbb{R}^m \times \{c\})$, $c \in \mathbb{R}^{n-m}$.)

4.5. Structural stability of hyperbolic sets

First we consider sections of the tangent bundle.

Let $\Lambda \subset M$ be a compact invariant set of f. As usual, by a *section* (or *vector field*) of $T_\Lambda M$ we mean a map $\gamma : \Lambda \to T_\Lambda M$ such that $\gamma(x) \in T_x M$ for every $x \in \Lambda$. Denote by $\Gamma^0(T_\Lambda M)$ the Banach space of all continuous sections of $T_\Lambda M$, endowed with the C^0 norm

$$|\gamma| = \sup_{x \in \Lambda} |\gamma(x)|.$$