Chapter 4

Curves and surfaces in Euclidean space

4.1. Introduction

In this chapter, we get to the heart of the matter: Cartan's method of moving frames. This method is used to study the geometry of submanifolds of homogeneous spaces; in this chapter, we will see how it applies to curves and surfaces in $\mathbb{E}^3$. The main idea goes something like this: By associating a frame to each point of a submanifold in some geometrically natural way and then studying how the frame varies along the submanifold, we can construct a complete set of *invariants* for a given class of submanifolds. Invariants are quantities associated to a submanifold (such as curvature and torsion for curves in $\mathbb{E}^3$) that remain unchanged when the submanifold is acted on by an element of the symmetry group of the homogeneous space. A *complete* set of invariants contains enough information to determine a submanifold uniquely up to the group action. This perspective naturally leads to two questions:

(1) How can we tell when we have found a complete set of invariants? This is a question about *uniqueness:* Given two submanifolds of a homogeneous space, when is it possible to transform one into the other via an element of the symmetry group? This is also known as the *equivalence problem:* When are two submanifolds equivalent under the action of the symmetry group?

(2) Once we know what a complete set of invariants should look like, can they be prescribed arbitrarily? This is a question about *existence:* Given prescribed values for the invariants, does there necessarily exist a submanifold whose invariants coincide with the given values?

In §4.2, we will address the theory underlying the first question, and in §4.3 we will show how it applies to curves in $\mathbb{E}^3$. Then in §4.4 and §4.5, we will take up the second question and show how the theory as a whole applies to surfaces in $\mathbb{E}^3$.

4.2. Equivalence of submanifolds of a homogeneous space

We will approach the equivalence problem for submanifolds of a homogeneous space G/H by considering the restriction of certain frames on the underlying space G/H to the submanifold in question.

Remark 4.1. If $M \subset G/H$ and $f : U \to G/H$ is an immersion with $f(U) = M$ (typically U is some open, connected, and simply connected region in $\mathbb{R}^n$ and f is a parametrization of M), then "restriction to M" really means pullback to U. The *pullback bundle* (or *induced bundle*) f^*G of the principal bundle $\pi : G \to G/H$ is the bundle over U whose fiber over a point $\mathbf{u} \in U$ is just the fiber of G over the point $f(\mathbf{u}) \in G/H$:

$$f^*G = \{(\mathbf{u}, \mathbf{f}) \in U \times G \mid f(\mathbf{u}) = \pi(\mathbf{f})\}.$$

The bundle f^*G is a principal bundle over U, with fiber group H. There is a natural map $\hat{f} : f^*G \to G$ defined by

$$\hat{f}(\mathbf{u}, \mathbf{f}) = \mathbf{f}.$$

When $\pi : G \to G/H$ is regarded as a frame bundle, the image $\hat{f}(\mathbf{u}, \mathbf{f})$ of any element $(\mathbf{u}, \mathbf{f}) \in f^*G$ may be thought of as a frame based at the point $f(\mathbf{u}) \in G/H$. These maps may be represented by the following commutative diagram:

$$\begin{tikzcd} f^*G \arrow[r, "\hat{f}"] \arrow[d, "\bar{\pi}"'] & G \arrow[d, "\pi"] \\ U \arrow[r, "f"] & G/H. \end{tikzcd}$$

We will generally be interested in choosing a frame at each point of $M \subset G/H$—i.e., a "frame field" on M—according to certain geometric considerations. Technically, this means choosing a section of the bundle f^*G over U, but we will usually regard it as choosing a *lifting* $\tilde{f} : U \to G$, i.e., a function $\tilde{f}$ with the property that for any $\mathbf{u} \in U$,

$$(\pi \circ \tilde{f})(\mathbf{u}) = f(\mathbf{u}) \in M \subset G/H.$$

In other words, we choose $\tilde{f}$ so that the following diagram commutes:

$$\begin{tikzcd} & G \arrow[d, "\pi"] \\ U \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & G/H. \end{tikzcd}$$

When choosing a lifting $\tilde{f} : U \to G$, we will want to choose frame fields that are *adapted* to M. This means that, instead of just choosing arbitrary frame fields, we will use the geometry of M to choose "nice" frame fields. This is somewhat analogous to choosing "nice" coordinates on a neighborhood of a point on a surface to study the geometry at that point; the beauty of the method of moving frames is that we can do this at all points simultaneously.

Once we have chosen a nice lifting (called an *adapted frame field*, or sometimes simply an *adapted frame*) $\tilde{f} : U \to G$, we can consider the pullback $\tilde{f}^*\omega$ of the Maurer-Cartan form ω of G and its structure equations to U. The pulled-back Maurer-Cartan form $\tilde{f}^*\omega$ will generally contain quantities that are *invariants* of M: If we act on M by a symmetry of the ambient space G/H, then these quantities remain unchanged. (The invariance of $\tilde{f}^*\omega$ under such an action follows from the left-invariance of ω under action by an element of G.) Typical examples of invariants are quantities such as arc length, curvature, etc.

In order for the adapted frame field $\tilde{f} : U \to G$ to contain useful information about the invariants of M, the algorithm for choosing $\tilde{f}$ should be completely determined in some canonical way by the geometry of M. Moreover, the adapted frame field itself should be *equivariant*; this means that

$$\widetilde{(g \cdot f)} = g \cdot \tilde{f}$$

for any $g \in G$. If such an equivariant adapted frame field exists, then the question of equivalence is completely answered by the following important lemma:

Lemma 4.2. *Let $U \subset \mathbb{R}^n$ be a connected, open set, and let $\tilde{f}_1, \tilde{f}_2 : U \to G$ be two immersions. Then there exists an element $g \in G$ such that*

$$\tilde{f}_1(\mathbf{u}) = g \cdot \tilde{f}_2(\mathbf{u})$$

for all $\mathbf{u} \in U$ if and only if $\tilde{f}_1^*\omega = \tilde{f}_2^*\omega$, where ω is the Maurer-Cartan form of G.

Proof. First, observe that for any map $\tilde{f} : U \to G$, we have

$$\tilde{f}^*\omega = \tilde{f}^{-1}d\tilde{f}. \tag{4.1}$$

Remark 4.3. What does equation (4.1) really mean? Recall that for any $g \in G$, ω is a linear map from T_gG to $T_eG = \mathfrak{g}$ defined by

$$\omega(\mathbf{w}) = (L_{g^{-1}})_*(\mathbf{w})$$

for $\mathbf{w} \in T_gG$. Now, if $\tilde{f} : U \to G$ is a differentiable map, then $\tilde{f}^*\omega$ is a linear map from $T_{\mathbf{u}}U$ to $\mathfrak{g}$ defined by

$$\tilde{f}^*\omega(\mathbf{v}) = \omega(\tilde{f}_*(\mathbf{v})) = (L_{(\tilde{f}(\mathbf{u}))^{-1}})_*(\tilde{f}_*(\mathbf{v})) = (\tilde{f}(\mathbf{u}))^{-1} \cdot d\tilde{f}(\mathbf{v})$$

for $\mathbf{v} \in T_{\mathbf{u}}U$. Therefore,

$$\tilde{f}^*\omega = \tilde{f}^{-1}d\tilde{f}.$$

Now, given $\tilde{f}_1, \tilde{f}_2 : U \to G$, there exists a unique function $g : U \to G$ satisfying the condition that

$$\tilde{f}_2(\mathbf{u}) = g(\mathbf{u})\tilde{f}_1(\mathbf{u}) \tag{4.2}$$

for all $\mathbf{u} \in U$—specifically, $g(\mathbf{u}) = \tilde{f}_2(\mathbf{u})(\tilde{f}_1(\mathbf{u}))^{-1}$. Differentiating (4.2) yields

$$d\tilde{f}_2 = dg\,\tilde{f}_1 + g\,d\tilde{f}_1;$$

therefore,

$$\begin{aligned}
\tilde{f}_2^*\omega &= \tilde{f}_2^{-1}d\tilde{f}_2 \\
&= \tilde{f}_2^{-1}dg\,\tilde{f}_1 + \tilde{f}_2^{-1}g\,d\tilde{f}_1 \\
&= \tilde{f}_2^{-1}dg\,\tilde{f}_1 + (g\tilde{f}_1)^{-1}g\,d\tilde{f}_1 \\
&= \tilde{f}_2^{-1}dg\,\tilde{f}_1 + \tilde{f}_1^{-1}g^{-1}g\,d\tilde{f}_1 \\
&= \tilde{f}_2^{-1}dg\,\tilde{f}_1 + \tilde{f}_1^{-1}d\tilde{f}_1 \\
&= \tilde{f}_2^{-1}dg\,\tilde{f}_1 + \tilde{f}_1^*\omega.
\end{aligned}$$

It follows that $\tilde{f}_1^*\omega = \tilde{f}_2^*\omega$ if and only if $dg = 0$, i.e., if and only if $g(\mathbf{u})$ is constant. □

This lemma is more powerful than it looks; it says that:

(1) Whatever geometric information is contained in $\tilde{f}^*\omega$ remains unchanged when M is transformed by a symmetry g of the ambient homogeneous space G/H.

(2) Conversely, $\tilde{f}^*\omega$ contains enough information about the geometry of M to *completely* determine it up to a symmetry of the ambient space.

So, our approach from here on will go something like this: Given an immersion $f : U \to G/H$, we will look for an equivariant method of constructing a canonical adapted frame field $\tilde{f} : U \to G$. Then we will examine the pulled-back Maurer-Cartan form $\tilde{f}^*\omega$, which will contain a complete set of geometric invariants for the original immersion f. This is known as the *method of moving frames*, and we will start by demonstrating how to carry it out for curves in $\mathbb{E}^3$.

4.3. Moving frames for curves in $\mathbb{E}^3$

Consider a smooth, parametrized curve $\alpha : I \to \mathbb{E}^3$ that maps some open interval $I \subset \mathbb{R}$ into Euclidean space. $\mathbb{E}^3$ has the structure of the homogeneous space $E(3)/SO(3)$, so an adapted frame field along α should be a lifting $\tilde{\alpha} : I \to E(3)$. Any such lifting can be written as

$$\tilde{\alpha}(t) = (\alpha(t); \mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t)),$$

where for each $t \in I$, $(\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t))$ is an oriented, orthonormal basis for the tangent space $T_{\alpha(t)}\mathbb{E}^3$. Such an adapted frame field is usually called an *orthonormal frame field* along α. If the curve is "nice enough" (the precise meaning of this will become clear shortly), then we will be able to choose such a frame field in a canonical way, based on the geometry of the curve.

Remark 4.4. While the orthonormal frame field is technically the image of the map $\tilde{\alpha}$ and so includes the position vector $\alpha(t)$ at each point, it is common to refer to the triple of vector fields $(\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t))$ as an "orthonormal frame field along α". Hopefully this terminology will not cause any confusion.

Recall that α is *regular* if $\alpha'(t) \neq \mathbf{0}$ for every $t \in I$. The first condition that we will require in order for α to be "nice enough" is that α must be a regular curve. With this assumption, we can make our first frame adaptation by setting

$$\mathbf{e}_1(t) = \frac{\alpha'(t)}{|\alpha'(t)|};$$

i.e., we require that $\mathbf{e}_1(t)$ be the unit tangent vector to the curve at $\alpha(t)$.

Exercise 4.5. Show that this choice of $\mathbf{e}_1(t)$ is equivariant under the action of $E(3)$: If we replace α by $g \cdot \alpha$ for some $g \in E(3)$, then $\mathbf{e}_1(t) \in T_{\alpha(t)}\mathbb{E}^3$ will be replaced by $(L_g)_*(\mathbf{e}_1(t)) \in T_{g\cdot\alpha(t)}\mathbb{E}^3$.

The vector $\mathbf{e}_1(t)$ is now uniquely determined, but we still have the freedom to vary the pair $(\mathbf{e}_2(t), \mathbf{e}_3(t))$ by an arbitrary rotation in $SO(2)$. We will need to delve deeper into the geometry of the curve α in order to determine how to choose the remainder of the adapted frame field.

Here we make an observation that will simplify the remainder of our computations. Fix $t_0 \in I$ and define the *arc length* function along α to be

$$s(t) = \int_{t_0}^{t} |\alpha'(u)|\, du.$$

Exercise 4.6. Show that $s(t)$ is invariant under the action of $E(3)$; that is, for any $g \in E(3)$, the curves α and $g \cdot \alpha$ have the same arc length function.

Since $\alpha'(t) \neq \mathbf{0}$ for all $t \in I$, the inverse function theorem implies that $s(t)$ has a differentiable inverse function $t(s)$. By setting $\alpha(s) = \alpha(t(s))$, we may assume that α is parametrized by arc length, so that $|\alpha'(s)| = 1$ and $\mathbf{e}_1(s) = \alpha'(s)$.

In order to make the next adaptation, we need to make another assumption about the curve. We will say that α is *nondegenerate* if α is regular and, in addition, $\mathbf{e}_1'(s) \neq \mathbf{0}$ for all $s \in I$. In this case, differentiating the equation

$$\langle \mathbf{e}_1(s), \mathbf{e}_1(s) \rangle = 1$$

with respect to s yields

$$\langle \mathbf{e}_1'(s), \mathbf{e}_1(s) \rangle = 0.$$

Thus, $\mathbf{e}_1'(s)$ is orthogonal to $\mathbf{e}_1(s)$, and we can make our second adaptation by setting

$$\mathbf{e}_2(s) = \frac{\mathbf{e}_1'(s)}{|\mathbf{e}_1'(s)|}.$$

This vector is called the *unit normal vector* to the curve at $\alpha(s)$.

Exercise 4.7. Show that $\mathbf{e}_2(s)$ is equivariant under the action of $E(3)$: If we replace α by $g \cdot \alpha$ for some $g \in E(3)$, then $\mathbf{e}_2(s) \in T_{\alpha(s)}\mathbb{E}^3$ will be replaced by $(L_g)_*(\mathbf{e}_2(s)) \in T_{g\cdot\alpha(s)}\mathbb{E}^3$.

The adapted frame field is now uniquely determined: Because the frame must be oriented and orthonormal, $\mathbf{e}_3(s)$ is uniquely determined by the condition that

$$\mathbf{e}_3(s) = \mathbf{e}_1(s) \times \mathbf{e}_2(s).$$

The vector $\mathbf{e}_3(s)$ is called the *binormal vector* to the curve at $\alpha(s)$. The adapted frame field $(\mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s))$ is called the *Frenet frame* of the curve $\alpha(s)$; it determines a canonical, left-invariant lifting $\tilde{\alpha} : I \to E(3)$ given by

$$\tilde{\alpha}(s) = (\alpha(s); \mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s))$$

for any nondegenerate curve α parametrized by arc length. (See Figure 4.1.)

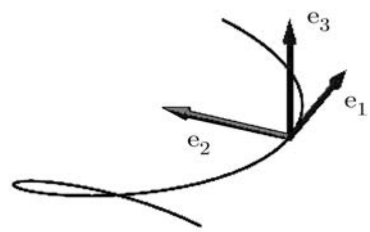

Figure 4.1. Frenet frame at a point of a curve in $\mathbb{E}^3$

Now consider the pullbacks of equations (3.1) to I via $\tilde{\alpha}$. We have

$$\tilde{\alpha}^*(\mathbf{x}) = \alpha(s), \qquad \tilde{\alpha}^*(\mathbf{e}_i) = \mathbf{e}_i(s).$$

Therefore,

$$\begin{aligned}\tilde{\alpha}^*(d\mathbf{x}) &= d(\tilde{\alpha}^*(\mathbf{x})) = \alpha'(s)ds,\\ \tilde{\alpha}^*(d\mathbf{e}_i) &= d(\tilde{\alpha}^*(\mathbf{e}_i)) = \mathbf{e}_i'(s)ds.\end{aligned}$$

As in Chapter 3, write $(\bar{\omega}^i, \bar{\omega}^i_j)$ for the pulled-back forms $(\tilde{\alpha}^*\omega^i, \tilde{\alpha}^*\omega^i_j)$. Note that, since these are all 1-forms on I, they must all be multiples of ds.

We can write the pullbacks of equations (3.1) as

$$\begin{aligned}\alpha'(s)ds &= \mathbf{e}_i(s)\,\bar{\omega}^i,\\ \mathbf{e}_i'(s)ds &= \mathbf{e}_j(s)\,\bar{\omega}^j_i.\end{aligned} \tag{4.3}$$

Now recall how we constructed our adapted frame field. First, we chose $\mathbf{e}_1(s)$ so that $\alpha'(s) = \mathbf{e}_1(s)$; therefore, the first equation in (4.3) implies that

$$\bar{\omega}^1 = ds, \qquad \bar{\omega}^2 = \bar{\omega}^3 = 0.$$

Then we chose $\mathbf{e}_2(s)$ so that $\mathbf{e}_1'(s)$ is a multiple of $\mathbf{e}_2(s)$, say $\mathbf{e}_1'(s) = \kappa(s)\mathbf{e}_2(s)$. The function $\kappa(s)$ is called the *curvature* of α at s; note that α is nondegenerate if and only if $\kappa(s) > 0$ for all $s \in I$. So the equation for $\mathbf{e}_1'(s)$ in (4.3) implies that

$$\bar{\omega}^2_1 = \kappa(s)ds, \qquad \bar{\omega}^3_1 = 0.$$

(Recall that $\bar{\omega}^1_1 = 0$ by the skew-symmetry of the (ω^i_j).) The only remaining Maurer-Cartan form is $\bar{\omega}^2_3$; it must be equal to some multiple of ds, so define a function $\tau(s)$ by the condition that

$$\bar{\omega}^2_3 = -\tau(s)ds.$$

The function $\tau(s)$ is called the *torsion* of α at s.

Remark 4.8. The minus sign in the definition of $\tau(s)$ is a convention preferred by some authors, but it is not universal. This choice of sign has the feature that it results in a positive value of τ for the standard right-handed helix

$$\alpha(s) = {}^t[a\cos(s), a\sin(s), b],$$

where $a, b > 0$ and $a^2 + b^2 = 1$.

Using the skew-symmetry of the $(\bar{\omega}^i_j)$, the remaining two equations in (4.3) become

$$\begin{aligned}
\mathbf{e}_2'(s)ds &= \mathbf{e}_1(s)\,\bar{\omega}^1_2 + \mathbf{e}_3(s)\,\bar{\omega}^3_2 = (-\mathbf{e}_1(s)\kappa(s) + \mathbf{e}_3(s)\tau(s))ds,\\
\mathbf{e}_3'(s)ds &= \mathbf{e}_1(s)\,\bar{\omega}^1_3 + \mathbf{e}_2(s)\,\bar{\omega}^2_3 = -\mathbf{e}_2(s)\tau(s)ds.
\end{aligned}$$

Thus, we have the familiar *Frenet equations*:

$$\begin{bmatrix}\alpha'(s) & \mathbf{e}_1'(s) & \mathbf{e}_2'(s) & \mathbf{e}_3'(s)\end{bmatrix} = \begin{bmatrix}\alpha(s) & \mathbf{e}_1(s) & \mathbf{e}_2(s) & \mathbf{e}_3(s)\end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -\kappa(s) & 0 \\ 0 & \kappa(s) & 0 & -\tau(s) \\ 0 & 0 & \tau(s) & 0 \end{bmatrix}.$$

Note that if we regard $\tilde{\alpha}(s)$ as the matrix

$$\tilde{\alpha}(s) = \begin{bmatrix}\alpha(s) & \mathbf{e}_1(s) & \mathbf{e}_2(s) & \mathbf{e}_3(s)\end{bmatrix},$$

then the matrix on the right multiplied by the 1-form ds is equal to

$$\tilde{\alpha}(s)^{-1}d(\tilde{\alpha}(s)),$$

and so it is exactly the pullback of the Maurer-Cartan form $\omega = g^{-1}dg$ on $E(3)$ via $\tilde{\alpha}$.

Applying Lemma 4.2 yields the following theorem:

Theorem 4.9. *Two nondegenerate curves $\alpha_1, \alpha_2 : I \to \mathbb{E}^3$ parametrized by arc length differ by a rigid motion if and only if they have the same curvature $\kappa(s)$ and torsion $\tau(s)$.*

This is the uniqueness portion of the fundamental theorem of space curves (cf. Theorem 3.1). We will address the existence portion in §4.4.

Exercise 4.10. Repeat the analysis of this section for curves in $\mathbb{E}^4$. Here are some things to think about along the way:

- Is there a natural choice of parametrization for the curve?
- How should you choose the vectors $(\mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s), \mathbf{e}_4(s))$ of the frame field? (And how do you ensure that these vectors form an orthonormal frame field?) Prove that your choice is equivariant

under the action of $E(4)$. (Hint: The tricky part is how to choose $\mathbf{e}_3(s)$ so that it is orthogonal to both $\mathbf{e}_1(s)$ and $\mathbf{e}_2(s)$. For guidance, use the Frenet equations to convince yourself that for curves in $\mathbb{E}^3$,

$$\mathbf{e}_3(s) = \frac{\mathbf{e}_2'(s) - \langle \mathbf{e}_2'(s), \mathbf{e}_1(s)\rangle \mathbf{e}_1(s)}{|\mathbf{e}_2'(s) - \langle \mathbf{e}_2'(s), \mathbf{e}_1(s)\rangle \mathbf{e}_1(s)|}.$$

In other words, $\mathbf{e}_3(s)$ is obtained by taking the orthogonal projection of $\mathbf{e}_2'(s)$ onto the orthogonal complement of $\mathbf{e}_1(s)$ and $\mathbf{e}_2(s)$ and then normalizing it to have unit length.)

- What is the right definition of "nondegenerate" for curves in $\mathbb{E}^4$?
- Where do invariants appear in the pullbacks of equations (3.1)? What can you conclude from the skew-symmetry of the connection forms?
- What is the 4-dimensional analog of the Frenet equations?
- How do you think the analysis would go for curves in $\mathbb{E}^n$?

4.4. Compatibility conditions and existence of submanifolds with prescribed invariants

In §4.3, we saw that a curve $\alpha : I \to \mathbb{E}^3$ parametrized by arc length s is completely determined up to rigid motions of $\mathbb{E}^3$ by its curvature $\kappa(s)$ and torsion $\tau(s)$. We may express this by saying that the curvature and torsion form a *complete set of invariants* for curves in $\mathbb{E}^3$.

In general, Lemma 4.2 tells us when we have found a complete set of invariants for a "nice" immersion $f : U \to G/H$: Assuming that we can find a canonical, equivariant way of choosing a lifting $\tilde{f} : U \to G$ (this is what "nice" means), a complete set of invariants is contained in $\tilde{f}^*\omega$, the pullback via $\tilde{f}$ of the Maurer-Cartan form ω of G.

For curves in $\mathbb{E}^3$, it is now natural to ask whether the functions $\kappa(s)$ and $\tau(s)$ may be prescribed arbitrarily. In other words, given arbitrary functions $\kappa(s), \tau(s)$ with $\kappa(s) > 0$, does there necessarily exist a curve $\alpha : I \to \mathbb{E}^3$ that is parametrized by arc length and has curvature $\kappa(s)$ and torsion $\tau(s)$?

Exercise 4.11. Why must we require $\kappa(s) > 0$?

The answer to this existence question is yes, but this result is particular to 1-dimensional submanifolds of homogeneous spaces G/H. It follows from the following lemma:

Lemma 4.12. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and suppose that $\bar{\omega}$ is a $\mathfrak{g}$-valued 1-form on a connected and simply connected manifold U. Then there exists a smooth map $\tilde{f} : U \to G$ with $\tilde{f}^*\omega = \bar{\omega}$ if and only if $\bar{\omega}$ satisfies the Maurer-Cartan equation*

$$d\bar{\omega} = -\bar{\omega} \wedge \bar{\omega}. \tag{4.4}$$

Outline of Proof. The full proof of this lemma requires the Frobenius theorem and is beyond the scope of this book. (If you're curious, the proof may be found in [**Gri74**].) However, the main idea goes something like this: $\tilde{f}^*\omega$ contains quantities involving derivatives of the unknown function $\tilde{f} : U \to G$, and for any given $\mathfrak{g}$-valued 1-form $\bar{\omega}$ on U, the equation $\tilde{f}^*\omega = \bar{\omega}$ may be regarded as a system of partial differential equations for $\tilde{f}$. In general, this system is overdetermined and may have no solutions. However, equation (4.4) is precisely the compatibility condition that must be satisfied in order to guarantee that solutions exist, at least locally. (In this case, it turns out that a solution is uniquely determined by specifying an initial condition $\tilde{f}(\mathbf{u}_0)$ for any $\mathbf{u}_0 \in U$.) Once we know that local solutions exist, a patching argument can be used to construct a solution $\tilde{f}$ on the entire domain U. □

Remark 4.13. Even without the hypothesis that U is simply connected, the result of Lemma 4.12 holds in some neighborhood of any point $\mathbf{u} \in U$; simple connectivity is only necessary to ensure that these local solutions can be patched together to form a single solution that is globally defined on U. For simplicity of exposition, we will not explicitly state topological hypotheses on the domain U every time we introduce an immersion $f : U \to G/H$. But keep in mind that if U is topologically nontrivial, then many of our constructions may be possible only locally and not globally on U. For example, because a frame bundle over a topologically nontrivial base space may have no global sections, it might not be possible to construct an adapted frame field globally on U. Because of these limitations, the method of moving frames is a tool best suited to the study of the *local* geometry of submanifolds of homogeneous spaces; it has very little to say about global properties.

Assuming that the conditions of Lemma 4.12 are satisfied, composing the map $\tilde{f}$ with the natural projection $\pi : G \to G/H$ gives a smooth map $f : U \to G/H$ that, in most cases of interest, realizes $M = f(U)$ as a submanifold of the homogeneous space G/H. According to Lemma 4.2, specifying a $\mathfrak{g}$-valued 1-form $\bar{\omega}$ on U is equivalent to prescribing the values of a complete set of invariants for an unknown submanifold $M \subset G/H$;

Lemma 4.12 then gives a necessary and sufficient condition for the existence of a smooth map $f : U \to G/H$ whose image has the prescribed invariants. Moreover, Lemma 4.2 implies that any such f is unique up to left action by an element $g \in G$.

***Exercise 4.14.** Let $(\omega^1, \ldots, \omega^n)$ be a basis for the left-invariant 1-forms on G that are semi-basic for the projection $\pi : G \to G/H$, and let $(\bar{\omega}^1, \ldots, \bar{\omega}^n)$ be the corresponding components of the $\mathfrak{g}$-valued 1-form $\bar{\omega}$ on U. Show that the map $f = \pi \circ \tilde{f}$ of Lemma 4.12 is an immersion if and only if $(\bar{\omega}^1, \ldots, \bar{\omega}^n)$ span the cotangent space $T^*_{\mathbf{u}}U$ at every point $\mathbf{u} \in U$. (Note that typically the dimension of U is less than n, so the forms $(\bar{\omega}^1, \ldots, \bar{\omega}^n)$ will generally *not* be linearly independent.)

Corollary 4.15. *Let $I \subset \mathbb{R}$ be an open interval, and let $\kappa, \tau : I \to \mathbb{R}$ be any differentiable functions satisfying $\kappa(s) > 0$ for all $s \in I$. Then there exists a nondegenerate curve $\alpha : I \to \mathbb{E}^3$, parametrized by arc length, with curvature $\kappa(s)$ and torsion $\tau(s)$.*

***Exercise 4.16.** Show how Corollary 4.15 follows from Lemma 4.12. (Hint: Observe that both sides of equation (4.4) are 2-forms on I.)

Corollary 4.15 applies more generally to curves in any homogeneous space G/H: Once we know how to construct equivariant frame fields and find a complete set of invariants, Lemma 4.12 implies that these invariants may be prescribed arbitrarily. But for surfaces (and generally for submanifolds of any dimension greater than one), equation (4.4) will give compatibility conditions that a prescribed set of invariants must satisfy in order for an immersed submanifold with the given invariants to exist.

4.5. Moving frames for surfaces in $\mathbb{E}^3$

Let U be an open set in $\mathbb{R}^2$ (assumed here and throughout the remainder of the book to be connected and simply connected; cf. Remark 4.13), and let $\mathbf{x} : U \to \mathbb{E}^3$ be an immersion whose image is a regular surface $\Sigma = \mathbf{x}(U)$. Just as for curves, an adapted frame field along Σ should be a lifting $\tilde{\mathbf{x}} : U \to E(3)$ of the form

$$\tilde{\mathbf{x}}(\mathbf{u}) = (\mathbf{x}(\mathbf{u}); \mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))\,,$$

where for each $\mathbf{u} \in U$, $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$ is an oriented, orthonormal basis for the tangent space $T_{\mathbf{x}(\mathbf{u})}\mathbb{E}^3$.

Since $\mathbf{x}$ is an immersion, there is a well-defined tangent plane $T_{\mathbf{x}(\mathbf{u})}\Sigma$ for each point $\mathbf{x}(\mathbf{u}) \in \Sigma$. Thus, we can make our first frame adaptation by requiring that $\mathbf{e}_3(\mathbf{u})$ be orthogonal to $T_{\mathbf{x}(\mathbf{u})}\Sigma$. This determines $\mathbf{e}_3(\mathbf{u})$ uniquely up to sign, and an orthonormal frame field satisfying this condition will be called *adapted.*

Exercise 4.17. Show that this choice of $\mathbf{e}_3(\mathbf{u})$ is equivariant (up to sign) under the action of $E(3)$.

Having chosen $\mathbf{e}_3(\mathbf{u})$ in this way, $\mathbf{e}_1(\mathbf{u})$ and $\mathbf{e}_2(\mathbf{u})$ must form a basis for $T_{\mathbf{x}(\mathbf{u})}\Sigma$ no matter how we choose them. We will explore how we might refine our choices later, but for now, we allow $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}))$ to be an arbitrary orthonormal basis of $T_{\mathbf{x}(\mathbf{u})}\Sigma$.

Now consider the pullbacks of equations (3.1) to U via $\tilde{\mathbf{x}}$. (As in Chapter 3, write $(\bar{\omega}^i, \bar{\omega}^i_j)$ for the pulled-back forms $(\tilde{\mathbf{x}}^*\omega^i, \tilde{\mathbf{x}}^*\omega^i_j)$ on U.) Our first observation about these forms is the following:

Proposition 4.18. *Let $U \subset \mathbb{R}^2$ be an open set, and let $\mathbf{x} : U \to \mathbb{E}^3$ be an immersion. For any adapted orthonormal frame field $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$ along $\Sigma = \mathbf{x}(U)$, the associated dual and connection forms $(\bar{\omega}^i, \bar{\omega}^i_j)$ have the property that $\bar{\omega}^3 = 0$.*

Proof. The pullback of the first equation in (3.1) is

$$d\mathbf{x} = \mathbf{e}_i \bar{\omega}^i.$$

Let $\mathbf{u} \in U$. Then $d\mathbf{x}_\mathbf{u}$ is a linear map from $T_\mathbf{u}U$ to $T_{\mathbf{x}(\mathbf{u})}\Sigma$, and so for any $\mathbf{v} \in T_\mathbf{u}U$, we must have

$$d\mathbf{x}_\mathbf{u}(\mathbf{v}) = \mathbf{e}_i(\mathbf{u})\bar{\omega}^i(\mathbf{v}) \in T_{\mathbf{x}(\mathbf{u})}\Sigma.$$

Since $T_{\mathbf{x}(\mathbf{u})}\Sigma$ is spanned by $\mathbf{e}_1(\mathbf{u})$ and $\mathbf{e}_2(\mathbf{u})$, the $\mathbf{e}_3(\mathbf{u})$ term in this sum must vanish; therefore, $\bar{\omega}^3(\mathbf{v}) = 0$. And since $\mathbf{v} \in T_\mathbf{u}U$ is arbitrary, it follows that $\bar{\omega}^3 = 0$. $\square$

***Exercise 4.19.** Show that $(\bar{\omega}^1, \bar{\omega}^2)$ are linearly independent 1-forms on U. Therefore, they form a basis for the 1-forms on U. (Hint: Evaluate $\bar{\omega}^1$ and $\bar{\omega}^2$ on vectors $\mathbf{v}_1, \mathbf{v}_2 \in T_\mathbf{u}U$ with the property that $d\mathbf{x}(\mathbf{v}_i) = \mathbf{e}_i(\mathbf{u})$ for $i = 1, 2$.)

You may recall that the metric properties of a regular surface in $\mathbb{E}^3$ are encapsulated in the *first fundamental form* of the surface.

Definition 4.20. Let $U \subset \mathbb{R}^2$ be an open set, and let $\mathbf{x} : U \to \mathbb{E}^3$ be an immersion. The *first fundamental form* of $\Sigma = \mathbf{x}(U)$ is the quadratic form I on TU defined by

$$\mathrm{I}(\mathbf{v}) = \langle d\mathbf{x}(\mathbf{v}), d\mathbf{x}(\mathbf{v})\rangle$$

for $\mathbf{v} \in T_\mathbf{u}U$.

In other words, I is just the restriction of the Euclidean metric on $\mathbb{E}^3$ to vectors which are tangent to Σ. Its primary function is to describe how to compute this metric in terms of the local coordinates $\mathbf{u}$ on Σ that are given by the parametrization $\mathbf{x} : U \to \mathbb{E}^3$.

***Exercise 4.21.** Show that for any $\mathbf{v} \in T_{\mathbf{u}}U$,

$$\mathrm{I}(\mathbf{v}) = \left(\bar{\omega}^1(\mathbf{v})\right)^2 + \left(\bar{\omega}^2(\mathbf{v})\right)^2.$$

This is often written more concisely as

$$\mathrm{I} = (\bar{\omega}^1)^2 + (\bar{\omega}^2)^2,$$

and each term in the sum should be interpreted as the *symmetric* product $\bar{\omega}^i \circ \bar{\omega}^i$.

While the first fundamental form is defined as a function of a single tangent vector, it can be used to define an inner product $\langle \cdot, \cdot \rangle_{\mathbf{u}}$ on each tangent space $T_{\mathbf{u}}U$ through a process called *polarization*.

Definition 4.22. The inner product $\langle \cdot, \cdot \rangle_{\mathbf{u}}$ is defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{u}} = \tfrac{1}{4}\left(\mathrm{I}(\mathbf{v} + \mathbf{w}) - \mathrm{I}(\mathbf{v} - \mathbf{w})\right)$$

for $\mathbf{v}, \mathbf{w} \in T_{\mathbf{u}}U$.

***Exercise 4.23.** (a) Show that

$$\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{u}} = \bar{\omega}^1(\mathbf{v})\, \bar{\omega}^1(\mathbf{w}) + \bar{\omega}^2(\mathbf{v})\, \bar{\omega}^2(\mathbf{w}).$$

(b) Convince yourself that $\langle \cdot, \cdot \rangle_{\mathbf{u}}$ is a section of the symmetric tensor bundle $S^2(T^*U)$. Any section of this bundle defines a symmetric bilinear form

$$B : TU \times TU \to \mathbb{R},$$

which in turn defines a quadratic form

$$Q : TU \to \mathbb{R}$$

by setting $Q(\mathbf{v}) = B(\mathbf{v}, \mathbf{v})$.

***Exercise 4.24.** If you've seen the first fundamental form before, you probably saw it written as

$$\mathrm{I} = E\, du^2 + 2F\, du\, dv + G\, dv^2,$$

where (u, v) are local coordinates on U and

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \qquad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \qquad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle.$$

Suppose that $\mathbf{x} : U \to \mathbb{E}^3$ is an immersion with $F = 0$. (Such a parametrization for a given surface Σ always exists, at least locally; the proof is beyond the scope of this book but can be found in [**dC76**]. This assumption isn't necessary, but it keeps the calculations simpler.)

(a) Show that the frame field

$$\mathbf{e}_1(\mathbf{u}) = \frac{1}{\sqrt{E}}\,\mathbf{x}_u, \qquad \mathbf{e}_2(\mathbf{u}) = \frac{1}{\sqrt{G}}\,\mathbf{x}_v, \qquad \mathbf{e}_3(\mathbf{u}) = \mathbf{e}_1(\mathbf{u}) \times \mathbf{e}_2(\mathbf{u})$$

is an oriented, orthonormal frame field along $\Sigma = \mathbf{x}(U)$, with $\mathbf{e}_3(\mathbf{u})$ orthogonal to $T_{\mathbf{x}(\mathbf{u})}\Sigma$.

(b) Show that the dual forms of this frame field are

$$\bar\omega^1 = \sqrt{E}\,du, \qquad \bar\omega^2 = \sqrt{G}\,dv, \qquad \bar\omega^3 = 0$$

and that

$$\mathrm{I} = (\bar\omega^1)^2 + (\bar\omega^2)^2 = E\,du^2 + G\,dv^2.$$

(c) Show that if $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$ is replaced by another adapted frame field $(\tilde{\mathbf{e}}_1(\mathbf{u}), \tilde{\mathbf{e}}_2(\mathbf{u}), \tilde{\mathbf{e}}_3(\mathbf{u}))$ of the form

$$\begin{aligned}\tilde{\mathbf{e}}_1 &= \cos(\theta)\,\mathbf{e}_1 + \sin(\theta)\,\mathbf{e}_2,\\ \tilde{\mathbf{e}}_2 &= -\sin(\theta)\,\mathbf{e}_1 + \cos(\theta)\,\mathbf{e}_2,\\ \tilde{\mathbf{e}}_3 &= \mathbf{e}_3,\end{aligned}$$

then the dual forms $(\tilde{\bar\omega}^1, \tilde{\bar\omega}^2)$ of the new adapted frame field are

$$\begin{aligned}\tilde{\bar\omega}^1 &= \cos(\theta)\,\bar\omega^1 + \sin(\theta)\,\bar\omega^2,\\ \tilde{\bar\omega}^2 &= -\sin(\theta)\,\bar\omega^1 + \cos(\theta)\,\bar\omega^2.\end{aligned}$$

Moreover,

$$\mathrm{I} = (\bar\omega^1)^2 + (\bar\omega^2)^2 = (\tilde{\bar\omega}^1)^2 + (\tilde{\bar\omega}^2)^2.$$

Now let's see what we can learn by differentiating! Since $\bar\omega^3 = 0$, we must have $d\bar\omega^3 = 0$ as well. According to the Cartan structure equations (3.8), this implies that

$$d\bar\omega^3 = -\bar\omega^3_1 \wedge \bar\omega^1 - \bar\omega^3_2 \wedge \bar\omega^2 = 0.$$

Since $(\bar\omega^1, \bar\omega^2)$ are linearly independent 1-forms, Cartan's lemma (cf. Lemma 2.49) implies that there exist real-valued functions h_{11}, h_{12}, h_{22} on U such that

$$\begin{bmatrix}\bar\omega^3_1\\ \bar\omega^3_2\end{bmatrix} = \begin{bmatrix}h_{11} & h_{12}\\ h_{12} & h_{22}\end{bmatrix}\begin{bmatrix}\bar\omega^1\\ \bar\omega^2\end{bmatrix}.$$

How should we interpret the functions (h_{ij})? Recall that

$$d\mathbf{e}_3 = \mathbf{e}_1\omega^1_3 + \mathbf{e}_2\omega^2_3 = -(\mathbf{e}_1\omega^3_1 + \mathbf{e}_2\omega^3_2).$$

For any tangent vector $\mathbf{w} \in T_{\mathbf{x}}\Sigma$, $d\mathbf{e}_3(\mathbf{w})$ measures the directional derivative of the normal vector field $\mathbf{e}_3$ in the direction of $\mathbf{w}$. So, up to sign, $\omega^3_1(\mathbf{w})$ measures the $\mathbf{e}_1$ component of this directional derivative, and $\omega^3_2(\mathbf{w})$ measures its $\mathbf{e}_2$ component. In other words, $\omega^3_i(\mathbf{w})$ measures how rapidly $\mathbf{e}_3$

rotates towards $\mathbf{e}_i$ if we move in the direction $\mathbf{w}$. When we pull everything back to U via the parametrization $\mathbf{x}$ and express $\bar{\omega}_i^3$ as a linear combination of $\bar{\omega}^1$ and $\bar{\omega}^2$, we see that h_{ij} measures how rapidly $\mathbf{e}_3$ rotates towards $\mathbf{e}_i$ if we move in the direction $\mathbf{e}_j$.

Recall that, in addition to the metric properties of a regular surface, there are various types of curvature that arise from the geometry of the *Gauss map* of the surface. This is the map from the surface to the unit sphere $\mathbb{S}^2 \subset \mathbb{E}^3$ that sends any point of the surface to the unit normal vector of the surface at that point. In our context, it can be defined as follows:

Definition 4.25. Let $U \subset \mathbb{R}^2$ be an open set, and let $\mathbf{x} : U \to \mathbb{E}^3$ be an immersion with image $\Sigma = \mathbf{x}(U)$. The *Gauss map* of $\Sigma = \mathbf{x}(U)$ is the map $N : \Sigma \to \mathbb{S}^2$ defined by

$$N(\mathbf{x}(\mathbf{u})) = \mathbf{e}_3(\mathbf{u}),$$

where $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$ is any adapted frame field on $\Sigma = \mathbf{x}(U)$. (Note that N is well-defined up to sign.)

Notions of curvature typically associated with surfaces (Gauss curvature, mean curvature, etc.) arise as linear-algebraic properties of the differential dN of the Gauss map, also known as the *shape operator* of the surface. The relevant information is contained in the *second fundamental form* of the surface.

Definition 4.26. Let $U \subset \mathbb{R}^2$ be an open set, and let $\mathbf{x} : U \to \mathbb{E}^3$ be an immersion. The *second fundamental form* of $\Sigma = \mathbf{x}(U)$ is the quadratic form II on TU defined by

$$\mathrm{II}(\mathbf{v}) = -\langle d\mathbf{e}_3(\mathbf{v}), d\mathbf{x}(\mathbf{v})\rangle$$

for $\mathbf{v} \in T_{\mathbf{u}}U$, where $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$ is any adapted frame field on $\Sigma = \mathbf{x}(U)$.

Since curvature is related to how rapidly the normal vector varies as we move around the surface, we might expect the functions (h_{ij}) to show up in the second fundamental form.

***Exercise 4.27.** (a) Show that for any $\mathbf{v} \in T_{\mathbf{u}}U$,

$$\begin{aligned}\mathrm{II}(\mathbf{v}) &= \bar{\omega}_1^3(\mathbf{v})\,\bar{\omega}^1(\mathbf{v}) + \bar{\omega}_2^3(\mathbf{v})\,\bar{\omega}^2(\mathbf{v}) \\ &= h_{11}\,(\bar{\omega}^1(\mathbf{v}))^2 + 2h_{12}\,\bar{\omega}^1(\mathbf{v})\,\bar{\omega}^2(\mathbf{v}) + h_{22}\,(\bar{\omega}^2(\mathbf{v}))^2.\end{aligned}$$

This is often written more concisely as

$$\mathrm{II} = \bar{\omega}_1^3\,\bar{\omega}^1 + \bar{\omega}_2^3\,\bar{\omega}^2 = h_{11}\,(\bar{\omega}^1)^2 + 2h_{12}\,\bar{\omega}^1\,\bar{\omega}^2 + h_{22}\,(\bar{\omega}^2)^2.$$

(b) Suppose that $\mathbf{x} : U \to \mathbb{E}^3$ is a parametrization with $F = 0$, and let $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$ be the frame field in part (a) of Exercise 4.24. Show that

$$\mathrm{II} = Eh_{11}\, du^2 + 2\sqrt{EG}h_{12}\, du\, dv + Gh_{22}\, dv^2.$$

(c) The second fundamental form is more commonly written as

$$\mathrm{II} = e\, du^2 + 2f\, du\, dv + g\, dv^2,$$

where

$$\begin{aligned} e &= \langle \mathbf{e}_3, \mathbf{x}_{uu} \rangle = -\langle (\mathbf{e}_3)_u, \mathbf{x}_u \rangle = -\langle d\mathbf{e}_3\left(\tfrac{\partial}{\partial u}\right), d\mathbf{x}\left(\tfrac{\partial}{\partial u}\right) \rangle, \\ f &= \langle \mathbf{e}_3, \mathbf{x}_{uv} \rangle = -\langle (\mathbf{e}_3)_v, \mathbf{x}_u \rangle = -\langle d\mathbf{e}_3\left(\tfrac{\partial}{\partial v}\right), d\mathbf{x}\left(\tfrac{\partial}{\partial u}\right) \rangle \\ &\qquad\qquad\;\; = -\langle (\mathbf{e}_3)_u, \mathbf{x}_v \rangle = -\langle d\mathbf{e}_3\left(\tfrac{\partial}{\partial u}\right), d\mathbf{x}\left(\tfrac{\partial}{\partial v}\right) \rangle, \\ g &= \langle \mathbf{e}_3, \mathbf{x}_{vv} \rangle = -\langle (\mathbf{e}_3)_v, \mathbf{x}_v \rangle = -\langle d\mathbf{e}_3\left(\tfrac{\partial}{\partial v}\right), d\mathbf{x}\left(\tfrac{\partial}{\partial v}\right) \rangle. \end{aligned}$$

(Some authors use ℓ, m, n or L, M, N in place of e, f, g.) Show that this agrees with Definition 4.26, and conclude that

$$h_{11} = \frac{e}{E}, \qquad h_{12} = \frac{f}{\sqrt{EG}}, \qquad h_{22} = \frac{g}{G}.$$

Now, we still haven't figured out how we should choose the vectors $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}))$, except that they should form an orthonormal basis for $T_{\mathbf{x}(\mathbf{u})}\Sigma$ at each point. In order to refine our adapted frame field further, we will examine how the matrix $[h_{ij}]$ changes if we vary the frame. So, let $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$ be any adapted frame field, with associated Maurer-Cartan forms $(\bar{\omega}^i, \bar{\omega}^i_j)$. Any other adapted frame field $(\tilde{\mathbf{e}}_1(\mathbf{u}), \tilde{\mathbf{e}}_2(\mathbf{u}), \tilde{\mathbf{e}}_3(\mathbf{u}))$ has the form (up to sign)

$$\begin{bmatrix} \tilde{\mathbf{e}}_1(\mathbf{u}) & \tilde{\mathbf{e}}_2(\mathbf{u}) & \tilde{\mathbf{e}}_3(\mathbf{u}) \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1(\mathbf{u}) & \mathbf{e}_2(\mathbf{u}) & \mathbf{e}_3(\mathbf{u}) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for some function θ on U. Let $(\tilde{\bar{\omega}}^i, \tilde{\bar{\omega}}^i_j)$ be the Maurer-Cartan forms associated to the new frame field, and set $B = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

***Exercise 4.28.** (a) Show that the result in part (c) of Exercise 4.24 can be expressed as

$$\begin{bmatrix} \tilde{\bar{\omega}}^1 \\ \tilde{\bar{\omega}}^2 \end{bmatrix} = B^{-1} \begin{bmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \end{bmatrix}. \tag{4.5}$$

(b) Show that

$$\begin{bmatrix} \tilde{\bar{\omega}}^3_1 \\ \tilde{\bar{\omega}}^3_2 \end{bmatrix} = B^{-1} \begin{bmatrix} \bar{\omega}^3_1 \\ \bar{\omega}^3_2 \end{bmatrix} = {}^tB \begin{bmatrix} \bar{\omega}^3_1 \\ \bar{\omega}^3_2 \end{bmatrix}. \tag{4.6}$$

(Hint: Use the equation for $d\mathbf{e}_3$ in (3.1).)

(c) Cartan's lemma implies that there exist functions $\tilde{h}_{11}, \tilde{h}_{12}, \tilde{h}_{22}$ on U such that

$$\begin{bmatrix} \tilde{\bar{\omega}}^3_1 \\ \tilde{\bar{\omega}}^3_2 \end{bmatrix} = \begin{bmatrix} \tilde{h}_{11} & \tilde{h}_{12} \\ \tilde{h}_{12} & \tilde{h}_{22} \end{bmatrix} \begin{bmatrix} \tilde{\bar{\omega}}^1 \\ \tilde{\bar{\omega}}^2 \end{bmatrix}.$$

Show that

$$\begin{bmatrix} \tilde{h}_{11} & \tilde{h}_{12} \\ \tilde{h}_{12} & \tilde{h}_{22} \end{bmatrix} = B^{-1} \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} B = {}^tB \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} B. \tag{4.7}$$

Recall from linear algebra that any symmetric matrix can be transformed to a *diagonal* matrix by just such an orthogonal change of basis. Therefore, for each $\mathbf{u} \in U$ there exists an adapted frame $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$ at the point $\mathbf{x}(\mathbf{u}) \in \Sigma$ with the property that

$$\begin{bmatrix} h_{11}(\mathbf{u}) & h_{12}(\mathbf{u}) \\ h_{12}(\mathbf{u}) & h_{22}(\mathbf{u}) \end{bmatrix} = \begin{bmatrix} \kappa_1(\mathbf{u}) & 0 \\ 0 & \kappa_2(\mathbf{u}) \end{bmatrix} \tag{4.8}$$

for some real numbers $\kappa_1(\mathbf{u}), \kappa_2(\mathbf{u})$.

***Exercise 4.29.** Let $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$ be an adapted frame satisfying (4.8), and let $\mathbf{v}_1, \mathbf{v}_2 \in T_{\mathbf{u}}U$ be vectors with the property that $d\mathbf{x}(\mathbf{v}_i) = \mathbf{e}_i(\mathbf{u})$ for $i = 1, 2$. Show that

$$d(\mathbf{e}_3)_{\mathbf{u}}(\mathbf{v}_i) = dN_{\mathbf{x}(\mathbf{u})}(\mathbf{e}_i(\mathbf{u})) = -\kappa_i(\mathbf{u})\mathbf{e}_i(\mathbf{u}), \quad i = 1, 2.$$

This implies that $\mathbf{e}_1(\mathbf{u})$ and $\mathbf{e}_2(\mathbf{u})$ are eigenvectors for the linear transformation $dN_{\mathbf{x}(\mathbf{u})}$, the differential of the Gauss map $N : \Sigma \to \mathbb{S}^2$ at the point $\mathbf{x}(\mathbf{u}) \in \Sigma$, with eigenvalues $-\kappa_1(\mathbf{u}), -\kappa_2(\mathbf{u})$, respectively.

You may recall the following definition:

Definition 4.30. The eigenvectors for $-dN_{\mathbf{x}(\mathbf{u})}$ are called *principal vectors* or *principal directions* at the point $\mathbf{x}(\mathbf{u}) \in \Sigma$. The associated eigenvalues $\kappa_1(\mathbf{u})$, $\kappa_2(\mathbf{u})$ are called the *principal curvatures* of Σ at $\mathbf{x}(\mathbf{u})$.

Therefore, there exists an adapted frame $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$ at each point $\mathbf{x}(\mathbf{u}) \in \Sigma$ with the property that $\mathbf{e}_1(\mathbf{u})$ and $\mathbf{e}_2(\mathbf{u})$ are principal vectors at $\mathbf{x}(\mathbf{u})$. Such a frame will be called a *principal adapted frame* at the point $\mathbf{x}(\mathbf{u}) \in \Sigma$, and an adapted frame field on Σ which has this property at every point $\mathbf{x}(\mathbf{u}) \in \Sigma$ will be called a *principal adapted frame field* on Σ.

Definition 4.31. If $\kappa_1(\mathbf{u}) = \kappa_2(\mathbf{u})$ for some point $\mathbf{u} \in U$, then the corresponding point $\mathbf{x}(\mathbf{u})$ of Σ is called an *umbilic point* of Σ.

If Σ has no umbilic points, then a principal adapted frame field can be determined uniquely (up to sign) by requiring that $\kappa_1 > \kappa_2$; moreover, this frame field determines a smooth map $\tilde{\mathbf{x}} : U \to \mathbb{E}(3)$. However, it can happen that a principal adapted frame field cannot be chosen smoothly in a neighborhood of an umbilic point; for this reason, umbilic points can be somewhat problematic.

***Exercise 4.32.** (a) Show that if Σ has no umbilic points, then the choice of a principal adapted frame field $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$ is equivariant (up to sign) under the action of $E(3)$.

(b) Show that for a principal adapted frame field, the second fundamental form is given by

$$\mathrm{II} = \kappa_1(\bar{\omega}^1)^2 + \kappa_2(\bar{\omega}^2)^2.$$

Remark 4.33. Exactly how much freedom does the phrase "up to sign" represent? Given any principal adapted frame field $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$, we can

(1) replace $\mathbf{e}_3(\mathbf{u})$ by $-\mathbf{e}_3(\mathbf{u})$;

(2) depending on whether or not we changed the sign of $\mathbf{e}_3(\mathbf{u})$, replace one or both of $\mathbf{e}_1(\mathbf{u})$ and $\mathbf{e}_2(\mathbf{u})$ by their opposites so as to preserve the orientation of the basis $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$. (We might also exchange $\mathbf{e}_1(\mathbf{u})$ and $\mathbf{e}_2(\mathbf{u})$ with appropriately chosen signs, but for the most part, we will ignore this option.)

So the other choices for a principal adapted frame field are

$$(-\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), -\mathbf{e}_3(\mathbf{u})),$$
$$(\mathbf{e}_1(\mathbf{u}), -\mathbf{e}_2(\mathbf{u}), -\mathbf{e}_3(\mathbf{u})),$$
$$(-\mathbf{e}_1(\mathbf{u}), -\mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u})).$$

***Exercise 4.34.** For each of the principal adapted frame fields in Remark 4.33, how do the sign changes to the frame vectors affect the Maurer-Cartan forms $(\bar{\omega}^i, \bar{\omega}^i_j)$?

Suppose that $\Sigma = \mathbf{x}(U)$ has no umbilic points. Now that we (finally!) have a way of defining a canonical adapted frame field along Σ, we can apply Lemma 4.2 to find a complete set of invariants for the surface.

Theorem 4.35 (Bonnet)**.** *Let $U \subset \mathbb{R}^2$ be an open set. Two immersions $\mathbf{x}_1, \mathbf{x}_2 : U \to \mathbb{E}^3$ without umbilic points differ by a rigid motion if and only if they have the same first and second fundamental forms.*

Remark 4.36. This theorem is true even for surfaces with umbilic points, but the proof is slightly more involved due to the issue of how to choose a canonical adapted frame field near umbilic points.

PROOF. One direction is clear: Since all our constructions are equivariant under the action of $E(3)$, any two surfaces that differ by a rigid motion must have the same first and second fundamental forms.

Conversely, suppose that $\mathbf{x}_1, \mathbf{x}_2$ have the same first and second fundamental forms. Let $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2 : U \to E(3)$ be principal adapted frame fields for $\mathbf{x}_1, \mathbf{x}_2$, respectively; let $(\bar{\omega}^i, \bar{\omega}^i_j)$ denote the pulled-back dual and connection forms for $\mathbf{x}_1$ and let $(\bar{\Omega}^i, \bar{\Omega}^i_j)$ denote those for $\mathbf{x}_2$. By hypothesis,

$$\begin{aligned} \mathrm{I}_{\mathbf{x}_1} &= (\bar{\omega}^1)^2 + (\bar{\omega}^2)^2 = (\bar{\Omega}^1)^2 + (\bar{\Omega}^2)^2 = \mathrm{I}_{\mathbf{x}_2}, \\ \mathrm{II}_{\mathbf{x}_1} &= (\kappa_1)_{\mathbf{x}_1}(\bar{\omega}^1)^2 + (\kappa_2)_{\mathbf{x}_1}(\bar{\omega}^2)^2 = (\kappa_1)_{\mathbf{x}_2}(\bar{\Omega}^1)^2 + (\kappa_2)_{\mathbf{x}_2}(\bar{\Omega}^2)^2 = \mathrm{II}_{\mathbf{x}_2}. \end{aligned}$$

Equality of the first fundamental forms implies that

$$\begin{bmatrix} \bar{\Omega}^1 \\ \bar{\Omega}^2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \pm\sin(\theta) & \pm\cos(\theta) \end{bmatrix} \begin{bmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \end{bmatrix}$$

for some function θ on U, where the signs on the bottom row of the matrix are the same. Substituting this relation into the equation for the second fundamental forms yields

$$\begin{aligned} (\kappa_1)_{\mathbf{x}_1}(\bar{\omega}^1)^2 + (\kappa_2)_{\mathbf{x}_1}(\bar{\omega}^2)^2 = &\left((\kappa_1)_{\mathbf{x}_2}\cos^2(\theta) + (\kappa_2)_{\mathbf{x}_2}\sin^2(\theta)\right)(\bar{\omega}^1)^2 \\ &+ 2\left(((\kappa_2)_{\mathbf{x}_2} - (\kappa_1)_{\mathbf{x}_2})\sin(\theta)\cos(\theta)\right)\bar{\omega}^1\bar{\omega}^2 \\ &+ \left((\kappa_1)_{\mathbf{x}_2}\sin^2(\theta) + (\kappa_2)_{\mathbf{x}_2}\cos^2(\theta)\right)(\bar{\omega}^2)^2. \end{aligned}$$

Since $(\kappa_1)_{\mathbf{x}_2} > (\kappa_2)_{\mathbf{x}_2}$, the vanishing of the middle term on the right-hand side implies that θ is a multiple of $\frac{\pi}{2}$. Then the remaining terms, together with the inequality $\kappa_1 > \kappa_2$ on both sides, imply that θ is a multiple of π. Therefore, $(\kappa_1)_{\mathbf{x}_2} = (\kappa_1)_{\mathbf{x}_1}$, $(\kappa_2)_{\mathbf{x}_2} = (\kappa_2)_{\mathbf{x}_1}$, and

$$\bar{\Omega}^1 = \pm\bar{\omega}^1, \qquad \bar{\Omega}^2 = \pm\bar{\omega}^2.$$

By making one of the permissible frame changes described in Remark 4.33 on one side or the other if necessary, we can arrange that both signs above are positive.

Since we now have $\bar{\Omega}^1 = \bar{\omega}^1$, we must have $d\bar{\Omega}^1 = d\bar{\omega}^1$. According to the Cartan structure equations (3.8), this implies that

$$(\bar{\Omega}^1_2 - \bar{\omega}^1_2) \wedge \bar{\omega}^2 = 0.$$

By Cartan's lemma, $\bar{\Omega}^1_2 - \bar{\omega}^1_2$ must be a multiple of $\bar{\omega}^2$. But the same reasoning applied to the equation $d\bar{\Omega}^2 = d\bar{\omega}^2$ implies that $\bar{\Omega}^1_2 - \bar{\omega}^1_2$ must also be a multiple of $\bar{\omega}^1$. Since $(\bar{\omega}^1, \bar{\omega}^2)$ are linearly independent, it follows that

$$\bar{\Omega}^1_2 = \bar{\omega}^1_2.$$

Finally, since $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ are both principal adapted frame fields, we have

$$\begin{aligned}\bar{\Omega}^3_1 &= (\kappa_1)_{\mathbf{x}_2}\bar{\Omega}^1 = (\kappa_1)_{\mathbf{x}_1}\bar{\omega}^1 = \bar{\omega}^3_1,\\ \bar{\Omega}^3_2 &= (\kappa_2)_{\mathbf{x}_2}\bar{\Omega}^2 = (\kappa_2)_{\mathbf{x}_1}\bar{\omega}^2 = \bar{\omega}^3_2.\end{aligned}$$

The theorem now follows from Lemma 4.2. □

Now we consider the question discussed in §4.4; namely, can the first and second fundamental forms be prescribed arbitrarily? We must require that I be a *positive definite* quadratic form (i.e., that $\mathrm{I}(\mathbf{v}) > 0$ for every $\mathbf{v} \neq \mathbf{0} \in TU$) in order to define a metric on the surface. And in order to avoid the issue of umbilic points, we will assume that I and II are prescribed in such a way that $\mathrm{II}_{\mathbf{u}}$ is not a scalar multiple of $\mathrm{I}_{\mathbf{u}}$ at any point $\mathbf{u} \in U$.

Exercise 4.37. Why is this the right condition to impose on I and II in order to avoid umbilic points?

We saw in the proof of Theorem 4.35 that prescribing these fundamental forms determines the 1-forms $(\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3_1, \bar{\omega}^3_2)$ associated to a principal adapted frame field up to sign and that these forms will have the properties that $(\bar{\omega}^1, \bar{\omega}^2)$ are linearly independent and that $\bar{\omega}^3_i$ is a multiple of $\bar{\omega}^i$ for $i = 1, 2$. So, suppose that we are given 1-forms $(\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3_1, \bar{\omega}^3_2)$ on an open set $U \subset \mathbb{R}^2$ that satisfy these conditions. What additional conditions must these forms satisfy in order that there exist an embedding $\mathbf{x} : U \to \mathbb{E}^3$ whose first and second fundamental forms are

$$\begin{aligned}\mathrm{I} &= (\bar{\omega}^1)^2 + (\bar{\omega}^2)^2,\\ \mathrm{II} &= \bar{\omega}^3_1\,\bar{\omega}^1 + \bar{\omega}^3_2\,\bar{\omega}^2?\end{aligned}$$

Lemma 4.12 gives the answer: The forms $\bar{\omega}^1$, $\bar{\omega}^2$, $\bar{\omega}^3_1 = -\bar{\omega}^1_3$, $\bar{\omega}^3_2 = -\bar{\omega}^2_3$, together with the form $\bar{\omega}^3 = 0$ and some additional form $\bar{\omega}^1_2 = -\bar{\omega}^2_1$, must satisfy the structure equations (3.8) for the Maurer-Cartan forms on $E(3)$. Because $\bar{\omega}^3 = 0$, the first three of these equations may be written as

$$\begin{aligned}d\bar{\omega}^1 &= -\bar{\omega}^1_2 \wedge \bar{\omega}^2,\\ d\bar{\omega}^2 &= \bar{\omega}^1_2 \wedge \bar{\omega}^1,\\ d\bar{\omega}^3 &= 0 = -\bar{\omega}^3_1 \wedge \bar{\omega}^1 - \bar{\omega}^3_2 \wedge \bar{\omega}^2.\end{aligned} \tag{4.9}$$

***Exercise 4.38.** Show that the first two equations in (4.9) uniquely determine the 1-form $\bar{\omega}^1_2$. (Hint: $\bar{\omega}^1_2$ must be equal to some linear combination of $(\bar{\omega}^1, \bar{\omega}^2)$. Show that each of the first two equations determines one of the unknown coefficients.)

The form $\bar\omega_2^1$ determined by the first two equations in (4.9) is called the *Levi-Civita connection form* of the metric defined by the first fundamental form $\mathrm{I} = (\bar\omega^1)^2 + (\bar\omega^2)^2$. The third equation just says that $(\bar\omega_1^3, \bar\omega_2^3)$ must be symmetric linear combinations of $(\bar\omega^1, \bar\omega^2)$, which will automatically be true under our assumptions.

The remaining structure equations may be written as

$$\begin{aligned} d\bar\omega_2^1 &= \bar\omega_1^3 \wedge \bar\omega_2^3, \\ d\bar\omega_1^3 &= \bar\omega_2^3 \wedge \bar\omega_2^1, \\ d\bar\omega_2^3 &= -\bar\omega_1^3 \wedge \bar\omega_2^1. \end{aligned} \tag{4.10}$$

The first of these equations is called the *Gauss equation*, and the last two are called the *Codazzi-Mainardi equations*, or simply the *Codazzi equations*. By Lemma 4.12, we have the following theorem:

Theorem 4.39 (Bonnet). *Let $(\bar\omega^1, \bar\omega^2, \bar\omega_1^3, \bar\omega_2^3)$ be 1-forms on a connected and simply connected open set $U \subset \mathbb{R}^2$ satisfying the conditions that $(\bar\omega^1, \bar\omega^2)$ are linearly independent at each point of U and that $\bar\omega_i^3$ is a scalar multiple of $\bar\omega^i$ for $i = 1, 2$. Suppose that, together with the Levi-Civita connection form $\bar\omega_2^1$ determined by $\bar\omega^1$ and $\bar\omega^2$, these forms satisfy the Gauss and Codazzi equations* (4.10). *Then there exists an immersed surface $\mathbf{x} : U \to \mathbb{E}^3$, unique up to rigid motion, whose first and second fundamental forms are*

$$\begin{aligned} \mathrm{I} &= (\bar\omega^1)^2 + (\bar\omega^2)^2, \\ \mathrm{II} &= \bar\omega_1^3\, \bar\omega^1 + \bar\omega_2^3\, \bar\omega^2. \end{aligned}$$

Because of this result, the Gauss and Codazzi equations are also referred to as the *compatibility equations* of the theory of surfaces in $\mathbb{E}^3$.

Exercise 4.40. Let (u, v) be local coordinates on $\mathbb{R}^2$. Use the following steps to determine whether there exists an immersion $\mathbf{x} : \mathbb{R}^2 \to \mathbb{E}^3$ with first and second fundamental forms

$$\begin{aligned} \mathrm{I} &= \cosh^2(v)\,(du^2 + dv^2), \\ \mathrm{II} &= du^2 - dv^2. \end{aligned}$$

(a) Show that the 1-forms $(\bar\omega^1, \bar\omega^2, \bar\omega_1^3, \bar\omega_2^3)$ determined by I and II according to the conditions of Theorem 4.39 are

$$\begin{aligned} \bar\omega^1 &= \cosh(v)\, du, & \bar\omega^2 &= \cosh(v)\, dv, \\ \bar\omega_1^3 &= \frac{1}{\cosh(v)}\, du, & \bar\omega_2^3 &= -\frac{1}{\cosh(v)}\, dv. \end{aligned}$$

(b) Show that the Levi-Civita connection form is

$$\bar{\omega}_2^1 = \tanh(v)\, du.$$

(Hint: Set

$$\bar{\omega}_2^1 = a\, du + b\, dv$$

for some unknown functions a, b on $\mathbb{R}^2$. Use the structure equations for $d\bar{\omega}^1$ and $d\bar{\omega}^2$ to determine a and b.)

(c) Check that these forms satisfy the Gauss and Codazzi equations.

Therefore, Theorem 4.39 implies that the desired surface exists. (In fact, it is a catenoid.)

***Exercise 4.41.** This exercise is a continuation of Exercise 4.24. Suppose that $\mathbf{x} : U \to \mathbb{E}^3$ is an immersion whose coordinate curves are all principal curves. (This means that $\mathbf{x}_u, \mathbf{x}_v$ are both principal vectors at each point of $\Sigma = \mathbf{x}(U)$.)

(a) Show that the frame field in part (a) of Exercise 4.24 is a principal adapted frame field along Σ.

(b) Show that the condition that all coordinate curves of $\mathbf{x}$ are principal curves is equivalent to the condition that the first and second fundamental forms

$$\mathrm{I} = E\, du^2 + 2F\, du\, dv + G\, dv^2,$$
$$\mathrm{II} = e\, du^2 + 2f\, du\, dv + g\, dv^2$$

have the property that $F = f = 0$.

(c) Use the structure equations for the dual forms in part (b) of Exercise 4.24 to show that

$$\bar{\omega}_2^1 = \frac{1}{2\sqrt{EG}}\,(E_v\, du - G_u\, dv).$$

(d) Show that

$$\bar{\omega}_1^3 = \frac{e}{E}\,\bar{\omega}^1 = \frac{e}{\sqrt{E}}\, du,$$
$$\bar{\omega}_2^3 = \frac{g}{G}\,\bar{\omega}^2 = \frac{g}{\sqrt{G}}\, dv.$$

(e) Show that the Gauss equation is equivalent to

$$\frac{eg}{EG} = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]. \tag{4.11}$$

(f) Show that the Codazzi equations are equivalent to

$$\begin{aligned} e_v &= \frac{1}{2}E_v\left(\frac{e}{E}+\frac{g}{G}\right), \\ g_u &= \frac{1}{2}G_u\left(\frac{e}{E}+\frac{g}{G}\right). \end{aligned} \tag{4.12}$$

While isolated umbilic points on a surface can be problematic, it is natural to ask whether we can categorize those surfaces that are *totally umbilic*, i.e., surfaces with the property that every point is an umbilic point.

Exercise 4.42. Suppose that the surface $\mathbf{x} : U \to \mathbb{E}^3$ is totally umbilic.

(a) Show that any adapted frame field $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$ is a principal adapted frame field.

(b) Let $(\bar{\omega}^i, \bar{\omega}^i_j)$ be the Maurer-Cartan forms for an adapted frame field on $\Sigma = \mathbf{x}(U)$. Show that there exists a smooth function $\lambda : U \to \mathbb{R}$ such that

$$\bar{\omega}^3_1 = \lambda\bar{\omega}^1, \qquad \bar{\omega}^3_2 = \lambda\bar{\omega}^2. \tag{4.13}$$

Conclude that the second fundamental form of Σ is a scalar multiple of the first fundamental form, i.e., that

$$\mathrm{II} = \lambda \mathrm{I}.$$

(c) Prove that λ is constant. (Hint: Use the structure equations to differentiate equations (4.13), taking into account the fact that we must have

$$d\lambda = \lambda_1\bar{\omega}^1 + \lambda_2\bar{\omega}^2$$

for some functions λ_1, λ_2 on U. Then use Cartan's lemma.)

(d) Show that if $\lambda = 0$, then $d\mathbf{e}_3 = 0$. Conclude that the normal vector field of Σ is constant and that Σ is therefore contained in a plane.

(e) Show that if $\lambda \neq 0$, then $d(\mathbf{x}+\frac{1}{\lambda}\mathbf{e}_3) = 0$. Conclude that the vector-valued function $\mathbf{x} + \frac{1}{\lambda}\mathbf{e}_3 : U \to \mathbb{E}^3$ is equal to some constant point $\mathbf{q} \in \mathbb{E}^3$ and that Σ is therefore contained in the sphere of radius $\frac{1}{|\lambda|}$ centered at $\mathbf{q}$.

Thus, the only totally umbilic surfaces in $\mathbb{E}^3$ are (open subsets of) planes and spheres.

One of the conclusions of Exercise 4.42 is that if the principal curvatures κ_1, κ_2 of a regular surface Σ are equal at every point of Σ, then they must in fact be constant. This suggests a related question: Can we categorize those surfaces for which κ_1, κ_2 are constants, but not necessarily equal?

Exercise 4.43. Suppose that the surface $\mathbf{x} : U \to \mathbb{E}^3$ has the property that both principal curvatures κ_1, κ_2 are constants. We know from Exercise 4.42 that if $\kappa_1 = \kappa_2$, then $\Sigma = \mathbf{x}(U)$ is contained in either a plane or a sphere, so assume that $\kappa_1 \neq \kappa_2$. Let $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$ be a principal adapted frame field on Σ, and let $\tilde{\mathbf{x}} : U \to E(3)$ denote the corresponding lifting of $\mathbf{x} : U \to \mathbb{E}^3$, with associated Maurer-Cartan forms $(\bar{\omega}^i, \bar{\omega}^i_j)$. Then we have

$$\bar{\omega}^3_1 = \kappa_1 \bar{\omega}^1, \qquad \bar{\omega}^3_2 = \kappa_2 \bar{\omega}^2. \tag{4.14}$$

(a) Differentiate equations (4.14) to obtain

$$(\kappa_1 - \kappa_2)\bar{\omega}^1_2 \wedge \bar{\omega}^1 = (\kappa_1 - \kappa_2)\bar{\omega}^1_2 \wedge \bar{\omega}^2 = 0.$$

Use Cartan's lemma to conclude that $\bar{\omega}^1_2 = 0$.

(b) Differentiate the equation $\bar{\omega}^1_2 = 0$ and show that $\kappa_1\kappa_2 = 0$. Without loss of generality, we may assume that $\kappa_1 = 0$, $\kappa_2 \neq 0$.

In the remainder of this exercise, we will see how the structure equations can be *integrated* in order to determine the surface Σ.

(c) Show that $d\bar{\omega}^1 = d\bar{\omega}^2 = 0$. Apply the Poincaré lemma (cf. Theorem 2.31) to conclude that there exist functions u, v on U such that

$$\bar{\omega}^1 = du, \qquad \bar{\omega}^2 = dv.$$

Thus the pullbacks of the structure equations (3.1) to U via $\tilde{\mathbf{x}}$ can be written as

$$\begin{aligned} d\mathbf{x} &= \mathbf{e}_1\, du + \mathbf{e}_2\, dv, \\ d\mathbf{e}_1 &= 0, \\ d\mathbf{e}_2 &= \mathbf{e}_3(\kappa_2\, dv), \\ d\mathbf{e}_3 &= -\mathbf{e}_2(\kappa_2\, dv). \end{aligned} \tag{4.15}$$

(d) Integrate equations (4.15) (beginning with the equations for $d\mathbf{e}_1, d\mathbf{e}_2, d\mathbf{e}_3$ and working backwards) to show that there exist *constant* vectors $\bar{\mathbf{e}}_1$, $\bar{\mathbf{e}}_2$, $\bar{\mathbf{e}}_3$, $\bar{\mathbf{x}} \in \mathbb{E}^3$ such that

$$\begin{aligned} \mathbf{e}_1(u,v) &= \bar{\mathbf{e}}_1, \\ \mathbf{e}_2(u,v) &= \cos(\kappa_2 v)\, \bar{\mathbf{e}}_2 + \sin(\kappa_2 v)\, \bar{\mathbf{e}}_3, \\ \mathbf{e}_3(u,v) &= -\sin(\kappa_2 v)\, \bar{\mathbf{e}}_2 + \cos(\kappa_2 v)\, \bar{\mathbf{e}}_3, \\ \mathbf{x}(u,v) &= \bar{\mathbf{x}} + u\, \bar{\mathbf{e}}_1 + \frac{1}{\kappa_2}\sin(\kappa_2 v)\, \bar{\mathbf{e}}_2 - \frac{1}{\kappa_2}\cos(\kappa_2 v)\, \bar{\mathbf{e}}_3. \end{aligned} \tag{4.16}$$

(e) Use equations (4.16) and the fact that $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$ is an orthonormal frame field to show that $(\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3)$ is an orthonormal frame.

Conclude that via a Euclidean transformation, we can arrange that $\bar{\mathbf{x}} = \mathbf{0}$ and

$$\bar{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \bar{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \bar{\mathbf{e}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and hence that

$$\mathbf{x}(u, v) = \begin{bmatrix} u \\ \frac{1}{\kappa_2}\sin(\kappa_2 v) \\ -\frac{1}{\kappa_2}\cos(\kappa_2 v) \end{bmatrix}. \tag{4.17}$$

Equation (4.17) describes a parametrization for the cylinder of radius $\frac{1}{|\kappa_2|}$ centered along the x^1-axis; therefore, $\Sigma = \mathbf{x}(U)$ is contained in a cylinder of radius $\frac{1}{|\kappa_2|}$.

Together, Exercises 4.42 and 4.43 prove the following classification theorem:

Theorem 4.44. *Let Σ be a connected, regular surface in $\mathbb{E}^3$ whose principal curvatures are constant. Then Σ is contained in either a plane, sphere, or cylinder.*

Any invariant of an immersed surface $\mathbf{x} : U \to \mathbb{E}^3$ that can be expressed purely in terms of the first fundamental form

$$\mathrm{I} = (\bar{\omega}^1)^2 + (\bar{\omega}^2)^2$$

is called an *intrinsic* invariant of the surface. For instance, arc length and area are intrinsic quantities on $\Sigma = \mathbf{x}(U)$. The principal curvatures κ_1, κ_2, however, are not intrinsic; they depend not only on the metric, but also on how the surface is immersed.

Two important notions of curvature for surfaces are given in the following definition:

Definition 4.45. The function $K = \kappa_1\kappa_2$ on Σ is called the *Gauss curvature* of Σ. The function $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ on Σ is called the *mean curvature* of Σ.

Remark 4.46. It is not necessary that an adapted frame field be principal in order to compute the Gauss and mean curvatures. For any adapted frame field on Σ with associated matrix $[h_{ij}]$, we have

$$K = \det[h_{ij}], \qquad H = \tfrac{1}{2}\mathrm{tr}[h_{ij}].$$

Even though κ_1, κ_2 are not intrinsic quantities, Gauss's "Theorema Egregium" states that their product K is, in fact, intrinsic. (The mean curvature H, however, is *not* intrinsic.) In the following exercise, we will prove this in several steps. (For simplicity, we will assume that the surface is oriented, meaning that a choice of $\mathbf{e}_3$ has been specified.)

Exercise 4.47. Let $\mathbf{x} : U \to \mathbb{E}^3$ be an immersed surface. The 1-forms $(\bar{\omega}^1, \bar{\omega}^2)$ are determined by the first fundamental form of $\mathbf{x}$ up to a transformation of the form

$$\begin{bmatrix} \tilde{\bar{\omega}}^1 \\ \tilde{\bar{\omega}}^2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \end{bmatrix}$$

for some function θ on U.

(a) Show that the area form

$$dA = \bar{\omega}^1 \wedge \bar{\omega}^2$$

is an intrinsic quantity, i.e., that

$$d\tilde{A} = \tilde{\bar{\omega}}^1 \wedge \tilde{\bar{\omega}}^2 = dA.$$

(Note: The notation dA for the area form is traditional, but the d does *not* signify that dA is the exterior derivative of some 1-form.)

(b) Show that if $\bar{\omega}^1_2$ is the Levi-Civita connection form corresponding to $(\bar{\omega}^1, \bar{\omega}^2)$, then

$$\tilde{\bar{\omega}}^1_2 = \bar{\omega}^1_2 - d\theta.$$

Conclude that $d\bar{\omega}^1_2$ is an intrinsic quantity.

(c) Show that

$$d\bar{\omega}^1_2 = \bar{\omega}^3_1 \wedge \bar{\omega}^3_2 = K\,\bar{\omega}^1 \wedge \bar{\omega}^2 = K\,dA.$$

Conclude that K must be an intrinsic quantity. (Note that this is simply another version of equation (4.11), which expresses the Gauss curvature $K = \frac{eg}{EG}$ as a function of E, G, and their derivatives.)

Surfaces for which the mean curvature H is identically zero are called *minimal surfaces*; these surfaces are of considerable interest and will be treated in detail in Chapter 8. Surfaces for which the Gauss curvature K is identically zero are called *flat*, and we will conclude this chapter with a brief exploration of their local theory.

Because the Gauss curvature of a regular surface Σ is an intrinsic quantity, it is not changed by any deformation of Σ that preserves the first fundamental form of Σ. Intuitively, this means that the surface may be smoothly bent and/or twisted, but not stretched or contracted. So for instance, any surface that can be obtained by smoothly bending a sheet of paper must be flat.

Since any flat surface Σ must have $K = \kappa_1\kappa_2 = 0$, one of the principal curvatures κ_1, κ_2 must be identically zero on Σ. As we saw in Exercise 4.42 that any surface with $\kappa_1 = \kappa_2 = 0$ must be contained in a plane, we will disregard this case and, to keep things simple, we will assume that Σ has no umbilic points. (In practice, this simply means that we restrict our attention to the open subset of Σ consisting of the non-umbilic points.)

Exercise 4.48. Suppose that the surface $\mathbf{x} : U \to \mathbb{E}^3$ is flat and has no umbilic points. Without loss of generality, we may assume that the principal curvatures of $\Sigma = \mathbf{x}(U)$ satisfy $\kappa_1 = 0$, $\kappa_2 \neq 0$. Let $(\mathbf{e}_1(\mathbf{u}), \mathbf{e}_2(\mathbf{u}), \mathbf{e}_3(\mathbf{u}))$ be a principal adapted frame field on Σ, and let $\tilde{\mathbf{x}} : U \to E(3)$ denote the corresponding lifting of $\mathbf{x} : U \to \mathbb{E}^3$, with associated Maurer-Cartan forms $(\bar{\omega}^i, \bar{\omega}^i_j)$. Then since $\kappa_1 = 0$, we have

$$\bar{\omega}^3_1 = 0, \qquad \bar{\omega}^3_2 = \kappa_2 \bar{\omega}^2. \tag{4.18}$$

(a) Differentiate the equation $\bar{\omega}^3_1 = 0$ and use Cartan's lemma to conclude that

$$\bar{\omega}^1_2 = \mu\, \bar{\omega}^2 \tag{4.19}$$

for some function $\mu : U \to \mathbb{R}$.

(b) Show that $d\bar{\omega}^1 = 0$, and apply the Poincaré lemma (cf. Theorem 2.31) to conclude that there exists a function u on U such that $\bar{\omega}^1 = du$.

(c) Use the structure equation for $d\bar{\omega}^2$ and the Frobenius theorem (cf. Theorem 2.33) to conclude that for any point $\mathbf{u} \in U$, there exist a neighborhood $V \subset U$ of $\mathbf{u}$ and differentiable functions $\lambda, v : V \to \mathbb{R}$ (with $\lambda \neq 0$) such that the restriction of $\bar{\omega}^2$ to V is given by $\bar{\omega}^2 = \lambda\, dv$. (For simplicity, we will shrink U if necessary and assume that these functions are defined on the entire open set U.)

(d) Since $\bar{\omega}^1 \wedge \bar{\omega}^2 \neq 0$, the functions (u, v) form a local coordinate system on U, and we may regard λ, μ as functions of u and v. Show that the structure equation for $d\bar{\omega}^2$ implies that $\mu = -\frac{\lambda_u}{\lambda}$, and therefore

$$\bar{\omega}^1_2 = -\lambda_u\, dv. \tag{4.20}$$

(e) Use the structure equation for $d\bar{\omega}^1_2$ to conclude that $\lambda_{uu} = 0$, and therefore

$$\lambda(u, v) = u f_1(v) + f_0(v) \tag{4.21}$$

for some smooth functions $f_0(v), f_1(v)$.

(f) Use the structure equation for $d\bar{\omega}_2^3$ to conclude that

$$(\kappa_2\lambda)_u = 0.$$

Integrate and use equation (4.21) to conclude that

$$\kappa_2(u,v) = \frac{f_2(v)}{uf_1(v) + f_0(v)} \tag{4.22}$$

for some smooth functions $f_0(v), f_1(v), f_2(v)$.

(g) The pullbacks of the structure equations (3.1) to U via $\tilde{\mathbf{x}}$ can now be written as

$$\begin{aligned} d\mathbf{x} &= \mathbf{e}_1\, du + \mathbf{e}_2\,(uf_1(v) + f_0(v))dv, \\ d\mathbf{e}_1 &= \mathbf{e}_2\, f_1(v)\, dv, \\ d\mathbf{e}_2 &= -\mathbf{e}_1\, f_1(v)\, dv + \mathbf{e}_3\, f_2(v)\, dv, \\ d\mathbf{e}_3 &= -\mathbf{e}_2\, f_2(v)\, dv. \end{aligned} \tag{4.23}$$

Conclude that the u-parameter curves are straight line segments (and that u is an arc-length parameter along these curves) and hence that Σ is a ruled surface.

We have now proved the following theorem, keeping in mind that with the notation of Exercise 4.48, the mean curvature H of Σ is given by $H = \frac{1}{2}\kappa_2$ with κ_2 as in equation (4.22):

Theorem 4.49. *Let Σ be a flat surface whose mean curvature H is nonzero everywhere. Then for each point $\mathbf{x} \in \Sigma$, there exists a unique straight line $\ell_{\mathbf{x}}$ in $\mathbb{E}^3$ such that $\mathbf{x} \in \ell_{\mathbf{x}}$ and $\ell_{\mathbf{x}} \cap \Sigma$ is an open neighborhood of $\mathbf{x}$ in $\ell_{\mathbf{x}}$. Moreover, the restriction of the function $\frac{1}{H}$ to the open interval $\ell_{\mathbf{x}} \cap \Sigma$ is an affine linear function of the arc length parameter along this interval.*

A more traditional proof of this result is given in [**MR05**].

4.6. Maple computations

In order to get set up to use Maple for some of the exercises in this chapter, begin by loading the `Cartan` and `LinearAlgebra` packages into Maple:

```
> with(Cartan);
> with(LinearAlgebra);
```

Next, introduce the Maurer-Cartan forms on the frame bundle $\mathcal{F}(\mathbb{E}^3)$; these need to be declared so that Maple will recognize them as 1-forms. It suffices to declare a linearly independent subset; we'll define the others in terms of these shortly.

```
> Form(omega[1], omega[2], omega[3]);
  Form(omega[1,2], omega[3,1], omega[3,2]);
```

Next, tell MAPLE about the symmetries in the connection forms:

```
> omega[1,1]:= 0;
  omega[2,2]:= 0;
  omega[3,3]:= 0;
  omega[2,1]:= -omega[1,2];
  omega[1,3]:= -omega[3,1];
  omega[2,3]:= -omega[3,2];
```

Tell MAPLE how to differentiate these forms according to the Cartan structure equations (3.8):

```
> for i from 1 to 3 do
    d(omega[i]):= -add('omega[i,j] &^ omega[j]', j=1..3);
  end do;

  d(omega[1,2]):= -add('omega[1,k] &^ omega[k,2]', k=1..3);
  d(omega[3,1]):= -add('omega[3,k] &^ omega[k,1]', k=1..3);
  d(omega[3,2]):= -add('omega[3,k] &^ omega[k,2]', k=1..3);
```

Now consider the pullbacks of the Maurer-Cartan forms to the surface via an adapted frame field. The first condition that these forms must satisfy is $\bar{\omega}^3 = 0$.

In MAPLE, it's often useful to impose such conditions via a substitution rather than by simply setting $\bar{\omega}^3$ equal to zero. The reason for this is that if we make the assignment $\bar{\omega}^3 = 0$, we lose the ability to use the structure equation for $d\bar{\omega}^3$ because MAPLE will just evaluate $d\bar{\omega}^3$ as $d(0) = 0$. Using a substitution allows us to choose when we want MAPLE to be aware that $\bar{\omega}^3 = 0$ and when we don't.

So, introduce the following substitution for the Maurer-Cartan forms associated to an adapted frame field. (We'll add more information to this substitution as we learn more about the Maurer-Cartan forms.)

```
> adaptedsub1:= [omega[3]=0];
```

Now, since we have $\bar{\omega}^3 = 0$, we must have $d\bar{\omega}^3 = 0$ as well. So the following quantity must be zero:

```
> zero1:= Simf(subs(adaptedsub1, Simf(d(omega[3]))));
```

$$zero1 := (\omega_1)\ \&\hat{}\ (\omega_{31}) + (\omega_2)\ \&\hat{}\ (\omega_{32})$$

Note that we first computed $d\bar{\omega}^3$ and *then* applied the substitution to tell MAPLE that $\bar{\omega}^3 = 0$. In this case the knowledge that $\bar{\omega}^3 = 0$ didn't affect the computation of $d\bar{\omega}^3$, but it's a good idea to get in the habit of applying such substitutions when you intend for them to be in effect.

Applying Cartan's lemma tells us that $(\bar{\omega}_1^3, \bar{\omega}_2^3)$ must be symmetric linear combinations of $(\bar{\omega}^1, \bar{\omega}^2)$, so we add this information to our substitution:

```
> adaptedsub1:= [op(adaptedsub1),
  omega[3,1] = h[1,1]*omega[1] + h[1,2]*omega[2],
  omega[3,2] = h[1,2]*omega[1] + h[2,2]*omega[2]];
```

EXERCISE 4.28: In order to keep up with both the original Maurer-Cartan forms $(\bar{\omega}^i, \bar{\omega}_j^i)$ and the transformed forms $(\tilde{\bar{\omega}}^i, \tilde{\bar{\omega}}_j^i)$, introduce new 1-forms to represent the transformed forms, with the same symmetry conditions as the original forms:

```
> Form(Omega[1], Omega[2], Omega[3]);
  Form(Omega[1,2], Omega[3,1], Omega[3,2]);
  Omega[1,1]:= 0;
  Omega[2,2]:= 0;
  Omega[3,3]:= 0;
  Omega[2,1]:= -Omega[1,2];
  Omega[1,3]:= -Omega[3,1];
  Omega[2,3]:= -Omega[3,2];
```

(It won't be necessary to assign their exterior derivatives because these will be computed in terms of the exterior derivatives of the original forms when needed.)

We can introduce the relations (4.5), (4.6) via the following substitution:

```
> framechangesub:= [
  Omega[1] = cos(theta)*omega[1] + sin(theta)*omega[2],
  Omega[2] = -sin(theta)*omega[1] + cos(theta)*omega[2],
  Omega[3,1] = cos(theta)*omega[3,1] + sin(theta)*omega[3,2],
  Omega[3,2] = -sin(theta)*omega[3,1] + cos(theta)*omega[3,2]];
```

We'll also need the reverse substitution so that we can go back and forth between the two sets of Maurer-Cartan forms:

```
> framechangebacksub:= makebacksub(framechangesub);
```

In order to compare the functions $(\tilde{h}_{ij})$ associated to the transformed forms to the functions (h_{ij}) associated to the original forms, introduce another substitution describing the adaptations of the transformed frame:

```
> adaptedsub2:= [Omega[3]=0,
  Omega[3,1] = H[1,1]*Omega[1] + H[1,2]*Omega[2],
  Omega[3,2] = H[1,2]*Omega[1] + H[2,2]*Omega[2]];
```

Now combine all these substitutions to see how the $(\tilde{h}_{ij})$ are expressed in terms of the (h_{ij}): First, write $\tilde{\bar{\omega}}^3_1$ in terms of $(\bar{\omega}^3_1, \bar{\omega}^3_2)$:

```
> Simf(subs(framechangesub, Omega[3,1]));
```

$$\cos(\theta)\,\omega_{3,1} + \sin(\theta)\,\omega_{3,2}$$

Next, convert this to an expression in terms of the (h_{ij}) and $(\bar{\omega}^1, \bar{\omega}^2)$:

```
> Simf(subs(adaptedsub1, %));
```

$$(\cos(\theta)\,h_{1,1} + \sin(\theta)\,h_{1,2})\,\omega_1 + (\cos(\theta)\,h_{1,2} + \sin(\theta)\,h_{2,2})\,\omega_2$$

Finally, convert this to an expression in terms of $(\tilde{\bar{\omega}}^1, \tilde{\bar{\omega}}^2)$:

```
> Simf(subs(framechangebacksub, %));
```

$$\begin{aligned}&(\cos(\theta)^2\,h_{1,1} + 2\cos(\theta)\sin(\theta)\,h_{1,2} + h_{2,2} - h_{2,2}\cos(\theta)^2)\,\Omega_1\\&\quad + (-\cos(\theta)\sin(\theta)\,h_{1,1} + \cos(\theta)\sin(\theta)\,h_{2,2} + 2\cos(\theta)^2\,h_{1,2} - h_{1,2})\Omega_2\end{aligned}$$

Of course, this sequence of operations can be combined into a single command:

```
> Simf(subs(framechangebacksub, Simf(subs(adaptedsub1,
  Simf(subs(framechangesub, Omega[3,1]))))));
```

Now, the coefficients of $(\tilde{\bar{\omega}}^1, \tilde{\bar{\omega}}^2)$ in the output are, of course, equal to $\tilde{h}_{11}, \tilde{h}_{12}$, respectively. But in order to illustrate how we might handle a slightly more complicated situation, we will let MAPLE do the work of comparing this expression to our original expression for $\tilde{\bar{\omega}}^3_1$:

```
> zero2:= Simf(subs(adaptedsub2, Omega[3,1]) - %);
```

The coefficients of this expression must both be zero, which gives us two equations that can be solved for $\tilde{h}_{11}$ and $\tilde{h}_{12}$. These equations can be extracted as follows:

```
> eqns:= {op(ScalarForm(zero2))};
```

Before solving these equations, we might as well compute the analogous equations that result from consideration of $\tilde{\bar{\omega}}^3_2$. We can add these to our system of equations as follows:

```
> zero3:= Simf(subs(adaptedsub2, Omega[3,2])
  - Simf(subs(framechangebacksub, Simf(subs(adaptedsub1,
  Simf(subs(framechangesub, Omega[3,2])))))));
> eqns:= eqns union {op(ScalarForm(zero3))};
```

Now solve these equations for the functions $(\tilde{h}_{ij})$:

```
> solve(eqns, {H[1,1], H[1,2], H[2,2]});
```

Then, we might as well actually assign these values to the $(\tilde{h}_{ij})$:

```
> assign(%);
```

Now, it's not entirely obvious how to recognize these expressions as those of equation (4.7), but we can at least check that our computations are consistent with these expressions. First, define matrices $[h_{ij}], [\tilde{h}_{ij}], B$ as follows:

```
> hmatrix:= Matrix([[h[1,1], h[1,2]], [h[1,2], h[2,2]]]);
  Hmatrix:= Matrix([[H[1,1], H[1,2]], [H[1,2], H[2,2]]]);
  B:= Matrix([[cos(theta), -sin(theta)],
  [sin(theta), cos(theta)]]);
```

If everything has gone according to plan, the following matrix should be zero:

```
> Hmatrix - simplify(Transpose(B).hmatrix.B);
```

EXERCISE 4.40: Set up a substitution for the forms that we know from part (a), together with an expression for $\bar{\omega}_2^1$ with coefficients to be determined later:

```
> examplesub:= [omega[1] = cosh(v)*d(u),
  omega[2] = cosh(v)*d(v), omega[3]=0,
  omega[3,1] = d(u)/cosh(v), omega[3,2] = -d(v)/cosh(v),
  omega[1,2] = a*d(u) + b*d(v)];
```

Now, compute $d\bar{\omega}^1$ in two ways: by first making the substitution into $\bar{\omega}^1$ and then differentiating, and by applying the structure equations and then making the substitution. Then the difference of the resulting expressions must be equal to zero:

```
> Simf(d(Simf(subs(examplesub, omega[1])))
  - subs(examplesub, Simf(d(omega[1]))));
```

$$(\sinh(v) - \cosh(v)\, a)\, d(v) \ \&\hat{}\ d(u)$$

```
> a:= solve(%, a);
```

$$a := \frac{\sinh(v)}{\cosh(v)}$$

An analogous computation for $d\bar{\omega}^2$ yields $b = 0$; once we have made this assignment, we will have $\bar{\omega}_2^1 = \tanh(v)\, du$, as expected.

Finally, verifying the Gauss and Codazzi equations simply involves checking that both ways of computing the structure equations for each of the $(d\bar{\omega}_j^i)$ yield the same result:

```
> Simf(d(Simf(subs(examplesub, omega[1,2])))
  - subs(examplesub, Simf(d(omega[1,2]))));
```

$$0$$

```
> Simf(d(Simf(subs(examplesub, omega[3,1])))
  - subs(examplesub, Simf(d(omega[3,1]))));
```

$$0$$

```
> Simf(d(Simf(subs(examplesub, omega[3,2])))
  - subs(examplesub, Simf(d(omega[3,2]))));
```

$$0$$

EXERCISE 4.41: For a principal parametrization as in Exercise 4.24, we have

$$\bar{\omega}^1 = \sqrt{E}\, du, \qquad \bar{\omega}^2 = \sqrt{G}\, dv.$$

Then, in order for the second fundamental form to have the desired form, we must have

$$\bar{\omega}_1^3 = \frac{e}{\sqrt{E}}\, du, \qquad \bar{\omega}_2^3 = \frac{g}{\sqrt{G}}\, dv.$$

Moreover, $\bar{\omega}_2^1$ must be equal to some linear combination of du and dv.

Start by unassigning the variables a, b so that we can use them again and declaring that E, G, e, g are functions of u and v. (This declaration isn't strictly necessary, but it will make the output of some computations look nicer.)

```
> unassign('a', 'b');
> PDETools[declare](E(u,v), G(u,v), e(u,v), g(u,v));
```

Introduce a substitution for the Maurer-Cartan forms in terms of the coordinate 1-forms:

```
> coordsub:= [omega[3]=0, omega[1] = sqrt(E(u,v))*d(u),
  omega[2] = sqrt(G(u,v))*d(v),
```

```
  omega[3,1] = (e(u,v)/sqrt(E(u,v)))*d(u),
  omega[3,2] = (g(u,v)/sqrt(G(u,v)))*d(v),
  omega[1,2] = a*d(u) + b*d(v)];
```

Compute the coefficients in $\bar{\omega}_2^1$ as we did in the previous exercise:

```
> Simf(d(Simf(subs(coordsub, omega[1])))
  - subs(coordsub, Simf(d(omega[1]))));
> a:= solve(%, a);
> Simf(d(Simf(subs(coordsub, omega[2])))
  - subs(coordsub, Simf(d(omega[2]))));
> b:= solve(%, b);
```

The Gauss equation comes from comparing the two expressions for $d\bar{\omega}_2^1$:

```
> Simf(d(Simf(subs(coordsub, omega[1,2])))
  - subs(coordsub, Simf(d(omega[1,2]))));
> Gausseq1:= pick(%, d(u), d(v));
```

Now, you'll probably notice that this expression doesn't look quite like the one in part (e) of the exercise. But we can ask MAPLE to compare the two expressions to confirm that they are, in fact, equivalent. First, give a name to the expression that results from moving all the terms in equation (4.11) to the left-hand side:

```
> Gausseq2:= (e(u,v)*g(u,v))/(E(u,v)*G(u,v))
  + (1/(2*sqrt(E(u,v)*G(u,v))))*
  (diff(diff(E(u,v), v)/sqrt(E(u,v)*G(u,v)), v)
  + diff(diff(G(u,v), u)/sqrt(E(u,v)*G(u,v)), u));
```

Solve this equation for one of the variables (say, g), and then substitute this expression into the first version of the Gauss equation. If the two equations are equivalent, then the result should be zero.

```
> solve(Gausseq2, {g(u,v)});
> Simf(subs(%, Gausseq1));
```

$$0$$

Similar manipulations involving $d\bar{\omega}_1^3$ and $d\bar{\omega}_2^3$ will confirm that their structure equations are equivalent to the Codazzi equations (4.12).

Now, in fact, there's nothing special about assuming that $F = f = 0$, except that it makes the computations simpler. For a challenge, you might try redoing this exercise without this assumption. You'll need to start by applying the Gram-Schmidt algorithm to the basis $(\mathbf{x}_u, \mathbf{x}_v)$ in order to obtain an orthonormal frame field and then compute the dual forms $(\bar{\omega}^1, \bar{\omega}^2)$ for this frame field. (This part isn't too bad to do by hand.) Details are given in the MAPLE worksheet for this chapter on the AMS webpage.