Orderable groups and their algebraic properties

In this chapter we will discuss some of the special algebraic properties enjoyed by orderable groups, which come in two basic flavors: left-orderable and the more special bi-orderable groups. As we'll soon see, a group is right-orderable if and only if it is left-orderable. The literature is more or less evenly divided between considering right- and left-invariant orderings. Some authors (including those of this book) have flip-flopped on the issue of right vs. left. Of course results from the "left" school have dual statements in the right-invariant world but, as with driving, one must be consistent.

There are several useful reference books on ordered groups, such as *Fully ordered groups* by Kokorin and Kopytov [**59**], *Orderable groups* by Mura and Rhemtulla [**8**], *Right-ordered groups* by Kopytov and Medvedev [**61**], A. M. W. Glass' *Partially ordered groups* [**37**] and *Groups, orders, and dynamics* [**30**] by Deroin, Navas and Rivas. Many interesting results and examples on orderability of groups which won't be discussed here can be found in these books. We will focus mostly on groups of special topological interest and results relevant to topological applications. On the other hand, we try to include enough material to provide context and to make the core development of ideas in this book reasonably self-contained.

1.1. Invariant orderings

By a *strict ordering* of a set X we mean a binary relation $<$ which is transitive ($x < y$ and $y < z$ imply $x < z$) and such that $x < y$ and $y < x$ cannot both hold. It is a strict *total* ordering if for every $x, y \in X$ exactly one of $x < y$, $y < x$ or $x = y$ holds. A strict total ordering is sometimes also called a *linear* ordering.

A group G is called *left-orderable* if its elements can be given a strict total ordering $<$ which is left invariant, meaning that $g < h$ implies $fg < fh$ for all $f, g, h \in G$. We will say that G is *bi-orderable* if it admits a total ordering which is simultaneously left and right invariant (historically, this has been called simply "orderable"). We refer to the pair $(G, <)$ as the ordered group. We shall usually use the symbol 1 to denote the identity element of a group G. However, for abelian groups in which the group operation is denoted by addition, the identity element may be denoted by 0. In an ordered group the symbols $\leq$ and $>$ have the obvious meaning: $g \leq h$ means $g < h$ or $g = h$; $g > h$ means $h < g$. Note that the opposite ordering can also be considered an ordering, also invariant.

Problem 1.1. *Show that*:

(1) *In a left-ordered group one has $1 < g$ if and only if $g^{-1} < 1$.*
(2) *In a left-ordered group, if $1 < g$ and $1 < h$, then $1 < gh$.*
(3) *A left-ordering is a bi-ordering if and only if the ordering is invariant under conjugation.*

As already mentioned, the class of right-orderable groups is the same as the class of left-orderable groups. In fact, a concrete correspondence can be given as follows.

Problem 1.2. *If $<$ is a left-invariant ordering of the group G, show the recipe*

$$g \prec h \iff h^{-1} < g^{-1}$$

defines a right-invariant ordering $\prec$ which has the same "positive cone"—that is, $1 \prec g \iff 1 < g$.

The following shows that left-orderable groups are infinite, with the exception of the trivial group, consisting of the identity alone.

Proposition 1.3. *A left-orderable group has no elements of finite order. In other words, it is torsion-free.*

Proof. If g is an element of the left-ordered group G and $1 < g$, then $g < g^2$, $g^2 < g^3$ and so on, and by transitivity we conclude that $1 < g^n$ for all positive integers n. The case $g < 1$ is similar. □

Problem 1.4. *Show that if f and g are elements of a left-ordered group and $f \neq 1$, then g is strictly between fg and $f^{-1}g$ and also strictly between gf and gf^{-1}.*

1.2. Examples

Example 1.5. The additive reals $(\mathbb{R}, +)$, rationals $(\mathbb{Q}, +)$ and integers $(\mathbb{Z}, +)$ are bi-ordered groups, under their usual orderings. On the other hand, the multiplicative group of nonzero reals, $(\mathbb{R} \setminus \{0\}, \cdot)$, cannot be bi-ordered. The element -1 has order two; by Proposition 1.3 this is impossible in a left-orderable group.

Example 1.6. Both left- and bi-orderability are clearly preserved under taking subgroups. If G and H are left- or bi-ordered groups, then so is their direct product $G \times H$ using lexicographic ordering, which declares that $(g, h) < (g', h')$ if and only if $g <_G g'$ or else $g = g'$ and $h <_H h'$.

Example 1.7. Consider the additive group $\mathbb{Z}^2$. It can be ordered lexicographically as just described, taking $G = H = \mathbb{Z}$. Another way to order $\mathbb{Z}^2$ is to think of it sitting in the plane $\mathbb{R}^2$ in the usual way, and then choose a vector $\vec{v} = (v_1, v_2) \in \mathbb{R}^2$ which has irrational slope. We can order $\vec{m} = (m_1, m_2), \vec{n} = (n_1, n_2) \in \mathbb{Z}^2$ according to their dot product with $\vec{v}$, that is,

$$\vec{m} < \vec{n} \iff m_1 v_1 + m_2 v_2 < n_1 v_1 + n_2 v_2.$$

We leave the reader to check that this is an invariant strict total ordering, and that one obtains uncountably many different orderings of $\mathbb{Z}^2$ in this way. If $\vec{v}$ has rational slope, then one may also compare as above, but using lexicographically the dot product with v and then with some pre-chosen vector orthogonal to $\vec{v}$. Higher-dimensional spaces can be invariantly ordered in a similar manner.

Problem 1.8. *Suppose G is a group with normal subgroup K and quotient group $H \cong G/K$. In other words, suppose there is an exact sequence*

$$1 \to K \hookrightarrow G \xrightarrow{p} H \to 1.$$

Further suppose $(H, <_H)$ and $(K, <_K)$ are left-ordered groups. Verify that we can then give G a left-ordering defined in a sort of lexicographic way: declare that $g < g'$ if and only if either $p(g) <_H p(g')$ or else $p(g) = p(g')$ (so $g^{-1}g' \in K$) and $1 <_K g^{-1}g'$.

Example 1.9. The Klein bottle is a nonorientable surface, which can be considered as a square with opposite sides identified with each other in the directions indicated in Figure 1.1. We see that its fundamental group has

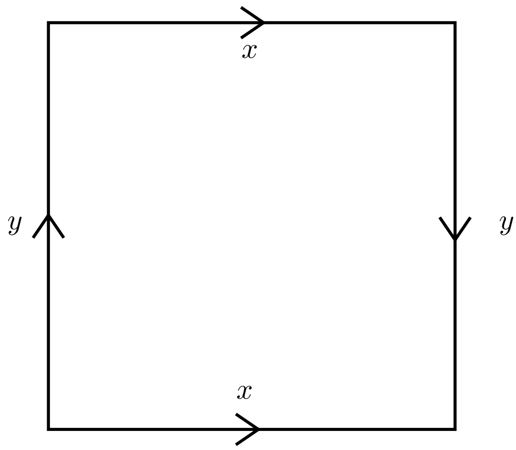

Figure 1.1. The Klein bottle as a square with opposite sides identified as shown.

the presentation with two generators x and y and the relation $yxyx^{-1} = 1$. In other words,

$$K = \pi_1(\text{Klein bottle}) \cong \langle x, y \mid xyx^{-1} = y^{-1} \rangle.$$

Problem 1.10. *Show that the subgroup $\langle y \rangle$ of the Klein bottle group K which is generated by y is a normal subgroup isomorphic to $\mathbb{Z}$ and that the quotient subgroup $K/\langle y \rangle$ is also isomorphic with $\mathbb{Z}$. Use this to show that K is left-orderable. Finally, conclude that K cannot be given a bi-invariant ordering, by using the defining relation to derive a contradiction.*

Example 1.11. Let $\mathrm{Homeo}_+(\mathbb{R})$ denote the group of all order-preserving homeomorphisms of the real line—that is, continuous functions with continuous inverses and which preserve the usual order of the reals. This is a group under composition. It can be left-ordered in the following way. Let $x_1, x_2, \dots$ be a countable dense set of real numbers. For two functions $f, g \in \mathrm{Homeo}_+(\mathbb{R})$, compare them by choosing $m = m(f, g)$ to be the minimum i for which $f(x_i) \neq g(x_i)$ and then declare that $f \prec g$ if and only if $f(x_m) < g(x_m)$ (in the usual ordering of $\mathbb{R}$).

Problem 1.12. *Verify that $\prec$ is a left-ordering of* $\mathrm{Homeo}_+(\mathbb{R})$. (*Hint*: *To show that $f \prec g, g \prec h \implies f \prec h$, consider the cases $m(f,g) = m(g,h)$ and $m(f,g) \neq m(g,h)$ separately.*)

We will see later that $\mathrm{Homeo}_+(\mathbb{R})$ is universal for countable left-orderable groups, in the sense that any countable left-orderable group embeds in $\mathrm{Homeo}_+(\mathbb{R})$.

Problem 1.13. *Suppose that G is a path-connected topological group, which as a space has universal cover $\widetilde{G}$. Show that there is a multiplication on $\widetilde{G}$*

that is compatible with the multiplication on G*, meaning that the covering map* $p : \widetilde{G} \to G$ *becomes a group homomorphism.*

Recall that a (left) *action* of a group G on a set X is a function $G \times X \to X$ which satisfies $1x = x$ and $(gh)x = g(hx)$ for all $g, h \in G, x \in X$.

Problem 1.14. *Suppose that* G *and* $\widetilde{G}$ *are as above and* G *acts on a space* X*. Show that if* $\widetilde{X}$ *is the universal cover of* X*, then* $\widetilde{G}$ *acts on* $\widetilde{X}$*.*

Example 1.15. The group

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

is naturally a subgroup of $\mathrm{SL}(2,\mathbb{C})$, and it is conjugate to the subgroup

$$\mathrm{SU}(1,1) = \left\{ \left(\begin{array}{cc} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{array} \right) \middle| \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

The conjugacy is given by sending each matrix $A \in \mathrm{SL}(2,\mathbb{R})$ to the matrix $JAJ^{-1} \in \mathrm{SU}(1,1)$, where $J = \left(\begin{array}{cc} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{array} \right) \in \mathrm{SL}(2,\mathbb{C})$. Now thinking of the group in this way, we can observe a faithful action of $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ on the unit circle $S^1 \subset \mathbb{C}$ by homeomorphisms. An element of $\mathrm{PSL}(2,\mathbb{R})$ acts on $z \in S^1$ by first choosing a representative $A \in \mathrm{SL}(2,\mathbb{R})$, converting A to an element of $\mathrm{SU}(1,1)$ and then applying the associated Möbius transformation. In other words if $A = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right)$, then $JAJ^{-1} = \left(\begin{array}{cc} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{array} \right)$ for some $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 - |\beta|^2 = 1$, and then we can define

$$A(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}.$$

By considering $\mathrm{SL}(2,\mathbb{R})$ as a subspace of $\mathbb{R}^4$, we can think of it as a 3-manifold and its quotient $\mathrm{PSL}(2,\mathbb{R})$ is also a manifold. Thus it admits a universal covering space $p : \widetilde{\mathrm{PSL}}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$, and the universal covering space has a group structure that is lifted from the base space, as in Problem 1.13. The action of $\mathrm{PSL}(2,\mathbb{R})$ on the circle lifts to an action of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ on $\mathbb{R}$ by orientation-preserving homeomorphisms by Problem 1.14, so we can think of $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ as a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ (see [**57**] for details). Since $\mathrm{Homeo}_+(\mathbb{R})$ is left-orderable, so is $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$.

Problem 1.16. *Check that the definition in the previous example yields an action of* $\mathrm{PSL}(2,\mathbb{R})$ *on* S^1*, by checking that* $A \mapsto JAJ^{-1}$ *defines an* *isomorphism of* SL$(2,\mathbb{R})$ *with* SU$(1,1)$, *and that* $\left|\frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}}\right| = 1$ *whenever* $|z| = 1$.

Problem 1.17. *Show that, as a subspace of* $\mathbb{R}^4$, SL$(2,\mathbb{R})$ *is homeomorphic with an open solid torus*: SL$(2,\mathbb{R}) \cong S^1 \times \mathbb{C}$. *Moreover show that the action on* SL$(2,\mathbb{R})$ *given by* $M \to -M$ *is fixed-point free, and so* PSL$(2,\mathbb{R})$ *is a manifold, in fact also an open solid torus, and the projection map* SL$(2,\mathbb{R}) \to$ PSL$(2,\mathbb{R})$ *is a covering space.*

Problem 1.18. *Conclude that* $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ *is homeomorphic with* $\mathbb{R}^3$.

1.3. Bi-orderable groups

We summarize a few algebraic facts about bi-orderable groups, which do not hold in general for left-orderable groups, and leave their proofs to the reader. For example, inequalities multiply:

Problem 1.19. *In a bi-ordered group* $g_1 < h_1$ *and* $g_2 < h_2$ *imply* $g_1g_2 < h_1h_2$.

Problem 1.20. *Bi-orderable groups have unique roots, that is, if* $g^n = h^n$ *for some* $n > 0$, *then* $g = h$.

The following was observed by B. H. Neumann [**82**].

Problem 1.21. *In a bi-orderable group* G, g^n *commutes with* h *if and only if* g *commutes with* h. (*Hint*: *For the nontrivial direction, assume* g *and* h *do not commute, say* $g < h^{-1}gh$, *and multiply this inequality by itself several times to conclude* g^n *cannot commute with* h. *Show more generally that if* g^n *and* h^m *commute for some nonzero integers* m *and* n, *then* g *and* h *must commute.*)

Problem 1.22. *Bi-orderable groups do not have generalized torsion*: *any product of conjugates of a nontrivial element must be nontrivial. In particular,* $x^{-1}yx = y^{-1}$ *implies* $y = 1$.

On the down side, bi-orderable groups do not behave as nicely under extension as left-orderable groups do. As seen in Problem 1.10 we have a group K which is flanked by bi-orderable groups in a short exact sequence (and is left-orderable for that reason) but it is not bi-orderable.

Problem 1.23. *Consider groups* K, G *and* $H = G/K$ *as in Problem* 1.8, *with*

$$1 \to K \hookrightarrow G \xrightarrow{p} H \to 1$$

exact. Suppose K and H are bi-ordered. Then the recipe of Problem 1.8 *defines a bi-ordering of G if and only if the conjugation action of G upon K preserves the given ordering of K.*

1.4. Positive cone

Theorem 1.24. *A group G is left-orderable if and only if there exists a subset $P \subset G$ such that*

(1) *$P \cdot P \subset P$ and*

(2) *for every $g \in G$, exactly one of $g = 1$, $g \in P$ or $g^{-1} \in P$ holds.*

Proof. Given such a P, the recipe $g < h$ if and only if $g^{-1}h \in P$ is easily seen to define a left-invariant strict total order, and conversely such an ordering defines the set $P = \{g \in G \mid 1 < g\}$, called the *positive cone.* □

Problem 1.25. *Verify the details of this proof.*

Problem 1.26. *Show that G is bi-orderable if and only if it admits a subset P satisfying* (1), (2) *above, and in addition*

(3) *$gPg^{-1} \subset P$ for all $g \in G$.*

Example 1.27. The positive cone for the ordering of $\mathbb{Z}^2$ described in Problem 1.7 is the set of all points in the plane which lie to one side of the line through the origin which is orthogonal to $\vec{v}$, if $\vec{v}$ has irrational slope. If the slope is rational, one must also include points of $\mathbb{Z}^2$ on one half of that orthogonal line to lie in the positive cone.

Example 1.28. In Problem 1.8, the positive cone for the ordering described for G is the union of the positive cone of (the ordering of) K and the pullback $p^{-1}(P_H)$ of the positive cone of H. That is, $P_G = P_K \cup p^{-1}(P_H)$.

Problem 1.29. *Let $(G, <)$ be a left-ordered group. Then the following are equivalent*:

(1) *The ordering $<$ is also right-invariant.*

(2) *For every $g, h \in G$, if $g < h$, then $h^{-1} < g^{-1}$.*

(3) *For every $g, h \in G$, if $g < gh$, then $g < hg$.*

(4) *If $g_1 < h_1$ and $g_2 < h_2$, then $g_1 g_2 < h_1 h_2$.*

Problem 1.30. *Show that the Klein bottle group discussed above is isomorphic with the group $\langle a, b \mid a^2 = b^2 \rangle$. Define an explicit function $h : \langle a, b \mid a^2 = b^2 \rangle \to \langle x, y \mid xyx^{-1} = y^{-1} \rangle$ by assigning $h(a)$ and $h(b)$ expressions as words in x and y and show that the relation $a^2 = b^2$ in the domain implies $xyx^{-1} = y^{-1}$ in the range, so that h is a homomorphism.*

Similarly define a homomorphism in the other direction and verify that it is inverse to h.

Another way of seeing this isomorphism is to observe that the Klein bottle is the union of two Möbius bands, glued along their boundaries, and apply the theorem of Seifert and van Kampen.

Problem 1.31. *Show that the Klein bottle group does* not *have unique roots. Indeed, we have $a \neq b$ (why?) but $a^2 = b^2$. This gives another proof that it is not bi-orderable.*

1.5. Topology and the spaces of orderings

To understand the totality of all orderings of a given group, topology will be quite useful. We recall that a topological space is a set X and a collection of subsets of X, called open sets, for which finite intersections and arbitrary unions of open sets are open. The space X itself and the empty set $\emptyset$ are always considered open. A subset is closed if its complement is open. Any subset A of X inherits a topology from a topology on X by taking sets of the form $A \cap U$, where U is an open subset of X, to be open in A. The discrete topology on a set is the one in which *every* subset is open.

An open covering of a space is a collection of open sets whose union is the whole space. A space is *compact* if every open covering has a finite subcollection whose union is the space. A *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X such that the open sets are exactly all unions of sets in $\mathcal{B}$.

1.5.1. Topology on the power set. For any set X, one may consider the collection of all its subsets—that is, its power set—often denoted $\mathcal{P}(X)$ or 2^X. This latter notation indicates that the power set may be identified with the set of all functions $X \to \{0,1\}$ (using von Neumann's definition $2 := \{0,1\}$), via the characteristic function $\chi_A : X \to \{0,1\}$ associated to a subset $A \subset X$ defined by

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A. \end{cases}$$

The set 2^X is a special case of a product space: one gives $\{0,1\}$ the discrete topology, and 2^X is considered the product of copies of $\{0,1\}$ indexed by the set X. The product topology is the smallest topology on the set 2^X such that for each $x \in X$ the sets $\{f \in 2^X \mid f(x) = 0\}$ and $\{f \in 2^X \mid f(x) = 1\}$ are open. In other notation, the subsets of $\mathcal{P}(X)$ of the form

$$U_x = \{A \subset X \mid x \in A\} \quad \text{and} \quad U_x^c = \{A \subset X \mid x \notin A\}$$

are open in the "Tychonoff" topology on the power set. Note that the sets U_x and U_x^c are also closed, as they are each other's complement. A basis for the topology can be obtained by taking finite intersections of various U_x and U_x^c. A famous theorem of Tychonoff asserts that an arbitrary product of compact spaces is again compact. Since the space $\{0, 1\}$ is compact, we conclude:

Theorem 1.32. *The power set $\mathcal{P}(X)$ of any set X, with the Tychonoff topology, is compact.*

Problem 1.33. *A space is said to be* totally disconnected *if for each pair of points, there is a set which is both closed and open and which contains one of the points and not the other. Show that $\mathcal{P}(X)$, with the Tychonoff topology, is totally disconnected.*

If X is finite, then so is 2^X and the Tychonoff topology is just the discrete topology. If X is countably infinite, then 2^X is homeomorphic to the Cantor space obtained by deleting middle thirds successively of the interval $[0, 1]$. In particular, the Tychonoff topology on $\mathcal{P}(X)$ is metrizable when X is countable. A useful characterization of the Cantor space is that any nonempty compact metric space which is totally disconnected will be homeomorphic with the Cantor space if and only if it has no isolated points. A point is *isolated* if it has an open neighborhood disjoint from the rest of the space. See [**46**, Corollary 2.98] for details.

Problem 1.34. *If $A \subset X$ is a fixed subset, there is a natural inclusion $\mathcal{P}(A) \subset \mathcal{P}(X)$. Show that $\mathcal{P}(A)$ is a closed subset.*

Problem 1.35. *Consider the complementation function $C : \mathcal{P}(X) \to \mathcal{P}(X)$ on the power set of the set X defined by $C(Y) = X \setminus Y$. Show that C is a fixed-point free involution—that is, C is a homeomorphism of $\mathcal{P}(X)$ with C^2 the identity map and $C(Y) \neq Y$ for all $Y \in \mathcal{P}(X)$.*

Example 1.36. Let G be a group and define $\mathcal{S}(G)$ to be the collection of all subsemigroups of G. That is, $\mathcal{S}(G) = \{S \subset G \mid g, h \in S \implies gh \in S\}$. Note that $\mathcal{S}(G) \subset \mathcal{P}(G)$. We will argue that $\mathcal{S}(G)$ is in fact a *closed* subset of $\mathcal{P}(G)$. Consider the complement $\mathcal{P}(G) \setminus \mathcal{S}(G)$. A subset Y of X belongs to $\mathcal{P}(G) \setminus \mathcal{S}(G)$ if and only if there exist $g, h \in Y$ with $gh \notin Y$. Therefore

$$\mathcal{P}(G) \setminus \mathcal{S}(G) = \bigcup_{g,h \in G} (U_g \cap U_h \cap U_{gh}^c).$$

Each term in the parentheses is an open set, by definition, and therefore so is the intersection of the three, and so $\mathcal{P}(G) \setminus \mathcal{S}(G)$ is a union of open sets. It follows that $\mathcal{S}(G)$ is closed.

1.5.2. The spaces of orderings. In this section we will show how to topologize the set of all orderings of a group, so as to make a compact space of orderings.

Definition 1.37. The space of left-orderings of a group G, denoted $LO(G)$, is the collection of all subsets $P \subset G$ such that (1) P is a subsemigroup, (2) $P \cap P^{-1} = \emptyset$ and (3) $P \cup P^{-1} = G \setminus \{1\}$.

Problem 1.38. *Show that $LO(G)$ is a closed subset of $\mathcal{P}(G \setminus \{1\})$ and of $\mathcal{P}(G)$, and is therefore a compact and totally disconnected space (with the subspace topology).*

Problem 1.39. *Suppose $<$ is a left-invariant ordering of the group G, and suppose we have a finite string of inequalities $g_1 < g_2 < \cdots < g_n$ which hold. Show that the set of all left-orderings in which all these inequalities hold forms an open neighborhood of $<$ in $LO(G)$. The set of all such neighborhoods is a basis for the topology of $LO(G)$. Equivalently, a basic open set in $LO(G)$ consists of all orderings in which some specified finite set of elements of G are all positive.*

In particular, an ordering of G is *isolated* in $LO(G)$ if it is the only ordering satisfying some finite set of inequalities. This property is also known as "finitely determined" in the literature. Some groups G have isolated points in $LO(G)$, while others do not, as we will see in Chapter 10.

Similarly, we can define the set $O(G)$ of bi-invariant orderings on the group G to be the collection of subsets $P \subset G$ satisfying (1), (2) and (3) above; and also $g^{-1}Pg \subset P$.

Problem 1.40. *Show that $O(G)$ is a closed subset of $LO(G)$, so it is also a compact totally disconnected space.*

To our knowledge, this definition of $LO(G)$ first appeared in [**105**]. We will discuss the structure of $LO(G)$, some of Sikora's results and other applications in greater detail in Chapter 10.

Problem 1.41. *Suppose a countable left-orderable group G has its non-identity elements enumerated, so $G \setminus \{1\} = \{g_1, g_2, \dots\}$. If $<$ and $<'$ are two left-orderings of G, define*

$$d(<, <') = 2^{-n},$$

where n is the first index at which $<$ and $<'$ differ on g_n (i.e., either $1 < g_n$ and $g_n <' 1$ or else $1 <' g_n$ and $g_n < 1$). In other words, g_n is in the symmetric difference of their respective positive cones. Show that this really is a metric (the triangle inequality is the only nontrivial part). Moreover, verify that the topology generated by this metric is the Tychonoff topology.

1.6. Testing for orderability

Suppose we wish to determine if a given group G is left-orderable. Consider a set S of generators of G, which may be infinite. That is, each $g \in G$ may be written as a finite product of elements of S and their inverses. The length $l(g)$ of a group element (relative to the choice of generators) is the smallest integer k such that

$$g = g_1^{\epsilon_1} \cdots g_k^{\epsilon_k}$$

where each $g_i \in S$ and $\epsilon_i = \pm 1$. Let G_k denote the set of all elements of G of length at most k. If S is finite, G_k is also a finite set, which includes the identity (length zero) and also is invariant under taking inverses. It can be regarded as the k-ball of the Cayley graph of G, relative to the given generators.

Now let us define a subset Q of G_k to be a *proper k-partition* if (1) whenever $g, h \in Q$ and $gh \in G_k$, then $gh \in Q$, (2) $Q \cap Q^{-1} = \emptyset$ and (3) $Q \cup Q^{-1} = G_k \setminus \{1\}$.

Notice that if P is a positive cone (of a left-ordering) of G, then $P \cap G_k$ is a proper k-partition. So the following is clear:

Proposition 1.42. *Suppose G is a group with generating set S, with respect to which there is no proper k-partition for some positive integer k. Then G is not left-orderable.*

Perhaps surprisingly, there is a converse.

Theorem 1.43. *Suppose G is generated by $S \subset G$ with respect to which, for all $k \geq 1$, there is a proper k-partition. Then G is left-orderable.*

Proof. We will prove this using compactness of $\mathcal{P}(G)$. Consider the set $\mathcal{P}_k$ of all subsets of G whose intersection with G_k is a proper k-partition. One argues as usual that $\mathcal{P}_k$ is a closed subset of $\mathcal{P}(G)$, and by hypothesis $\mathcal{P}_k$ is nonempty. Note also that for all k we have $\mathcal{P}_{k+1} \subset \mathcal{P}_k$. Thus the $\mathcal{P}_k$ form a nested descending sequence of nonempty closed subsets of $\mathcal{P}(X)$. In a compact space such a sequence has nonempty intersection, so we conclude

$$\bigcap_{k=1}^{\infty} \mathcal{P}_k \neq \emptyset.$$

Also observing that if g, h belong to G_k, then gh is in G_{2k}, we see that if $P \in \bigcap_{k=1}^{\infty} \mathcal{P}_k$, then $P \in LO(G)$ and we conclude that in fact

$$LO(G) = \bigcap_{k=1}^{\infty} \mathcal{P}_k \neq \emptyset,$$

completing the proof. □

In the case of a finitely-generated group G with solvable word problem, it is a finite task to check whether or not there exists a proper k-partition for a particular fixed k. So there is an algorithm to decide whether a proper k-partition exists. If G is not left-orderable, then the algorithm will discover that fact in finite time (although one does not know when). Moreover, one can design the algorithm to supply a proof of non-left-orderability if it finds a G_k having no proper partition. On the other hand, if the group under scrutiny *is* left-orderable, the algorithm will never end. An example of such an algorithm, due to Nathan Dunfield, is described in [**15**] and is available from his website. In [**15**] this algorithm was used to discover Example 5.11, showing a certain torsion-free group (the fundamental group of the Weeks manifold, the compact hyperbolic 3-manifold of least volume) is not left-orderable.

Theorem 1.44. *A group is left-orderable if and only if each of its finitely-generated subgroups is left-orderable.*

The "only if" part is trivial. The proof in the other direction will use the following version of compactness. A collection of sets is said to have the *finite intersection property* if every finite subcollection of the sets has a nonempty intersection.

Problem 1.45. *A topological space is compact if and only if every collection of closed subsets with the finite intersection property has a nonempty total intersection.*

To prove the nontrivial part of Theorem 1.44, consider any finite subset F of the given group G and let $\langle F\rangle$ denote the subgroup of G generated by F. Define

$$\mathcal{Q}(F) = \{Q \subset G \mid Q \cap \langle F\rangle \text{ is a positive cone for } \langle F\rangle\}.$$

For each finite $F \subset G$, $\mathcal{Q}(F)$ is a closed subset of $\mathcal{P}(G)$. The family of all $\mathcal{Q}(F)$, for finite $F \subset G$, is a collection of closed sets which has the finite intersection property, because

$$\mathcal{Q}(F_1 \cup F_2 \cup \cdots \cup F_n) \subset \mathcal{Q}(F_1) \cap \mathcal{Q}(F_2) \cap \cdots \cap \mathcal{Q}(F_n).$$

By compactness, $\bigcap_{F\subset G \text{ finite}} \mathcal{Q}(F) \neq \emptyset$.

Problem 1.46. *Verify that any element of $\bigcap_{F\subset G \text{ finite}} \mathcal{Q}(F)$ is a left-ordering of G, completing the proof. In fact*

$$\bigcap_{F\subset G \text{ finite}} \mathcal{Q}(F) = LO(G).$$

Theorem 1.47. *An abelian group G is bi-orderable if and only if it is torsion-free.*

Proof. We need only show that torsion-free abelian groups are left-orderable (which in this case is equivalent to bi-orderable). But any finitely-generated subgroup is isomorphic to $\mathbb{Z}^n$ for some n, which we have already seen to be bi-orderable (Example 1.7). The result follows from Theorem 1.44. □

1.7. Characterization of left-orderable groups

Following [**24**], we have a number of characterizations of left-orderability of a group G. If $X \subset G$, we let $S(X)$ denote the semigroup generated by X, that is, all elements of G expressible as (nonempty) products of elements of X (no inverses allowed). If $X = x_1, \dots, x_n$ is a finite set, we drop the brackets and write $S(X) = S(x_1, \dots, x_n)$.

Theorem 1.48. *A group G can be left-ordered if and only if for every finite subset $\{x_1, \dots, x_n\}$ of G which does not contain the identity, there exist $\epsilon_i = \pm 1$ such that $1 \notin S(x_1^{\epsilon_1}, \dots, x_n^{\epsilon_n})$.*

One direction is clear, for if $<$ is a left-ordering of G, just choose ϵ_i so that $x_i^{\epsilon_1}$ is greater than the identity. For the converse, by Theorem 1.44 we may assume that G is finitely generated, and by Theorem 1.43 we need only show that each k-ball G_k, with respect to a fixed finite generating set, has a proper k-partition. To do this, let $\{x_1, \dots, x_n\}$ denote the entire set $G_k \setminus \{1\}$, and choose $\epsilon_i = \pm 1$ such that $1 \notin S(x_1^{\epsilon_1}, \dots, x_n^{\epsilon_n})$.

Problem 1.49. *Show that the set $G_k \cap S(x_1^{\epsilon_1}, \dots, x_n^{\epsilon_n})$ is a proper k-partition, completing the proof of Theorem* 1.48.

Another characterization of left-orderability is due to Burns and Hale [**13**].

Theorem 1.50 (Burns-Hale). *A group G is left-orderable if and only if for every finitely-generated subgroup $H \neq \{1\}$ of G, there exists a left-orderable group L and a nontrivial homomorphism $H \to L$.*

Proof. One direction is obvious. To prove the other direction, assume the subgroup condition. According to Theorem 1.48, the result will follow if one can show:

Claim: For every finite subset $\{x_1, \dots, x_n\}$ of $G \setminus \{1\}$, there exist $\epsilon_i = \pm 1$ such that $1 \notin S(x_1^{\epsilon_1}, \dots, x_n^{\epsilon_n})$.

We will establish this claim by induction on n. It is certainly true for $n = 1$, for $S(x_1)$ cannot contain the identity unless x_1 has finite order, which is impossible since the cyclic subgroup $\langle x_1 \rangle$ must map nontrivially to a left-orderable group.

Next assume the claim is true for all finite subsets of $G \setminus \{1\}$ having fewer than n elements, and consider $\{x_1, \ldots, x_n\} \subset G \setminus \{1\}$. By hypothesis, there is a nontrivial homomorphism

$$h : \langle x_1, \ldots, x_n \rangle \to L$$

where $(L, \prec)$ is a left-ordered group. Not all the x_i are in the kernel since the homomorphism is nontrivial, so we may assume they are numbered so that

$$h(x_i) \begin{cases} \neq 1 \text{ if } i = 1, \ldots, r, \\ = 1 \text{ if } r < i \leq n. \end{cases}$$

Now choose $\epsilon_1, \ldots, \epsilon_r$ so that $1 \prec h(x_i^{\epsilon_i})$ in L for $i = 1, \ldots, r$. For $i > r$, the induction hypothesis allows us to choose $\epsilon_i = \pm 1$ so that $1 \notin S(x_{r+1}^{\epsilon_{r+1}}, \ldots, x_n^{\epsilon_n})$. We now check that $1 \notin S(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$ by contradiction. Suppose that 1 *is* a product of some of the $x_i^{\epsilon_i}$. If all the i are greater than r, this is impossible, as $1 \notin S(x_{r+1}^{\epsilon_{r+1}}, \ldots, x_n^{\epsilon_n})$. On the other hand if some i is less than or equal to r, we see that h must send the product to an element strictly greater than the identity in L, again a contradiction. □

A group is said to be *indicable* if it has the group of integers $\mathbb{Z}$ as a quotient, and *locally indicable* if each of its nontrivial finitely-generated subgroups is indicable. This notion was introduced by Higman [**41**] to study zero divisors and units in group rings (see Section 1.8).

Corollary 1.51. *Locally indicable groups are left-orderable.*

Corollary 1.52. *Suppose G is a group which has a (finite or infinite) family of normal subgroups $\{G_\alpha\}$ such that $\bigcap_\alpha G_\alpha = \{1\}$. If all the factor groups G/G_α are left-orderable, then G is left-orderable.*

Proof. If H is a finitely-generated nontrivial subgroup of G, one can choose α for which $H \setminus G_\alpha$ is nonempty. Then the composition of homomorphisms $H \to G \to G/G_\alpha$ is a nontrivial homomorphism of H to a left-orderable group. □

Problem 1.53. *Show that each of the following conditions on a group G is equivalent to left-orderability*:

(1) *For each element $g \neq 1$ in G, there exists a subsemigroup S_g of G which contains g but not 1 and such that $G \setminus S_g$ is also a semigroup.*

(2) *For each finite subset $x_1, \ldots, x_n$ of G, the intersection of the 2^n subsemigroups $S(1, x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$ is equal to $\{1\}$, where the ϵ_i are ± 1.*

(3) *There exists a set $\mathbf{S}$ of subsemigroups of G whose intersection is $\{1\}$ and such that for every $g \in G$ and $S \in \mathbf{S}$, either $g \in S$ or $g^{-1} \in S$.*

See [**24**] *if you get stuck, but note that the author uses the right-ordering convention.*

A subset Q of a group G is called a *partial left-order* if it is a subsemigroup ($Q \cdot Q \subset Q$) such that $Q \cap Q^{-1} = \emptyset$. Such a subset Q can be regarded as the positive cone of a left-invariant *partial* order of the group. In particular, Q corresponds to a total left-order if and only if $G \setminus \{1\} = Q \cup Q^{-1}$. If Q and Q' are partial left-orders such that $Q \subset Q'$, then Q' is called an *extension* of Q. The following is a useful criterion for a partial order to extend to a total one.

Problem 1.54. *A partial left-order Q on G has an extension to a total left-order if and only if whenever $\{x_1, \dots, x_n\}$ is a finite subset of G which does not contain the identity 1 of G, there exist $\epsilon_i = \pm 1$ such that $1 \notin S(Q \cup \{x_1^{\epsilon_1}, \dots, x_n^{\epsilon_n}\})$.*

1.8. Group rings and zero divisors

We will now discuss one of the algebraic reasons why it is worth knowing that a group is left-orderable.

If R is a ring with identity and G is a group (written multiplicatively), then the group ring RG is defined to be the free left R-module generated by the elements of G, endowed with a natural multiplication analogous to products of polynomials. That is, a typical element of RG is a finite formal linear combination

$$\sum_{i=1}^{m} r_i g_i$$

with $r_i \in R$ and $g_i \in G$. The product is defined by the formula

$$\left(\sum_{i=1}^{m} r_i g_i\right)\left(\sum_{j=1}^{n} s_j h_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} r_i s_j (g_i h_j). \tag{1.1}$$

Of course, on the right-hand side of equation (1.1), cancellations may be possible, and this leads to some mischief, as the example below illustrates. If 1 is the identity of G, then the group ring element $r1$ is customarily denoted simply as r, and likewise for the ring identity, also denoted by 1, $1g$ may be abbreviated as g.

Group rings (known as group algebras if R is a field) arise naturally in representation theory, algebraic topology, Galois theory, etc. An important problem is the so-called zero-divisor conjecture, which dates back at least to the 1940's, often attributed to Kaplansky. It remains unsolved even for the case $R = \mathbb{Z}$. Recall that an element $\alpha \neq 0$ of a ring is called a *zero divisor* if there exists another ring element $\beta \neq 0$ such that $\alpha\beta = 0$.

Conjecture 1.55 (Zero divisor conjecture)**.** *If R is a ring without zero divisors and G is a torsion-free group, then RG has no zero divisors.*

One of the strongest reasons for knowing whether a group is orderable is that the zero divisor conjecture is true for left-orderable groups. Before proving this, let us discuss by example how zero divisors, and nontrivial units (elements with inverses), can arise in group rings. If r is an invertible element of R and g an arbitrary element of G, then the "monomial" rg is clearly a unit of RG: $(rg)(r^{-1}g^{-1}) = 1$. Such a unit is called a *trivial* unit of RG.

Example 1.56. Consider the ring of integers $R = \mathbb{Z}$ and the cyclic group of order five, $G = \langle x \mid x^5 = 1\rangle$. Define the following elements of RG:

$$\alpha = 1 + x + x^2 + x^3 + x^4, \quad \beta = 1 - x, \quad \gamma = 1 - x^2 - x^3, \quad \delta = 1 - x - x^4.$$

Problem 1.57. *Verify that $\alpha\beta = 0$ and $\gamma\delta = 1$. Therefore, the group ring in this example has zero divisors and nontrivial units as well.*

The existence of nontrivial units in group rings, like the zero divisor problem, is a notoriously difficult problem in algebra. However, for left-orderable groups the answer is straightforward.

Theorem 1.58. *If R is a ring without zero divisors and G is a left-orderable group, then the group ring RG does not have zero divisors or nontrivial units.*

Proof. Consider a product, as in equation (1.1), where we assume that the r_i and s_j are all nonzero, the g_i are distinct and the h_j are written in strictly ascending order, with respect to a given left-ordering of G. At least one of the group elements g_ih_j on the right-hand side of (1.1) is minimal in the left-ordering. If $j > 1$ we have, by left-invariance, that $g_ih_1 < g_ih_j$ and g_ih_j is not minimal. Therefore we must have $j = 1$. On the other hand, since we are in a group and the g_i are distinct, we have that $g_ih_1 \neq g_kh_1$ for any $k \neq i$. We have established that there is exactly one minimal term on the right-hand side of (1.1), and similarly there is exactly one maximal term. It follows that they survive any cancellation, and so the right-hand side cannot be zero (because $r_is_1 \neq 0$). Thus RG has no zero divisors. If one of n or m is greater than one, there are at least two terms on the right-hand side of (1.1) which do not cancel, so the product cannot equal 1. This implies that all units of RG are trivial. □

1.9. Torsion-free groups which are not left-orderable

Left-orderable groups are torsion-free, but there are many examples to show the converse is far from true. One of the simplest examples, which has appeared several times in the literature, is the following.

Example 1.59. We will consider a crystallographic group G which is torsion-free but not left-orderable. Specifically consider the group G with generators a, b, c acting on $\mathbb{R}^3$ with coordinates (x, y, z) by the rigid motions:

$$a(x, y, z) = (x + 1, 1 - y, -z),$$
$$b(x, y, z) = (-x, y + 1, 1 - z),$$
$$c(x, y, z) = (1 - x, -y, z + 1).$$

One can easily check the relations $a^2ba^2 = b, b^2ab^2 = a$ and $abc = id$. By the last relation we see that one generator may be eliminated. In fact G has the presentation $G = \langle a, b \mid a^2ba^2 = b, b^2ab^2 = a\rangle$.

Problem 1.60. *Check the relations cited above. Argue that the group G is torsion-free.*

Problem 1.61. *Argue that G is not left-orderable as follows. First show that for all choices of $m, n \in \{-1, +1\}$ one has $a^{2m}b^na^{2m} = b^n$ and $b^{2n}a^mb^{2n} = a^m$. Then argue that*

$$\begin{aligned}(a^mb^n)^2(b^na^m)^2 &= a^mb^{-n}b^{2n}a^mb^{2n}a^mb^na^m \\ &= a^mb^{-n}a^{2m}b^na^{2m}a^{-m} \\ &= a^mb^{-n}b^na^{-m} = 1.\end{aligned}$$

Conclude that if G were left-orderable, all choices of sign for a and b would lead to a contradiction.

Problem 1.62. *Show that the subgroup $A = \langle a^2, b^2, c^2\rangle$ is generated by shifts (by even integral amounts) in the directions of the coordinate axes, and so is a free abelian group of rank 3. Moreover A is normal in G and of finite index. Therefore G is virtually bi-orderable, in the sense that a finite index subgroup is bi-orderable.*

Next we will construct an infinite family of examples. Consider the Klein bottle group $K = \langle a, b \mid a^2 = b^2\rangle$.

Problem 1.63. *Verify that a^2 and ab commute, that the subgroup $H = \langle a^2, ab\rangle$ is an index two subgroup of K and that $H \cong \mathbb{Z}^2$.*

In fact, H can be regarded as the fundamental group of the 2-dimensional torus which double-covers the Klein bottle as in Figure 1.2, the so-called oriented double cover.

Alternatively, we can realize K as the 2-dimensional crystallographic group generated by the glide reflections

$$a(x, y) = (x + 1, -y), \quad b(x, y) = (x + 1, 1 - y)$$

and H as the subgroup of orientation-preserving motions.

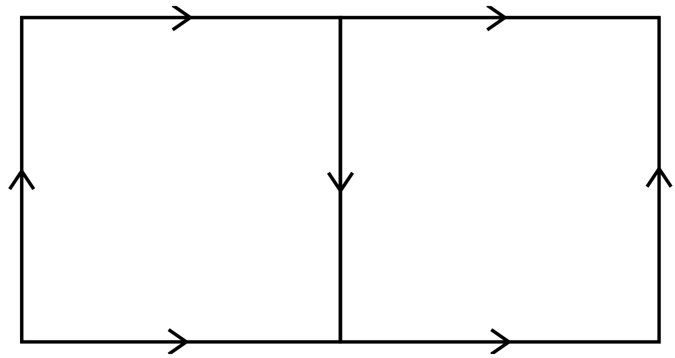

Figure 1.2. The torus as a rectangle with opposite sides identified, which we can subdivide into two Klein bottles as shown.

Now take two copies K_1 and K_2 of the Klein bottle group, and amalgamate them along their corresponding subgroups H_1 and H_2. An isomorphism $\phi : H_1 \to H_2$ is given by a 2×2 matrix (using the bases $\{a_i^2, a_i b_i\}$)

$$\phi \sim \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

with determinant ± 1. We take this to mean, in multiplicative notation,

$$\phi(a_1^2) = (a_2^2)^p (a_2 b_2)^q; \quad \phi(a_1 b_1) = (a_2^2)^r (a_2 b_2)^s.$$

This identification defines an amalgamated free product

$$G_\phi := K_1 *_\phi K_2$$

which has the presentation

$$G_\phi = \langle a_1, b_1, a_2, b_2 \mid a_1^2 = b_1^2,\ a_2^2 = b_2^2,\\ a_1^2 = (a_2^2)^p (a_2 b_2)^q,\ a_1 b_1 = (a_2^2)^r (a_2 b_2)^s \rangle.$$

The groups G_ϕ are torsion-free, since they are amalgamated products of torsion-free (in fact left-orderable) groups. This can be seen by considering the normal form for elements of an amalgamated free product; see for example [**102**], Section 1.3, Corollary 2.

Example 1.64. Suppose $p, q \geq 0$ and $r, s \leq 0$ (or vice versa). Then G_ϕ is *not* left-orderable.

To see this, suppose for contradiction that G_ϕ is left-orderable. Then the first relation implies that a_1 and b_1 must have the same sign (either both are positive or both are negative) and the second implies a_2 and b_2 also have the same sign. The third relation implies that a_1 (and hence b_1) has the same sign as a_2 and b_2 (note that one of p or q must be strictly positive).

But then the last relation implies a_1b_1 has the opposite sign as a_2 and b_2, the desired contradiction. □

Problem 1.65. *Calculate that the abelianization of* G_ϕ *is a finite group of order* $16|p+q-r-s|$*, and therefore this construction provides infinitely many nonisomorphic groups which are torsion-free but not left-orderable.*

It will be seen later (Problem 6.23) that the G_ϕ are the fundamental groups of an interesting class of 3-manifolds: the union of two twisted I-bundles over the Klein bottle. Further examples of torsion-free groups which are not left-orderable are discussed in Chapter 5.

Finally, we mention a useful result, due independently to Brodskiĭ [**12**] and Howie [**49**]. See also [**50**] for a simpler proof. The difficult direction is to show that torsion-free implies locally indicable.

Theorem 1.66. *If* G *is a group which has a presentation with a single relation, the following are equivalent*:

(1) G *is torsion-free.*
(2) G *is locally indicable.*
(3) G *is left-orderable.*

Note that the examples of torsion-free non-left-orderable groups described above have two or more defining relations.

We end this chapter with an open question. Chehata [**19**] constructed a bi-orderable group which is simple. But the example is uncountable, and therefore not finitely generated. In fact, every bi-orderable simple group must be infinitely generated, because finitely-generated bi-orderable groups have infinite abelianization (for a proof of this fact, see Theorem 2.19).

Question 1.67. *Is there a finitely-generated left-orderable simple group*?