Preface

Perhaps the earliest example of a moving frame is the Frenet frame along a nondegenerate curve in the Euclidean space $\mathbb{R}^3$, consisting of a triple of orthonormal vectors (T, N, B) based at each point of the curve. First introduced by Bartels in the early nineteenth century [**Sen31**] and later described by Frenet in his thesis [**Fre47**] and Serret in [**Ser51**], the frame at each point is chosen based on properties of the geometry of the curve near that point, and the fundamental geometric invariants of the curve—curvature and torsion—appear when the derivatives of the frame vectors are expressed in terms of the frame vectors themselves.

In the late nineteenth century, Darboux studied the problem of constructing moving frames on surfaces in Euclidean space [**Dar72a**], [**Dar72b**], [**Dar72c**], [**Dar72d**]. In the early twentieth century, Élie Cartan generalized the notion of moving frames to other geometries (for example, affine and projective geometry) and developed the theory of moving frames extensively. A very nice introduction to Cartan's ideas may be found in Guggenheimer's text [**Gug77**].

More recently, Fels and Olver [**FO98**], [**FO99**] have introduced the notion of an "equivariant moving frame", which expands on Cartan's construction and provides new algorithmic tools for computing invariants. This approach has generated substantial interest and spawned a wide variety of applications in the last several years. This material will not be treated here, but several surveys of recent results are available; for example, see [**Man10**], [**Olv10**], and [**Olv11a**].

The goal of this book is to provide an introduction to Cartan's theory of moving frames at a level suitable for beginning graduate students, with an emphasis on curves and surfaces in various 3-dimensional homogeneous spaces. This book assumes a standard undergraduate mathematics background, including courses in linear algebra, abstract algebra, real analysis, and topology, as well as a course on the differential geometry of curves and surfaces. (An appropriate differential geometry course might be based on a text such as [**dC76**], [**O'N06**], or [**Opr07**].) There are occasional references to additional topics such as differential equations, but these are less crucial.

The first two chapters contain background material that might typically be taught in a graduate differential geometry course; Chapter 1 contains general material from differential geometry, while Chapter 2 focuses more specifically on differential forms. Students who have taken such a course might safely skip these chapters, although it might be wise to skim them to get accustomed to the notation that will be used throughout the book.

Chapters 3–7 are the heart of the book. Chapter 3 introduces the main ingredients for the method of moving frames: homogeneous spaces, frame bundles, and Maurer-Cartan forms. Chapters 4–7 show how to apply the method of moving frames to compute local geometric invariants for curves and surfaces in 3-dimensional Euclidean, Minkowski, affine, and projective spaces. These chapters should be read in order (with the possible exception of Chapter 5), as they build on each other.

Chapters 8–10 show how the method of moving frames may be applied to several classical problems in differential geometry. The first half of Chapter 8, all of Chapter 9, and the last half of Chapter 10 may be read anytime after Chapter 4; the remainder of these chapters may be read anytime after Chapter 6.

Chapters 11 and 12 give a brief introduction to the method of moving frames on non-flat Riemannian manifolds and the additional issues that arise when the underlying space has nonzero curvature. These chapters may be read anytime after Chapter 4.

Exercises are embedded in the text rather than being presented at the end of each chapter. Readers are strongly encouraged to pause and attempt the exercises as they occur, as they are intended to engage the reader and to enhance the understanding of the text. Many of the exercises contain results which are important for understanding the remainder of the text; these exercises are marked with a star and should be given particular attention. (Even if you don't do them, you should at least read them!)

A special feature of this book is that it includes guidance on how to use the mathematical software package MAPLE to perform many of the computations involved in the exercises. (If you do not have access to MAPLE, rest assured that, with very few exceptions, the exercises can be done perfectly well by hand.) The computations here make use of the custom MAPLE package `Cartan`, which was written by myself and Yunliang Yu of Duke University. The `Cartan` package can be downloaded either from the AMS webpage

`www.ams.org/bookpages/gsm-178`

or from my webpage at

`http://euclid.colorado.edu/~jnc/Maple.html.`

(Installation instructions are included with the package.) The last section of Chapter 2 contains an introduction to the `Cartan` package, and beginning with Chapter 3, each chapter includes a section at the end describing how to use MAPLE and the `Cartan` package for some of the exercises in that chapter. Additional exercises are worked out in MAPLE worksheets for each chapter that are available on the AMS webpage.

Remark. As of MAPLE 16 and above, much of `Cartan`'s functionality is now available as part of the `DifferentialGeometry` package, which is included in the standard MAPLE installation and covers a wide range of applications. The two packages have very different syntax, and no attempt will be made here to translate—but interested readers are encouraged to do so!