Chapter 5

General Aspects of Holomorphic and Nonholomorphic Modular Forms

5.1. Introduction

In Chapter 1 we introduced some basic notions and tools that we need, and we gave a number of motivations for studying modular forms. In the present chapter we start afresh the study of modular forms and related objects and we will revisit the definitions and concepts introduced in Section 1.1.

For most parts of this book, and in particular throughout this section, we will consider functions on the upper half-plane $\mathfrak{H}$ which are *weakly modular* (in the sense of Definition 1.1.2) with respect to the modular group Γ or one of its finite index subgroups except for a few general results which will be formulated in the setting of general cofinite Fuchsian groups of the first kind.

Some other important settings which extends the current one will be briefly mentioned in Chapter 15, for instance half-integral weight, Jacobi, Maass, Hilbert, and Bianchi modular forms.

The functions we are interested in will always be assumed to be C^∞ outside of possible mild (nonessential) singularities. This still leaves a wide range of possibilities and the most important ones are the following:

- Holomorphic or meromorphic functions. This leads, respectively, to *modular forms* and to *modular functions*.
- *Almost holomorphic modular forms*, which behave modularly but are *polynomials* in $1/y = 1/\Im(\tau)$ with holomorphic coefficients.
- *Nonholomorphic modular forms*, among which important examples are *nonholomorphic Eisenstein series* and more generally *Maass forms*, which are eigenfunctions of the *hyperbolic Laplacian*.

In addition to the upper half-plane it is also necessary to describe the singular behavior at the boundary. More precisely, if G is a cofinite noncocompact Fuchsian group of the first kind, we assume, without loss of generality, that G has a cusp at $i\infty$, that is, there exists $P \in G$ with $P(i\infty) = i\infty$, and we say that the *cusps of* G are the elements of $G(i\infty)$, the G-orbit of $i\infty$. We define the extended upper half-plane $\overline{\mathfrak{H}}_G = \mathfrak{H} \cup G(i\infty) \subseteq \mathfrak{H} \cup \mathbb{R} \cup \{i\infty\}$ in the same way as for Γ in Section 4.2. For any space of weakly modular functions on G to be interesting we need to prescribe the singular behavior on the whole of $\overline{\mathfrak{H}}_G$.

Recall first that the upper half-plane is isomorphic to the open unit ball B via the *Cayley transform*, $\tau \mapsto z = (\tau - i)/(\tau + i)$, which sends the point i to the center of B and the boundary of $\mathfrak{H}$ to the boundary of B. Let z_0 be a rational point on the boundary of B. We will ask that as $z \to z_0$ the corresponding function $g(z) = f(\tau) = f(-i(z+1)/(z-1))$ does not tend too rapidly to ∞, more precisely that it is bounded by some negative power of $1-|z|$. We easily compute that $(1+|\tau|^2)/y = 2(1+|z|^2)/(1-|z|^2)$, and since $2(\sqrt{2}-1) \leq (1+|z|^2)/(1+|z|) \leq 2$, we are led to the following definition:

Definition 5.1.1. A function f on $\mathfrak{H}$ is said to be *polynomially bounded* if there exists a constant $N > 0$ such that $f(\tau) = O(((1+|\tau|^2)/y)^N)$ as $y \to \infty$ and $y \to 0$.

If we restrict ourselves to a vertical strip of bounded width, such as $|\Re(\tau)| \leq 1/2$, this means that $f(\tau) = O(y^N)$ as $y \to \infty$ and $f(\tau) = O(y^{-N})$ as $y \to 0$.

We will *always* assume that our functions are polynomially bounded if they have no singularities.

5.1.1. Basic Definitions. We are now ready to introduce the basic definitions. We give them for a general cofinite Fuchsian group G of the first kind (see Section 4.6), but the reader may assume on a first reading that G is a subgroup of finite index of the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. In addition, we will assume that the *weight* k is an integer. In the general case, some definitions must be slightly modified; see Section 15.1 for the case where $k \in (1/2) + \mathbb{Z}$. We recall that the main reason for using subgroups of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and not of $\overline{\Gamma} = \mathrm{PSL}_2(\mathbb{Z})$ is to allow for *odd* weights.

Definition 5.1.2. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$, we define the *slash operator* of (integer) weight k by

$$(f\big|_k\gamma)(\tau) = (ad - bc)^{k/2}(c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\gamma\tau)\left(\frac{d(\gamma\tau)}{d\tau}\right)^{k/2}\;,$$

where this last equality makes sense only if k is a nonnegative even integer.

This is a notation that we will use throughout this book, and when the weight k is implicit and/or fixed in a section, it may be omitted. Note that the factor $(ad - bc)^{k/2}$ is included so that the definition is homogeneous, in other words (at least when k is even), only depends on the linear fractional transformation corresponding to γ, and not on the matrix γ itself.

As mentioned in Chapter 1 we will also use this notation when k is a half-integer, and in that case all the square roots are considered to be taken using the principal branch.

Remark 5.1.3. The group $\mathrm{GL}_2^+(\mathbb{R})$ has a *right* action on functions f via the slash operator; in other words,

$$f\big|_k\gamma_1\big|_k\gamma_2 = f\big|_k\gamma_1\gamma_2\;.$$

Note, however, that the operators which we will construct using the slash operator are usually written on the left, for instance $T(n)f$, where $T(n)$ is a Hecke operator (see Section 10.2). This may lead to some confusion: for instance $T(n)f\big|_k\gamma$ can either be interpreted as $(T(n)f)\big|_k\gamma$ or as $T(n)(f\big|_k\gamma)$, which are in general quite different. Thus, it will be necessary to be careful.

Definition 5.1.4. Let G be a Fuchsian group of the first kind and let $k \in \mathbb{Z}$. A map v from G to $\mathbb{C}^\times$ is called a *multiplier system* of weight k on G if v is a *group homomorphism* from G to the group S^1 of complex numbers of modulus 1, and if $-I \in G$, then $v(-I) = (-1)^k$.

Note that if the weight k is not an integer, then the definition above needs to be modified in a way we will describe in Section 15.1.

Remarks 5.1.5.

(a) The restriction that $|v| = 1$ is not absolutely necessary but very useful. In fact, in practice v will even be of *finite order*, in other words, such that there exists $m > 0$ with $v(\gamma)^m = 1$ for all $\gamma \in G$.

(b) The three basic examples of multiplier systems are: the η-multiplier system (see Theorem 5.8.1), the θ-multiplier system (see Proposition 15.1.1), both for weight 1/2 (so simply raise them to an even power to obtain multiplier systems for integral weight), and

$v(\gamma) = \chi(a)$, when $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and χ is a Dirichlet character modulo N.

(c) Theorem 4.2.1 immediately implies that there exist exactly 6 multiplier systems on the full modular group $\overline{\Gamma}$, defined by $v(S) = (-1)^j$ and $v(ST) = \rho^j$ for $0 \le j \le 5$. On the contrary, because of Proposition 6.2.18, which we will prove later, there exist infinitely many multiplier systems on the principal congruence subgroup $\Gamma(2)$.

Definition 5.1.6. Let G be a Fuchsian group of the first kind, k an integer, and v a multiplier system of weight k for G.

(a) A function f from $\mathfrak{H}$ to $\mathbb{C}$ is said to be *weakly modular of weight* k and *multiplier system* v if for any $\gamma \in G$ we have $f\big|_k\gamma = v(\gamma)f$, in other words, if for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and $\tau \in \mathfrak{H}$ we have

$$f(\gamma\tau) = v(\gamma)(c\tau + d)^k f(\tau)\ .$$

(b) A *modular form* is a weakly modular function which is holomorphic and polynomially bounded in $\overline{\mathfrak{H}}_G$; the vector space of modular forms is denoted $M_k(G, v)$.

(c) A *modular function* is a weakly modular function of weight 0 which is meromorphic in $\overline{\mathfrak{H}}_G$.

(d) An *almost holomorphic modular form* is a weakly modular function f of the form $f = P(1/y)$, where P is a polynomial whose coefficients are polynomially bounded holomorphic functions; the degree of P is called the *depth* of f. The vector space of such forms with $\deg(P) \le p$ is denoted by $M_k^{\mathrm{ah},p}(G, v)$.

Also related to these definitions is the following:

Definition 5.1.7. A function f from $\mathfrak{H}$ to $\mathbb{C}$ is said to be a *quasi-modular form* of weight k and multiplier system v with respect to G if f is holomorphic and polynomially bounded and if for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and $\tau \in \mathfrak{H}$

$$(f\big|_k\gamma)(\tau) = v(\gamma)P\Big(\tau, \frac{c}{c\tau + d}\Big)\ ,$$

where P is a polynomial in two variables whose coefficients depend only on f and not on γ; the degree of P will be called the *depth* of f. The vector space of such forms with $\deg(P) \le p$ will be denoted $M_k^{\mathrm{qm},p}(G, v)$.

Remarks 5.1.8.

(a) The holomorphy and meromorphy conditions are of course given with respect to the *local coordinates* as in Theorem 4.4.3, in particular with respect to $q = e^{2\pi i\tau}$ in a neighborhood of $i\infty$.

(b) We use the term *modular function* for weight 0 since this case corresponds to functions on $G\backslash\mathfrak{H}$ when the multiplier system v is trivial.

(c) In contrast, a holomorphic or meromorphic weakly modular function of (even) weight k and trivial multiplier system v is not invariant under G but instead corresponds to a holomorphic or meromorphic differential form on $G\backslash\mathfrak{H}$ since $f(z)(dz)^{k/2}$ is invariant under G.

(d) Sometimes we use the term *meromorphic modular form* to denote a meromorphic weakly modular function of nonzero weight.

(e) When $v = 1$, the trivial multiplier system, it will be omitted from the notation and the weight must be even if $-I \in G$.

(f) Since $v(I) = 1$ (where I is the identity), choosing $\gamma = I$ in the definition of quasi-modular forms shows that $P(\tau, 0) = f(\tau)$, hence that $f(\tau + m) = f(\tau)$ if $T^m \in G$ and $v(T^m) = 1$.

(g) We will mainly study the spaces $M_k(G, v)$ of modular *forms*, essentially because they are finite-dimensional and have the nicest properties. It is, however, important to introduce other spaces from the start, even though they are in general infinite-dimensional.

(h) It is clear that we have

$$M_k(G,v) = M_k^{\mathrm{ah},0}(G,v) \subset M_k^{\mathrm{ah},1}(G,v) \subset \cdots \subset M_k^{\mathrm{ah},p}(G,v) \subset \cdots \quad \text{and}$$
$$M_k(G,v) = M_k^{\mathrm{qm},0}(G,v) \subset M_k^{\mathrm{qm},1}(G,v) \subset \cdots \subset M_k^{\mathrm{qm},p}(G,v) \subset \cdots .$$

Some of the following examples were briefly introduced in Chapter 1 and we also introduce a few new ones:

Examples 5.1.9.

(1) When $k \geq 4$ is even, the Eisenstein series E_k and G_k are modular forms of weight k on Γ.

(2) The discriminant function $\Delta = (E_4^3 - E_6^2)/1728$ is a modular form of weight 12 on Γ.

(3) The modular invariant, $j = E_4^3/\Delta$, is a modular function on Γ.

(4) The Dedekind eta function, η, is a modular form of weight 1/2 on Γ with a (complicated) multiplier system.

(5) The theta function, $\theta_{0,0}$, is a modular form of weight 1/2 on the group Γ_θ with multiplier system.

(6) The Eisenstein series, E_2, is a quasi-modular form of weight 2 and depth 1 on Γ.

(7) It is immediate to see that if $f \in M_k(G, v)$, then $f' \in M_{k+2}^{\mathrm{qm},1}(G, v)$ is quasi-modular of weight $k + 2$ and depth 1, and more generally

that if f is quasi-modular of weight k and depth p, then f' is quasi-modular of weight $k+2$ and depth $p+1$.

(8) We will see below, but it is immediate to prove, that if $f \in M_k(G,v)$, then $f' + (k/(2iy))f \in M_{k+2}^{\mathrm{ah},1}(G,v)$ is an almost holomorphic modular form of weight $k+2$ and depth 1, and more generally that if $f \in M_k^{\mathrm{ah},p}(G,v)$, then $f' + (k/(2iy))f \in M_{k+2}^{\mathrm{ah},p+1}(G,v)$.

(9) Similarly, $E_2 - (3/(\pi y)) \in M_2^{\mathrm{ah},1}(\Gamma)$ is an almost holomorphic modular form of weight 2 and depth 1 on Γ.

5.1.2. Polynomially Bounded Functions. We will first need to study the set of polynomially bounded functions in a little more detail.

Lemma 5.1.10.

(a) *A function f is polynomially bounded if and only if there exist positive constants N_1 and N_2 such that $f(\tau) = O((1+|\tau|^2)^{N_1}/y^{N_2})$.*

(b) *If f is polynomially bounded, then so is $f\big|_k\gamma$ for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in$ $\mathrm{SL}_2(\mathbb{R})$ and any k.*

Proof. (a) If f is polynomially bounded, we choose $N_1 = N_2 = N$. Conversely, assume the condition of (a) is satisfied. We claim that $f(\tau) = O(((1+|\tau|^2)/y)^{\max(N_2, 2N_1 - N_2)})$. Indeed, assume first that $N_1 \geq N_2$ so that $\max(N_2, 2N_1 - N_2) = 2N_1 - N_2$ and write

$$(1+|\tau|^2)^{N_1}/y^{N_2} = ((1+|\tau|^2)/y)^{2N_1-N_2}(y^2/(1+|\tau|^2))^{N_1-N_2}\ ,$$

proving the claim since $y^2 \leq 1 + |\tau|^2$ and $N_1 - N_2 \geq 0$. Otherwise, if $N_1 < N_2$, then $\max(N_2, 2N_1 - N_2) = N_2$ and because $1 + |\tau|^2 \geq 1$ we are done since

$$(1+|\tau|^2)^{N_1}/y^{N_2} \leq ((1+|\tau|^2)/y)^{N_2}\ .$$

(b) With a suitable constant A, depending on γ, we have

$$(1+|\gamma\tau|^2)/\Im(\gamma\tau) = (|a\tau+b|^2 + |c\tau+d|^2)/y \leq A(1+|\tau|^2)/y$$

and the result follows by (a) since $\min(1, y^2) \leq |c\tau+d|^2 \leq B(1+|\tau|^2)$. $\square$

Lemma 5.1.11. *If f is a polynomially bounded holomorphic function on $\mathfrak{H}$ which is periodic with period m for some $m \in \mathbb{R}_{>0}$, then, in fact, as $y \to \infty$ we have $f(\tau) = O(1)$. More precisely, there exists a constant a_0 such that*

$$f(\tau) = a_0 + O(e^{-2\pi y/m}) \quad \text{as} \quad y \to \infty.$$

Proof. Set $f_1(\tau) = f(m\tau)$, so that f_1 is holomorphic and 1-periodic; in particular, it is equal to the sum of its Fourier series $f_1(\tau) = \sum_{n\in\mathbb{Z}} c_n(y)e^{2\pi inx}$ (as usual $\tau = x + iy$), and again since f_1 is holomorphic, we have

$$c_n(y) = \int_0^1 f_1(x+iy)e^{-2\pi inx}\,dx = a_n e^{-2\pi ny}$$

for some constant a_n. Since f_1 is polynomially bounded, it follows that $a_n e^{-2\pi ny} = O(y^N)$ when $y \to \infty$, which means that $a_n = 0$ when $n < 0$. Thus, $f(\tau) = \sum_{n\geq 0} a_n e^{2\pi in\tau/m}$, which tends to a_0 when $y \to \infty$. $\square$

We now specialize to the very important case of weakly modular functions on a subgroup G of the full modular group Γ. We begin with the following lemma from elementary group theory:

Lemma 5.1.12. *If G_1 is a subgroup of finite index of some other group G_2, then for any $\gamma \in G_2$ there exists $m \neq 0$ such that $\gamma^m \in G_1$.*

Proof. If γ has finite order n, the result is clear since $\gamma^n = 1 \in G_1$. Otherwise, there are infinitely many γ^n's; thus at least two distinct ones are in the same right coset of G_2 modulo G_1. In other words, there exists $i \neq j$ and $g \in G_1$ such that $\gamma^j = g\gamma^i$, so that $\gamma^{j-i} = g \in G_1$ with $j - i \neq 0$. $\square$

Note that m cannot always be taken to be equal to the index $[G_2 : G_1]$: for instance if $G_2 = S_3$, the symmetric group on $\{1, 2, 3\}$, and G_1 is the subgroup $\{(1)(2)(3), (1\,2)\}$, whose index is 3, then if $\gamma = (1\,3)$, we must take $m = 2$ or more generally an even number, not necessarily a multiple of 3. However, we have the following more precise result when $G_2 = \Gamma$:

Lemma 5.1.13. *If G is a subgroup of finite index s of Γ and $\gamma \in \Gamma$, then either the image of γ in $\overline{\Gamma}$ has order less than or equal to 3, or there exists an integer $m \leq s$ such that $\gamma^m \in G$.*

Proof. The proof of the preceding lemma shows that if γ has infinite order, then there exists $m \leq s$ such that $\gamma^m \in G$. Thus, assume that γ has finite order. If γ has order $n \geq 2$, say, then $\gamma^n = I$ and hence the minimal polynomial of γ divides $X^n - 1$ and in particular it has no multiple roots. This is equivalent to γ being *diagonalizable* over $\mathbb{C}$; in other words, we can write $\gamma = \beta D \beta^{-1}$ for some diagonal matrix D with diagonal entries ζ and ζ^{-1} where ζ is an nth root of unity. Now the trace t of γ, which is an integer, is equal to the trace of D; in other words, $t = \zeta + \zeta^{-1}$. Thus, $|t| \leq 2$ and the cases $t = 0$, ± 1, or ± 2 correspond to γ being conjugate to $\pm S$, $\pm ST$, or $\pm I$, respectively, proving the lemma. $\square$

Corollary 5.1.14. *Let G be a subgroup of finite index s of Γ.*

(a) *There exists m with $0 < m \leq s$ such that $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \in G$.*

(b) *If f is weakly modular of weight k for G, then for any $\gamma \in \Gamma$, the function $f\big|_k\gamma$ is periodic of some period m_γ with $0 < m_\gamma \leq s$.*

Proof. (a) follows immediately from the lemma since $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ has infinite order. For (b), we note that $f\big|_k\gamma$ is weakly modular of weight k for the conjugate group $\gamma^{-1}G\gamma \subset \Gamma$, which has the same index as G in Γ, so by (a) there exists $m_\gamma > 0$ such that $\left(\begin{smallmatrix} 1 & m_\gamma \\ 0 & 1 \end{smallmatrix}\right) \in \gamma^{-1}G\gamma$, so the modular property implies that $f\big|_k\gamma$ is periodic of period (dividing) m_γ. □

Proposition 5.1.15. *Let ϕ be a continuous Γ-invariant function on $\mathfrak{H}$ satisfying $|\phi(\tau)| \le B(y)$ as $y \to \infty$ for some continuous function B. Then:*

(a) *If $B(y)$ is bounded as $y \to \infty$, then ϕ is bounded on $\mathfrak{H}$.*

(b) *If $B(y)$ is increasing as $y \to \infty$, then $|\phi(\tau)| \le \max(A, B(y), B(1/y))$ for some constant A.*

Proof. (a) Let $\mathfrak{F}$ denote the standard fundamental domain of $\Gamma\backslash\mathfrak{H}$. If $B(y)$ is bounded as $y \to \infty$, then there exists a y_0 such that $\phi(\tau)$ is bounded for $\Im(\tau) > y_0$ and since ϕ is also bounded on the compact set $\mathfrak{F} \cap \{\tau \mid \Im(\tau) \le y_0\}$, it follows that ϕ is bounded on $\mathfrak{F}$ and hence on $\mathfrak{H}$ since it is Γ-invariant.

(b) By assumption there exists y_0 such that $\phi(\tau) \le B(y)$ for $y \ge y_0$ and since ϕ is bounded on the compact subset of $\mathfrak{F}$ with $\Im(\tau) \le y_0$, there is a constant A such that $\phi(\tau) \le \max(A, B(y))$ for all $\tau \in \mathfrak{F}$. Now assume that $\tau = x + iy \notin \mathfrak{F}$. By periodicity we may assume that $-1/2 < \Re(\tau) \le 1/2$. Since $\mathfrak{F}$ is a fundamental domain, there exists $\gamma \in \Gamma$ such that $\gamma(\tau) = \tau' \in \mathfrak{F}$ and γ is not a translation. Thus, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \ne 0$ and in particular $|c\tau+d|^2 = (cx+d)^2 + c^2y^2 \ge c^2y^2 \ge y^2$, so that $\Im(\tau') = \Im(\tau)/|c\tau+d|^2 \le 1/y$. Since ϕ is Γ-invariant, we thus have

$$|\phi(\tau)| = |\phi(\gamma\tau)| = |\phi(\tau')| \le \max(A, B(\Im(\tau'))) \le \max(A, B(1/y))$$

since B is an increasing function, proving the proposition. □

Corollary 5.1.16. *Let G be a subgroup of Γ of finite index and let $(\gamma_i)_{1 \le i \le n}$ be a system of representatives of right cosets of $G\backslash\Gamma$; in other words, $\Gamma = \bigsqcup_{1 \le i \le n} G\gamma_i$. Let f be a continuous function from $\mathfrak{H}$ to $\mathbb{C}$ which is weakly modular of weight k for G and set $g(\tau) = \max_i(|(f\big|_k\gamma_i)(\tau)|)$. Assume that as $y \to \infty$ we have $y^{k/2}g(\tau) \le B(y)$ for some continuous function B. Then:*

(a) *If $B(y)$ is bounded as $y \to \infty$, then there is a constant A such that*

$$|g(\tau)| \le Ay^{-k/2} \quad \text{for all } \tau \in \mathfrak{H}.$$

(b) *If $B(y)$ is increasing as $y \to \infty$, then there is a constant A such that*

$$|g(\tau)| \le \max(Ay^{-k/2}, y^{-k/2}B(y), y^{-k/2}B(1/y)) \quad \text{for all } \tau \in \mathfrak{H}$$

and in particular, if $g(\tau)$ is bounded as $y \to \infty$, then

$$|g(\tau)| \le \max(A, y^{-k}) \quad \text{for all } \tau \in \mathfrak{H}.$$

Proof. Recall that $f\big|_k\gamma_i$ does not depend on the chosen representative γ_i since $f\big|_k\gamma\gamma_i = f\big|_k\gamma_i$ for $\gamma \in G$. Now if $\gamma \in \Gamma$, we have by definition

$\gamma_i\gamma = \gamma'\gamma_j$ for some j and some $\gamma' \in G$, so that

$$f\big|_k\gamma_i\gamma = f\big|_k\gamma'\gamma_j = f\big|_k\gamma_j.$$

Since the map sending γ_i to γ_j is a permutation of the representatives, it follows that g is weakly modular of weight k for Γ, or equivalently that $\phi(\tau) = y^{k/2}g(\tau)$ is Γ-invariant. The result now follows immediately from the proposition. □

The following is an important consequence:

Corollary 5.1.17. *Let G be a subgroup of Γ of finite index, let $(\gamma_i)_{1\le i\le n}$ be a system of representatives of right cosets of $G\backslash\Gamma$, and let f be a continuous function from $\mathfrak{H}$ to $\mathbb{C}$ which is weakly modular of weight k for G. Then the following are equivalent:*

(a) *The function f is polynomially bounded.*

(b) *For all i the functions $f\big|_k\gamma_i$ are polynomially bounded.*

(c) *There exists N such that for all i we have $(f|_k\gamma_i)(\tau) = O(y^N)$ as $y \to \infty$.*

If, in addition, f is holomorphic, then the above conditions are equivalent to:

(d) *For all i the functions $(f\big|_k\gamma_i)(\tau)$ are bounded when $y \to \infty$.*

(e) *The function $f(\tau)$ is bounded when $y \to \infty$ and there exists σ such that $f(\tau) = O(y^{-\sigma})$ uniformly in x as $y \to 0$.*

In addition, if $\sigma < k$ in (e), *then for all i the functions $(f\big|_k\gamma_i)(\tau)$ tend to* 0 *when $y \to \infty$; in other words, f is a* cusp form (*see Definition* 5.1.20).

Proof. By Lemma 5.1.10(b) we know that if f is polynomially bounded, then so is $f\big|_k\gamma_i$, and in particular $(f\big|_k\gamma_i)(\tau) = O(y^N)$ as $y \to \infty$ and hence (a) implies (b) implies (c). Conversely, if we assume (c), then the previous corollary, with $B(y) = y^{N+k/2}$, implies that for all i we have $f\big|_k\gamma_i = O(\max(y^{-k/2}, y^N, y^{-N-k}))$, independently of the sign of $N + k/2$. It follows that $f\big|_k\gamma_i = O(y^M) + O(y^{-M})$ for a sufficiently large M. By Corollary 5.1.14 the function $f\big|_k\gamma_i$ is periodic in x and we may therefore restrict it to a finite vertical strip and hence by definition it is indeed polynomially bounded. Hence (c) implies (b), and (b) trivially implies (a). The equivalence of (c) and (d) when f is holomorphic follows from Lemma 5.1.11, again using the periodicity of $f\big|_k\gamma_i$. Corollary 5.1.16 shows that (d) implies (e). Conversely, assume (e) and write $\gamma_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$. If $c_i = 0$, then $\gamma_i = T^b$ for some b, so that $(f\big|_k\gamma_i)(\tau) = f(\tau + b)$ which is clearly bounded since f is. Thus, we may assume $c_i \ne 0$. Since by Corollary 5.1.14 the function $f_i = f\big|_k\gamma_i$

is periodic of some period w_i and since it is holomorphic, we can write $f_i = \sum_{n\in\mathbb{Z}} a_i(n)e^{2\pi in\tau/w_i}$, so that for any $\tau_0 = x_0 + iy_0 \in \mathfrak{H}$ we have

$$a_i(n) = \frac{1}{w_i}\int_{\tau_0}^{\tau_0+w_i} f_i(\tau)e^{-2\pi in\tau/w_i}\,d\tau\ .$$

For $\tau = x + iy_0 \in [\tau_0, \tau_0 + w_i]$ we have

$$\Im(\gamma_i\tau) = \frac{\Im(\tau)}{|c_i\tau + d_i|^2} = \frac{y_0}{(c_i x + d_i)^2 + c_i^2 y_0^2} = O(1/y_0)\ ,$$

as $y_0 \to \infty$, since x is in the compact interval $[x_0, x_0 + w_i]$ and since we have assumed $c_i \neq 0$. It follows by (e) that $f(\gamma_i\tau) = O(y_0^\sigma)$ as $y_0 \to \infty$, so that

$$f_i(\tau) = f\big|_k\gamma_i(\tau) = (c_i\tau + d_i)^{-k} f(\gamma_i\tau) = O(y_0^{\sigma-k})$$

(again because $c_i \neq 0$). Thus, by the integral formula above, we have

$$a_i(n) = O(y_0^{\sigma-k}e^{2\pi ny_0/w_i})$$

when $y_0 \to \infty$. This implies that $a_i(n) = 0$ for $n < 0$, in other words, that $f_i(\tau)$ is bounded as $\tau \to i\infty$, as claimed. In addition $a_i(0) = O(y_0^{\sigma-k})$ and therefore if we also have $\sigma < k$, then it follows that $a_i(0) = 0$. □

Note that conditions (d) or (e) are those which are almost always encountered in the usual definitions of modular forms. In fact:

Proposition 5.1.18. *Keep the same assumptions on G and (γ_i) as in Corollary* 5.1.17 *and assume that f is a modular form of weight $k \geq 0$ on G.*

(a) *There exists a constant A such that*

$$|(f\big|_k\gamma_i)(\tau)| \leq A\max(1, y^{-k}) \quad \textit{for all } \tau \in \mathfrak{H},$$

and in particular $|f(\tau)| \leq A\max(1, y^{-k})$.

(b) *If $(f\big|_k\gamma_i)(\tau)$ tends to 0 as $y \to \infty$ for all i, then*

$$|(f\big|_k\gamma_i)(\tau)| \leq Ay^{-k/2} \quad \textit{for all } \tau \in \mathfrak{H} \textit{ and all } i,$$

and in particular $|f(\tau)| \leq Ay^{-k/2}$, for some constant A.

Proof. By the above corollary and its proof, we see that $(f\big|_k\gamma_i)(\tau) = O(1)$ when $y \to \infty$, and also, using $N = 0$, that there is a constant A such that

$$|(f\big|_k\gamma_i)(\tau)| \leq A\max(y^{-k/2}, y^0, y^{-k}) \leq A\max(1, y^{-k})\ ,$$

for all $\tau \in \mathfrak{H}$, proving (a). For (b) note that by Lemma 5.1.11 and the assumption we have $(f\big|_k\gamma_i)(\tau) = O(e^{-2\pi y/m_{\gamma_i}})$ for some m_{γ_i}. Hence $B(y) = y^{k/2}e^{-2\pi y/m_{\gamma_i}}$ is bounded as $y \to \infty$ and we deduce from Corollary 5.1.16 that there is an A such that $|(f\big|_k\gamma_i)(\tau)| \leq Ay^{-k/2}$ for all $\tau \in H$. □

Remark 5.1.19. It is important to note that the result of (b) is true only if we assume that the function $(f\big|_k\gamma_i)(\tau)$ tends to 0 when $y \to \infty$ for *all* i: for instance, if we only assume that $f(\tau)$ tends to 0, the result would be false, except of course if $n = 1$; in other words, $G = \Gamma$.

The above proposition leads to the following very important definition which we will study in much greater detail later:

Definition 5.1.20. Let G be as above. We will say that a modular form of weight k for G is a *cusp form* if for all i the function $(f\big|_k\gamma_i)(\tau)$ tends to 0 when $y \to \infty$. The subspace of cusp forms of $M_k(G, v)$ is denoted $S_k(G, v)$.

The letter "S" comes from the German "Spitze", meaning cusp.

Remark 5.1.21. It is clear that all of the results of this section are valid for weakly modular forms with multiplier system, and since $f\big|_k(-\gamma) = (-1)^k f\big|_k\gamma$, that we can replace systems of representatives of $G\backslash\Gamma$ by systems of representatives of $\overline{G}\backslash\overline{\Gamma}$.

5.1.3. Almost Holomorphic and Quasi-Modular Forms. Before continuing with the study of modular forms we will prove some general results about the spaces of almost holomorphic and quasi-modular forms. These results are taken directly from a course of D. Zagier at the Collège de France and some of them are also mentioned in [**Zag08**]. For simplicity we assume a trivial multiplier system, but all results can be generalized immediately.

The first important fact here is that quasi-modular forms are essentially just constant terms of almost holomorphic modular forms and as such they in fact determine the corresponding almost holomorphic modular form completely (Corollary 5.1.23 below).

Another important fact is that if G is a cofinite noncocompact Fuchsian group of the first kind, then there exists a quasi-modular form ϕ of weight 2 and depth 1 (which is not modular) such that every quasi-modular form on G can be written in the following two ways:

- as a polynomial in ϕ with modular forms as coefficients and
- as a linear combination of ϕ and derivatives of modular forms.

We will not prove this fact (see [**Zag08**]) but observe that it provides us with an efficient method for computing with quasi-modular forms using modular forms. For the modular group and its subgroups the function ϕ can be taken to be the weight 2 Eisenstein series, E_2.

For the remainder of this subsection we assume that k is an even integer and G a cofinite Fuchsian group of the first kind.

Theorem 5.1.22 (Zagier). *Let $(f_i)_{0\le i\le p}$ be a finite sequence of polynomially bounded holomorphic functions on $\mathfrak{H}$. The following properties of this sequence are equivalent:*

(a) $\left(f_0\big|_k\gamma\right)(\tau)=\sum_{0\le n\le p}\frac{f_n(\tau)}{n!}\left(\frac{c}{c\tau+d}\right)^n$ *for all* $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in G$.

(b) $F(\tau)=\sum_{0\le n\le p}\frac{f_n(\tau)}{n!}(2iy)^{-n}$ *satisfies* $F\big|_k\gamma=F$ *for all* $\gamma\in G$.

(c) $P(\tau,T)=\sum_{0\le n\le p}\frac{f_n(\tau)}{n!}T^n$ *satisfies* $P\big|_k\gamma=P$ *for all* $\gamma\in G$, *where, by definition,*

$$(P\big|_k\gamma)(\tau,T)=(c\tau+d)^{-k}P(\gamma\tau,(c\tau+d)^2T-c(c\tau+d))\ .$$

(d) *For all m such that $0\le m\le p$ we have*

$$(f_m\big|_{k-2m}\gamma)(\tau)=\sum_{0\le n\le p-m}\frac{f_{n+m}(\tau)}{n!}\left(\frac{c}{c\tau+d}\right)^n\quad \textit{for all } \gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in G\ .$$

Proof. We show that (c) is equivalent to (d), (b) is equivalent to (c), and (a) is equivalent to (d).

(c) $\Longleftrightarrow$ (d). Assuming (d) we have

$$\begin{aligned}P(\gamma\tau,&(c\tau+d)^2T-c(c\tau+d))=\sum_{0\le m\le p}\frac{f_m(\gamma\tau)}{m!}((c\tau+d)^2T-c(c\tau+d))^m\\&=(c\tau+d)^k\sum_{0\le m\le p}\frac{(f_m|_{k-2m}\gamma)(\tau)}{m!}\left(T-\frac{c}{c\tau+d}\right)^m\\&=(c\tau+d)^k\sum_{0\le m+n\le p}\frac{f_{m+n}(\tau)}{m!n!}\left(\frac{c}{c\tau+d}\right)^n\left(T-\frac{c}{c\tau+d}\right)^m\\&=(c\tau+d)^k\sum_{0\le N\le p}\frac{f_N(\tau)}{N!}\left(\frac{c}{c\tau+d}+T-\frac{c}{c\tau+d}\right)^N\\&=(c\tau+d)^kP(\tau,T)\ ,\end{aligned}$$

proving (c). Conversely, if we assume (c), then we have

$$(P\big|_k\gamma)(\tau,T)=\sum_{0\le m\le p}\frac{(f_m|_{k-2m}\gamma)(\tau)}{m!}\left(T-\frac{c}{c\tau+d}\right)^m=\sum_{0\le n\le p}\frac{f_n(\tau)}{n!}T^n\ ,$$

and if we change T to $T+c/(c\tau+d)$, expand $(T+c/(c\tau+d))^n$ by the binomial theorem, and then identify the coefficients of T^m, we recover (d).

For future reference, note that if we do *not* change T to $T+c/(c\tau+d)$ before identifying coefficients, and make evident changes in the indices, we obtain the following identity, which is therefore also equivalent to (d):

$$(\star) \qquad f_m(\tau) = \sum_{0\le n\le p-m} \frac{(f_{m+n}\big|_{k-2(m+n)}\gamma)(\tau)}{n!}\left(\frac{-c}{c\tau+d}\right)^n .$$

(c) $\Longleftrightarrow$ (b). Assume (c). We evidently have $F(\tau) = P(\tau, 1/(2iy))$, and

$$F\big|_k\gamma = (c\tau+d)^{-k}P(\gamma\tau, 1/(2i\Im(\gamma\tau))) ,$$

and therefore (b) follows since we check immediately that

$$\begin{aligned}\frac{1}{2i\Im(\gamma\tau)} &= \frac{1}{2iy}(c\tau+d)(c\overline{\tau}+d)\\ &= \frac{1}{2iy}(c\tau+d)(c\tau+d-2iyc) = \frac{(c\tau+d)^2}{2iy} - c(c\tau+d) .\end{aligned}$$

Conversely, by the same formula, (b) clearly means that $Q(\tau, 1/(2iy)) = P(\tau, 1/(2iy))$, where $Q(\tau, T) = P\big|_k\gamma(\tau, T)$ is still a polynomial in T with holomorphic coefficients. Since the nonholomorphic function y is not algebraic over the field of meromorphic functions (see Exercise 5.2), we must have $Q = P$ and (c) follows.

(a) $\Longleftrightarrow$ (d). It is clear that (a) is the special case of (d) with $m = 0$, so conversely assume (a). For $i = 1, 2$ let $\gamma_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) \in G$ and $\gamma = \gamma_1\gamma_2$. We now use the following crucial identity given in Lemma 1.1.4:

$$\frac{c_1}{c_1\gamma_2\tau + d_1} = (c_2\tau+d_2)^2\left(\frac{c}{c\tau+d} - \frac{c_2}{c_2\tau+d_2}\right) .$$

This allows us to express $g(\tau) = (f_0\big|_k\gamma_1\gamma_2)(\tau)$ in two different ways. First, by (a) applied to γ we have

$$g(\tau) = \sum_{0\le m\le p} \frac{f_m(\tau)}{m!}\left(\frac{c}{c\tau+d}\right)^m .$$

Second, we have $g = (f_0\big|_k\gamma_1)\big|_k\gamma_2$; hence by (a) applied to γ_1 we obtain

$$\begin{aligned} g(\tau) &= (c_2\tau+d_2)^{-k}\sum_{0\le n\le p}\frac{f_n(\gamma_2\tau)}{n!}\left(\frac{c_1}{c_1\gamma_2\tau+d_1}\right)^n\\ &= (c_2\tau+d_2)^{-k}\sum_{0\le n\le p}\frac{(c_2\tau+d_2)^{2n}f_n(\gamma_2\tau)}{n!}\left(\frac{c}{c\tau+d} - \frac{c_2}{c_2\tau+d_2}\right)^n\\ &= \sum_{0\le m\le n\le p}\frac{1}{m!(n-m)!}\left(\frac{c}{c\tau+d}\right)^m\left(\frac{-c_2}{c_2\tau+d_2}\right)^{n-m}(f_n\big|_{k-2n}\gamma_2)(\tau) .\end{aligned}$$

Since this holds for any $\gamma_2 \in G$, we can identify the coefficients of $(c/(c\tau+d))^m$ in the two expressions and deduce, after changing n to $n+m$,

that
$$f_m(\tau) = \sum_{0\le n\le p-m} \frac{1}{n!}\left(\frac{-c_2}{c_2\tau+d_2}\right)^n (f_{n+m}\big|_{k-2(n+m)}\gamma_2)(\tau)\ ,$$
which is equivalent to (d) by the identity ($\star$), thus finishing the proof. ☐

Corollary 5.1.23.

(a) *The spaces* $M_k^{\mathrm{ah},p}(G)$ *and* $M_k^{\mathrm{qm},p}(G)$ *are canonically isomorphic.*

(b) *If* $f_0 \in M_k^{\mathrm{qm},p}(G)$*, then* $f_m \in M_{k-2m}^{\mathrm{qm},p-m}(G)$*, where* f_m *is defined by* (a) *of the theorem.*

Proof. (a) is a rephrasing of the equivalence of (a) and (b) of the theorem, and (b) is a rephrasing of the equivalence of (a) and (d). ☐

Definition 5.1.24. We denote by δ the *shift* operator which sends $f_0 \in M_k^{\mathrm{qm},p}(G)$ to $f_1 \in M_{k-2}^{\mathrm{qm},p-1}(G)$.

It is clear from the theorem that we have $f_m = \delta^m(f_0)$. Thus, $\delta^p(f_0) = f_p \in M_{k-2p}^{\mathrm{qm},0}(G) = M_{k-2p}(G)$ is a true modular form (possibly equal to 0).

Proposition 5.1.25. *The sequence*
$$0 \longrightarrow M_k^{\mathrm{qm},p-1}(G) \longrightarrow M_k^{\mathrm{qm},p}(G) \xrightarrow{\delta^p} M_{k-2p}(G) \longrightarrow 0$$
is exact, except when $k = 2p$*,* $p > 0$*, and* G *is cocompact, in which case* δ^p *is the zero map.*

Proof. The map from $M_k^{\mathrm{qm},p-1}(G)$ to $M_k^{\mathrm{qm},p}(G)$ is the natural injection, and it is clear by definition that $\delta^p(f_0) = f_p = 0$ if and only if, in fact, $f_0 \in M_k^{\mathrm{qm},p-1}(G)$. Thus, we only need to prove surjectivity of δ^p in the given cases. For this we will assume a few (easy) results and refer to [**Zag16**] for the complete proof. First, it is not difficult to show that there do not exist any nonzero modular *forms* of strictly negative weight (we will prove this for subgroups of finite index of Γ in Section 5.6); hence we may assume that $k \ge 2p$ since otherwise $M_{k-2p}(G) = \{0\}$. Thus, let $f \in M_{k-2p}(G)$, and consider $g = f^{(p)}$ the pth derivative of f. By what we have said above, we have $g \in M_k^{\mathrm{qm},p}(G)$, and an immediate induction argument shows that
$$(g\big|_k\gamma)(\tau) = \sum_{0\le n\le p} \frac{g_n(\tau)}{n!}\left(\frac{c}{c\tau+d}\right)^n$$
with, in particular, $g_p(\tau) = \delta^p(g) = (k-2p)(k-2p+1)\cdots(k-p-1)f$ (the general formula for $g_n(\tau)$ will be given in Corollary 5.3.19 below). It follows that if $k > 2p$, a preimage of f is given by $f^{(p)}/((k-2p)\cdots(k-p-1))$.

In the case where G is a finite index subgroup of Γ we can construct a different preimage of f which avoids the restriction $k > 2p$. Indeed, in that case $E_2 \in M_2^{\mathrm{qm},1}$, hence clearly $E_2^p f \in M_k^{\mathrm{qm},p}$, and from the transformation formula for E_2 (Corollary 2.1.18) we see immediately that $\delta^p(E_2^p f) = (12/(2\pi i))^p f$ and again we find a preimage of f.

In the remaining cases, where $k = 2p$ and G is not a subgroup of Γ, it is possible to show that if G is not cocompact, then we can construct a form in $M_2^{\mathrm{qm},1}(G) \smallsetminus M_2(G)$ which will play the same role as E_2 above. In contrast, if G is cocompact, it can be shown that such a form cannot exist. Since we will not need the general case, we refer the reader to Exercises 5.3 and 5.4 for a sketch of the proof (see also [**Zag16**]). ☐

5.2. Examples of Modular Forms: Eisenstein Series

Before continuing the general study of modular forms, both holomorphic and nonholomorphic, we would like to give a few more examples extending the ones we gave in Chapter 1 in a more systematic fashion.

To simplify the exposition we will limit ourselves to the case of the full modular group, Γ, in this section. The more general case of modular forms on subgroups will be considered in Chapter 7.

5.2.1. The Weight k Nonholomorphic Eisenstein Series.

Definition 5.2.1. For an even integer $k \geq 0$ and a complex number s satisfying $\Re(2s+k) > 2$ we define $G_k(s)(\tau)$, the nonholomorphic Eisenstein series of weight k, for $\tau \in \mathfrak{H}$, by

$$G_k(s)(\tau) = \frac{1}{2} \sum_{(c,d)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}} \frac{y^s}{(c\tau+d)^k|c\tau+d|^{2s}}.$$

Proposition 5.2.2.

(a) *When $\Re(2s+k) > 2$ the above series converges uniformly on any compact subset of $\mathfrak{H}$.*

(b) *For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have $G_k(s)|_k\gamma = G_k(s)$; that is,*

$$G_k(s)(\gamma\tau) = (c\tau+d)^k G_k(s)(\tau)\ .$$

(c) *We have the Fourier series expansion*

$$\begin{aligned} G_k(s)(\tau) &= \zeta(2s+k)y^s + A_k(s)\zeta(2s+k-1)y^{1-s-k} \\ &\quad + B_k(s)y^s \sum_{n\neq 0} \sigma_{2s+k-1}(n) W_k(2\pi n\tau; s)\ , \end{aligned}$$

where $\sigma_t(n) = \sum_{d\mid n,\, d>0} d^t$ is the divisor function, W_k is given in Definition 3.5.5, and

$$A_k(s) = i^k \frac{\pi^{\frac{1}{2}}\Gamma(s+\frac{k-1}{2})\Gamma(s+\frac{k}{2})}{\Gamma(s+k)\Gamma(s)} \quad \text{and} \quad B_k(s) = 2\pi^s \frac{(2\pi)^{s+k-1/2}}{\Gamma(s+k)}.$$

(d) *If the variable* $\tau \in \mathfrak{H}$ *is fixed, then the above Fourier series gives a meromorphic continuation of* $G_k(s)$ *to the whole of* $\mathbb{C}$, *and if we set* $\mathcal{E}_k(s) := \pi^{-s}\Gamma(s)G_k(s)$, *then* $\mathcal{E}_k(s)$ *satisfies the functional equation:*

$$\mathcal{E}_k(1-s-k) = \mathcal{E}_k(s).$$

Proof. (a) By Lemma 2.1.6 and its proof we know that this series is absolutely convergent and majorized by $A^{-s-k/2}S$, where S is a convergent series independent of τ and $A = y^2/(x^2+y^2+1)$. Now if $K \subset \mathfrak{H}$ is compact, $\inf_{\tau\in K} \Im(\tau) = y_0$ is attained and thus is strictly positive, and furthermore $\sup_{\tau\in K} |\Re(\tau)| = x_0 < \infty$ and $\sup_{\tau\in K} \Im(\tau) = y_1 < \infty$ for some x_0, y_1 and $A \geq y_0^2/(x_0^2+y_1^2+1) > 0$, proving uniform convergence on K.

(b) Keeping the absolute convergence in mind, note that we can write

$$\begin{aligned}2G_k(s)(\tau) &= \sum_{e\geq 1}\sum_{\gcd(c,d)=e} \frac{y^s}{(c\tau+d)^k|c\tau+d|^{2s}}\\ &= \sum_{e\geq 1} e^{-2s-k} \sum_{\gcd(c,d)=1} \frac{y^s}{(c\tau+d)^k|c\tau+d|^{2s}} = 2\zeta(2s+k)E_k(s)(\tau).\end{aligned}$$

It is easy to see that we can write $E_k(s)(\tau)$ as

$$E_k(s)(\tau) = \sum_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma} \frac{y^s}{(c\tau+d)^k|c\tau+d|^{2s}} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Im(\gamma\tau)^s j(\gamma,\tau)^{-k}$$

where $j(\alpha,\tau) = c\tau+d$, since right cosets of Γ modulo Γ_∞ are determined by pairs (c,d) such that $\gcd(c,d)=1$, modulo the equivalence of (c,d) with $(-c,-d)$. Recall from Lemma 1.1.4 that j satisfies a cocycle property; that is, $j(\alpha,\gamma\tau) = j(\gamma,\tau)^{-1}j(\alpha\gamma,\tau)$. It follows immediately that $E_k(s)(\gamma\tau) = j(\gamma,\tau)^k E_k(s)(\tau)$ and hence $G_k(s)(\gamma\tau) = j(\gamma,\tau)^k G_k(s)(\tau)$.

(c) To find the Fourier expansion we separate the terms with $c = 0$, $c > 0$, and $c < 0$ (the latter two giving equal sums), thus obtaining

$$\begin{aligned}G_k(s)(\tau) &= \frac{y^s}{2}\sum_{d\in\mathbb{Z}\backslash\{0\}} \frac{1}{|d|^{k+2s}} + \sum_{c\geq 1}\sum_{d\in\mathbb{Z}} \frac{y^s}{(c\tau+d)^k|c\tau+d|^{2s}}\\ &= y^s\zeta(k+2s) + \sum_{c\geq 1} \frac{1}{c^s} S_k(s)(c\tau)\ ,\end{aligned}$$

where

$$S_k(s)(\tau) = \sum_{n\in\mathbb{Z}} \frac{\Im(\tau)^s}{(\tau+n)^k|\tau+n|^{2s}}\ .$$

By Corollary 3.5.7 we see that

$$S_k(s)(\tau) = A_k(s)y^{1-s-k} + B_k(s)\sum_{m\neq 0} |m|^{2s+k-1} y^s W_k(2\pi m\tau; s)\ ,$$

where $A_k(s)$, $B_k(s)$, and W_k are as given and (c) now follows since

$$\sum_{c\geq 1}\frac{1}{c^s}S_k(s)(c\tau) = A_k(s)\zeta(2s+k-1)y^{1-s-k} + B_k(s)y^s\sum_{n\neq 0}\sigma_{2s+k-1}(n)W_k(2\pi n\tau;s)\ .$$

(d) We can write the Fourier expansion of $\mathcal{E}_k(s)$ as

$$\mathcal{E}_k(s)(\tau) = c(0;s) + y^{1/2}\sum_{n\neq 0}c(n;s)(2\pi|n|y)^{s-\frac{1}{2}}W_k(2\pi n\tau;s)$$

where the constant term is given in terms of $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ as

$$c(0;s) = \pi^{\frac{k}{2}}\frac{\Gamma(s)}{\Gamma\left(s+\frac{k}{2}\right)}\Lambda(2s+k)y^s + (-1)^{\frac{k}{2}}\frac{\Gamma(s+\frac{k}{2})}{\Gamma(s+k)}\pi^{\frac{k}{2}}\Lambda(2s+k-1)y^{1-s-k}$$

and by using the functional equation $\Lambda(1-s) = \Lambda(s)$ (see Theorem 3.3.15) we see that $c(0;1-s-k)$ is given by

$$(-1)^{\frac{k}{2}}\frac{\Gamma(1-s-\frac{k}{2})}{\Gamma(1-s)}\pi^{\frac{k}{2}}\Lambda(2s+k)y^s + \pi^{\frac{k}{2}}\frac{\Gamma(1-s-k)}{\Gamma\left(1-s-\frac{k}{2}\right)}\Lambda(2s+k-1)y^{1-s-k}.$$

From the reflection formula for the gamma function (Theorem 3.3.9(a)) we have

$$\frac{\Gamma(1-s-\frac{k}{2})}{\Gamma(1-s)} = \frac{\Gamma(s)\sin\pi\left(s+\frac{k}{2}\right)}{\Gamma\left(s+\frac{k}{2}\right)\sin\pi s} = (-1)^{\frac{k}{2}}\frac{\Gamma(s)}{\Gamma\left(s+\frac{k}{2}\right)},$$

which shows that $c(0;1-s-k) = c(0;s)$. For $n\neq 0$ the coefficient is

$$c(n;s) = \frac{2(2\pi)^k\Gamma(s)}{\Gamma(s+k)}\frac{\sigma_{2s+k-1}(n)}{|n|^{s-1/2}}$$

and it is easy to see that both $\sigma_{1-2s-k}(n)/|n|^{s-1/2}$ and $\Gamma(s)/\Gamma(s+k)$ are invariant under the transformation $s\to 1-s-k$. The functional equation $\mathcal{E}_k(1-s-k) = \mathcal{E}_k(s)$ now follows from Lemma 3.5.6, which shows that the function $(2\pi|n|y)^{s-1/2}W_k(2\pi n\tau;s)$ is also invariant under $s\to 1-s-k$. $\square$

The weight zero nonholomorphic Eisenstein series plays a very important role in the spectral theory of hyperbolic surfaces (see e.g. [**Iwa02**]). We therefore present its main properties in the following corollary.

Corollary 5.2.3 (Nonholomorphic, weight 0)**.** *The function $\mathcal{E}(s)(\tau)$ defined by $\mathcal{E}(s)(\tau) := \pi^{-s}\Gamma(s)G_0(s)(\tau)$ has the following properties:*

(a) *For $\Re(s) > 1$ we have the Fourier expansion*

$$\mathcal{E}(s)(\tau) = \Lambda(2s)y^s + \Lambda(2(1-s))y^{1-s} + 4 \cdot y^{1/2} \sum_{n \geq 1} \frac{\sigma_{2s-1}(n)}{n^{s-1/2}} K_{s-1/2}(2\pi n y)\cos(2\pi n x) ,$$

which gives an analytic continuation to the whole complex s-plane.

(b) *The function $\mathcal{E}(s)$ is meromorphic with exactly two poles, which are simple, at $s = 0$ and $s = 1$, with (constant) residues $-1/2$ and $1/2$.*

(c) *It satisfies the functional equation $\mathcal{E}(1-s) = \mathcal{E}(s)$.*

(d) *For fixed $s \in \mathbb{C}$ the function $\mathcal{E}(s)(\tau)$, as a function of τ, satisfies*
   (i) *$\mathcal{E}(s)(\gamma\tau) = \mathcal{E}(s)(\tau)$ for all $\gamma \in \Gamma$ and $\tau \in \mathfrak{H}$ and*
   (ii) *$\Delta\mathcal{E}(s)(\tau) = s(1-s)\mathcal{E}(s)(\tau)$ for all $\tau \in \mathfrak{H}$ where Δ is the hyperbolic Laplacian: $\Delta = -y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$.*

Proof. It is easy it verify directly that $\Delta\mathcal{E}(s) = s(1-s)\mathcal{E}(s)$ and all other properties follow immediately from the previous proposition and the definition of W_0. We leave the details to the reader. □

In the theory of holomorphic modular forms the Eisenstein series play several important roles. They serve as important examples with coefficients that are easy to compute but they in fact also generate large spaces of so-called cusp forms. For the modular group, for example, it can be shown that any modular form can be expressed as a polynomial in the Eisenstein series of weight 4 and 6. We will see more results in this direction in Section 10.6.

Corollary 5.2.4 (Holomorphic, weight $k \geq 4$)**.** *For $k \geq 4$ even we have the Fourier expansion*

$$G_k(\tau) := G_k(0)(\tau) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) e^{2\pi i n \tau} ,$$

and G_k is a modular form of weight k; in other words, $G_k \in M_k(\Gamma)$.

Proof. Again immediate from Lemma 3.5.6 and left to the reader. □

Corollary 5.2.5 (Almost holomorphic, weight $k = 2$)**.** *We have the Fourier expansion*

$$G_2^*(\tau) := G_2(0)(\tau) = \frac{\pi^2}{6} - \frac{\pi}{2y} - 4\pi^2 \sum_{n \geq 1} \sigma_1(n) e^{2\pi i n \tau} ,$$

and G_2^* is an almost holomorphic modular form of weight 2 and depth 1.

Proof. For $k \geq 4$, the term $A_2(s,k)\Lambda(2s+k-1)y^{1-s-k}$ vanishes when $s=0$ because of the factor $\Gamma(s)$ in the denominator of $A(s,k)$. However, for $k=2$, the function $\zeta(2s+k-1)$ also has a pole at $s=0$, so computing the residues gives the result. □

5.2.2. Holomorphic Eisenstein Series. There are several different natural normalizations of the holomorphic Eisenstein series. Instead of summing over pairs $(c,d) \neq (0,0)$ we can also sum over relatively prime c, d and obtain

Proposition 5.2.6. *For an even integer $k \geq 4$ we set*

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ \gcd(m,n)=1}} (m\tau + n)^{-k} \ .$$

(a) *The function E_k is a modular form of weight k, and $E_k(\tau) = G_k(\tau)/\zeta(k)$. In particular, $E_k(i\infty) = 1$.*

(b) *If we let $\Gamma_\infty = \{ \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \mid n \in \mathbb{Z}\} = \{T^n \mid n \in \mathbb{Z}\}$, then*

$$E_k(\tau) = \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma} (c\tau + d)^{-k} \ .$$

Proof. Immediate and left to the reader as Exercise 5.6. □

It follows that this is also a reasonable normalization (although not the only one) of the Eisenstein series.

Proposition 5.2.7.

(a) *For all even integers $k \geq 2$ we have*

$$\zeta(k) = -\frac{(2\pi i)^k}{k!} \frac{B_k}{2} \ .$$

Thus, for instance $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, and $\zeta(6) = \pi^6/945$.

(b) *The Fourier expansion of E_k is given by*

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{m \geq 1} \sigma_{k-1}(m) q^m \ .$$

Proof. Statement (a) is a famous result due to Euler and has already been proved in Theorem 3.3.15 but also follows immediately from the power series expansion of $\pi\tau \operatorname{cotan}(\pi\tau)$ obtained in Proposition 3.1.17, and (b) is an immediate consequence, left to the reader (Exercise 5.5). □

Corollary 5.2.8. *Let $k \geq 4$ be an even integer. If $4 \nmid k$, we have $E_k(i) = 0$, and if $6 \nmid k$, we have $E_k(\rho) = 0$, where we recall that $\rho = e^{2\pi i/3}$.*

Proof. Setting $\tau = i$ in the modular identity $E_k(-1/\tau) = \tau^k E_k(\tau)$ gives $E_k(i) = (-1)^{k/2} E_k(i)$, proving the first result, and similarly the second result is obtained by substituting $\tau = \rho$ in $E_k(-1/(\tau+1)) = (\tau+1)^k E_k(\tau)$. □

Example 5.2.9. Choosing $k = 6$ gives the identity

$$\sum_{n\ge1} \sigma_5(n) e^{-2\pi n} = \sum_{n\ge1} \frac{n^5}{e^{2\pi n}-1} = \frac{1}{504}\ .$$

We summarize the different normalizations for Eisenstein series and introduce still another one in the following definition:

Definition 5.2.10. For any even $k \ge 2$ (including $k = 2$) we define

$$E_k = 1 - \frac{2k}{B_k} \sum_{m\ge1} \sigma_{k-1}(m) q^m\ ,$$

$$F_k = -\frac{B_k}{2k} E_k = \sum_{m\ge0} \sigma_{k-1}(m) q^m\ , \quad \text{and}$$

$$G_k = \zeta(k) E_k = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{m\ge1} \sigma_{k-1}(m) q^m\ ,$$

where, by convention, we set $\sigma_{k-1}(0) = -\dfrac{B_k}{2k}$.

We will mainly use the normalization E_k, but for instance when dealing with Hecke operators (see Chapter 10) the natural normalization is F_k and in this context we will call it *the* normalized Eisenstein series of weight k.

5.2.3. Nonholomorphic Eisenstein Series.

In the same way as for holomorphic Eisenstein series we can also use different normalizations for the nonholomorphic Eisenstein series of weight 0. The most common normalization of this series is the following.

Definition 5.2.11. For $\Re(s) > 1$ we define the *nonholomorphic Eisenstein series* of weight k by $E_k(s)(\tau) = G_k(s)(\tau)/\zeta(2s+k)$ and in particular for weight 0 we usually set $E(s) = E_0(s)$ and have the formula

$$E(s)(\tau) = \sum_{\gamma=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma} \frac{y^s}{|c\tau+d|^{2s}} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Im(\gamma\tau)^s.$$

Proposition 5.2.12. *The above series is absolutely convergent for* $\Re(s) > 1$ *and we have the alternative expressions*

$$E(s) = G_0(s)/\zeta(2s) = \mathcal{E}(s)/\Lambda(2s)$$

where $G_0(s)$ is as given by Definition 5.2.1 *and $\mathcal{E}(s)$ by Corollary* 5.2.3. *In particular, $E(s)$ has a Fourier expansion*

$$\begin{aligned} E(s)(\tau) &= y^s + \frac{\Lambda(2(1-s))}{\Lambda(2s)} y^{1-s} \\ &\quad + \frac{4y^{1/2}}{\Lambda(2s)} \sum_{n \geq 1} \frac{\sigma_{2s-1}(n)}{n^{s-1/2}} K_{s-1/2}(2\pi n y) \cos(2\pi n x) \,, \end{aligned}$$

which gives a meromorphic continuation to the whole complex s-plane.

Proof. The convergence has already been proved in the context of the series $G_k(s)(\tau)$ and the relation between the two series is proved in exactly the same way as in the holomorphic case:

$$\begin{aligned} 2G(s)(\tau) &= \sum_{e \geq 1} \sum_{(c,d)\,,\ \gcd(c,d)=e} \frac{y^s}{|c\tau + d|^{2s}} \\ &= \sum_{e \geq 1} e^{-2s} \sum_{\gcd(c_1,d_1)=1} \frac{y^s}{|c_1\tau + d_1|^{2s}} = 2\zeta(2s) E(s)(\tau). \end{aligned}$$

□

By using the properties of $G_0(s)$ and $\mathcal{E}(s) = \pi^{-s}\Gamma(s)G_0(s) = \Lambda(2s)E(s)$ we obtain the corresponding properties of the function $E(s)$.

Corollary 5.2.13. *Consider $E(s)$ as a function of s for fixed $\tau \in \mathfrak{H}$. Then:*

(a) *The function $E(s)$ has a meromorphic continuation to the whole complex plane and satisfies the functional equation*

$$E(1-s) = E(s) \frac{\Lambda(2s)}{\Lambda(2s-1)}.$$

(b) *In the half-plane $\Re(s) \geq \frac{1}{2}$ the function $E(s)$ has only a simple pole at $s = 1$ with residue $3/\pi$.*

(c) *Any other pole of $E(s)$ must be a nonreal zero of $\zeta(2s)$ which, assuming the Riemann hypothesis, has real part equal to* 1/4.

(d) *In addition, we have $E(1/2)(\tau) = 0$, $E(0)(\tau) = 1$, and $E(k)(\tau) \neq 0$ for all $k \in \mathbb{Z}$.*

Proof. The meromorphic continuation, functional equation, positions of (possible) poles, and the values at $s = 0$ and $1/2$ follow directly from the corresponding properties of $\mathcal{E}(s) = \pi^{-s}\Gamma(s)\zeta(2s)E(s)$ together with properties of $\Gamma(s)$ and $\zeta(2s)$. The details are left to the reader.

Finally, observe that the function $\zeta(2s)$ is nonzero for $\Re(s) > 1/2$ by absolute convergence of the Euler product, and also for $\Re(s) = 1/2$ by the well-known result, which we will not prove here, that $\zeta(1+it)$ never vanishes.

In addition, by absolute convergence and positivity, it is clear that $E(k)(\tau)$ is strictly positive, and in particular nonzero, when $k \in \mathbb{Z}_{\geq 2}$. Using the functional equation we conclude that $E(k)(\tau)$ is nonzero for all $k \in \mathbb{Z}$. □

Remark 5.2.14. It is in general *not* true that $G_0(s)(\tau)$ vanishes when $\zeta(2s) = 0$, so that the possible poles of $E(s)(\tau)$ on the line $\Re(s) = 1/4$ do exist. See, however, Remarks 11.12.4.

For future reference, note the following lemma:

Lemma 5.2.15. *Assume that* $m \in \mathbb{Z}_{\geq 2}$. *Then*

$$\begin{aligned}\mathcal{E}(m)(\tau) &= \Lambda(2m)y^m + \Lambda(2-2m)y^{1-m} \\ &\quad + \sum_{n\in\mathbb{Z},\ n\neq 0} \frac{\sigma_{2m-1}(|n|)}{|n|^m} P_{m-1}(1/(2\pi|n|y))e^{-2\pi|n|y}e^{2\pi inx} \\ &= \Lambda(2m)y^m + \Lambda(2-2m)y^{1-m} \\ &\quad + \sum_{n\geq 1} \frac{\sigma_{2m-1}(n)}{n^m} P_{m-1}(1/(2\pi ny))\left(e^{2\pi in\tau} + e^{-2\pi in\overline{\tau}}\right)\ ,\end{aligned}$$

where P_{m-1} *is the polynomial defined in Corollary* 3.2.16.

Proof. Immediate and left to the reader. □

5.2.4. Eisenstein Series of Weight 2. For $k = 2$, the series defining E_k is not absolutely convergent, or in other words, the series $G_2(s)$ is not absolutely convergent for $s = 0$. There are several methods available to bypass this problem, but we will never obtain a (nonzero) modular form of weight 2 since, as we will see later, the space $M_2(\Gamma)$ is equal to zero. We use a standard method due to Hecke which essentially amounts to analytically continuing $G_2(s)$ to $s = 0$ by introducing a limit.

Theorem 5.2.16. *The function* $G_2(s)$, *which is analytic for* $\Re(s) > 0$, *has an analytic continuation to* $s = 0$ *given by*

$$G_2^*(\tau) = \lim_{s\to 0,\ \Re(s)>0} G_2(s)(\tau)$$

and G_2^* *is an almost holomorphic modular form of weight* 2 *and depth* 1 *with Fourier expansion*

$$G_2^*(\tau) = \frac{\pi^2}{6} - \frac{\pi}{2y} - 4\pi^2 \sum_{n\geq 1} \sigma_1(n)q^n\ .$$

Proof. By Proposition 5.2.2 we know that $G_2(s)$ has a Fourier expansion:

$$G_2(s)(\tau) = \zeta(2s+2)y^s - \frac{\pi^{1/2}\Gamma(s+1)\Gamma(s+1/2)}{\Gamma(s+2)\Gamma(s)}\zeta(2s+1)y^{-s-1}$$
$$+ 2\pi^s \frac{(2\pi)^{s+3/2}}{\Gamma(s+2)} y^s \sum_{n\neq 0} \sigma_{2s+1}(n) W_2(2\pi n\tau; s) \,,$$

for $\Re(s) > 0$. As we let $s \to 0$ in the region $\Re(s) > 0$ we see that the limit exists for all factors involved except for the term $A_2(s,2)\Lambda(2s+1)$ where we have to evaluate the limit $\lim_{s\to 0} \zeta(2s+1)/\Gamma(s) = 1/2$ and we obtain

$$G_2^*(\tau) = \zeta(2) - \frac{\pi^{1/2}\Gamma(1/2)}{y\Gamma(2)} \lim_{s\to 0} \frac{\zeta(2s+1)}{\Gamma(s)} + 2\frac{(2\pi)^{3/2}}{\Gamma(2)} \sum_{n\neq 0} \sigma_1(n) W_2(2\pi n\tau; 0)$$
$$= \frac{\pi^2}{6} - \frac{\pi}{2y} + 4\pi^2 \sum_{n\geq 1} \sigma_1(n) e^{2\pi i n\tau} \,.$$

From Proposition 5.2.2 it is clear that $G_2^*(s)$ is weakly modular and from the Fourier series above we see that it is an almost holomorphic modular form of weight 2 and depth 1. □

The most common normalization of the series G_2^* is once again obtained by dividing by $\zeta(2)$ and if we set

$$E_2(\tau) = 1 - 24\sum_{n\geq 1} \sigma_1(n) q^n \quad \text{and}$$
$$E_2^*(\tau) = G_2^*(\tau)/\zeta(2) = E_2(\tau) - \frac{3}{\pi y} = 1 - \frac{3}{\pi y} - 24\sum_{n\geq 1}\sigma_1(n)q^n \,,$$

then we have the following:

Corollary 5.2.17.

(a) *The function* $E_2^*(\tau)$ *is an almost holomorphic modular form of weight* 2 *and depth* 1.

(b) *The function* $E_2(\tau)$ *is a quasi-modular form of weight* 2 *and depth* 1 *and for all* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma$ *we have the transformation formula*

$$E_2(\gamma\tau) = (c\tau+d)^2 E_2(\tau) - \frac{6i}{\pi} c(c\tau+d) \,.$$

Proof. (a) is a restatement of the theorem, and the transformation formula in (b) follows by a simple computation. The fact that E_2 is quasi-modular follows directly from the definition. □

Note that we have already proved the transformation formula for E_2 in Chapter 1 using the theory of quasi-elliptic functions.

Corollary 5.2.18. *We have the identity*

$$\sum_{n\geq 1}\sigma_1(n)e^{-2\pi n}=\sum_{n\geq 1}\frac{n}{e^{2\pi n}-1}=\frac{1}{24}-\frac{1}{8\pi}\ .$$

Proof. As in Corollary 5.2.8 we have $E_2^*(i)=0$, which gives the identity. □

5.3. Differential Operators

We will now explore the intimate connection between modular forms and differential operators. One of the main goals of the present section is to show how to construct new modular forms from existing ones using *differential operators*. Most of the constructions work for general cofinite Fuchsian groups of the first kind G but in a number of places we will in fact assume that G is a subgroup of finite index of Γ.

5.3.1. Introduction. Recall that for $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{GL}_2^+(\mathbb{R})$ we defined

$$f\big|_k\gamma=(ad-bc)^{k/2}(c\tau+d)^{-k}f(\gamma\tau)\ .$$

Definition 5.3.1. If f is a C^1 function defined on $\mathfrak{H}$, we set

$$D_\tau(f):=\frac{1}{2\pi i}\frac{\partial f}{\partial\tau}\quad\text{and}\quad D_{\overline{\tau}}(f):=\frac{1}{2\pi i}\frac{\partial f}{\partial\overline{\tau}}\ .$$

Lemma 5.3.2. *We have*

$$D_\tau(f)\big|_{k+2}\gamma=D_\tau(f\big|_k\gamma)+\frac{k}{2\pi i}\frac{c}{c\tau+d}f\big|_k\gamma\ .$$

Proof. For simplicity denote by ∂_τ the partial derivative with respect to τ. Differentiating the definition of $f\big|_k\gamma$ with respect to τ and using $\partial_\tau((a\tau+b)/(c\tau+d))=(ad-bc)/(c\tau+d)^2$ gives

$$\begin{aligned}\partial_\tau(f\big|_k\gamma)(\tau)&=(ad-bc)^{(k+2)/2}(c\tau+d)^{-k-2}\partial_\tau(f)(\gamma\tau)\\&\quad-kc(ad-bc)^{k/2}(c\tau+d)^{-k-1}f(\gamma\tau)\ ;\end{aligned}$$

in other words,

$$\partial_\tau(f)\big|_{k+2}\gamma=\partial_\tau(f\big|_k\gamma)+\frac{kc}{c\tau+d}f\big|_k\gamma\ ,$$

proving the lemma after division by $2\pi i$. □

Thus, if f is a weakly modular function of weight k, the function $D_\tau(f)$ is a quasi-modular form of weight $k+2$ and depth less than or equal to 1. It is only weakly modular if $k=0$. For instance, $j'(\tau)$ is a weakly modular function of weight 2; see Exercise 5.19 for additional information.

The fact that the derivative of a modular function of weight 0 is still modular, of weight 2, has important consequences also for modular functions of *nonzero* weight. For instance:

Proposition 5.3.3. *Let G be a subgroup of finite index of Γ, and let $f \in M_k(G,v)$ and $g \in M_\ell(G,w)$. Then*

$$F = \ell g f' - k f g'$$

is a cusp form *of weight $k+\ell+2$ and multiplier vw on G.*

Proof. If we set $h = f^\ell/g^k$ (assuming g is not identically zero), then clearly h is a modular function of weight 0 and h' is modular of weight 2; hence

$$F = g^{k+1}h'/f^{\ell-1} = \ell g f' - k f g'$$

is a modular function of weight $\ell(k+1)+2-k(\ell-1) = k+\ell+2$. In addition, $F = \ell g f' - k f g'$ is clearly holomorphic on $\overline{\mathfrak{H}}$ and polynomially bounded.

Let us show that it is a cusp form: let $(\gamma_i)_{1\le i\le n}$ be a system of representatives of the right cosets of $G\backslash\Gamma$, set $f_i = f\big|_k\gamma_i$, $g_i = g\big|_\ell\gamma_i$ and $h_i = h\big|_0\gamma_i$, so that $h_i = f_i^\ell/g_i^k$ is a modular function of weight 0 on the conjugate group $G_i = \gamma_i^{-1}G\gamma_i$. As above, $F_i = \ell g_i f_i' - k f_i g_i'$ is modular of weight $k+\ell+2$ on G_i. Furthermore, by definition $h_i' = h'\big|_2\gamma_i$, so that $F\big|_{k+\ell+2}\gamma_i = F_i$, and looking at the Fourier expansions, it is clear that this function tends to 0 when $y \to \infty$, proving that F is a cusp form. $\square$

This is in fact the first step in a ladder of differential operators, which we will study below.

5.3.2. Modifications of the Differentiation Operator. If the weight k is nonzero, it is not difficult to modify the differentiation operator so that modularity is preserved. This can be done in at least two ways. The first way has the advantage of preserving holomorphy when the initial form is holomorphic. The second only preserves almost holomorphy (see Definition 5.1.6) but has the advantage of being easier to handle. We first need to introduce some notation. There is no agreement on the notation to use, so the reader should be warned that the notation encountered in other texts might be different.

Definition 5.3.4. Let f be a function defined on $\mathfrak{H}$ and $k \in \mathbb{R}$. We set

$$\begin{aligned}
Y(\tau) &:= -\frac{1}{4\pi\Im(\tau)} = -\frac{1}{4\pi y}\ ,\\
Y_2(\tau) &:= -\frac{1}{12}E_2(\tau) = -\frac{1}{12} + 2\sum_{n\ge1}\sigma_1(n)q^n\ ,\\
Y_2^*(\tau) &:= -\frac{1}{12}E_2^*(\tau) = Y_2(\tau) - Y(\tau)\ ,
\end{aligned}$$

and for any function Z defined on $\mathfrak{H}$, such as, for example, Y or Y_2, we set

$$\mathfrak{D}_Z(f) = \mathfrak{D}_{Z,k}(f) := D_\tau(f) + kZf \;,$$

where the index k is omitted when it is implicit.

Remarks 5.3.5. Note the following immediate facts:

(a) By Corollary 5.2.17 we have $Y(\tau) - Y_2(\tau) = E_2^*(\tau)/12 = -Y_2^*$ which is an almost holomorphic modular form and in particular is weakly modular. It follows that we have the identity $\mathfrak{D}_{Y,k}(f) - \mathfrak{D}_{Y_2,k}(f) = (k/12)E_2^* f = -Y_2^* f$.

(b) The operator $\mathfrak{D}_{Y_2,k}$ clearly preserves holomorphy, while $\mathfrak{D}_{Y,k}$ only preserves almost holomorphy. On the other hand, we are going to see that they both preserve (weak) modularity.

We begin by the nonholomorphic modification $\mathfrak{D}_{Y,k}$.

Proposition 5.3.6.

(a) *We have* $D_\tau(Y) = -Y^2$, *and for any* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ *we have*

$$(Y\big|_2\gamma)(\tau) = Y(\tau) - \frac{1}{2\pi i}\frac{c}{c\tau + d} \;.$$

(b) *If f is a C^1 function on $\mathfrak{H}$, for any* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ *we have*

$$\mathfrak{D}_{Y,k}(f)\big|_{k+2}\gamma = \mathfrak{D}_{Y,k}(f\big|_k\gamma) \;.$$

In particular, if f is an almost holomorphic modular form of weight k and depth p for some group G, then $\mathfrak{D}_{Y,k}(f)$ is an almost holomorphic modular form of weight $k+2$ and depth $p+1$ for G.

Proof. (a) From $Y(\tau) = (1/(2\pi i))(1/(\tau - \overline{\tau}))$ we deduce easily that $\partial_\tau Y(\tau) = -2\pi i Y^2(\tau)$. For the second formula recall that $\Im(\gamma\tau) = (ad - bc)\Im(\tau)|c\tau + d|^{-2}$ and hence

$$Y(\gamma\tau) = -\frac{1}{4\pi\Im(\gamma\tau)} = \frac{|c\tau + d|^2}{ad - bc}Y(\tau) \;,$$

so that

$$(Y\big|_2\gamma)(\tau) = \frac{c\overline{\tau} + d}{c\tau + d}Y(\tau) = \left(1 - \frac{2icy}{c\tau + d}\right)Y(\tau) = Y(\tau) - \frac{1}{2\pi i}\frac{c}{c\tau + d} \;.$$

(b) This is again a simple verification: by Lemma 5.3.2 we have

$$D_\tau(f)\big|_{k+2}\gamma = D_\tau(f\big|_k\gamma) + \frac{k}{2\pi i}\frac{c}{c\tau + d}f\big|_k\gamma \;.$$

Thus, since $\mathfrak{D}_{Y,k}(f) = D_{\mathcal{T}}(f) + kYf$ and using (a), we have

$$\begin{aligned}\mathfrak{D}_{Y,k}(f)\big|_{k+2}\gamma &= D_{\mathcal{T}}(f)\big|_{k+2}\gamma + kY\big|_2\gamma f\big|_k\gamma \\ &= D_{\mathcal{T}}(f\big|_k\gamma) + \frac{k}{2\pi i}\frac{c}{c\tau+d}f\big|_k\gamma + k\left(Y - \frac{1}{2\pi i}\frac{c}{c\tau+d}\right)f\big|_k\gamma \\ &= \mathfrak{D}_{Y,k}(f\big|_k\gamma)\ ,\end{aligned}$$

as claimed. The last statement concerning the depth is also clear: since $D_{\mathcal{T}}(y^{-p}) = -(i/2)py^{-(p+1)}$, it is clear that the operator $\mathfrak{D}_{Y,k}$ increases the depth by 1. ☐

Proposition 5.3.7. *If f is a C^1 function on $\mathfrak{H}$, then*

$$\mathfrak{D}_{Y_2,k}(f)\big|_{k+2}\gamma = \mathfrak{D}_{Y_2,k}(f\big|_k\gamma) \quad \textit{for all } \gamma \in \Gamma.$$

In particular, if f is weakly modular of weight k for some subgroup G of Γ, then $\mathfrak{D}_{Y_2,k}(f)$ is modular of weight $k+2$, and if f is a modular form or a cusp form for G, then so is $\mathfrak{D}_{Y_2,k}(f)$.

Proof. It is immediate once again to check the result directly, but we can also simply remark that by Proposition 5.3.6 the function $\mathfrak{D}_{Y,k}(f) = D_{\mathcal{T}}(f) + kYf$ satisfies $\mathfrak{D}_{Y,k}(f)\big|_{k+2}\gamma = \mathfrak{D}_{Y,k}(f\big|_k\gamma)$, and since we have mentioned that $\mathfrak{D}_{Y,k}(f) - \mathfrak{D}_{Y_2,k}(f) = (k/12)fE_2^* = -kY_2^*f$ and Y_2^* is weakly modular of weight 2, the behavior of $\mathfrak{D}_{Y_2,k}$ under γ follows from that of $\mathfrak{D}_{Y,k}$. In addition, since Y_2 is holomorphic and polynomially bounded, it is clear that modular forms are preserved. Finally, if f is a modular form, then for all $\gamma \in \Gamma$ the function $f\big|_k\gamma$ has a Fourier expansion at infinity of the form $\sum_{n\geq 0} a_\gamma(n)e^{2\pi i n\tau/m}$, so its derivative will vanish at infinity; thus, if f is a cusp form, then so is $\mathfrak{D}_{Y_2,k}(f)$. ☐

Remarks 5.3.8.

(a) Since E_2 is only quasi-modular for Γ, the transformation formula for $\mathfrak{D}_{Y_2,k}$ is valid only for $\gamma \in \Gamma$, and not for any $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ in contrast to the transformation formula for the operator $\mathfrak{D}_{Y,k}$.

(b) We have seen in the proof of Proposition 5.1.25, and more precisely in Exercise 5.3, that if G is not a cocompact subgroup of $\mathrm{PSL}_2(\mathbb{R})$, there exists a function analogous to E_2, hence also a similar modification of the differentiation operator.

Lemma 5.3.9. *We have*

$$\frac{1}{2\pi i}E_2' - \frac{1}{12}E_2^2 = -\frac{1}{12}E_4 \in M_4(\Gamma)\ , \quad \textit{or equivalently}$$

$$D_{\mathcal{T}}(Y_2) + Y_2^2 = \frac{1}{144}E_4 \in M_4(\Gamma)\ .$$

Proof. A similar computation to the one done for the preceding proposition using Corollary 5.2.17(b) shows that the left-hand side is weakly modular of weight 4 on Γ, and it is clearly holomorphic and polynomially bounded, so it is in $M_4(\Gamma)$. Since this space is of dimension 1, the result follows by comparing the constant terms of the Fourier expansions. $\square$

Proposition 5.3.10. *The algebra* $\mathbb{C}[E_2, E_4, E_6]$*, which contains*

$$\bigoplus_{k\geq 0} M_k(\Gamma) = \mathbb{C}[E_4, E_6]\ ,$$

is stable under the operator D_τ*. More precisely, we have*

$$D_\tau(E_2) = \frac{E_2^2 - E_4}{12}\ ,\ D_\tau(E_4) = \frac{E_2E_4 - E_6}{3}\ ,\ \textit{and}\ D_\tau(E_6) = \frac{E_2E_6 - E_4^2}{2}\ .$$

Proof. Clear from the above formulas. $\square$

Corollary 5.3.11. *Any element of* $\mathbb{C}[E_2, E_4, E_6]$ (*and in particular any modular form for* Γ) *is a solution of a nonlinear third-order differential equation with constant coefficients.*

Proof. If $f \in R = \mathbb{C}[E_2, E_4, E_6]$, then by the proposition, f, f', f'', and f''' are in R, and since the transcendence degree of R over $\mathbb{C}$ is at most equal to 3 (and in fact is equal to 3 as is easily seen), these four functions must be algebraically dependent over $\mathbb{C}$. $\square$

The above proof is in fact completely constructive, and it is easy to find the explicit differential equation for any given f. We will give it below in Corollary 5.3.29 for $f = E_2$.

Corollary 5.3.12. *The results of Proposition* 5.3.10 *are valid exactly as stated if we replace* E_2 *by* E_2^* *and* D_τ *by* $\mathfrak{D}_{Y,k}$.

Proof. This is an immediate exercise. $\square$

Corollary 5.3.13. *If* f *is a modular function of weight* k *on* Γ*, then* $\mathfrak{D}_{Y,k}^n(f)$ *is a rational function of* E_2^**,* E_4*, and* E_6*. Furthermore, if* f *has Fourier coefficients which are rational or algebraic numbers, then the rational function also has coefficients which are rational or algebraic numbers.*

Proof. Indeed, $f/(E_6/E_4)^{k/2}$ is a modular function of weight 0, hence a rational function of $j = 1728E_4^2/(E_4^3 - E_6^2)$, and thus f is a rational function of E_4 and E_6. The result now follows from the preceding corollary, including the rationality and algebraicity statements. $\square$

5.3.3. Iteration of Modified Differential Operators. In several cases we will need to iterate the differential operators introduced above. This is of course only a combinatorial problem, but we need to handle it in a suitable manner. Evidently, the iteration of the operator $D_{\mathcal{T}}$ (or of $\partial/\partial\tau$) is considered given. We are going to see that the iteration of the operator $\mathfrak{D}_{Y,k}$ is easy, but that of $\mathfrak{D}_{Y_2,k}$ is more subtle. Throughout this section we let f denote a C^n function on $\mathfrak{H}$ unless otherwise stated.

Definition 5.3.14. With the integer k being implicit we set, by a slight abuse of notation, for any integer n,

$$\mathfrak{D}^n_{Y,k}(f) = (\mathfrak{D}_{Y,k+2n-2} \circ \cdots \circ \mathfrak{D}_{Y,k+2} \circ \mathfrak{D}_{Y,k})(f)\ .$$

Proposition 5.3.15. *The operator $\mathfrak{D}^n_{Y,k}$ has the following properties:*

(a) *It can be explicitly expressed as*

$$\mathfrak{D}^n_{Y,k}(f) = \sum_{j=0}^{n} \binom{n}{j} \frac{(n+k-1)!}{(j+k-1)!} Y^{n-j} D^j_{\mathcal{T}}(f)\ ,$$

or equivalently

$$\frac{\mathfrak{D}^n_{Y,k}(f)}{n!} = \sum_{\substack{j+\ell=n\\ j,\ell\geq 0}} \binom{n+k-1}{\ell} Y^{\ell} \frac{D^j_{\mathcal{T}}(f)}{j!}\ .$$

(b) *For any $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ we have*

$$\mathfrak{D}^n_{Y,k}(f)\big|_{k+2n}\gamma = \mathfrak{D}^n_{Y,k}(f\big|_k\gamma)\ .$$

Proof. (a) follows from a simple computation: denoting the right-hand side by $F_n(f)$ and using $D_{\mathcal{T}}(Y) = -Y^2$, we find that

$$\begin{aligned}
&D_{\mathcal{T}}(F_n(f)) + (k+2n)YF_n(f)\\
&\quad= \sum_{j=0}^{n} \binom{n}{j} \frac{(n+k-1)!}{(k+j-1)!} (-(n-j)Y^{n-j+1}D^j_{\mathcal{T}}(f)\\
&\qquad + Y^{n-j}D^{j+1}_{\mathcal{T}}(f) + (k+2n)Y^{n-j+1}D^j_{\mathcal{T}}(f))\\
&\quad= \sum_{j=0}^{n+1} Y^{n+1-j} D^j_{\mathcal{T}}(f) c(n,j)\ ,
\end{aligned}$$

where

$$\begin{aligned}
c(n,j) &= (n+k+j)\binom{n}{j}\frac{(n+k-1)!}{(k+j-1)!} + \binom{n}{j-1}\frac{(n+k-1)!}{(k+j-2)!}\\
&= \frac{(n+k)!}{(k+j-1)!}\binom{n+1}{j}
\end{aligned}$$

and (a) now follows immediately since

$$\mathfrak{D}_{Y,k+2n}(F_n(f)) = D_\tau(F_n(f)) + (k+2n)YF_n(f) = F_{n+1}(f)\ .$$

(b) follows directly from (a) and Proposition 5.3.6 by induction. □

We can reinterpret the above proposition in terms of *generating series*:

Definition 5.3.16. We define the following two *formal* power series:

$$CK_D(f;\tau,T) = \sum_{n\geq 0} \frac{(D_\tau^n(f))(\tau)}{n!(n+k-1)!} T^n \quad \text{and}$$
$$CK_Y(f;\tau,T) = \sum_{n\geq 0} \frac{(\mathfrak{D}_{Y,k}^n(f))(\tau)}{n!(n+k-1)!} T^n\ .$$

These series were essentially introduced by Kuznetsov [**Kuz75**] and the first author [**Coh75**]. Recall that the weight-k slash-action for $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ is given by

$$(f\big|_k\gamma)(\tau) = (ad-bc)^{k/2}(c\tau+d)^{-k}f(\gamma\tau) \quad \text{where} \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

We generalize this action to formal power series whose coefficients are functions of τ (as the ones defined above) as follows:

Definition 5.3.17. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ and $CK(\tau,T)$ is a formal power series in T with coefficients which are functions of τ, we set

$$\begin{aligned}(CK\big|_k\gamma)(\tau,T) &= (ad-bc)^{k/2}(c\tau+d)^{-k}CK\left(\gamma\tau, \frac{d(\gamma\tau)}{d\tau}T\right) \\ &= (ad-bc)^{k/2}(c\tau+d)^{-k}CK\left(\frac{a\tau+b}{c\tau+d}, \frac{ad-bc}{(c\tau+d)^2}T\right)\ .\end{aligned}$$

Proposition 5.3.18. *The formal power series CK_Y and CK_D satisfy the following:*

(a) *They are related through*

$$CK_Y(f;\tau,T) = e^{TY(\tau)}CK_D(f;\tau,T)\ .$$

(b) *For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ we have*

$$(CK_D\big|_k\gamma)(f;\tau,T) = e^{(T/(2\pi i))(c/(c\tau+d))}CK_D(f|_k\gamma;\tau,T) \quad \textit{and}$$
$$(CK_Y\big|_k\gamma)(f;\tau,T) = CK_Y(f|_k\gamma;\tau,T)\ .$$

Proof. (a) By direct computation of the product of two formal power series we see that $e^{TY(\tau)}CK_D(f;\tau,T)=\sum_{n\ge 0}c(n)T^n$, with

$$\begin{aligned}c(n)&=\sum_{j=0}^{n}\frac{(D_T^j)(f)(\tau)Y(\tau)^{n-j}}{(n-j)!j!(k+j-1)!}\\&=\frac{1}{n!(n+k-1)!}\sum_{j=0}^{n}\binom{n}{j}\frac{(n+k-1)!}{(k+j-1)!}(D_T^j(f))(\tau)Y(\tau)^{n-j}\ ,\end{aligned}$$

and hence (a) follows from Proposition 5.3.15.

(b) The relation for CK_Y is easily seen to be equivalent to the equality $(\mathfrak{D}_{Y,k}^n(f))\big|_{k+2n}\gamma=\mathfrak{D}_{Y,k}^n(f\big|_k\gamma)$ and is in fact independent of the coefficients chosen for the formal power series defining CK_Y (these coefficients were chosen to satisfy (a)). Furthermore, replacing (τ,T) by $((a\tau+b)/(c\tau+d),$ $(ad-bc)T/(c\tau+d)^2)$ changes $e^{-TY(\tau)}$ to

$$e^{-(ad-bc)(c\tau+d)^{-2}Y(\gamma\tau)T}=e^{-(Y|_2\gamma)(\tau)T}=e^{-TY(\tau)}e^{(T/(2\pi i))(c/(c\tau+d))}\ ,$$

using the transformation formula for Y given above in Proposition 5.3.6, proving the relation for CK_D since by (a) we have

$$CK_D(f;\tau,T)=e^{-TY(\tau)}CK_Y(f;\tau,T)\ .$$ □

Corollary 5.3.19. *With $\partial_\tau=\partial/\partial\tau$ and $\delta=ad-bc=\det(\gamma)$ we have*

$$(\partial_\tau^n(f))(\gamma\tau)=\delta^{-n-k/2}\sum_{j=0}^{n}\binom{n}{j}\frac{(n+k-1)!}{(k+j-1)!}c^{n-j}(c\tau+d)^{n+k+j}(\partial_\tau^j(f|_k\gamma))(\tau)\ .$$

Proof. Observing that $\partial_\tau=2\pi iD_T$ we get from the proposition that

$$CK_D(f;\gamma\tau,\delta T/(c\tau+d)^2)=\delta^{-\frac{k}{2}}(c\tau+d)^k e^{(T/(2\pi i))(c/(c\tau+d))}CK_D(f|_k\gamma;\tau,T)\ .$$

By identifying the coefficients of T^n on both sides we see that

$$\begin{aligned}&\frac{(D_T^n(f))(\gamma\tau)}{n!(n+k-1)!}\frac{\delta^n}{(c\tau+d)^{2n}}\\&\quad=\delta^{-k/2}(c\tau+d)^k\sum_{j=0}^{n}\frac{(D_T^j(f|_k\gamma))(\tau)}{j!(k+j-1)!}\frac{c^{n-j}}{(2\pi i)^{n-j}(n-j)!(c\tau+d)^{n-j}}\ ;\end{aligned}$$

in other words,

$$(D_T^n(f))(\gamma\tau)=\delta^{-n-k/2}\sum_{j=0}^{n}\frac{n!(n+k-1)!c^{n-j}(c\tau+d)^{n+k+j}}{(2\pi i)^{n-j}j!(k+j-1)!(n-j)!}(D_T^j(f|_k\gamma))(\tau)\ ,$$

which proves the corollary after we replace D_T by $(1/(2\pi i))\partial_\tau$. □

As we already mentioned, the iteration of $\mathfrak{D}_{Y_2,k}$ is slightly more subtle. Our approach closely follows [**Zag16**] and we begin with a general lemma.

Lemma 5.3.20. *For any C^1 function Z defined on $\mathfrak{H}$ we define the formal power series CK_Z and the functions f_n by*

$$CK_Z(f;\tau,T) = e^{TZ(\tau)}CK_D(f;\tau,T) = \sum_{n\geq 0} \frac{f_n(\tau)}{n!(n+k-1)!}T^n \ .$$

We have $f_0 = f$, $f_1 = D_\tau(f) + kZf = \mathfrak{D}_Z(f)$, and the recursion

$$f_{n+1} = \mathfrak{D}_{Z,k+2n}(f_n) - n(n+k-1)(Z^2 + D_\tau(Z))f_{n-1} \ .$$

Proof. For simplicity, we introduce the operator

$$\Delta_k = T\frac{\partial^2}{\partial T^2} + k\frac{\partial}{\partial T} \ .$$

It is clear that $\Delta_k(T^n) = n(n+k-1)T^{n-1}$, so that by definition the series CK_D satisfies the partial differential equations $(\Delta_k - D_\tau)(CK_D) = 0$. Thus, omitting τ and f and denoting by $'$ the derivative with respect to T, we have

$$\Delta_k(CK_Z) = e^{TZ}(TZ^2CK_D + 2TZCK_D' + TCK_D'' + kZCK_D + kCK_D') \ ,$$

and since $(\Delta_k - D_\tau)(CK_D) = 0$, this gives

$$\Delta_k(CK_Z) = e^{TZ}(TZ^2CK_D + 2TZCK_D' + kZCK_D + D_\tau(CK_D)) \ .$$

Since

$$\mathfrak{D}_{Z,k}(CK_Z) = e^{TZ}(D_\tau(CK_D) + TD_\tau(Z)CK_D + kZCK_D),$$

we obtain

$$\begin{aligned}(\Delta_k - \mathfrak{D}_{Z,k})(CK_Z) &= e^{TZ}(T(Z^2 - D_\tau(Z))CK_D + 2TZCK_D') \\ &= (2TZ(CK_Z)' - T(Z^2 + D_\tau(Z))CK_Z) \ .\end{aligned}$$

Replacing CK_Z and CK_D by their formal power series expansions and identifying the coefficients of $T^n/(n!(n+k-1)!)$ gives the recursion

$$f_{n+1} - D_\tau(f_n) - (k+2n)Zf_n + n(n+k-1)(Z^2 + D_\tau(Z))f_{n-1} = 0 \ ,$$

proving the lemma. □

As a special case of this lemma, we note that if we choose $Z = Y$, then, since $D_\tau(Y) = -Y^2$, the recursion is simply $f_{n+1} = \mathfrak{D}_{Y,k+2n}(f_n)$, which is essentially the statement of Proposition 5.3.6(b).

Definition 5.3.21. We define

(a) the formal power series associated with Y_2 by

$$CK_{Y_2}(f;\tau,T) = e^{TY_2(\tau)}CK_D(f;\tau,T) = e^{TY_2^*(\tau)}CK_Y(f;\tau,T),$$

(b) the *canonical sequence* $(f_n)_{n\ge0}$ of functions attached to f by

$$CK_{Y_2}(f;\tau,T)=\sum_{n\ge0}\frac{f_n(\tau)}{n!(n+k-1)!}T^n\ .$$

Note that the canonical sequence is implicitly attached to Y_2, but we will omit this indication from now on.

Proposition 5.3.22. *Let f be a function, let $(f_n)_{n\ge0}$ be the canonical sequence attached to f, and let $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$. Then*

(a) $$CK_{Y_2}(f;\tau,T)\big|_k\gamma=CK_{Y_2}\left(f\big|_k\gamma;\tau,T\right)\ ,$$

(b) $$f_n\big|_{k+2n}\gamma=(f\big|_k\gamma)_n\ .$$

The functions f_n satisfy the recursion defined by $f_0=f$, $f_1=\mathfrak{D}_{Y_2,k}(f)$, and

$$f_{n+1}=\mathfrak{D}_{Y_2,k+2n}(f_n)-\frac{n(n+k-1)}{144}E_4f_{n-1}\ .$$

Furthermore, if $G\subseteq\Gamma$ and $f\in M_k(G,v)$, then $f_n\in M_{k+2n}(G,v)$.

Proof. (a) follows immediately from the formula

$$CK_{Y_2}(f;\tau,T)=e^{TY_2^*(\tau)}CK_Y(f;\tau,T)\ ,$$

the transformation property of CK_Y given by Proposition 5.3.18, and the fact that Y_2^* is weakly modular of weight 2. (b) is trivially equivalent to (a). The recursion formula is an immediate consequence of Lemma 5.3.20 since by Lemma 5.3.9 we have $D_{\mathcal{T}}(Y_2)+Y_2^2=E_4/144$. Finally, note that thanks to the *other* formula $CK_{Y_2}(f;\tau,T)=e^{TY_2(\tau)}CK_D(f;\tau,T)$ it is clear that the f_n are holomorphic and polynomially bounded, hence modular forms. □

5.3.4. The Rankin–Cohen Operators. In the preceding section we have introduced *linear* operators such as $\mathfrak{D}_{Y_2,k}$ and $\mathfrak{D}_{Y,k}$ and their iterates, which we used to construct new modular forms or almost holomorphic modular forms from a given one. In the present section we introduce *bilinear* operators that do the same, starting from two given forms.

Definition 5.3.23. If f and g are C^∞ functions on $\mathfrak{H}$ and k and l are fixed integers, we define the nth *Rankin–Cohen* bracket by

$$[f,g]_n=\sum_{j=0}^{n}(-1)^j\binom{n+k-1}{j}\binom{\ell+n-1}{n-j}D_{\mathcal{T}}^{n-j}(f)D_{\mathcal{T}}^{j}(g)\ .$$

The precise definition of the brackets $[f,g]_n$ was first given by the first author [**Coh75**] but it was also implicit in earlier work by Rankin [**Ran56**], and Zagier [**Zag94**] therefore named them *Rankin–Cohen* brackets.

Theorem 5.3.24. *The Rankin–Cohen brackets have the following properties:*

(a) *For any* $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$ *we have*

$$([f,g]_n)\big|_{k+\ell+2n}\gamma = \left[f\big|_k\gamma, g\big|_\ell\gamma\right]_n \;.$$

(b) *If* f *and* g *are weakly modular on* G *of weights* k *and* ℓ *and multiplier systems* v *and* w, *then* $[f,g]_n$ *is weakly modular on* G *of weight* $k+\ell+2n$ *and multiplier system* vw.

(c) *If* f *and* g *are modular forms, then* $[f,g]_n$ *is a modular form, and in fact a cusp form as soon as* $n \geq 1$.

(d) *We also have* $[g,f]_n = (-1)^n[f,g]_n$, *and in particular* $[f,f]_n = 0$ *if* n *is odd.*

Note that when f and g have rational Fourier coefficients, so does $[f,g]_n$ since the operator $D_\mathcal{T}$ preserves rationality of coefficients. Beware also of the position of j and $n-j$ in the definition of $[f,g]_n$.

Proof. (a) It is immediate to compute that

$$CK_D(f;\tau,T)CK_D(g;\tau,-T) = \sum_{n\geq 0} T^n \sum_{j=0}^{n} (-1)^j c(n,j) D_\mathcal{T}^j(g) D_\mathcal{T}^{n-j}(f) \;,$$

with

$$\begin{aligned} c(n,j) &= \frac{1}{(n-j)!(n+k-j-1)!j!(\ell+j-1)!} \\ &= \frac{1}{(n+k-1)!(\ell+n-1)!}\binom{n+k-1}{j}\binom{\ell+n-1}{n-j} \;, \end{aligned}$$

so by definition of $[f,g]_n$ we have

$$CK_D(f;\tau,T)CK_D(g;\tau,-T) = \sum_{n\geq 0} \frac{1}{(n+k-1)!(\ell+n-1)!}[f,g]_n T^n \;.$$

By Proposition 5.3.18, changing (τ,T) to $((a\tau+b)/(c\tau+d), (ad-bc)T/(c\tau+d)^2)$ introduces the factors $(c\tau+d)^k e^{(T/(2\pi i))c/(c\tau+d)}$ and $(c\tau+d)^\ell e^{-(T/(2\pi i))c/(c\tau+d)}$ which multiply out to $(c\tau+d)^{k+\ell}$ and the result immediately follows, again independently of the coefficient $1/((n+k-1)!(\ell+n-1)!)$.

(b) follows immediately from (a) and the proof is left to the reader (see the proof of Proposition 5.3.3 for instance). □

This theorem was first proved by the first author, although it was also implicit in the previously mentioned earlier work of Rankin. The Rankin–Cohen brackets have many interesting properties, some of which had not been noticed when they were invented. We will see one of the most important in Corollary 9.4.6, related to the Petersson scalar product.

Examples 5.3.25. The first few brackets are explicitly given by $[f,g]_0 = fg$,

$$[f,g]_1 = kfD_{\mathcal{T}}(g) - \ell D_{\mathcal{T}}(f)g, \quad \text{and}$$
$$[f,g]_2 = \left(\frac{k(k+1)}{2}fD_{\mathcal{T}}^2(g) - (k+1)(\ell+1)D_{\mathcal{T}}(f)D_{\mathcal{T}}(g) + \frac{\ell(\ell+1)}{2}D_{\mathcal{T}}^2(f)g\right).$$

Note that we already proved in Proposition 5.3.3 that $[f,g]_1$ is modular if f and g are modular.

Lemma 5.3.26. *Consider the function Y as having weight 2, even though it is only quasi-modular and almost holomorphic. If f is of weight k, we have*

$$[Y,f]_n = \frac{(-1)^n}{n+k}(\mathfrak{D}_Y^{n+1}(f) - D_{\mathcal{T}}^{n+1}(f))\ ,$$

or equivalently

$$[E_2^*,f]_n = [E_2,f]_n + (-1)^n\frac{12}{n+k}(\mathfrak{D}_Y^{n+1}(f) - D_{\mathcal{T}}^{n+1}(f))\ .$$

Proof. By definition

$$[Y,f]_n = \sum_{j=0}^{n}(-1)^j\binom{n+1}{j}\binom{n+k-1}{n-j}D_{\mathcal{T}}^{n-j}(Y)D_{\mathcal{T}}^j(f)\ .$$

Since $D_{\mathcal{T}}(Y) = -Y^2$, it follows that $D_{\mathcal{T}}^m(Y) = (-1)^m m! Y^{m+1}$ and hence

$$\begin{aligned}[Y,f]_n &= (-1)^n\sum_{j=0}^{n}(n-j)!\binom{n+1}{j}\binom{n+k-1}{n-j}Y^{n-j+1}D_{\mathcal{T}}^j(f)\\ &= \frac{(-1)^n}{n+k}\sum_{j=0}^{n}\binom{n+1}{j}\frac{(n+k)!}{(k+j-1)!}Y^{n+1-j}D_{\mathcal{T}}^j(f)\\ &= \frac{(-1)^n}{n+k}(\mathfrak{D}_Y^{n+1}(f) - D_{\mathcal{T}}^{n+1}(f))\end{aligned}$$

by definition of $\mathfrak{D}_Y^n(f)$, proving the first formula, and the second follows from the fact that $E_2^* = E_2 - 3/(\pi y) = E_2 + 12Y$. ∎

We can now modify the construction of the operators $[f,g]_n$ so that they give true modular forms even with E_2, although E_2 is only quasi-modular:

Proposition 5.3.27. *Let f be a modular form of weight k and set*

$$[E_2,f]_n^* = [E_2,f]_n - (-1)^n\frac{12}{n+k}D_\mathcal{T}^{n+1}(f)\ ,$$
$$[f,E_2]_n^* = [f,E_2]_n - \frac{12}{n+k}D_\mathcal{T}^{n+1}(f)\ ,\quad \textit{and}$$
$$[E_2,E_2]_n^{**} = [E_2,E_2]_n - \frac{12}{n+2}(1+(-1)^n)D_\mathcal{T}^{n+1}(E_2)\ .$$

The functions $[E_2,f]_n^*$ and $[f,E_2]_n^*$ are modular forms of weight $k+2+2n$ on Γ, $[E_2,E_2]_n^{**}$ is a modular form of weight $2n+4$, and they are all cusp forms as soon as $n>0$.

Remarks 5.3.28.

(a) The case $n=0$ is exactly given by Proposition 5.3.7 since

$$[f,E_2]_0^* = E_2 f - (12/k)D_\mathcal{T}(f) = -(12/k)(D_\mathcal{T}(f)+kY_2) = -(12/k)\mathfrak{D}_{Y_2,k}(f)\ .$$

(b) By the preceding theorem we have $[f,E_2]_n^* = (-1)^n[E_2,f]_n^*$, and also $[E_2,E_2]_n^{**}=0$ when n is odd.

Proof. We could again reason as above using Proposition 5.3.18 and the transformation property of E_2 from Corollary 5.2.17. However, by Proposition 5.3.15 and Theorem 5.3.24 we know that $\mathfrak{D}_Y^{n+1}(f)$ and $[E_2^*,f]_n$ are almost holomorphic modular forms of weight $k+2+2n$. Thus, by Lemma 5.3.26 it follows that $[E_2,f]_n-(-1)^n 12/(n+k)D_\mathcal{T}^{n+1}(f)$ is also modular of weight $k+2+2n$, and since E_2 is holomorphic and polynomially bounded, it is clear that the above function is in fact a modular form, which is a cusp form for $n>0$. The other formulas are proved similarly. □

Corollary 5.3.29. *We have the identity*

$$2E_2D_\mathcal{T}^2(E_2) - 3(D_\mathcal{T}(E_2))^2 - 2D_\mathcal{T}^3(E_2) = 0\ .$$

Equivalently, the function $y=E_2$ satisfies the differential equation

$$2yy'' - 3y'^2 - \frac{2}{2\pi i}y''' = 0\ .$$

Proof. By the proposition, we have $[E_2,E_2]_2^{**}\in S_8(\Gamma)$, and since there are no cusp forms of weight 8 on Γ, we deduce that $[E_2,E_2]_2^{**}=0$, which is equivalent to the given identity. □

Note that this proves Corollary 5.3.11 in the case of $f=E_2$, with an explicit equation. The above differential equation is called a *Chazy equation.*

5.3.5. An Antiholomorphic Differential Operator. Another operator which we will study and which is seldom seen in textbooks is the *adjoint* of $\mathfrak{D}_{Y,k}$ for the Petersson scalar product. It appears most naturally in the context of Maass operators (see Section 12.2.2) but here we will study it independently.

Definition 5.3.30. Let f be a function defined on $\mathfrak{H}$ and set

$$\mathfrak{D}_{\overline{Y}}(f) := -Y^{-2}D_{\overline{\tau}}(f) = -(4\pi y)^2 D_{\overline{\tau}}(f)\ .$$

Remarks 5.3.31.

(a) Evidently, these operators are only interesting for nonholomorphic functions; otherwise, they vanish identically.

(b) In contrast to the operator $\mathfrak{D}_{Y,k}$ the operator $\mathfrak{D}_{\overline{Y}}$ does not involve the "weight" k.

The following is the analogue of Proposition 5.3.6(b):

Proposition 5.3.32. *If f is a C^1 function on $\mathfrak{H}$ and $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$, then*

$$\mathfrak{D}_{\overline{Y}}(f)\big|_{k-2}\gamma = \mathfrak{D}_{\overline{Y}}(f\big|_k\gamma).$$

In particular, if f is an almost holomorphic modular form of weight k and depth p for some group G, then $\mathfrak{D}_{\overline{Y}}(f)$ is an almost holomorphic modular form of weight $k-2$ and depth $p-1$ for G.

Proof. Differentiating the identity $(f\big|_k\gamma)(\tau) = (ad-bc)^{k/2}(c\tau+d)^{-k}f(\gamma\tau)$ with respect to $\overline{\tau}$ and using $\partial_{\overline{\tau}}((a\overline{\tau}+b)/(c\overline{\tau}+d)) = (ad-bc)/(c\overline{\tau}+d)^2$ gives

$$\partial_{\overline{\tau}}(f\big|_k\gamma)(\tau) = (ad-bc)^{(k+2)/2}(c\tau+d)^{-k}(c\overline{\tau}+d)^{-2}\partial_{\overline{\tau}}(f)(\gamma\tau)\ ,$$

and since $\Im(\tau)^2 = (ad-bc)^{-2}(c\tau+d)^2(c\overline{\tau}+d)^2\Im(\gamma\tau)^2$, we deduce that

$$\begin{aligned}\mathfrak{D}_{\overline{Y}}(f\big|_k\gamma) &= 8\pi i\Im(\tau)^2\partial_{\overline{\tau}}(f\big|_k\gamma)(\tau)\\ &= 8\pi i(ad-bc)^{(k-2)/2}(c\tau+d)^{-(k-2)}\Im(\gamma\tau)^2\partial_{\overline{\tau}}(f)(\gamma\tau)\\ &= (\mathfrak{D}_{\overline{Y}}(f)|_{k-2}\gamma)(\tau)\ ,\end{aligned}$$

as claimed. Since $\partial_{\overline{\tau}}(y^{-p}) = (i/2)py^{-(p+1)}$, it is clear that the operator $\mathfrak{D}_{\overline{Y}} = -(4\pi y)^2 D_{\overline{\tau}}$ decreases the depth by 1 so the last statement follows. □

We will see that (up to a multiplicative constant) the operator $D_{\overline{\tau}}$ is the *adjoint* of the operator $\mathfrak{D}_{Y,k}$, but it is certainly not its *inverse*. In fact:

Proposition 5.3.33. *We have*

$$\begin{aligned}\mathfrak{D}_{\overline{Y}}(\mathfrak{D}_{Y,k}(f)) &= -(Y^{-2}D_\tau D_{\overline{\tau}}(f) + kY^{-1}D_{\overline{\tau}}(f) + kf)\ ,\\ \mathfrak{D}_{Y,k-2}(\mathfrak{D}_{\overline{Y}}(f)) &= -(Y^{-2}D_\tau D_{\overline{\tau}}(f) + kY^{-1}D_{\overline{\tau}}(f))\ ; \quad \textit{hence}\\ [\mathfrak{D}_{\overline{Y}}, \mathfrak{D}_Y](f) &= \mathfrak{D}_{\overline{Y}}(\mathfrak{D}_{Y,k}(f)) - \mathfrak{D}_{Y,k-2}(\mathfrak{D}_{\overline{Y}}(f)) = kf\ .\end{aligned}$$

Proof. This is a simple computation left as an exercise for the reader, using the fact that $D_{\overline{\tau}}(Y)=Y^2$. Note in passing that

$$Y^{-2}D_\tau D_{\overline{\tau}}=-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

is the hyperbolic Laplacian. □

The iterates of the operator $\mathfrak{D}_{\overline{Y}}$ are given as follows:

Proposition 5.3.34.

(a) *We have*

$$\mathfrak{D}_{\overline{Y}}^n(f)=(-1)^n\sum_{j=1}^n\binom{n}{j}\frac{(n-1)!}{(j-1)!}Y^{-n-j}D_{\overline{\tau}}^j(f)\ .$$

(b) *For any* $\gamma\in\mathrm{GL}_2^+(\mathbb{R})$ *we have*

$$\mathfrak{D}_{\overline{Y}}^n(f)\big|_{k-2n}\gamma=\mathfrak{D}_{\overline{Y}}^n(f\big|_k\gamma)\ .$$

Proof. Again left to the reader, who should also compare with Proposition 5.3.15, giving the corresponding properties of $\mathfrak{D}_{Y,k}^n(f)$. □

In terms of generating series this can be rewritten as follows, where for simplicity write Y instead of $Y(\tau)$, $D_{\overline{\tau}}^n(f)$ instead of $(D_{\overline{\tau}}^n(f))(\tau)$, etc.:

Corollary 5.3.35. *Define the two formal power series*

$$CK_{\overline{D}}(f;\tau,T)=\sum_{n\ge1}\frac{D_{\overline{\tau}}^n(f)}{n!(n-1)!}T^n\quad\text{and}$$
$$CK_{\overline{Y}}(f;\tau,T)=\sum_{n\ge1}\frac{\mathfrak{D}_{\overline{Y}}^n(f)Y^{2n}}{n!(n-1)!}T^n\ .$$

(a) *We have*

$$CK_{\overline{Y}}(f;\tau,T)=e^{TY}CK_{\overline{D}}(f;\tau,T)\ .$$

(b) *If* $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{GL}_2^+(\mathbb{R})$, *we write, by abuse of notation,*

$$CK\big|_k\overline{\gamma}(\tau,T)=(ad-bc)^{k/2}(c\tau+d)^{-k}CK\left(\gamma\tau,\frac{ad-bc}{(c\overline{\tau}+d)^2}T\right)\ .$$

Then

$$(CK_{\overline{D}}\big|_k\overline{\gamma})(f;\tau,T)=e^{-(T/(2\pi i))(c/(c\overline{\tau}+d))}CK_{\overline{D}}(f\big|_k\gamma;\tau,T)\quad\text{and}$$
$$(CK_{\overline{Y}}\big|_k\overline{\gamma})(f;\tau,T)=CK_{\overline{Y}}(f\big|_k\gamma;\tau,T)\ .$$

Proof. (a) By Proposition 5.3.34 we have

$$\begin{aligned}CK_{\overline{Y}}(f;\tau,T) &= \sum_{n\ge 1}\sum_{1\le j\le n}(-1)^j\frac{1}{j!(j-1)!(n-j)!}Y^{-n-j}D^j_{\overline{\tau}}(f)Y^{2n}T^n\\ &= \sum_{j\ge 1}(-1)^j\frac{D^j_{\overline{\tau}}(f)}{j!(j-1)!}\sum_{m\ge 0}\frac{1}{m!}Y^mT^{m+j}\\ &= e^{TY}CK_{\overline{D}}(f;\tau,T)\ .\end{aligned}$$

(b) As above for the series CK_Y, the formula for $CK_{\overline{Y}}\big|_k$ is equivalent to the formulas $\mathfrak{D}^n_{\overline{Y}}(f)\big|_{k-2n}\gamma = \mathfrak{D}^n_{Y,k}(f\big|_k\gamma)$, independently of the coefficients, using the fact that

$$Y^2(\gamma\tau) = (c\tau+d)^2(c\overline{\tau}+d)^2(ad-bc)^{-2}Y^2(\tau)\ .$$

Now changing (τ,T) to $(\gamma\tau,(ad-bc)T/(c\overline{\tau}+d)^2)$ changes TY to

$$\begin{aligned}(ad-bc)(c\overline{\tau}+d)^{-2}TY|c\tau+d|^2(ad-bc)^{-1} &= TY\frac{c\tau+d}{c\overline{\tau}+d}\\ &= TY\left(1+(\tau-\overline{\tau})\frac{c}{c\overline{\tau}+d}\right)\\ &= TY+\frac{T}{2\pi i}\frac{c}{c\overline{\tau}+d}\ ;\end{aligned}$$

hence the formula for $CK_{\overline{D}}$ follows from this and that of $CK_{\overline{Y}}$. □

5.3.6. Linear Differential Equations Satisfied by Modular Forms. Corollary 5.3.11 tells us that any modular form on Γ satisfies a third-order nonlinear differential equation. However, such differential equations are normally not very useful, and it is an important fact that modular forms also satisfy *linear* differential equations, when the variable is not taken to be τ itself but some *modular function* $t(\tau)$. The precise statement is as follows:

Proposition 5.3.36. *Let G be a subgroup of Γ of finite index and let t be a modular* function *on G. If f is a holomorphic or meromorphic modular function on G of positive weight k, we write locally $f(\tau) = F(t(\tau))$.*

Then the function F satisfies a linear differential equation of order $k+1$ with algebraic coefficients, and even with polynomial coefficients if $\mathfrak{H}\backslash G$ has genus 0 and the field of modular functions for G is equal to $\mathbb{C}(t)$.

Proof. In [**Zag08**] Zagier gives three different proofs, but we only give the third. We are going to *construct* the desired linear differential equation. Note that since t has weight 0, t' is a meromorphic modular form of weight

2. By the theory of the Rankin–Cohen operators, it is clear that if we set

$$A=\frac{[f,t']_1}{kft'^2}=\frac{kft''-2f't'}{kft'^2}\quad\text{and}$$
$$B=-\frac{[f,f]_2}{k^2(k+1)f^2t'^2}=-\frac{kff''-(k+1)f'^2}{k^2f^2t'^2},$$

then A and B are modular functions of weight 0, so they are algebraically related to t, and even rational functions of t if the field of modular functions for G is $\mathbb{C}(t)$. On the other hand, if we denote by D_t the differential operator defined by $D_t(g)=g'/t'$, one checks that if we set locally $h=f^{1/k}$, we have

$$\begin{aligned}&D_t^2(h)+A\cdot D_t(h)+B\cdot h\\&\quad=\frac{1}{t'}\left(\frac{h'}{t'}\right)'+\frac{kft''-2f't'}{kft'^2}\cdot\frac{h'}{t'}-\frac{kff''-(k+1)f'^2}{k^2f^2t'^2}\cdot h=0.\end{aligned}$$

A slightly less ad hoc method of obtaining this identity is to verify it for $k=1$, which is much easier, and then apply it to the function h, which is locally of weight 1. The final step is then to compute the Rankin–Cohen operators $[h,t']$ and $[h,h]$ in terms of f.

Thus, if we write (locally) $A(\tau)=a(t(\tau))$ and $B(\tau)=b(t(\tau))$ and since the derivative of $F(z)=f(t^{-1}(z))$ is $F'(z)=(1/t'(\tau))f'(\tau)=D_tf(\tau)$, it follows that $F^{1/k}$ satisfies the linear differential equation $LF:=F''(t)+a(t)F'(t)+b(t)F(t)=0$, where $L=d^2/dt^2+a(t)d/dt+b(t)$. Let V denote the vector space of solutions of $LF=0$. The vector space of linear combinations of k-fold products of elements of V is the vector space of solutions of the so-called *symmetric kth power* of the operator L, which in the case of a differential operator of order 2 has order $k+1$, proving the proposition. $\square$

Examples 5.3.37. Let $G=\Gamma$ be the full modular group, and let $t=(aj+b)/(cj+d)$ with $ad-bc\neq 0$ and j the usual modular invariant. The field of modular functions on Γ is equal to $\mathbb{C}(j)$, hence to $\mathbb{C}(t)$. The functions $E_{2k}^{1/(2k)}$ for $k\geq 2$ are modular of weight 1, so by the above proposition they satisfy a linear differential equation of order 2 whose coefficients can of course be explicitly computed. Upon comparing with the Gaussian hypergeometric functions

$${}_2F_1(a,b;c;t)=\sum_{n\geq 0}\frac{(a)_n(b)_n}{(c)_n}\frac{t^n}{n!}$$

and

$${}_3F_2(a,b,c;d,e;t)=\sum_{n\geq 0}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n}\frac{t^n}{n!}\ ,$$

where $(a)_n=a(a+1)\cdots(a+n-1)$, we find for instance that

$$E_4^{1/4}={}_2F_1\left(\frac{1}{12},\frac{5}{12};1;\frac{1728}{j}\right)\ ,\quad E_6^{1/6}={}_2F_1\left(\frac{1}{12},\frac{7}{12};1;\frac{1728}{1728-j}\right)\ .$$

Additionally, we can also use the following well-known identity of Clausen:

$${}_2F_1(a,b;a+b+1/2;t)^2 = {}_3F_2(2a,2b,a+b;2a+2b,a+b+1/2;t),$$

which is easily proved by showing that both sides satisfy the same linear differential equation, or we can use the proposition directly, to find the formula

$$E_4^{1/2} = {}_3F_2\left(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;\frac{1728}{j}\right)\,.$$

5.4. Taylor Coefficients of Modular Forms

This section is taken with little change from the corresponding section in a hopefully forthcoming book by D. Zagier (see also [**Zag08**]).

5.4.1. Introduction and Definitions. As we have seen in the introduction to this book, modular forms arise in many different ways. The most common way to handle these functions, or to get interesting information from them, is to use their *Fourier expansion*, typically $f=\sum_{n\ge0}a(n)q^n$ at infinity, or at other cusps. It is an important fact that we can also obtain interesting information from expansions around arbitrary points $\tau_0\in\mathfrak{H}$. In addition, we will see that such expansions are sometimes *more* useful than Fourier expansions at cusps. Finally, note that in the more general situation of modular forms on *cocompact* groups (i.e, groups such that $G\backslash\mathfrak{H}$ is compact), which we will not study, there are in fact no cusps at all.

If f is a holomorphic function on $\mathfrak{H}$, then it has a Taylor expansion $f(\tau)=\sum_{n\ge0}(f^{(n)}(\tau_0)/n!)(\tau-\tau_0)^n$ around $\tau=\tau_0$, and since the real line is a natural boundary for f (unless f is constant), the radius of convergence of this series will be equal to the distance of τ_0 to the real line, in other words, to $y_0=\Im(\tau_0)$. This is, however, not satisfactory since the power series will not represent f on the whole of $\mathfrak{H}$, in contrast to the Fourier expansion.

The natural thing to do is to use the conformal mapping ϕ sending $\mathfrak{H}$ to the unit disk and τ_0 to the origin, given by

$$\phi(\tau)=\eta=\frac{\tau-\tau_0}{\tau-\overline{\tau_0}}\;,\quad\text{so that}\quad\tau=\phi^{-1}(\eta)=\frac{\tau_0-\overline{\tau_0}\eta}{1-\eta}\;.$$

The Taylor expansion of

$$g(\eta)=f\circ\phi^{-1}(\eta)=f\left(\frac{\tau_0-\overline{\tau_0}\eta}{1-\eta}\right)$$

around $\eta=0$ now converges on the whole open unit disk $|\eta|<1$, so $f(\tau)=g\circ\phi(\tau)$ will be represented on the whole of $\mathfrak{H}$, as desired. Finally, recall that to define the Fourier series expansion of a modular form of weight k at some cusp $s=\gamma(i\infty)$, we did not use the function $f(\gamma\tau)$, but the more natural function $f\big|_k\gamma = (c\tau+d)^{-k}f(\gamma\tau)$. We proceed in the same manner here and use the following definition:

Definition 5.4.1. Let $\tau_0 \in \mathfrak{H}$.

(a) The coefficients $C_n = C_n(f;\tau_0)$ defined by
$$(1-\eta)^{-k}f\left(\frac{\tau_0-\overline{\tau_0}\eta}{1-\eta}\right) = \sum_{n\geq 0}\frac{C_n(f;\tau_0)}{n!}\eta^n\ ,$$
or equivalently by
$$f(\tau) = \left(\frac{\tau_0-\overline{\tau_0}}{\tau-\overline{\tau_0}}\right)^k\sum_{n\geq 0}\frac{C_n(f;\tau_0)}{n!}\left(\frac{\tau-\tau_0}{\tau-\overline{\tau_0}}\right)^n\ ,$$
are called the *canonical Taylor coefficients* of f at τ_0.

(b) Two sequences (C_n) and (C'_n) of complex numbers will be said to be equivalent if there exist nonzero complex numbers α and β such that $C'_n = \alpha^k\beta^nC_n$.

(c) Any sequence (C'_n) equivalent to $(C_n(f;\tau_0))$ will be called *a canonical Taylor sequence* associated to f at τ_0.

5.4.2. Main Properties. The fundamental result concerning these canonical Taylor coefficients is the following:

Proposition 5.4.2. *If $f \in M_k(\Gamma)$ and $\tau_0 \in \mathfrak{H}$, then*
$$C_n(f;\tau_0) = (-4\pi y_0)^n\mathfrak{D}^n_{Y,k}(f)(\tau_0)\ ,$$
where as usual $y_0 = \Im(\tau_0)$ and $\mathfrak{D}^n_{Y,k}(f) = (\mathfrak{D}_{Y,k+2n-2}\circ\cdots\circ\mathfrak{D}_{Y,k+2}\circ\mathfrak{D}_{Y,k})(f)$ as in Definition 5.3.14.

Proof. By Proposition 5.3.15, we have
$$\frac{\mathfrak{D}^n_{Y,k}(f)(\tau_0)}{n!} = \sum_{\substack{j+\ell=n\\ j,\ell\geq 0}}\binom{n+k-1}{\ell}Y(\tau_0)^\ell\frac{D^j_\tau(f)(\tau_0)}{j!}\ .$$
It follows that
$$\begin{aligned}\sum_{n\geq 0}\frac{\mathfrak{D}^n_{Y,k}(f)(\tau_0)(\eta/Y(\tau_0))^n}{n!} &= \sum_{j\geq 0}\frac{D^j_\tau(f)(\tau_0)\eta^j}{j!Y(\tau_0)^j}\sum_{\ell\geq 0}\binom{j+\ell+k-1}{\ell}\eta^\ell\\ &= \sum_{j\geq 0}\frac{f^{(j)}(\tau_0)}{j!}(2i\eta y_0)^j(1-\eta)^{-j-k}\\ &= (1-\eta)^{-k}f\left(\tau_0+\frac{2i\eta y_0}{1-\eta}\right)\\ &= (1-\eta)^{-k}f\left(\frac{\tau-\overline{\tau_0}\eta}{1-\eta}\right)\ ,\end{aligned}$$

and hence

$$C_n(f;\tau_0) = Y(\tau_0)^{-n}\mathfrak{D}^n_{Y,k}(f)(\tau_0) = (-4\pi y_0)^n\mathfrak{D}^n_{Y,k}(f)(\tau_0)\ . \qquad \square$$

Corollary 5.4.3. *Keep the same assumptions as in the proposition.*

(a) *The function* $\Im(\tau)^{-n}C_n(f;\tau)$ *is an almost holomorphic modular form of weight* $k+2n$ *and depth* n *for all* n*. Equivalently,*

$$C_n(f;\gamma\tau_0) = (c\tau_0+d)^k\left(\frac{c\tau_0+d}{c\overline{\tau_0}+d}\right)^n C_n(f;\tau_0) \quad \text{for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma.$$

(b) *The equivalence class of the canonical Taylor sequence of* f *at* τ_0 *does not depend on the class of* τ_0 *modulo* Γ*; in other words, it depends only on the image of* τ_0 *in* $\Gamma\backslash\mathfrak{H}$*.*

(c) *If* τ_0 *is a CM point (see Definition* 5.10.2*), then the equivalence class of the canonical Taylor sequence of* f *at* τ_0 *is* algebraic*; in other words, there exists a representative of the class such that all the elements of the sequence are algebraic.*

Proof. (a) follows immediately from Proposition 5.3.15(b) and the fact that $\Im(\gamma\tau_0) = \Im(\tau_0)/((c\tau_0+d)(c\overline{\tau_0}+d))$, (b) follows from (a) and the definition of equivalence and is the main reason for this definition, and (c) follows immediately from the proposition and Corollary 5.10.7. $\square$

5.4.3. Computing the Canonical Taylor Coefficients. We now explain how to compute the canonical Taylor coefficients in practice. One method is algebraic: it has the advantage of simplicity, but the disadvantage of needing the algebraic structure of the ring of modular forms. The other method uses differential equations satisfied by modular forms. This is not as simple as the first method but it does not have the above disadvantage.

Let us consider the simplest case of the full modular group Γ, for which the structure is easy (see Corollary 5.6.8). In the case $\tau_0 = i$, the result is the following:

Proposition 5.4.4. *Let* (P_n) *be the sequence of polynomials defined recursively from a given polynomial* $P_0(X)$ *by*

$$P_{n+1}(X) = -\frac{k+2n}{12}XP_n(X) + \frac{X^2-1}{2}P_n'(X) - \frac{n(n+k-1)}{144}P_{n-1}(X)$$

for $n \geq 0$ *(with* $P_{-1}(X) = 0$*). Let* $f \in M_k(\Gamma)$ *and let* $P_0(X)$ *be the unique polynomial such that* $f = E_4^{k/4}P_0(E_6/E_4^{3/2})$*. Then the sequence* $(P_n(0))$ *is a canonical Taylor sequence for* f *at* $\tau_0 = i$ *up to equivalence.*

More precisely, we have

$$f(\tau)=\left(\frac{-4C}{(\tau+i)^2}\right)^{k/2}\sum_{n\ge0}\frac{(-4\pi C)^n}{n!}P_n(0)\left(\frac{\tau-i}{\tau+i}\right)^n\ ,$$

with

$$C=E_4(i)^{1/2}=3^{1/2}\frac{\Gamma(1/4)^4}{(2\pi)^3}\ .$$

Proof. Since $Y_2^*(i)=-E_2^*(i)/12=0$, it follows from $CK_{Y_2}(f;\tau,T)=e^{TY_2^*(\tau)}CK_Y(f;\tau,T)$ that we have $\mathfrak{D}^n_{Y,k}(f)(i)=f_n(i)$, where f_n is the canonical sequence attached to f, as in Definition 5.3.21. Thus, by Proposition 5.4.2, up to equivalence of sequences we have $C_n(f;i)\sim f_n(i)$. Now by Proposition 5.3.22 we have the recursion

$$f_{n+1}=D_\tau(f_n)-\frac{k+2n}{12}E_2f_n-\frac{n(n+k-1)}{144}E_4f_{n-1}\ ,$$

and $f_n\in M_{k+2n}(\Gamma)$, and by Corollary 5.6.9 there exists a polynomial P_n such that $f_n=E_4^{(k+2n)/4}P_n(E_6/E_4^{3/2})$. From the action of D_τ on E_2, E_4, and E_6 given by Proposition 5.3.10, a short computation gives the recursion.

The precise formula for $f(\tau)$ also follows immediately from this, apart from the value of $E_4(i)$ which is a result coming from complex multiplication and which we will not prove here, but see the discussion following Corollary 5.10.7. □

When $\tau_0\neq i$, we must modify the above argument slightly. Let Z be a modular function of weight 2 on some finite index subgroup G of Γ such that $Z(\tau_0)\neq0$. Then if we set $Z_2=Y_2+\lambda Z$ and $Z_2^*=Y_2^*+\lambda Z=Z_2-Y$ for $\lambda=-Y_2^*(\tau_0)/Z(\tau_0)$, we have $Z_2^*(\tau_0)=0$; hence $Z_2(\tau_0)=Y(\tau_0)=-1/(4\pi y_0)$. Then Z_2^* will also be modular on G, and

$$CK_{Y_2+\lambda Z}(f;\tau,T)=e^{TZ_2^*(\tau)}CK_Y(f;\tau,T)\ .$$

Thus, if we write

$$CK_{Y_2+\lambda Z}(f;\tau,T)=\sum_{n\ge0}\frac{f_n(\tau)}{n!(n+k-1)!}T^n\ ,$$

then as in the case $\tau_0=i$ we have $C_n(f;\tau_0)\sim f_n(\tau_0)$. On the other hand, by Lemma 5.3.20 we have the recursion

$$f_{n+1}=D_\tau(f_n)+(k+2n)Z_2f_n-n(n+k-1)(Z_2^2+D_\tau(Z_2))f_{n-1}\ .$$

We then proceed as above, using the specific properties of the function Z.

The corresponding result for E_2^* is the following, which is slightly more complicated since we must now work with $\mathbb{C}[E_2^*, E_4, E_6]$ which has transcendence degree 3 instead of 2 for $\mathbb{C}[E_4, E_6]$:

Proposition 5.4.5. *Consider the following double sequence of polynomials defined by the recursion*

$$P_{n+1,j}(X) = \frac{n+j+1}{12}P_{n,j}(X) - \frac{n-j+3}{12}P_{n,j-2}(X)$$
$$- \frac{j-1}{6}XP_{n,j-1}(X) + \frac{X^2-1}{2}P'_{n,j-1}(X) \;,$$

together with the values $P_{0,0}(X) = 1$*,* $P_{0,1}(X) = 0$*, and* $P_{n,j}(X) = 0$ *for* $j < 0$ *or* $j > n+1$*. Then, up to equivalence, the sequence* $(P_{n,n+1}(0))$ *is a canonical Taylor sequence for* E_2^* *at* $\tau_0 = i$*.*

Proof. Since the algebra $\mathbb{C}[E_2^*, E_4, E_6]$ is stable under $\mathfrak{D}_Y$, it is immediate to show that $\mathfrak{D}_{Y,k}^n(E_2^*)$ is of the form

$$\mathfrak{D}_{Y,k}^n(E_2^*) = \sum_{0\le j\le n+1} (E_2^*)^{n+1-j}E_4^{j/2}P_{n,j}(E_6/E_4^{3/2}) \;,$$

and an even more tedious but straightforward computation, left to the reader, shows that the polynomials $P_{n,j}(X)$ satisfy the above recursion and boundary values. The result follows since $E_2^*(i) = E_6(i) = 0$. □

As an application of the above propositions, we give canonical Taylor sequences at $\tau_0 = i$ for E_2^* and E_k for $4 \le k \le 12$ even, as well as for Δ. Note that of course the normalization is not unique, and since $E_k(-1/\tau) = \tau^k E_k(\tau)$, we have $C_n(E_k; i) = 0$ if $n \not\equiv k/2 \pmod 2$, and similarly $C_n(\Delta; i) = 0$ if n is odd.

A different method to compute the canonical Taylor coefficients is to use the differential equations satisfied by modular forms: for instance, we have seen in Corollary 5.3.11 that any $f \in M_k(\Gamma)$ satisfies a nonlinear differential equation of order 3, and in Proposition 5.3.36 that if we change the variable to a suitable modular function, then f satisfies a *linear* differential equation. Both of these results can be used to compute the canonical Taylor coefficients, and we refer to [**Zag16**] for details.

Remark 5.4.6. It should be noted that thanks to the above results and algorithms, it is possible to compute as many Taylor coefficients as one likes using only the recursions and a few coefficients. This is in marked contrast to the *Fourier coefficients* of a modular form: even though a modular form in a given space is determined by a finite and usually small number of coefficients (see Corollary 5.6.14 below), there is in general no direct way to obtain the successive Fourier coefficients from the first few, or in fact from any finite number; see Exercise 5.37 for an application of this.

Table 5.1. Canonical Taylor Sequences at $\tau_0 = i$ for E_k with $k \leq 12$ and Δ.

$f \diagdown n$	0	1	2	3	4	5	6	7	8	9
E_2^*	0	$\frac{1}{12}$	0	$\frac{1}{8}$	0	$\frac{1}{8}$	0	$\frac{13}{16}$	0	$\frac{9}{8}$
E_4	$\frac{1}{5}$	0	$\frac{1}{3}$	0	$\frac{1}{2}$	0	3	0	$\frac{19}{2}$	0
E_6	0	$\frac{1}{14}$	0	$\frac{1}{6}$	0	1	0	$\frac{21}{4}$	0	$\frac{261}{4}$
E_8	$\frac{1}{5}$	0	$\frac{2}{3}$	0	$\frac{13}{3}$	0	31	0	$\frac{773}{2}$	0
E_{10}	0	$\frac{1}{22}$	0	$\frac{1}{3}$	0	$\frac{107}{36}$	0	$\frac{119}{3}$	0	$\frac{1191}{2}$
E_{12}	$\frac{441}{65}$	0	57	0	$\frac{1211}{2}$	0	$\frac{26614}{3}$	0	153517	0
Δ	1	0	-1	0	$\frac{3}{2}$	0	6	0	-99	0

5.5. Modular Forms on the Modular Group and Its Subgroups

Although in the preceding sections we have assumed several times that we work with modular forms for subgroups of the modular group Γ, in view of their importance we give again the basic definitions, which are immediately seen to be equivalent to the definitions given in the more general setting of cofinite Fuchsian groups of the first kind.

5.5.1. Specific Definitions. Let f be a meromorphic function defined on the upper half-plane $\mathfrak{H}$, periodic of period w for some integer w. Hence

$$f(\tau) = g\left(e^{2\pi i\tau/w}\right) ,$$

where $g(q)$ is a meromorphic function in the punctured unit disk $0 < |q| < 1$.

Definition 5.5.1. We will say that f is meromorphic or holomorphic at infinity if g is meromorphic or holomorphic at 0, respectively. If f is holomorphic at infinity, the value $g(0)$ is denoted $f(i\infty)$ and is called the value of f at infinity, and we say that f vanishes at infinity if $g(0) = 0$.

Thus, if f is meromorphic at infinity, there exists some integer n_0 such that we can write $f(\tau) = \sum_{n \geq n_0} a(n)e^{2\pi in\tau/w}$ for some coefficients $a(n)$, the Fourier coefficients of f. The function f is *holomorphic* at infinity if and only if we can take $n_0 \geq 0$, and in that case $f(i\infty) = a(0)$. Thus, f vanishes at infinity if and only if we can write $f(\tau) = \sum_{n \geq 1} a(n) e^{2\pi i n\tau/w}$.

Definition 5.5.2. Let f be a weakly modular function of weight k and multiplier system v on some subgroup G of Γ. We will say that f is meromorphic or holomorphic on $\overline{\mathfrak{H}} = \mathfrak{H} \cup \mathbb{P}_1(\mathbb{Q})$ if it is meromorphic or holomorphic, respectively, on $\mathfrak{H}$ and if for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ the function

$$f_\gamma(\tau) = (f\big|_k\gamma)(\tau) = (c\tau + d)^{-k} f(\gamma\tau)$$

is meromorphic or holomorphic at infinity. If it is holomorphic, the value $f_\gamma(i\infty)$ is called the value of f at the *cusp* $\gamma(i\infty) = a/c$ (although this value may depend on γ; see remarks below).

Remarks 5.5.3.

(a) Since we will always assume that G has finite index in Γ, it is important to note that there are only a *finite* number of functions f_γ (up to multiplication by a root of unity if there is a nontrivial multiplier system) since clearly

$$f_{g\gamma} = f\big|_k g\gamma = (f\big|_k g)\big|_k\gamma = v(g) f\big|_k\gamma = v(g) f_\gamma\ ,$$

and hence we can choose for γ a system of representatives of the right cosets of $G\backslash\Gamma$, or even of $\overline{G}\backslash\overline{\Gamma}$ since $f_{-\gamma} = (-1)^k f_\gamma$.

(b) Calling $f_\gamma(i\infty)$ the value of f at the cusp a/c is a slight abuse of language for two reasons. First, if v is a nontrivial multiplier system, and/or if k is odd, this value is only defined up to multiplication by a value of v, and/or up to sign (but of course this does not change the meromorphy, holomorphy, or vanishing). Second, although we will sometimes write it as $f(a/c)$, it is certainly *not* equal to the limit as $\tau \to 0$, $\tau \in \mathfrak{H}$, of $f(a/c + \tau)$: indeed, it is immediate to see that the identity $f(\gamma\tau) = (c\tau + d)^k f_\gamma(\tau)$ implies that as $\tau \to 0$, $\tau \in \mathfrak{H}$, we have

$$f(a/c + \tau) \sim (-1/(c\tau))^k f_\gamma(i\infty)\ .$$

Definition 5.5.4. Let G be a subgroup of Γ, let k be an integer, and let f be a function from $\mathfrak{H}$ to $\mathbb{C}$ which is weakly modular of weight k and multiplier system v for G. We say that f is a

(a) *modular function* if, in addition, f is meromorphic on $\overline{\mathfrak{H}}$, and

(b) a *modular form* if, in addition, f is holomorphic on $\overline{\mathfrak{H}}$, and

(c) a *cusp form* if, in addition, f vanishes at all the cusps, in other words, if $(f\big|_k\gamma)(i\infty) = 0$ for all $\gamma \in \Gamma$.

We denote by $M_k(G,v)$ (or $M_k(G)$ if $v=1$) the vector space of modular forms of weight k and multiplier system v, and by $S_k(G,v)$ (or $S_k(G)$) the subspace of cusp forms.

A modular function of weight 0 will simply be called a modular function. It is thus a function which is meromorphic on $\mathfrak{H}$ and at infinity and invariant under G; hence it is a meromorphic function on the *Riemann surface* $G\backslash\overline{\mathfrak{H}}$. More generally, since clearly $\frac{d}{d\tau}(\gamma\tau)=(c\tau+d)^{-2}$, it follows that a function which is meromorphic on $\overline{\mathfrak{H}}$ is modular of weight k if and only if the "differential of weight k"

$$f(\tau)\,(d\tau)^{k/2}$$

is G-invariant. In the case of the full modular group we have the following:

Proposition 5.5.5. *A function f which is meromorphic on $\overline{\mathfrak{H}}$ is a modular function of weight k on the full modular group Γ with trivial multiplier system if and only if the following two functional equations hold for all $\tau\in\mathfrak{H}$:*

$$f(\tau+1)=f(\tau)\quad\textit{and}\quad f(-1/\tau)=\tau^k f(\tau)\;.$$

Proof. Clear since Γ is generated by S and T. □

5.5.2. Homogeneous Modular Forms, Lattice Functions. Recall that in Chapter 1 we considered the Eisenstein series G_k both as a function on $\mathfrak{H}$ and as a homogeneous function of (ω_1,ω_2) depending only on the lattice $L=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ that they generate. This is the case more generally for any modular function f of weight k: if for $\omega_1/\omega_2\in\mathfrak{H}$ we set

$$F(\omega_1,\omega_2)=\omega_2^{-k}f(\omega_1/\omega_2)\;,$$

then the functional equations of f are equivalent to the following:

(a) F depends only on the lattice $\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$, or, in other words,

$$F(a\omega_1+b\omega_2,c\omega_1+d\omega_2)=F(\omega_1,\omega_2)\quad\text{for all}\ \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma.$$

(b) F is homogeneous of degree k; that is, for all $\lambda\in\mathbb{C}^\times$ we have

$$F(\lambda\omega_1,\lambda\omega_2)=\lambda^{-k}F(\omega_1,\omega_2)\;.$$

This second property explains why in older literature, modular forms of weight k were sometimes called "of degree $-k$" or something similar.

If we let the set of lattices in $\mathbb{C}$ be denoted by $\mathcal{R}$, then we see immediately that the map $(\omega_1,\omega_2)\mapsto\omega_1/\omega_2$ induces a bijection from $\mathcal{R}/\mathbb{C}^\times$ to $\Gamma\backslash\mathfrak{H}$. In addition, if we associate the elliptic curve $E_L=\mathbb{C}/L$ to a lattice L in $\mathbb{C}$, it is not difficult to check that the two elliptic curves E_L and $E_{L'}$ are isomorphic if and only if L and L' are proportional. Thus, $\Gamma\backslash\mathfrak{H}$ can also be considered as the set of isomorphism classes of elliptic curves defined over $\mathbb{C}$.

5.6. Zeros, Poles, and Dimension Formulas

5.6.1. Modular Functions on the Full Modular Group. Using the ideas that we already used in Chapter 1, we are going to integrate f'/f on the boundary of a fundamental domain $\mathfrak{F}$, more exactly in a finite portion of this domain. If $\tau_0 \in H$, we denote by $v_{\tau_0}(f)$ the order of the zero or pole of f at τ_0, poles being considered as zeros of negative order; in other words, it is the unique integer v such that $f(\tau)/(\tau-\tau_0)^v$ is holomorphic and nonzero at τ_0. If $f(\tau) = g(e^{2\pi i\tau})$, we will set $v_{i\infty}(f) = v_0(g)$.

It is immediate to check that if f is modular and $\gamma \in \Gamma$, then $v_{\gamma\tau}(f) = v_\tau(f)$, so that $v_\tau(f)$ depends only on the class of τ in $\Gamma\backslash\overline{\mathfrak{H}}$.

Finally, we denote by e_τ the order of the isotropy group of $\tau \in \mathfrak{H}$: recall from Theorem 4.3.2 that $e_\tau = 1$ except if τ is Γ-equivalent to i, in which case $e_\tau = 2$, or to ρ, in which case $e_\tau = 3$.

Theorem 5.6.1. *Let f be a modular function of weight k for Γ which is not identically zero. We have*

$$v_{i\infty}(f) + \sum_{\tau\in\Gamma\backslash\mathfrak{H}} \frac{v_\tau(f)}{e_\tau} = \frac{k}{12}\ .$$

Proof. First note that the sum on the left-hand side makes sense, i.e., is finite. Indeed, if $f(\tau) = g(e^{2\pi i\tau})$, by assumption g is meromorphic so there exists $r > 0$ such that $g(q)$ has no zero or pole for $0 < q < r$, or equivalently, $f(\tau)$ has no zero or pole for $\Im(\tau) > e^{2\pi r}$, $\tau \neq i\infty$. Thus, all the zeros and poles of f modulo the action of Γ are either at $i\infty$ or in the compact set

$$\mathfrak{F}_{e^{2\pi r}} = \{\tau \in \mathfrak{F} \mid \Im(\tau) \le e^{2\pi r}\}$$

and hence are finite in number, proving the claim.

Set $T = e^{2\pi r}$ and consider the contour, $\mathcal{C}$, indicated in Figure 5.1:

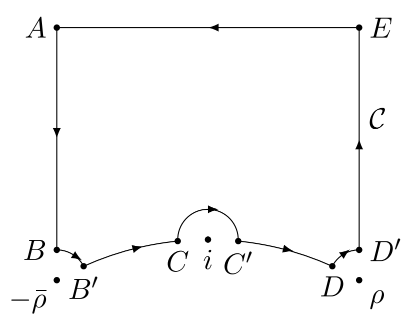

Figure 5.1. Contour of integration, $\mathcal{C}$.

The circular arcs BB', CC', DD' have been chosen so that f does not have zeros outside $\mathcal{C}$ apart from ρ, i, and $-\overline{\rho}$.

Assume first that f has no zero or pole *on* this contour. Then

$$\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{f'(\tau)}{f(\tau)}\,d\tau = \sum_{\substack{\tau\in\mathfrak{H}/\Gamma\\ \tau\not\equiv i,\rho\ (\mathrm{mod}\ \Gamma)}} v_\tau(f)$$

by the residue theorem. However, we can also evaluate the integral directly:

(a) With $q = e^{2\pi i\tau}$ and $f(\tau) = g(q)$ we have

$$\frac{1}{2\pi i}\int_E^A\frac{f'(\tau)}{f(\tau)}\,d\tau = -\frac{1}{2\pi i}\int_{|q|=1/T}\frac{g'(q)}{g(q)}\,dq = -v_0(g) = -v_{i\infty}(f)\ ,$$

the minus sign coming from the fact that the circle $|q| = 1/T$ is traversed in the negative direction.

(b) If C_ρ denotes the complete circle containing BB', we have

$$\frac{1}{2\pi i}\int_{C_\rho}\frac{f'(\tau)}{f(\tau)}\,d\tau = v_\rho(f)\ ,$$

and since the angle $\widehat{B\rho B'}$ tends to $2\pi/6$ when the radius tends to 0, we deduce that

$$\frac{1}{2\pi i}\int_B^{B'}\frac{f'(\tau)}{f(\tau)}\,d\tau \longrightarrow -\frac{1}{6}v_\rho(f)\ .$$

Similarly,

$$\frac{1}{2\pi i}\int_C^{C'}\frac{f'(\tau)}{f(\tau)}\,d\tau \longrightarrow -\frac{1}{2}v_i(f)$$

and

$$\frac{1}{2\pi i}\int_D^{D'}\frac{f'(\tau)}{f(\tau)}\,d\tau \longrightarrow -\frac{1}{6}v_{-\overline{\rho}}(f) = -\frac{1}{6}v_\rho(f)\ .$$

(c) Since $f(\tau+1) = f(\tau)$, the values of f'/f on AB and ED' are the same and since they are traversed in opposite directions, we see that

$$\frac{1}{2\pi i}\int_A^B\frac{f'(\tau)}{f(\tau)}\,d\tau + \frac{1}{2\pi i}\int_{D'}^E\frac{f'(\tau)}{f(\tau)}\,d\tau = 0\ .$$

(d) Finally, $S = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ transforms the arc $B'C$ onto the arc DC'. In addition, since $f(S(\tau)) = \tau^k f(\tau)$, we have

$$\frac{1}{\tau^2}\frac{f'(S(\tau))}{f(S(\tau))} = \frac{k}{\tau} + \frac{f'(\tau)}{f(\tau)}\ ,$$

and therefore

$$\frac{1}{2\pi i}\left(\int_{B'}^{C}\frac{f'(\tau)}{f(\tau)}\,d\tau+\int_{C'}^{D}\frac{f'(\tau)}{f(\tau)}\,d\tau\right)$$
$$=\frac{1}{2\pi i}\int_{B'}^{C}\left(\frac{f'(\tau)}{f(\tau)}-\frac{1}{\tau^2}\frac{f'(S(\tau))}{f(S(\tau))}\right)d\tau$$
$$=-\frac{k}{2\pi i}\int_{B'}^{C}\frac{d\tau}{\tau}\longrightarrow -k\left(-\frac{1}{12}\right)=\frac{k}{12}$$

when the radii of the three small circles tend to 0.

Regrouping the terms above, we deduce the theorem when f has no zero or pole on the contour $\mathcal{C}$.

If f does have a zero or pole on the contour, we simply deform it symmetrically to avoid them. For instance, if f has a zero or pole λ on the segment AB, we deform the upper part of the contour so as to obtain the contour $\mathcal{C}'$ in Figure 5.2 and the proof works as before. □

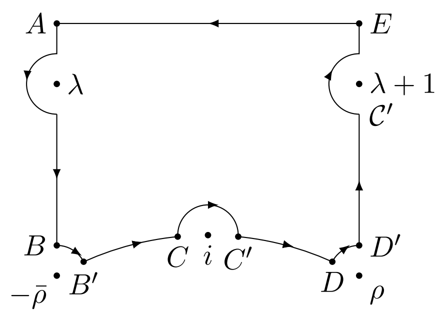

Figure 5.2. Modified contour of integration, $\mathcal{C}'$.

Remarks 5.6.2. (a) An equivalent but more conceptual proof of the above theorem uses the basic fact that a meromorphic function on a Riemann surface has as many zeros as poles counted with multiplicity; in our case the Riemann surface is $\Gamma\backslash\overline{\mathfrak{H}}$, and the points i and ρ must be counted with respective multiplicity 1/2 and 1/3. We then apply this fact to the function f^{12}/Δ^k and deduce the theorem since Δ does not vanish on the Riemann surface except at $i\infty$ with a zero of multiplicity 1. This same proof can of course also be applied to any subgroup of finite index of Γ.

(b) We will see in Exercise 5.1 that a multiplier system on Γ is necessarily of order dividing 12. Hence we can apply the theorem to f^{12} instead of f to show that it is still valid in the case of a nontrivial multiplier system.

Corollary 5.6.3. *Let f be a modular* form *of weight k, which is not identically zero. If $\tau_0 \in \overline{\mathfrak{H}}$ is not Γ-equivalent to i, ρ, or $i\infty$, then*

$$v_{\tau_0}(f) \le \begin{cases} \lfloor k/12 \rfloor & \text{if } k \not\equiv 2 \pmod{12}\ , \\ \lfloor k/12 \rfloor - 1 & \text{if } k \equiv 2 \pmod{12}\ . \end{cases}$$

Proof. Since f is a modular form, we have $v_\tau(f) \ge 0$ for all τ and hence

$$v_{\tau_0}(f) \le \frac{k}{12} - \frac{v_i(f)}{2} - \frac{v_\rho(f)}{3}$$

by the theorem. However, it also follows from the theorem that $3v_i(f) + 2v_\rho(f) \equiv k/2 \pmod 6$. Thus, if $k \equiv 2 \pmod 4$, then $v_i(f) \equiv 1 \pmod 2$ so that $v_i(f) \ge 1$; if $k \equiv 2 \pmod 6$, then $v_\rho(f) \equiv 2 \pmod 3$ so that $v_\rho(f) \ge 2$; and if $k \equiv 4 \pmod 6$, then $v_\rho(f) \equiv 1 \pmod 3$ so that $v_\rho(f) \ge 1$. The corollary now follows after studying each case separately. □

We can now prove the main theorem, which asserts that spaces of modular *forms* (hence holomorphic) are finite-dimensional. Recall that $M_k(\Gamma)$ is the space of modular forms of weight k for Γ and $S_k(\Gamma)$ is the subspace of cusp forms. Since $E_k(i\infty) = 1$, it is clear that any $f \in M_k(\Gamma)$ can be written uniquely as $f = a(0)E_k + f_1$ with $f_1 \in S_k(\Gamma)$, so that for $k \ge 4$ we have

$$M_k(\Gamma) = S_k(\Gamma) \oplus \mathbb{C}E_k\ . \tag{5.1}$$

In addition we define $\Delta \in S_{12}(\Gamma)$ by

$$\Delta(\tau) = \frac{1}{1728}(E_4^3(\tau) - E_6^2(\tau))\ .$$

Theorem 5.6.4. *The spaces $M_k(\Gamma)$ have the following properties:*

(a) *$M_k(\Gamma) = \{0\}$ if $k < 0$ or if k is odd,*
(b) *$M_0(\Gamma) = \mathbb{C}$, $M_2(\Gamma) = \{0\}$, and*
(c) *$M_k(\Gamma) = \mathbb{C}E_k$ for $k = 4, 6, 8$, and 10.*
(d) *Multiplication by Δ gives an isomorphism from $M_{k-12}(\Gamma)$ to $S_k(\Gamma)$.*

It is convenient to prove the following proposition at the same time:

Proposition 5.6.5. *We have the following vanishing and nonvanishing properties of the Eisenstein series and the Δ-function:*

(a) *$E_4(\rho) = 0$ (a simple zero) and $E_4(\tau) \ne 0$ for all $\tau \not\equiv \rho \pmod \Gamma$;*
(b) *$E_6(i) = 0$ (a simple zero) and $E_6(\tau) \ne 0$ for all $\tau \not\equiv i \pmod \Gamma$;*
(c) *$\Delta(i\infty) = 0$ (a simple zero) and $\Delta(\tau) \ne 0$ for all $\tau \in \mathfrak{H}$.*
(d) *Furthermore, we have the identities $E_8 = E_4^2$ and $E_{10} = E_4E_6$.*

Proof. If $f \in M_k(\Gamma)$, then $v_\tau(f) \geq 0$ for all $\tau \in \Gamma\backslash\overline{\mathfrak{H}}$, and by Theorem 5.6.1,

$$v_{i\infty}(f) + \frac{v_i(f)}{2} + \frac{v_\rho(f)}{3} + \sum_{\tau \not\equiv i,\rho \pmod{\Gamma}} v_\tau(f) = \frac{k}{12} ,$$

where the sum is over $\tau \in \Gamma\backslash\mathfrak{H}$ not in the Γ-orbits of i or ρ. Set

$$n_1(f) = v_{i\infty}(f) + \sum_{\tau \not\equiv i,\rho \pmod{\Gamma}} v_\tau(f) , \quad n_2(f) = v_i(f) , \text{ and } n_3(f) = v_\rho(f) ,$$

so that $n_1(f) + n_2(f)/2 + n_3(f)/3 = k/12$ and the $n_i(f)$ are nonnegative integers. This clearly implies that $k \geq 0$ and that k is even (if k was odd, then f would vanish identically), proving (a) of the theorem.

For $k = 0$, 2, 4, 6, 8, and 10 the equation $n_1 + n_2/2 + n_3/3 = k/12$ has only the solutions $(n_1, n_2, n_3) = (0,0,0)$, $\emptyset$, $(0,0,1)$, $(0,1,0)$, $(0,0,2)$, and $(0,1,1)$, respectively, proving (a) and (b) of the proposition.

By (a) and (b) of the proposition we thus have $\Delta(i) \neq 0$ and $\Delta(\rho) \neq 0$, and since $n_1(\Delta)+n_2(\Delta)/2+n_3(\Delta)/3 = 12/12 = 1$, it follows that $n_1(\Delta) = 1$, and since $v_{i\infty}(\Delta) \geq 1$, it follows from the definition of n_1 that $v_{i\infty}(\Delta) = 1$ and that $v_\tau(\Delta) = 0$ for all $\tau \not\equiv i\infty \pmod{\Gamma}$, proving (c) of the proposition.

We can now prove (c) of the theorem: it is clear that if $f \in M_{k-12}(\Gamma)$, then $\Delta f \in S_k(\Gamma)$. Conversely, if $f \in S_k(\Gamma)$ and if we set $f_1 = f/\Delta$, then f_1 will be of weight $k - 12$, holomorphic on $\mathfrak{H}$ since $\Delta \neq 0$ on $\mathfrak{H}$, and also at infinity since $v_{i\infty}(f) \geq 1 = v_{i\infty}(\Delta)$, proving (c). It follows that $S_k(\Gamma) = \{0\}$ for $k \leq 10$, so that $M_k(\Gamma) = S_k(\Gamma) \oplus \mathbb{C}E_k = \mathbb{C}E_k$ for $k = 4$, 6, 8, and 10, and $M_0(\Gamma) = S_0(\Gamma) \oplus \mathbb{C} = \mathbb{C}$; since we have seen above that $M_2(\Gamma) = \{0\}$, (b) of the theorem follows, and (d) of the proposition follows from this. $\square$

Remarks 5.6.6.

(a) The identities (d) of the proposition were proved in an equivalent but completely different way in Proposition 2.1.8.

(b) The proposition implies immediately the additional identities $E_{14} = E_6E_8 = E_6E_4^2$.

(c) To prove these identities directly, without appealing to elliptic functions or modular forms, is more difficult. For instance, the identity $E_4^2 = E_8$ is equivalent to the identity

$$\sigma_3(n) + 120 \sum_{1 \leq k \leq n-1} \sigma_3(k)\sigma_3(n-k) = \sigma_7(n) .$$

Proving this directly using the definition of σ_i as an arithmetic function can be done, but with some difficulty; see [**Sko93**].

Corollary 5.6.7. *For $k \in \mathbb{Z}_{\geq 0}$ we have*

$$\dim(M_k(\Gamma)) = \begin{cases} \lfloor k/12 \rfloor & \text{if } k \equiv 2 \pmod{12}\ , \\ \lfloor k/12 \rfloor + 1 & \text{if } k \not\equiv 2 \pmod{12}\ , \end{cases}$$

$$\dim(S_k(\Gamma)) = \begin{cases} \lfloor k/12 \rfloor - 1 + \delta_{k,2} & \text{if } k \equiv 2 \pmod{12}\ , \\ \lfloor k/12 \rfloor & \text{if } k \not\equiv 2 \pmod{12}\ , \end{cases}$$

where here and elsewhere $\delta_{i,j}$ is the Kronecker symbol.

Proof. By the theorem these formulas are true for $k = 0$, 2, 4, 6, 8, and 10. By (5.1) and (d) of the theorem it is clear that both sides increase by 1 when k is replaced by $k+12$ and hence formulas are true in general. □

Corollary 5.6.8. *The modular forms $E_4^a E_6^b$ with a and b nonnegative integers such that $4a + 6b = k$ form a* basis *of the space $M_k(\Gamma)$. In other words, the map $(X, Y) \mapsto (E_4, E_6)$ gives an isomorphism between $\mathbb{C}[X, Y]$ and $\bigoplus_{k \in \mathbb{Z}} M_k(\Gamma)$.*

Proof. We first show by induction on k that these forms generate $M_k(\Gamma)$ as a $\mathbb{C}$-vector space. By what we have seen above this is clearly true for $k \leq 6$. If $k \geq 8$, it is easy to check that we can find two nonnegative integers c and d such that $4c + 6d = k$. If $f \in M_k(\Gamma)$ is given and $a(0) = f(i\infty)$ is its 0th Fourier coefficient, we have $f - a(0)E_4^c E_6^d \in S_k(\Gamma)$ and by the theorem there exists $h \in M_{k-12}(\Gamma)$ such that

$$f - a(0)E_4^c E_6^d = \Delta h = \frac{1}{1728}(E_4^3 - E_6^2)h\ ,$$

which proves the claim by using induction on k.

We must now show that these forms are $\mathbb{C}$-linearly independent. For this, it is sufficient to show that the number of solutions in nonnegative integers of $4a + 6b = k$ is equal to $\dim(M_k(\Gamma))$ given in the above corollary. This is an easy exercise left to the reader. A more elegant proof using the results of the next section is the following: if these forms are linearly dependent, then E_6^2/E_4^3 is the solution of an algebraic equation with constant coefficients, or equivalently $j = 1728E_4^3/(E_4^3 - E_6^2)$ satisfies $P(j) = 0$ for some $P \in \mathbb{C}[X]$. But since the Fourier expansion of j begins with $j(\tau) = 1/q + O(1)$, we see that this is clearly impossible by looking at the highest degree term of P. □

Corollary 5.6.9. *If $f \in M_k(\Gamma)$, there exists a polynomial P such that $f = E_4^{k/4} P(E_6/E_4^{3/2})$ and $P(-X) = (-1)^{k/2} P(X)$.*

Proof. Since the algebra of modular forms for Γ is $\mathbb{C}[E_4, E_6]$, it follows that if $f \in M_k(\Gamma)$, there exist coefficients c_j such that

$$\begin{aligned} f &= \sum_{j \equiv k/2 \pmod 2,\ j \geq 0} c_j E_6^j E_4^{(k/2-3j)/2} \\ &= E_4^{k/4} \sum_{j \equiv k/2 \pmod 2,\ j \geq 0} c_j (E_6/E_4^{3/2})^j = E_4^{k/4} P(E_6/E_4^{3/2})\ , \end{aligned}$$

showing both the existence of P and that $P(-X) = (-1)^{k/2} P(X)$. □

5.6.2. Modular Functions on Subgroups. It is often possible to use results for Γ to obtain corresponding results for subgroups $G \subseteq \Gamma$. We must, however, be careful, in particular because $-I$ may or may not be an element of G. Recall that we consider $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and not the more natural $\overline{\Gamma} = \mathrm{PSL}_2(\mathbb{Z})$ only because we also want to work with modular forms of odd weight. However, in all other circumstances, we must work with the group $\overline{\Gamma}$ and the corresponding subgroup $\overline{G} = G/(\{\pm I\} \cap G)$. If $-I \in G$ (for instance, if $G = \Gamma_0(N)$), this makes little difference, but if $-I \notin G$ (for instance, if $G = \Gamma_1(N)$ or $\Gamma(N)$ for $N \geq 3$), the difference is crucial.

Let $m = [\overline{\Gamma} : \overline{G}]$, and let $(\gamma_i)_{1 \leq i \leq m}$ be a system of representatives of the right cosets of $\overline{G}\backslash\overline{\Gamma}$, so that $\overline{\Gamma} = \bigsqcup_{1 \leq i \leq m} \overline{G}\gamma_i$. It is clear that a fundamental domain for G (see Examples 4.6.3) can be taken to be $\bigcup_{1 \leq i \leq m} \gamma_i(\mathfrak{F})$, where the union is disjoint outside of a set of measure 0, and thus the covolume of G is $m(\pi/3)$. This would in general be false if we chose representatives of $G\backslash\Gamma$ when $-I \notin G$: for instance, we will see that $[\Gamma : \Gamma_1(3)] = 8$, while $[\overline{\Gamma} : \overline{\Gamma}_1(3)] = 4$, but the fundamental domain has covolume $4\pi/3$, and dimension formulas, zeros, etc., must be computed for $\overline{\Gamma}_1(3)$.

This being said, it is easy to generalize Theorem 5.6.1 to the case where G is a subgroup of finite index of Γ. As above, let $(\gamma_i)_{1 \leq i \leq m}$ be a system of representatives of the right cosets of $\overline{G}\backslash\overline{\Gamma}$ (and *not* of $G\backslash\Gamma$), thus defined up to sign, so that $\overline{\Gamma} = \bigsqcup_{1 \leq i \leq m} \overline{G}\gamma_i$. If f is a modular function of weight k for G with multiplier system v, set $f_i = f\big|_k \gamma_i$. It is clear that f_i depends on the choice of γ_i only up to multiplication by a nonzero constant since

$$f\big|_k g(\pm\gamma_i) = (\pm 1)^k (f\big|_k g)\big|_k \gamma_i = (\pm 1)^k v(g) f\big|_k \gamma_i = (\pm 1)^k v(g) f_i\ .$$

Furthermore, if $\gamma \in \Gamma$, then $\gamma_i\gamma = \pm g_i\gamma_j$ for some $g_i \in G$ and some j, so $f_i\big|_k \gamma = f\big|_k \gamma_i\gamma = (\pm 1)^k v(g_i) f_j$, and it is clear that the map $f_i \mapsto f_j$ is a permutation of the f_i's and we obtain the following (see also Lemma 6.3.1):

Lemma 5.6.10. *The function $h = \prod_{1 \leq i \leq m} f_i$ is modular of weight mk for the full modular group, possibly with a nontrivial multiplier system.*

Before stating the theorem, we must now explain the meaning of the *order* $v_\tau(f)$ of a nonzero modular function f at $\tau \in G\backslash\overline{\mathfrak{H}}$. If $\tau \in \mathfrak{H}$, this is the usual notion as defined previously. If $\tau = i\infty$, then there exists a smallest integer $w > 0$ such that $\pm T^w = \left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right) \in G$ for a suitable sign $\pm$, and the local parameter at $i\infty$ is then $e^{2\pi i\tau/w}$, so that if we write $f(\tau) = g(e^{2\pi i\tau/w})$, then we define $v_{i\infty}(f) = v_0(g)$. We will see that the stabilizer of $i\infty$ in G is either $\pm\langle T^w\rangle$, $\langle T^w\rangle$, or $\langle -T^w\rangle$ (see Lemma 6.3.9). In the last case, which can only occur when $-I \notin G$, we say that the cusp $i\infty$ is irregular (see Definition 6.3.10) and if f has odd weight, then $f(\tau + w) = -f(\tau)$, so that $v_0(g)$ is a *half-integer*. To treat all cases equally we can assume that f has period $2w$, write $f(\tau) = g_1(e^{2\pi i\tau/(2w)})$, and then define $v_{i\infty}(f) = v_0(g_1)/2$. Finally, for $\tau \in \mathbb{P}^1(\mathbb{Q})$ there exists $\gamma \in \Gamma$ such that $\tau = \gamma(i\infty)$, and we set $v_\tau(f) = v_{i\infty}(f\big|_k\gamma)$, and it is easy to check that this does not depend on the choice of γ.

Theorem 5.6.11. *Let G be a subgroup of Γ of finite index, set $m = [\overline{\Gamma} : \overline{G}]$, and let f be a modular function of weight k for G, which is not identically zero. We then have*

$$\sum_{\tau \in G\backslash\overline{\mathfrak{H}}} \frac{v_\tau(f)}{e_\tau} = \frac{mk}{12}\ ,$$

where e_τ is the order of the stabilizer of τ in $\overline{\Gamma}$ if $\tau \in \mathfrak{H}$ and $e_\tau = 1$ for $\tau \in \mathbb{P}^1(\mathbb{Q})$. In other words, $e_\tau = 2$ or 3 if τ is Γ-equivalent to i or ρ, respectively, and $e_\tau = 1$ otherwise.

Proof. As mentioned above, we apply Theorem 5.6.1 and Remarks 5.6.2(b) to the function $h = \prod_{1\le i\le m} f_i$, of weight mk. For any $\tau \in \overline{\mathfrak{H}}$ we then have

$$v_\tau(h) = \sum_{1\le i\le m} v_\tau(f_i) = \sum_{1\le i\le m} v_\tau(f\big|_k\gamma_i) = \sum_{1\le i\le m} v_{\gamma_i(\tau)}(f)\ ,$$

and it is clear that the $(\gamma_i(\tau))_{1\le i\le m}$ form a system of representatives modulo G of the elements of $\overline{\mathfrak{H}}$ which are Γ-equivalent to τ, proving the theorem. Equivalently, as already mentioned, a fundamental domain for $G\backslash\overline{\mathfrak{H}}$ is given by $\bigsqcup_{1\le i\le m} \gamma_i(\mathfrak{F})$. □

Corollary 5.6.12. *Let G be a subgroup of Γ of finite index and let $m = [\overline{\Gamma} : \overline{G}]$.*

(a) *The space $M_k(G)$ is finite-dimensional, and more precisely*

$$\dim(M_k(G)) \le 1 + \left\lfloor \frac{mk}{12} \right\rfloor\ .$$

(b) *In particular, if $k < 0$, there are no nonzero modular forms of weight k for G, and if $k = 0$, the only modular forms of weight k for G are the constant functions.*

Proof. First note that as above, since G has finite index, there exists $w > 0$ such that $\pm T^w \in G$, and if we choose w minimal, the local variable at $i\infty$ is $e^{2\pi i\tau/w}$, and as we have also mentioned, a modular function on G has period dividing $2w$ and not necessarily w. Now let $f_1, \dots, f_h$ be a system of h elements of $M_k(G)$ which are $\mathbb{C}$-linearly independent. In particular, the Fourier expansions at $i\infty$ of these functions are of the form

$$f_j(\tau) = \sum_{n \geq 0} a_j(n) e^{2\pi i n\tau/(2w)}$$

for some complex numbers $a_j(n)$. The homogeneous linear system

$$\sum_{1 \leq j \leq h} x_j a_j(n) = 0 \quad \text{for } 0 \leq n \leq h-2$$

is a system of $h-1$ equations in h unknowns, so it has a nontrivial solution. Since the f_j are linearly independent, it follows that the modular form

$$f = \sum_{1 \leq j \leq h} x_j f_j = \sum_{n \geq 0} \left(\sum_{1 \leq j \leq h} x_j a_j(n) \right) e^{2\pi i n\tau/(2w)}$$

is not identically zero and its Fourier coefficients vanish for $0 \leq n \leq h-2$; in other words, it has a zero of order at least $h-1$ at infinity. By the theorem we must have $mk/12 \geq h-1$, proving (a), and (b) is an immediate consequence, although it also follows directly from Liouville's theorem. ☐

The same proof also gives the following result, which gives an upper bound for the *Sturm bound* [**Stu87**](see also Remarks 12.6.2):

Definition 5.6.13. The *Sturm bound* for a space of modular forms $M_k(G, \chi)$ is a number s such that if $f = \sum_{n \geq 0} a(n)q^n$ and $g = \sum_{n \geq 1} b(n)q^n$ are both elements in $M_k(G, \chi)$ and $a(n) = b(n)$ for $0 \leq n \leq s+1$, then $f = g$.

Corollary 5.6.14. *Assume for simplicity that $T \in G$, for instance that $G = \Gamma_0(N)$. If two modular forms $f_i = \sum_{n \geq 0} a_i(n)q^n$ for $i = 1, 2$ are such that $a_1(n) = a_2(n)$ for $n \leq 1 + \lfloor mk/12 \rfloor$, then $f_1 = f_2$.*

Example 5.6.15. In Chapter 1 we have seen examples of functions which are modular for a subgroup of Γ and not for Γ itself, for instance the function $\theta_{0,0}(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$, which is modular of weight $1/2$ with multiplier system for the subgroup Γ_θ of Γ generated by $T^2 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. By Exercise 5.17, Γ_θ is the subgroup of index 3 consisting of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $a - d \equiv b - c \equiv 0 \pmod 2$, and a system of representatives (γ_i) of $\overline{\Gamma_\theta} \backslash \overline{\Gamma}$ is given by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. It follows from Lemma 5.6.10 that

$$h(\tau) = \tau^{-1/2} \theta_{0,0}(\tau) \theta_{0,0}(\tau+1) \theta_{0,0}\left(\frac{\tau - 1}{\tau}\right)$$

is modular of weight 3/2 with multiplier system on the full modular group Γ, and indeed, by Exercise 5.18 we find that

$$h(\tau) = 2\,e^{-\pi i/4}\eta^3(\tau) = 2^{1/2}(1-i)\eta^3(\tau)\ ,$$

thus giving an alternate proof of a result proved in Proposition 2.3.13. Since by Proposition 2.3.13 the function $\theta_{0,0}$ can be expressed in terms of η, this gives additional formulas satisfied by η (see again Exercise 5.18); these are closely related to so-called *modular equations*, which should not be confused with the modular functional equations of the type $f\big|_k\gamma = f$.

Unfortunately, it is in general impossible to deduce the dimension of $M_k(G)$ and of $S_k(G)$ from the above theorem as we have done for $G = \Gamma$. There are, however, at least two methods to compute these dimensions: one, which we will not discuss in this book but which is the most natural and elegant, is the use of the *Riemann–Roch theorem* and the *Riemann–Hurwitz formula* on the Riemann surface $G\backslash\overline{\mathfrak{H}}$. The other, which is more analytic in nature, will be sketched later (Theorem 12.4.11). In this section we will give some dimension formulas without proof, referring to [**Miy89**] or [**DS05**] for complete details.

Definition 5.6.16. Let G be a subgroup of finite index in Γ, set $m = [\overline{\Gamma} : \overline{G}]$, and let $(\gamma_i)_{1\le i\le m}$ be a system of representatives of right cosets of $\overline{G}\backslash\overline{\Gamma}$.

(a) If $\tau_0 \in \overline{\mathfrak{H}}$, we define $n(\tau_0)$ to be the number of G-equivalence classes of elements of $\overline{\mathfrak{H}}$ which are Γ-equivalent to τ_0, in other words, the number of G-inequivalent elements among the $\gamma_i(\tau_0)_{1\le i\le m}$.

(b) We define the number g by the formula

$$g = 1 + \frac{m}{12} - \frac{n(i)}{4} - \frac{n(\rho)}{3} - \frac{n(i\infty)}{2}\ .$$

Proposition 5.6.17. *The rational number g is in fact a nonnegative integer, equal to the* genus *of the Riemann surface $G\backslash\overline{\mathfrak{H}}$.*

There is no need for us to explain the notion of genus, and if desired, the reader can take the above formula as a definition. We can now give the dimensions of the spaces of modular forms. For simplicity we assume that we have a trivial multiplier system, and since there are no nonzero modular forms of weight less than zero and only constants of weight zero, we will also assume that the weight is positive.

It is possible to give formulas when k is half-integral (see [**CO77**]) but we restrict ourselves here to the case of integral weight. Still, as usual, it is necessary to distinguish the cases of odd and even weight.

Theorem 5.6.18. *Let G be as before and let k be an* even *positive integer. Using the above notation, we have*

$$\dim(M_k(G)) = (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor n(i) + \left\lfloor \frac{k}{3} \right\rfloor n(\rho) + \left\lfloor \frac{k}{2} \right\rfloor n(i\infty)\ ,$$
$$\dim(S_k(G)) = \dim(M_k(G)) - n(i\infty) + \delta_{k,2}\ .$$

In particular, $\dim(S_2(G)) = g$.

Remarks 5.6.19.

(a) If $n(i) = n(\rho) = 0$, we have $g = 1 + m/12 - n(i\infty)/2$, so that $m \equiv 6n(i\infty) \pmod{12}$ and if k is even,

$$\begin{aligned}\dim(M_k(G)) &= (k-1)(m/12 - n(i\infty)/2) + (k/2)n(i\infty)\\ &= (k-1)m/12 + n(i\infty)/2\ .\end{aligned}$$

Since we know from Corollary 5.6.12 that $\dim(M_k(G)) \le km/12 + 1$, it follows that $n(i\infty) \le (m/6)+2$, with $n(i\infty) \equiv (m/6) \pmod 2$.

(b) If $12 \mid k$, we have

$$\dim(M_k(G)) = (k-1)m/12 + n(i)/4 + n(\rho)/3 + n(i\infty)/2\ ,$$

and similarly we have $3n(i) + 4n(\rho) + 6n(i\infty) \le m + 12$.

Assume now that k is odd. If $-I \in G$, it is clear that there are no nonzero modular forms of weight k. We will therefore assume that $-I \notin G$, and before giving the formula, we must discuss the concept of irregular cusps in more detail (we will come back to this important notion in Subsection 6.3.3). We mentioned above that the stabilizer in G of $i\infty$ is either

$$G_{i\infty} = \{\pm T^{nw},\ n \in \mathbb{Z}\}, \quad \{T^{nw},\ n \in \mathbb{Z}\}, \quad \text{or} \quad \{(-T^w)^n,\ n \in \mathbb{Z}\}\ ,$$

where the *width*, w, is the smallest positive integer such that $\pm T^w \in G$ for a suitable sign. The first case is impossible here since we assume that $-I \notin G$. In the second case, if f is weakly modular of weight k for G, then since $T^w \in G$, it follows that f is periodic of period (dividing) w. However, in the third case, we only have $-T^w \in G$, so that

$$f(\tau) = (f\big|_k(-T^w))(\tau) = (-1)^k f(\tau + w) = -f(\tau + w)$$

since k is assumed to be odd, showing that f is periodic of period (dividing) $2w$ and not w. In this case we say that $i\infty$ is an *irregular* cusp. In general, if $\tau = \gamma(i\infty) \in \mathbb{P}^1(\mathbb{Q})$ for some $\gamma \in \Gamma$, we say that τ is irregular if $i\infty$ is an *irregular* cusp of the group $\gamma^{-1}G\gamma$, and otherwise we say that it is *regular*.

Theorem 5.6.20. *Let G be as before, denote by $n(i\infty)^{\text{reg}}$ the number of regular cusps of G, let k be an* odd *integer, and keep the above notation.*

(a) *If $k \geq 3$, then*

$$\dim(M_k(G)) = (k-1)(g-1) + \left\lfloor \frac{k}{3} \right\rfloor n(\rho) + \left\lfloor \frac{k}{2} \right\rfloor n(i\infty) + \frac{n(i\infty)^{\text{reg}}}{2}\ ,$$

$$\dim(S_k(G)) = \dim(M_k(G)) - n(i\infty)^{\text{reg}}\ .$$

(b) *If $k = 1$ and $n(i\infty)^{\text{reg}} > 2g-2$, then*

$$\dim(M_1(G)) = n(i\infty)^{\text{reg}}/2 \quad \textit{and} \quad \dim(S_1(G)) = 0.$$

(c) *If $k = 1$ and $n(i\infty)^{\text{reg}} \leq 2g-2$, then we can only say that*

$$\dim(M_1(G)) \geq n(i\infty)^{\text{reg}}/2 \quad \textit{and} \quad \dim(S_1(G)) = \dim(M_1(G)) - n(i\infty)^{\text{reg}}/2.$$

Giving general formulas for $k = 1$ when $n(i\infty)^{\text{reg}} \leq 2g-2$ is much more difficult and involves rather different objects. In the most important case of $G = \Gamma_0(N)$ (see Chapter 6), this has been done by Deligne–Serre [**DS74**], but even their theorem does not really give a formula, although it does give a method to compute the corresponding dimensions.

Note that in Section 7.4 we will give explicit formulas for the dimensions of the spaces $M_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$.

5.7. The Modular Invariant j

We define the function j on $\mathfrak{H}$ by

$$j(\tau) = \frac{E_4^3(\tau)}{\Delta(\tau)}\ .$$

Proposition 5.7.1.

(a) *The function j is a modular function of weight* 0 *on* Γ *which is holomorphic on $\mathfrak{H}$ and has a simple pole at $i\infty$.*

(b) *The function j induces a bijection from $\Gamma\backslash\overline{\mathfrak{H}}$ to $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.*

Proof. (a) follows immediately from Proposition 5.6.5. Thus, $j(\tau) = \infty$ if and only if τ is equivalent to $i\infty$ under Γ. Thus, if $\lambda \in \mathbb{C}$, we must show that $f_\lambda(\tau) = E_4^3(\tau) - \lambda\Delta(\tau)$ has a unique zero modulo Γ. If we let $n_i(f_\lambda)$ denote the number of zeros (counted with multiplicity) of f_λ at points in $\Gamma\backslash\mathfrak{H}$ with isotropy group of order i, then, by Theorem 5.6.11, we have

$$n_1(f_\lambda) + n_2(f_\lambda)/2 + n_3(f_\lambda)/3 = 1\ ,$$

so that $(n_1(f_\lambda), n_2(f_\lambda), n_3(f_\lambda)) = (1,0,0)$, $(0,2,0)$, or $(0,0,3)$, which indeed means that f_λ vanishes at a unique point of $\Gamma\backslash\mathfrak{H}$. ∎

Remark 5.7.2. We have in fact proved slightly more: we have shown that the order of the zero of f_λ is equal to the cardinality of the isotropy group of this zero, hence that in fact j is an isomorphism between the manifold $\Gamma\backslash\overline{\mathfrak{H}}$ and the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Theorem 5.7.3. *Let f be a meromorphic function on $\mathfrak{H}$. The following are equivalent:*

(a) *f is a modular function of weight* 0.

(b) *f is the quotient of two modular forms of equal weight.*

(c) *f is a rational function of j.*

Proof. The implications (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a) are immediate, so we must show (a) $\Longrightarrow$ (c). Thus, let f be a modular function of weight 0. Since j attains any possible value, if necessary after multiplying f by some polynomial in j, we may assume that f is holomorphic on $\mathfrak{H}$. Thus, there exists an integer n such that $g = \Delta^n f$ is holomorphic on $\mathfrak{H}$ and at infinity, hence a modular form of weight $12n$. By Corollary 5.6.8 we can thus write it as a linear combination of $E_4^a E_6^b$ with $4a + 6b = 12n$. It is thus sufficient to show the result for one of these forms, hence for $f = E_4^a E_6^b / \Delta^n$. But since $4a + 6b = 12n$ implies that $3 \mid a$ and $2 \mid b$, it follows that $f = (E_4^3)^{a/3}(E_6^2)^{b/2}/\Delta^{a/3+b/2}$, and since $E_4^3/\Delta = j$ and $E_6^2/\Delta = j - 1728$ are clearly rational functions of j, the result follows. $\square$

Note that this theorem means that the only meromorphic functions on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ are the rational functions, which is a well-known and fundamental result in complex analysis. Another fundamental result of complex analysis which can be shown to follow from the properties of j (in fact from the corresponding function on $\Gamma(2)$) is the little Picard theorem: if f is a nonconstant entire function on $\mathbb{C}$, then f attains every value except at most one (the example of e^z shows that f can miss one value).

We end this section with the following integrality result.

Proposition 5.7.4. *Set*

$$\Delta(\tau) = \sum_{n \geq 1} \tau(n) q^n \quad \textit{and}$$

$$j(\tau) = \frac{1}{q} + \sum_{n \geq 0} c(n) q^n \;.$$

Then for all n, $\tau(n)$ and $c(n)$ are integers.

Proof. Since $j = E_4^3/\Delta$, $\tau(1) = 1$, and the coefficients of E_4 are integers, it will be sufficient to prove our assertion for Δ. In other words, we must show that $E_4^3 - E_6^2 \equiv 0 \pmod{1728}$, the congruence being coefficientwise.

Now since $1728 = 2^6 3^3$, we have $3(240)^2 = 2^8 3^3 5^2 \equiv 0 \pmod{1728}$, $240^3 = 2^{12} 3^3 5^3 \equiv 0 \pmod{1728}$, and $504^2 = 2^6 3^4 7^2 \equiv 0 \pmod{1728}$. Since $1008 \equiv -720 \pmod{1728}$, it follows that

$$E_4^3 - E_6^2 \equiv -720 \sum_{n\ge1} (\sigma_5(n) - \sigma_3(n)) q^n \pmod{1728} .$$

Now

$$\sigma_5(n) - \sigma_3(n) = \sum_{d\mid n} d^3(d^2-1) \equiv 0 \pmod{12}$$

since for all d we have $d^3(d^2-1) \equiv 0 \pmod{12}$, so the result follows. $\square$

Note that $\tau(1) = 1$, $\tau(2) = -24$, $\tau(3) = 252$, $\tau(4) = -1472$, $c(0) = 744$, $c(1) = 196884$, $c(2) = 21493760$, $c(3) = 864299970$; using a modification of the circle method, Petersson [**Pet32**], and later independently Rademacher [**Rad38**], [**Rad39**] have shown the asymptotic formula

$$c(n) \sim \frac{e^{4\pi n^{1/2}}}{2^{1/2} n^{3/4}} .$$

5.8. The Dedekind η-Function and the Product Formula for Δ

In Chapter 1, we already introduced the Dedekind η-function

$$\eta(\tau) = q^{1/24} \prod_{n\ge1} (1-q^n) ,$$

where we recall that $q^{1/24}$ is shorthand for $e^{2\pi i\tau/24}$. Taking the logarithmic derivative of this, we deduce from Lemma 2.1.13 that

$$\frac{\eta'(\tau)}{\eta(\tau)} = 2\pi i \left(\frac{1}{24} - \sum_{n\ge1} \frac{nq^n}{1-q^n} \right) = \frac{2\pi i}{24} E_2(\tau) .$$

Thus, by Corollary 5.2.17, for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have

$$\frac{\eta'}{\eta}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 \frac{\eta'}{\eta}(\tau) + \frac{1}{2} c(c\tau+d) .$$

Now since $ad - bc = 1$, we have $\gamma'(\tau) = (c\tau+d)^{-2}$ so that

$$\frac{d}{d\tau}\left(\log\left(\eta\left(\gamma\tau\right)\right)\right) = \frac{d}{d\tau}(\log(\eta(\tau))) + \frac{1}{2}\frac{c}{c\tau+d} ,$$

and by integrating and exponentiating we deduce that there exists a constant $v(\gamma)$ independent of τ, but depending on $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, such that

$$\eta(\gamma\tau) = v(\gamma)(c\tau + d)^{1/2}\eta(\tau)\ .$$

In this formula, we define $(c\tau+d)^{1/2} = e^{\log(c\tau+d)/2}$ with the principal branch of the logarithm, or equivalently as the square root whose argument is in $]-\pi/2, \pi/2]$ (of course changing the branch of the square root would simply change $v(\gamma)$). More precisely, we have the following theorem.

Theorem 5.8.1. *The Dedekind η-function satisfies the following:*

(a) $$\eta(\tau+1) = e^{\pi i/12}\eta(\tau) \quad \text{and} \quad \eta\left(-\frac{1}{\tau}\right) = \left(\frac{\tau}{i}\right)^{1/2}\eta(\tau)$$

again with the principal branch of the square root. More generally, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we have

(b) $$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = v(\gamma)(c\tau+d)^{1/2}\eta(\tau)\ ,$$

where $v(\gamma)$ is given by the following formulas:

$$v(\gamma) = \begin{cases} \left(\dfrac{d}{|c|}\right)\exp\left(\dfrac{\pi i}{12}((a+d-3)c - bd(c^2-1))\right) & \text{if } 2 \nmid c, \\ \left(\dfrac{c}{|d|}\right)\exp\left(\dfrac{\pi i}{12}((a-2d)c - bd(c^2-1) + 3d - 3)\right)\varepsilon(c,d) & \text{if } 2 \mid c, \end{cases}$$

where $\left(\frac{c}{d}\right)$ is the Kronecker–Legendre symbol and $\varepsilon(c,d) = -1$ when $c \le 0$ and $d < 0$ and $\varepsilon(c,d) = 1$ otherwise.

Proof. The first formula of (a) is trivial. For the second, by what we have seen above we have $\eta(-1/\tau) = v(S)(\tau/i)^{1/2}\eta(\tau)$ for some constant $v(S)$. Since η is defined by a convergent infinite product, it does not vanish on $\mathfrak{H}$, so choosing $\tau = i$ we see that $v(S) = 1$, proving (a).

The proof of the explicit formula for $v(\gamma)$ in (b) is quite tedious, and we will not give it. It is given in a number of places; see for instance [**Kno70**]. Historically, the transformation formula for $\log(\eta)$, which is more precise, was given by Dedekind in terms of *Dedekind sums*. Many formulas for η itself were given later, and the above is due to Petersson. Note that if $c \neq 0$, then $\varepsilon(c,d) = (c,d)_\infty$, a *Hilbert symbol.* □

Corollary 5.8.2. *We have the following formulas for the Δ-function:*

$$\Delta(\tau) = q\prod_{n\ge1}(1-q^n)^{24} = \eta^{24}(\tau) \quad \text{and} \quad \frac{\Delta'(\tau)}{\Delta(\tau)} = 2\pi i E_2(\tau).$$

Proof. By the theorem we have $\eta^{24}(\tau+1)=\eta^{24}(\tau)$ and $\eta^{24}(-1/\tau)=\tau^{12}\eta(\tau)$. Since η^{24} is evidently holomorphic on $\mathfrak{H}$ and at ∞, it follows that η^{24} is a modular form of weight 12, and even a cusp form since $\eta^{24}(i\infty)=0$. Since the space of cusp forms of weight 12 is of dimension equal to 1 and generated by Δ, it follows that $\Delta=K\eta^{24}$ for some constant K, which we see is clearly equal to 1 by looking at the first Fourier coefficient. We recover the fact that Δ does not vanish on $\mathfrak{H}$, has a simple zero at infinity, and has integral Fourier coefficients. The second assertion is clear since $\eta'/\eta=(2\pi i/24)E_2$. □

For the application to eta quotients which we will give below, the following variant of the transformation formula for η due to Ligozat [**Lig75**] is particularly important:

Proposition 5.8.3. *If* $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$ *with* $\gcd(a,6)=1$ *and* $c\ge 0$*, then*

$$\eta(\gamma\tau)=w(\gamma)((c\tau+d)/i)^{1/2}\eta(\tau)\ ,$$

with

$$w(\gamma)=\left(\frac{c}{a}\right)\exp\left(\frac{\pi i}{12}(a(b-c+3))\right)\ .$$

Note that if $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$, then $T^n\gamma=\left(\begin{smallmatrix}a+nc&b+nd\\c&d\end{smallmatrix}\right)$, and since $\gcd(a,c)=1$, we can find n with $\gcd(a+nc,6)=1$, so we can apply Ligozat's theorem to $T^n\gamma$ and recover $\eta(\gamma(\tau))$ thanks to the relation

$$\eta\big|_{1/2}\gamma=\zeta_{24}^{-n}\eta\big|_{1/2}T^n\gamma\ .$$

5.9. Eta Quotients

Definition 5.9.1. An *eta quotient* is any function f of the form

$$f(\tau)=\prod_{1\le i\le s}\eta^{r_i}(m_i\tau)\ ,$$

where $m_i\in\mathbb{Z}_{\ge 1}$ and $r_i\in\mathbb{Z}$.

It is clear that we may assume that the m_i are all distinct and, by letting N be the least common multiple of the m_i, that all the m_i divide the same positive integer N. Thus, we will instead write

$$f(\tau)=\prod_{m\mid N}\eta^{r_m}(m\tau)\ .$$

It is clear from this definition that f is a modular function on a suitable subgroup of Γ, of weight $\sum_{m\mid N}r_m/2$, which may be integral or half-integral.

Proposition 5.9.2. *Let $f(\tau)=\prod_{m\mid N}\eta^{r_m}(m\tau)$ with $k=\sum_{m\mid N}r_m/2\in\mathbb{Z}$. Then f is a modular function of weight k for some $\Gamma_0(M)$ and character χ if and only if $\sum_{m\mid N}mr_m\equiv 0 \pmod{24}$ and $\sum_{m\mid N}(N/m)r_m\equiv 0 \pmod{24}$.*

We can choose M as the least common multiple of $N=\operatorname{lcm}(m_1,\dots,m_s)$ and the denominator of $\sum_{m\mid N}r_m/(24m)$, and

$$\chi(d)=\left(\frac{(-1)^kP}{d}\right)\;,\quad \textit{where}\quad P=\prod_{m\mid N}m^{r_m}\;.$$

Proof. By setting $r_m=0$ when $m\mid M$ and $m\nmid N$, we can change N to the least common multiple of N with the denominator of $\sum_{m\mid N}r_m/(24m)$ and thus assume that $M=N$. Since $\chi(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right))=1$, a first necessary condition is that $f(\tau+1)=f(\tau)$, and since $\eta(\tau+a)=\zeta_{24}^a\eta(\tau)$, where $\zeta_{24}=e^{\pi i/12}$ is a primitive 24th root of unity, this is equivalent to $\sum_{m\mid N}mr_m\equiv 0 \pmod{24}$, which is the first condition. Similarly, since $\chi(\left(\begin{smallmatrix}1&0\\N&1\end{smallmatrix}\right))=1$, a second necessary condition for f to be modular on $\Gamma_0(N)$ is that $f(\tau/(N\tau+1))=f(\tau)$. Now if we set $W=TST=\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, Theorem 5.8.1 implies that if $c>0$, then

$$\eta(W^c\tau)=\zeta_{24}^{-c}(c\tau+1)^{1/2}\eta(\tau)\;,$$

so that

$$\eta(m\tau/(N\tau+1))=\eta((m\tau)/((N/m)(m\tau)+1))=\zeta_{24}^{-N/m}(N\tau+1)^{1/2}\eta(m\tau)\;.$$

Thus,

$$f(\tau/(N\tau+1))=\zeta_{24}^{-\sum_{m\mid N}(N/m)r_m}(N\tau+1)^kf(\tau)\;,$$

and hence a second necessary condition is $\sum_{m\mid N}(N/m)r_m\equiv 0 \pmod{24}$. An equivalent way of proving this is as follows: since $W_N=\left(\begin{smallmatrix}0&-1\\N&0\end{smallmatrix}\right)$ normalizes $\Gamma_0(N)$, the function $\tau^{-k}f(-1/(N\tau))$ is also modular on $\Gamma_0(N)$, and the first necessary condition for this function is the second condition for f.

Conversely, assume that these two conditions are satisfied. Note that if $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(N)$, then $T^n\gamma=\left(\begin{smallmatrix}a+nc&b+nd\\c&d\end{smallmatrix}\right)$, and since $ad-bc=1$, we have $\gcd(a,c)=1$; hence in particular we can find n such that $\gcd(a+nc,6)=1$. Since $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, it follows that $\Gamma_0(N)$ is generated by matrices $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ with $\gcd(a,6)=1$, and changing γ to $-\gamma$, we may of course assume that $c\geq 0$. It is thus sufficient to prove modularity for such matrices.

Now, by Proposition 5.8.3, for such a matrix we have

$$\begin{aligned}\eta(m\gamma\tau) &= \eta\left(\frac{a(m\tau)+mb}{(c/m)(m\tau)+d}\right)\\ &= \left(\frac{c/m}{a}\right)\exp((\pi i/12)(a(mb-c/m+3)))((c\tau+d)/i)^{1/2}\eta(m\tau)\ ,\end{aligned}$$

so that

$$\begin{aligned}f(\gamma\tau) = &\left(\frac{c/N}{a}\right)^{\sum_{m\mid N} r_m}\prod_{m\mid N}\left(\frac{N/m}{a}\right)^{r_m}\exp((\pi i/12)aS(\gamma))\\ &\times((c\tau+d)/i)^k f(\tau)\\ = &\left(\frac{P_1}{a}\right)\exp((\pi i/12)aS(\gamma))((c\tau+d)/i)^k f(\tau)\ ,\end{aligned}$$

where

$$P_1 = \prod_{m\mid N}(N/m)^{r_m} \quad \text{and} \quad S(\gamma) = \sum_{m\mid N}(mb-c/m+3)r_m\ .$$

Now $P_1 = N^{2k}/\prod_{m\mid N} m^{r_m}$, and since $\gcd(a, N) = 1$, we thus have

$$\left(\frac{P_1}{a}\right) = \left(\frac{P}{a}\right) \quad \text{with} \quad P = \prod_{m\mid N} m^{r_m}\ .$$

Furthermore, by assumption we have

$$S(\gamma) = b\sum_{m\mid N} mr_m - (c/N)\sum_{m\mid N}(N/m)r_m + 3\sum_{m\mid N} r_m \equiv 6k \pmod{24}\ ,$$

and hence

$$f(\gamma\tau) = \left(\frac{P}{a}\right)i^{(a-1)k}(c\tau+d)^k f(\tau)\ .$$

Finally, since $\gcd(a, 6) = 1$, a is odd, so that

$$i^{(a-1)k} = (-1)^{((a-1)/2)k} = \left(\frac{-1}{a}\right)^k\ ,$$

and hence we obtain

$$f(\gamma\tau) = \chi(a)(c\tau + d)^k f(\tau) \ ,$$

with $\chi(a) = \left(\frac{(-1)^k P}{a}\right)$. This is true for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, so by Exercise 1.1, χ will be a Dirichlet character modulo N and $\chi(d) = \overline{\chi}(a) = \chi(a)$ since χ is real, proving the proposition. □

Recall that if $\tau \in \overline{\mathfrak{H}}$, then $v_\tau(f)$ is the order of f at τ, which is positive if f vanishes at τ and negative if f has a pole.

Proposition 5.9.3. *Let $a/c \in \mathbb{P}^1(\mathbb{Q})$ be a cusp with $\gcd(a,c) = 1$ and $c > 0$.*

(a) *The order of $\eta(m\tau)$ at a/c is given by*

$$v_{a/c}(\eta(m\tau)) = \frac{1}{24}\frac{\gcd(m,c)^2}{m} \ .$$

(b) *If $f(\tau) = \prod_{m \mid N} \eta^{r_m}(m\tau)$ is an arbitrary eta quotient, then the order of f at the cusp a/c is given by*

$$v_{a/c}(f) = \frac{1}{24}\sum_{m \mid N} \frac{\gcd(m,c)^2 r_m}{m} \ .$$

(c) *In particular, f is holomorphic at a/c if and only if the above sum is nonnegative, and f vanishes at a/c if and only if it is positive.*

Proof. If $c = 0$, the cusp is $i\infty$ and evidently $v_{i\infty}(\eta(m\tau)) = m/24$; hence (a) is true in this case. Now, assume that $c \neq 0$ and recall from Remark 4.4.4 that if a/c is a cusp with $\gcd(a,c) = 1$, the local parameter around this cusp is $e^{2\pi i/(c(a-c\tau))}$. Now, since $ad - bc = 1$, we have

$$\eta((d\tau - b)/(-c\tau + a)) = \zeta(-c\tau + a)^{1/2}\eta(\tau)$$

for some 24th root of unity ζ, and with $d\tau - b = -d/c(a - c\tau) + 1/c$ we have

$$\begin{aligned}\eta(\tau) &= \zeta^{-1}(-c\tau + a)^{-1/2}\eta(1/(c(a - c\tau)) - d/c) \\ &\sim \zeta^{-1}e^{-2\pi i d/(24c)}(-c\tau + a)^{-1/2}e^{2\pi i/(24c(a-c\tau))}\end{aligned}$$

when τ is in the neighborhood of a/c. It follows of course that $v_{a/c}(\eta) = 1/24$. More importantly, if $m \geq 1$, then $m\tau$ is in the neighborhood of

$ma/c = (m/\delta)a/(c/\delta)$ with $\delta = \gcd(m, c)$ and if we set $c' = c/\delta$, $a' = (m/\delta)a$ with b', d' such that $a'd' - b'c' = 1$, we thus have for some other root of unity ζ'

$$\eta(m\tau) \sim \zeta'^{-1} e^{-2\pi i d'/(24c')}(-c'm\tau + a')^{-1/2} e^{2\pi i/(24c'(a'-c'm\tau))} .$$

Now we see that

$$c'(a' - c'm\tau) = (c/\delta)((m/\delta)a - c(m/\delta)\tau) = (m/\delta^2)c(a - c\tau) ,$$

hence $v_{a/c}(m\tau) = \delta^2/(24m)$, proving (a), and (b) and (c) follow directly. □

Note that if f satisfies the conditions of Proposition 5.9.2, we can consider f as a modular function of weight k and character χ on $\Gamma_0(M)$. Then, since the *width* of the cusp a/c on $\Gamma_0(M)$ is $M/\gcd(c, M)^2$, it follows that the order of vanishing of f at a/c is given by

$$\frac{M}{24\gcd(c, M)^2} \sum_{m \mid N} \frac{\gcd(m, c)^2 r_m}{m} .$$

Eta quotients are particularly interesting for several reasons. First of all, thanks to the above two propositions it is easy to determine the group on which they are modular and whether they are modular forms or cusp forms. Second, they lead to an amazing number of explicit identities and they can also be easily computed both as q-expansions and numerically.

A large number of natural questions can be asked about eta quotients: for instance when are they eigenforms of Hecke operators? (See Exercise 5.29 for the answer in the case of eta *products*.) And, in particular, when do they correspond to elliptic curves, that is, normalized eigenforms of weight 2 with trivial character on some $\Gamma_0(N)$? (See [**MO97**].) When do they give theta functions, generalizing the formulas given in Section 2.3.2? When do they give modular forms of weight 1/2? (See Mersmann's theorem in [**Zag08**].) When are their expansions lacunary?; and so on. These questions are often related: for instance since by the Serre–Stark theorem all modular forms of weight 1/2 are linear combinations of theta series, this is necessarily the case for all eta quotients of Mersmann's list.

We will say that an eta quotient is *primitive* if the greatest common divisor of all the m which occur with $r_m \neq 0$ is equal to 1. It is clear that we can always reduce to the primitive case.

The following theorem of Mersmann gives all primitive holomorphic eta quotients of weight 1/2:

Theorem 5.9.4 (Mersmann)**.** *There are exactly* 14 *primitive eta quotients which are holomorphic modular forms of weight* 1/2*. They are the following, given both as eta quotients and as theta series:*

$$\eta(\tau) = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{12}{n}\right)q^{n^2/24}\ ,\quad \frac{\eta^2(\tau)}{\eta(2\tau)} = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}\ ,$$
$$\frac{\eta^2(2\tau)}{\eta(\tau)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{4}{n}\right)q^{n^2/8}\ ,\quad \frac{\eta(\tau)\eta(4\tau)}{\eta(2\tau)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{8}{n}\right)q^{n^2/8}\ ,$$
$$\frac{\eta^3(2\tau)}{\eta(\tau)\eta(4\tau)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{24}{n}\right)q^{n^2/24}\ ,\quad \frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)} = \sum_{n\in\mathbb{Z}}q^{n^2}\ ,$$
$$\frac{\eta^2(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}a(n)q^{n^2/8}\ ,\quad \frac{\eta(\tau)\eta^2(6\tau)}{\eta(2\tau)\eta(3\tau)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}b(n)q^{n^2/3}\ ,$$
$$\frac{\eta^2(2\tau)\eta(3\tau)}{\eta(\tau)\eta(6\tau)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}c(n)q^{n^2}\ ,\quad \frac{\eta(2\tau)\eta^2(3\tau)}{\eta(\tau)\eta(6\tau)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{36}{n}\right)q^{n^2/24}\ ,$$
$$\frac{\eta(\tau)\eta(4\tau)\eta^2(6\tau)}{\eta(2\tau)\eta(3\tau)\eta(12\tau)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}(-1)^n c(n)q^{n^2}\ ,$$
$$\frac{\eta(\tau)\eta(4\tau)\eta^5(6\tau)}{\eta^2(2\tau)\eta^2(3\tau)\eta^2(12\tau)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{72}{n}\right)q^{n^2/24},$$
$$\frac{\eta^2(2\tau)\eta(3\tau)\eta(12\tau)}{\eta(\tau)\eta(4\tau)\eta(6\tau)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{9}{n}\right)q^{n^2/3}\ ,$$
$$\frac{\eta^5(2\tau)\eta(3\tau)\eta(12\tau)}{\eta^2(\tau)\eta^2(4\tau)\eta^2(6\tau)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{8}{n}\right)a(n)q^{n^2/8},$$

where $a(n)$*,* $b(n)$*, and* $c(n)$ *are periodic functions of period* 6 *which take the values* $a(n) = \{0,1,0,-2,0,1\}$*,* $b(n) = \{0,1,-1,0,-1,1\}$*, and* $c(n) = \{2,1,-1,-2,-1,1\}$ *for* $n \equiv (0,1,2,3,4,5) \pmod 6$*, respectively.*

Among those, only $\eta(\tau)$ *and* $\eta^3(2\tau)/(\eta(\tau)\eta(4\tau))$ *are cusp forms.*

Note that exactly 8 of these eta quotients are "pure" theta series, $\theta(\chi,\tau)$, with (even) character χ and weight 1/2. We can also ask which eta quotients are pure theta series with *odd* character, hence of the form $\sum_{n\in\mathbb{Z}}\chi(n)nq^{n^2/N}$, of weight 3/2. Note that since n is a nonconstant spherical polynomial, all these series will in fact be *cusp forms*. The list of all such eta quotients has been given by Lemke Oliver [**LO13**]:

Theorem 5.9.5 (Lemke Oliver). *The only eta quotients which are pure theta functions with odd character are the following:*

$$\eta^3(\tau)=\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{-4}{n}\right)nq^{n^2/8}\;,\quad \frac{\eta^9(2\tau)}{\eta^3(\tau)\eta^3(4\tau)}=\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{-8}{n}\right)nq^{n^2/8},$$

$$\frac{\eta^2(\tau)\eta^2(4\tau)}{\eta(2\tau)}=\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{-3}{n}\right)nq^{n^2/3}\;,\quad \frac{\eta^{13}(2\tau)}{\eta^5(\tau)\eta^5(4\tau)}=\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{-24}{n}\right)nq^{n^2/24},$$

$$\frac{\eta^5(\tau)}{\eta^2(2\tau)}=\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{-12}{n}\right)nq^{n^2/24}\;.$$

In addition, we also have

$$\frac{\eta^5(2\tau)}{\eta^2(\tau)}=\frac{1}{2}\sum_{n\in\mathbb{Z}}(-1)^{n-1}\left(\frac{-3}{n}\right)nq^{n^2/3}\;.$$

An *eta product* is an eta quotient in which all the r_m are nonnegative. It is clear that apart from the constant function, all eta products vanish at all cusps. A necessary condition for an eta product $f(\tau)=\sum_{n\ge1}a(n)q^n$ to be a normalized Hecke eigenform is that $a(mn)=a(m)a(n)$ whenever $\gcd(m,n)=1$ and $a(1)=1$. This last condition is of course equivalent to $\sum_{m\mid N}mr_m=24$.

It is possible to show (see Exercise 5.29) that there are 1575 eta products of this form which are modular on $\Gamma_0(M)$ for some M and some character χ, that 793 of those have integral weight k, and that at most 28 of those can be normalized Hecke eigenforms: 7 in weight 1, 8 in weight 2, 4 in weight 3, 4 in weight 4, 1 in weight 5, 2 in weight 6, 1 in weight 8, and 1 in weight 12. The fact that they are all indeed eigenforms was proved in [**DKM85**], and a generalization to all eta *quotients* was given in [**Mar93**].

Many of the above forms have interesting interpretations; for example,

$$\eta(\tau)\eta(23\tau)=\frac{1}{2}\left(\sum_{(m,n)\in\mathbb{Z}^2}q^{m^2+mn+6n^2}-\sum_{(m,n)\in\mathbb{Z}^2}q^{2m^2+mn+3n^2}\right)\;,$$

which is linked to the fact that the class number of $\mathbb{Q}(\sqrt{-23})$ is equal to 3; see Exercise 5.32.

The 8 forms in weight 2 are

$$\eta^2(\tau)\eta^2(11\tau)\;,\quad \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)\;,\quad \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)\;,$$
$$\eta^2(2\tau)\eta^2(10\tau)\;,\quad \eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau)\;,\quad \eta^2(3\tau)\eta^2(9\tau)\;,$$
$$\eta^2(4\tau)\eta^2(8\tau)\;,\quad \text{and}\quad \eta^4(6\tau)\;,$$

and they are the modular forms associated to the unique elliptic curves E defined over $\mathbb{Q}$ up to isogeny, of conductors 11, 14, 15, 20, 24, 27, 32, and 36, respectively. Hence, in particular, their pth Fourier coefficient for p prime is equal to $p+1-|E(\mathbb{F}_p)|$. Note that in [**MO97**] the authors show that in addition to the above 8 eta *products*, there are exactly 4 more eta *quotients* which are also Hecke eigenforms of weight 2 with trivial character:

$$\frac{\eta^4(4\tau)\eta^4(12\tau)}{\eta(2\tau)\eta(6\tau)\eta(8\tau)\eta(24\tau)}\,,\quad \frac{\eta^8(8\tau)}{\eta^2(4\tau)\eta^2(16\tau)}\,,$$
$$\frac{\eta^6(4\tau)\eta^6(20\tau)}{\eta^2(2\tau)\eta^2(8\tau)\eta^2(10\tau)\eta^2(40\tau)}\,,\quad \frac{\eta^{12}(12\tau)}{\eta^4(6\tau)\eta^4(24\tau)}\,,$$

corresponding to elliptic curves of conductors 48, 64, 80, and 144, respectively, those of conductors 48 and 64 being unique up to isogeny.

5.10. A Brief Introduction to Complex Multiplication

The theory of complex multiplication (CM) is vast and important and would deserve a book in itself. In this brief section we state and prove the main statements that we will need later.

Lemma 5.10.1. *Let $\tau \in \mathfrak{H}$. The following are equivalent:*

(a) *There exist integers a, b, c with $a \neq 0$ such that $a\tau^2+b\tau+c=0$, or equivalently $\tau = (-b+\sqrt{D})/(2a)$ with $D=b^2-4ac$.*

(b) *τ belongs to an imaginary quadratic field.*

(c) *There exist an integer $n \geq 1$ and a matrix $\gamma \in \Gamma_n$, the set of 2×2 integral matrices with determinant n, such that $\gamma\tau=\tau$.*

Proof. Immediate and left to the reader. □

Definition 5.10.2. A *CM point* is an element of $\mathfrak{H}$ satisfying one of these equivalent conditions. In addition, in (a) the integers a, b, c can be chosen in a unique way such that $\gcd(a,b,c)=1$ and $a>0$, and for this choice $D=b^2-4ac$ is called the *discriminant* of τ.

Note that the discriminant of τ is not necessarily the discriminant of the quadratic field containing τ: for instance, $\tau = 2i$ has minimal equation $\tau^2+4=0$ and $D=-16$, although the discriminant of $\mathbb{Q}(\tau)$ is -4.

It is clear that if τ is CM, then for any $\gamma \in \Gamma$ we also have that $\gamma\tau$ is CM, so if desired we can speak of CM points in $\Gamma\backslash\mathfrak{H}$.

The general philosophy of the theory of complex multiplication is that if f is a modular *function* of weight 0, possibly nonholomorphic, which has algebraic Fourier coefficients in a suitable sense, then for any CM point τ which is not a pole of f, the *CM value* $f(\tau)$ will be an *algebraic number*, and in fact an element of a suitable *class field*. We will not enter into these considerations but simply state and prove the following:

Proposition 5.10.3. *If f is a meromorphic modular function of weight 0 for the modular group Γ with algebraic Fourier coefficients at $i\infty$, then for any CM point τ, the value $f(\tau)$ is an algebraic number.*

Proof. We have seen above that the field of meromorphic modular functions for Γ is $\mathbb{C}(j)$; since the Fourier coefficients of j are rational (in fact integral), it is clear that $f \in \mathbb{C}(j)$ has algebraic Fourier coefficients if and only if in fact $f \in \overline{\mathbb{Q}}(j)$. It is thus sufficient to prove the proposition in the special case $f = j$.

Consider the set Γ_n of 2×2 integer matrices of determinant n. It is clear that Γ has a left (and also a right) action on Γ_n and that the number of orbits is finite (we will study this action in much more detail in Section 6.5). Thus, write $\Gamma_n = \bigsqcup_{1 \le i \le r} \Gamma\gamma_i$ for some $\gamma_i \in \Gamma_n$. By Lemma 6.3.1 (or an immediate verification) we see that if $\gamma \in \Gamma$, then $\gamma_i\gamma = g_i\gamma_{\phi(i)}$ for some $g_i \in \Gamma$ and a *permutation* ϕ of $\{1, 2, \dots, r\}$. Thus, setting $j_i(\tau) = j(\gamma_i\tau)$ for all i, we have

$$j_i(\gamma\tau) = j(\gamma_i\gamma\tau) = j(g_i\gamma_{\phi(i)}\tau) = j(\gamma_{\phi(i)}\tau) = j_{\phi(i)}(\tau)$$

since j is invariant under Γ. Recall that the elementary symmetric functions are polynomials which are invariant under permutation of the variables. It follows that if we consider elementary symmetric functions σ_m with the functions $j_i(\tau) = j(\gamma_i\tau)$ as variables, then these are also invariant under Γ and hence are in $\mathbb{C}(j)$. Since the only poles of j are the cusps and since γ_i has rational coefficients, this is also true for the functions j_i, hence for σ_m. It follows that in fact $\sigma_m \in \mathbb{C}[j]$ is a *polynomial* in j. In addition, since the Fourier expansion of j at infinity has integral coefficients, it follows that the same is true for σ_m (for example, because we can always choose γ_i upper triangular; see Proposition 6.5.3), and since the expansion of j begins with $1/q$, it follows that in fact $\sigma_m \in \mathbb{Z}[j]$. Thus, there exists a nonzero polynomial $\Phi_n \in \mathbb{Z}[X, Y]$ such that $\Phi_n(j_i, j) = 0$ for all i.

Now let τ be a CM point. By Lemma 5.10.1, there exist $n \ge 1$ and $\gamma \in \Gamma_n$ such that $\gamma\tau = \tau$. Since $\gamma = g\gamma_i$ for some $g \in \Gamma$ and some i, we have $j(\gamma\tau) = j(g\gamma_i\tau) = j(\gamma_i\tau) = j_i(\tau)$, and since $\gamma\tau = \tau$, this is also equal to $j(\tau)$. It follows that $\Phi_n(j(\tau), j(\tau)) = 0$, so $j(\tau)$ is a root of the polynomial $\Phi_n(X, X) \in \mathbb{Z}[X]$ and hence is algebraic, proving the proposition. □

Corollary 5.10.4. *Let τ be a CM point. There exists a complex number Ω_τ such that $E_4(\tau)/\Omega_\tau^4$ and $E_6(\tau)/\Omega_\tau^6$ are algebraic integers. One can choose $\Omega_\tau = \eta^2(\tau)$, where as usual η is Dedekind's eta function. More generally, if f is a meromorphic modular function of weight k with algebraic Fourier coefficients, then $f(\tau)/\Omega_\tau^k$ is an algebraic number.*

Proof. The first statement is clear: the function E_4^3/E_6^2 is modular of weight 0 with integral Fourier coefficients, so $E_4^3(\tau)/E_6^2(\tau)$ is algebraic. In other words, if we set $\Omega_\tau = E_4(\tau)^{1/4}/\lambda$ (any 4th root) for some $\lambda \in \mathbb{Z}$, then $E_4(\tau)/\Omega_\tau^4$, so also $E_6^2(\tau)/\Omega_\tau^{12}$ and hence $E_6(\tau)/\Omega_\tau^6$ are algebraic. Choosing λ suitably ensures that they are in fact algebraic integers.

For the second statement, we use the fact (which we have not proved) that $j(\tau)$ is an algebraic *integer*. Since $j = E_4^3/\Delta$, it follows that $j^{1/3}(\tau) = (E_4/\eta^8)(\tau)$ is also an algebraic integer, so that we can indeed choose $\Omega_\tau = \eta^2(\tau)$.

Finally, if f is a modular of weight k, then $g = f(E_4/E_6)^{k/2}$ is a modular function of weight 0; hence $g(\tau)$ is algebraic and the last result follows. □

As already mentioned, this is only the beginning of a very long story: one can show that $j(\tau)$ is in fact an algebraic *integer*, that its minimal polynomial depends only on the discriminant D of τ, the degree of this polynomial is the *class number* $h(D)$ of the order of discriminant D, and in fact one can give an explicit formula for the conjugates of $j(\tau)$ (a special case of the *Shimura reciprocity law*). Finally, and most importantly, $j(\tau)$ generates an *abelian* extension of $\mathbb{Q}$ which is the *ring class field* of the order of discriminant D (the *Hilbert class field* when D is a fundamental discriminant).

Example 5.10.5. To perhaps better understand the above proof, we give the explicit example of $n = 2$. We have $\Gamma_2 = \bigsqcup_{1\le i\le 3}\Gamma\gamma_i$ with for instance $\gamma_1 = \left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)$, $\gamma_2 = \left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)$, and $\gamma_3 = \left(\begin{smallmatrix}1&1\\0&2\end{smallmatrix}\right)$. A small computation shows that

$$\begin{aligned}\Phi_2(X, j(\tau)) &= (X - j(2\tau))(X - j(\tau/2))(X - j((\tau+1)/2))\\ &= X^3 - A(\tau)X^2 + B(\tau)X - C(\tau)\ ,\end{aligned}$$

with the Fourier expansions

$$\begin{aligned}A(\tau) &= q^{-2} + 2232 + O(q)\ ,\\ B(\tau) &= 1488q^{-2} + 42987519q^{-1} + 40492979352 + O(q)\ ,\\ C(\tau) &= -q^{-3} + 159768q^{-2} - 8509195260q^{-1} + 151107596045760 + O(q)\ .\end{aligned}$$

Knowing that these functions are in $\mathbb{Z}[j]$, we easily find that

$$\begin{aligned}\Phi_2(X, Y) = X^3 &- (Y^2 - 1488Y + 162000)X^2\\ &+ (1488Y^2 + 40773375Y + 8748000000)X\\ &+ Y^3 - 162000Y^2 + 8748000000Y - 157464000000000\ .\end{aligned}$$

Thus, any τ which is a fixed point in $\mathfrak{H}$ of an integral matrix γ of determinant 2 will be a root of $\Phi_2(X) := \Phi_2(X,X)$ with

$$\begin{aligned}\Phi_2(X) &= -X^4 + 2978X^3 + 40449375X^2 + 17496000000X - 157464000000\\ &= -(X-1728)(X+3375)^2(X-8000)\ .\end{aligned}$$

Indeed, we find that for $\gamma = \left(\begin{smallmatrix}1 & -1\\ 1 & 1\end{smallmatrix}\right)$ we have $\tau = i$ and $j(i) = 1728$, for $\gamma = \left(\begin{smallmatrix}1 & -2\\ 1 & 0\end{smallmatrix}\right)$ we have $\tau = (1+\sqrt{-7})/2$ and $j((1+\sqrt{-7})/2) = -3375$, and for $\gamma = \left(\begin{smallmatrix}0 & -2\\ 1 & 0\end{smallmatrix}\right)$ we have $\tau = \sqrt{-2}$ and $j(\sqrt{-2}) = 8000$.

The above result is well known. Perhaps slightly less well known is that a similar result is also valid for nonholomorphic modular functions. For simplicity we restrict ourselves to forms constructed from E_2^* on Γ.

Proposition 5.10.6. *If τ is a CM point, then $((E_2^*)^2/E_4)(\tau)$ is an algebraic number. More precisely, if $\Omega = \Omega_\tau$ is as above, then $\sqrt{D}E_2^*(\tau)/\Omega^2$ is an algebraic* integer*, where D is the discriminant of τ.*

Proof. We use the same notation as in the preceding proof, and as above we only prove algebraicity, not the integrality. Once again, let $n \geq 1$, and now set $g_i = E_2^* - E_2^*\big|_2\gamma_i$. Since for $\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_n$ we have $1/y - (1/y)\big|_2\gamma = 2ci/(c\tau + d)$ which is holomorphic on $\mathfrak{H}$, it follows that the functions g_i are holomorphic on $\mathfrak{H}$. Exactly as in the preceding proof, the elementary symmetric functions σ_m of the g_i are now modular for the whole modular group Γ, are holomorphic on $\mathfrak{H}$, and are also clearly holomorphic at infinity since E_2 is. Thus, the σ_m are ordinary modular *forms* on Γ and clearly of weight $2m$. Since E_2 has integer coefficients, we have $\sigma_m \in \mathbb{Q}[E_4, E_6]$ (in fact σ_m belongs to $\mathbb{Z}[1/n][E_4, E_6]$, but *not* to $\mathbb{Z}[E_4, E_6]$ since the slash operator introduces denominators). For instance, considering, once again, the example $n = 2$, we find that

$$(X - g_1)(X - g_2)(X - g_3) = X^3 - (3/4)E_4X + (1/4)E_6\ .$$

Now if τ is a CM point and $\gamma \in \Gamma_n$ is such that $\gamma\tau = \tau$ for a suitable n, it follows from the previous proposition that $\sigma_m \in \mathbb{Q}\Omega_\tau^{2m}$, so that for all i the number $g_i(\tau)/\Omega_\tau^2$ is a root of a nonzero polynomial in $\mathbb{Q}[X]$ and hence is algebraic. On the other hand, as in the preceding proof we have

$$(E_2^*\big|_2\gamma)(\tau) = (E_2^*\big|_2 g\gamma_i)(\tau) = (E_2^*\big|_2\gamma_i)(\tau) = (E_2^* - g_i)(\tau)\ ,$$

and since $\gamma\tau = \tau$, if $\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, we have

$$(E_2^*\big|_2\gamma)(\tau) = n(c\tau + d)^{-2}E_2^*(\gamma\tau) = n(c\tau + d)^{-2}E_2^*(\tau)\ ,$$

so that

$$g_i(\tau) = E_2^*(\tau)(1 - n(c\tau + d)^{-2})\ ,$$

proving that $E_2^*(\tau)$ is algebraic. ∎

We can now generalize Corollary 5.10.4 to general modular functions:

Corollary 5.10.7. *Keep the same notation as in Corollary* 5.10.4. *If f is a modular function of weight k with algebraic Fourier coefficients, then $\mathfrak{D}^n_{Y,k}(f)(\tau)/\Omega_\tau^{k+2n}$ is algebraic.*

Proof. Immediate from the preceding corollary and Corollary 5.3.13. □

We have seen in Corollary 5.10.4 that one can choose $\Omega_\tau = \eta^2(\tau)$. Although Ω_τ is probably always a transcendental number, the *Lerch, Chowla–Selberg formula* gives an expression for Ω_τ in terms of the gamma function at rational arguments; see for instance Proposition 10.5.11 of [**Coh07b**]. For instance, we have the following values:

$$\begin{aligned}
\eta\left(\frac{-1+\sqrt{-3}}{2}\right) &= e^{-\pi i/24}2^{-1}3^{1/8}\pi^{-1}\Gamma(1/3)^{3/2}\ , \\
\eta(\sqrt{-1}) &= 2^{-1}\pi^{-3/4}\Gamma(1/4)\ , \\
\eta\left(\frac{-1+\sqrt{-7}}{2}\right) &= e^{-\pi i/24}2^{-1}7^{-1/8}\pi^{-1}\left(\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)\right)^{1/2}\ , \\
\eta(\sqrt{-2}) &= 2^{-11/8}\pi^{-3/4}\left(\Gamma(1/8)\Gamma(3/8)\right)^{1/2} \\
\eta\left(\frac{\sqrt{-1}}{2}\right) &= 2^{-7/8}\pi^{-3/4}\Gamma(1/4)\ .
\end{aligned}$$

Thus, for instance, since $E_4(i) = 12\Omega_i^4$, we deduce the identity

$$\sum_{n\geq 1}\frac{n^3}{e^{2\pi n}-1} = \frac{\Gamma(1/4)^8}{80(2\pi)^6} - \frac{1}{240}\ .$$

Exercises

5.1. Determine all possible multiplier systems for Γ.

5.2. Prove that as claimed in the text $y = \Im(\tau)$ is not algebraic over the field of meromorphic functions on $\mathbb{C}$.

5.3. Assume that G is not cocompact. The aim of this exercise is to show the existence of $\phi \in M_2^{\mathrm{qm},1}(G) \smallsetminus M_2(G)$.

   (i) With suitable changes of variables, show that we may assume that $i\infty$ is a cusp and also that its stabilizer is $\Gamma_\infty = \{\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right),\ n \in \mathbb{Z}\}$.

   (ii) For $\tau \in \mathfrak{H}$ and $\Re(s) > 1$ define the nonholomorphic Eisenstein series $E(\tau, s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\Im(\gamma\tau)^s$ (we will study these series in great detail later in Section 12.1, to which the reader should refer for hints).

Show that it converges for $\Re(s) > 1$, that it is invariant under G in τ and defines a holomorphic function of s for $\Re(s) > 1$

(iii) Show that as $s \to 1$ we have

$$E(\tau, s) = \frac{c}{s-1} + E_0(\tau) + O(s-1)$$

for some function E_0 and some nonzero constant $c = 1/\operatorname{covol}(G)$.

(iv) Let $\Delta = -y^2(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2})$ be the hyperbolic Laplace operator. Show that $\Delta(E) = s(1-s)E$.

(v) Deduce that $\Delta(E_0) = c$ is constant.

(vi) By integrating this formula, deduce that

$$\frac{\partial E_0}{\partial \tau} = \frac{c}{2iy} + \phi(\tau)$$

for some *holomorphic* function ϕ.

(vii) Deduce finally from the G-invariance of E_0 that $\frac{\partial E_0}{\partial \tau} \in M_2^{\mathrm{ah},1}(G)$ and that $\phi \in M_2^{\mathrm{qm},1}(G) \smallsetminus M_2(G)$.

(viii) Deduce the validity of Proposition 5.1.25 for all noncocompact groups G.

5.4. Assume now that G is cocompact. The aim of this exercise is to show that $M_2^{\mathrm{qm},1}(G) = M_2(G)$. Assume the contrary, so let $\phi \in M_2^{\mathrm{qm},1}(G) \smallsetminus M_2(G)$.

(i) Show that possibly after replacing G by a subgroup of finite index, we may assume that $M_2(G) \neq \{0\}$, so let f be a nonzero holomorphic modular form of weight 2.

(ii) Show that possibly after multiplying ϕ by a nonzero constant, we have $g = f' - \phi f \in M_4(G)$.

(iii) For all $\tau_0 \in \mathfrak{H}$ show that

$$v_{\tau_0}(f) = \operatorname{Res}_{\tau=\tau_0}\left(\frac{g(\tau)}{f(\tau)}\right) .$$

(iv) If $\mathfrak{F}$ is a fundamental domain for the action of G on $\mathfrak{H}$, show that

$$\sum_{\tau_0 \in \mathfrak{F}} v_{\tau_0}(f) = \frac{1}{2\pi} \operatorname{covol}(G) \neq 0 .$$

(v) On the other hand, using the fact that G is cocompact show that

$$\sum_{\tau_0 \in \mathfrak{F}} \operatorname{Res}_{\tau=\tau_0}\left(\frac{g(\tau)}{f(\tau)}\right) = 0 ,$$

thus obtaining a contradiction.

(vi) Deduce the validity of Proposition 5.1.25 for cocompact groups G.

5.5. Prove Proposition 5.2.7.

5.6. Prove Proposition 5.2.6.

5.7. Using the Fourier series expansion of the nonholomorphic Eisenstein series $E(s)$ and its modularity, prove the following functional equations:

(i) Set

$$S_1(x) = \gamma - \log(4\pi x) + 4\sum_{n\ge1}\sigma_0(n)K_0(2\pi n/x)\ .$$

Show that $S_1(1/x) = S_1(x)/x$.

(ii) Set

$$S_2(x) = \frac{\pi}{12}x + \sum_{n\ge1}\frac{\sigma_1(n)}{n}e^{-2\pi nx}\ .$$

Show that $S_2(1/x) = S_2(x) - \log(x)/2$, and deduce the functional equation for Dedekind's eta function.

(iii) Set

$$S_3(x) = \sum_{n\ge1}\frac{\sigma_2(n)}{n}K_1(2\pi nx)\ .$$

Show that

$$S_3(1/x) = xS_3(x) + \frac{\pi}{24}(1-x) + \frac{\zeta(3)}{8\pi}\left(x^2 - \frac{1}{x}\right)\ .$$

(iv) Set

$$S_4(x) = -\frac{\pi^3}{180}x^3 - \frac{\pi^3}{72}x + \frac{\zeta(3)}{2} + \sum_{n\ge1}\frac{\sigma_3(n)}{n^3}e^{-2\pi nx}\ .$$

Show that $S_4'''(x) = -(\pi^3/30)E_4(ix)$ and the functional equation $S_4(1/x) = -S_4(x)/x^2$. We will see in Section 11.5 that this is a special case of the theory of antiderivatives and periods of modular forms.

5.8. (Continuation)

(i) Prove the following two formulas involving K-Bessel functions:

$$\int_0^\infty \frac{dt}{t^{s+1}(e^{2\pi xt}-1)(e^{2\pi/t}-1)} = 2x^{s/2}\sum_{n\ge1}n^{s/2}\sigma_{-s}(n)K_s(4\pi\sqrt{nx})$$

and

$$\int_0^\infty K_{s/2}(2\pi xt)K_{s/2}(2\pi n/t)\,dt = K_s(4\pi\sqrt{nx})/(2x)\ ,$$

for $x > 0$.

(ii) In a manner similar to the previous exercise, prove formulas such as

$$\sum_{n\ge1}\sigma_0(n)K_0(4\pi\sqrt{nx}) = \frac{x}{2\pi^2}\sum_{n\ge1}\sigma_0(n)\frac{\log(n/x)}{n^2-x^2} - L(x)\ ,$$

with

$$L(x) = \frac{\gamma}{4} + \frac{\log(x)}{8} + \frac{\log(2\pi\sqrt{x})}{4\pi^2 x}\ .$$

5.9. Show that $M_{14}(\mathbb{C}) = \mathbb{C}E_{14}$, that $S_{14}(\mathbb{C}) = \{0\}$, and that $E_{14} = E_6E_8 = E_6E_4^2$.

5.10. Finish the proof of Corollary 5.6.8.

5.11. We have

$$E_{12}(\tau) = 1 + \frac{65520}{691}\sum_{n\geq 1}\sigma_{11}(n)q^n\ .$$

Deduce that

$$\frac{691}{65520}E_{12} - \Delta = \frac{691}{112320}E_4^3 + \frac{691}{157248}E_6^2\ ,$$

hence that $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ for all $n \geq 1$, where Ramanujan's tau function $\tau(n)$ is the nth Fourier coefficient of Δ.

5.12. (i) Generalizing Example 5.2.9, show that for $k \equiv 2 \pmod 4$, $k \geq 6$, we have

$$\sum_{n\geq 1}\frac{n^{k-1}}{e^{2\pi n}-1} = \frac{B_k}{2k}\ .$$

(ii) Prove that under the same assumptions we have

$$\int_0^\infty \frac{t^{k-1}}{e^{2\pi t}-1} = \frac{B_k}{2k}\ ,$$

and more generally compute this integral for all $k \in \mathbb{C}$ with $\Re(k) > 1$.

(iii) Show that the similarity of the two formulas is not a coincidence. More precisely, show that if f is holomorphic on $\Re(z) \geq 0$, even, satisfying $|f(z)| = o(\exp(2\pi|\Im(z)|))$ as $|\Im(z)| \to \infty$ uniformly in vertical strips of bounded width, and some additional growth conditions, then

$$\frac{f(0)}{2} + \sum_{m\geq 1} f(m) = \int_0^\infty f(t)\,dt$$

(for help, see the errata and addenda to [**Coh07b**] on the first author's home page).

5.13. Show that $E_2(i) = 3/\pi$, hence that

$$\sum_{n\geq 1}\frac{n}{e^{2\pi n}-1} = \frac{1}{24} - \frac{1}{8\pi}\ .$$

5.14. Using the formula $\eta(2i) = 2^{-11/8}\pi^{-3/4}\Gamma(1/4)$ coming from complex multiplication, show that

$$\sum_{n\ge1}\frac{n}{e^{2\pi n}+1} = -\frac{1}{24}+\frac{\Gamma(1/4)^4}{2^7\pi^3}\;,$$

$$\sum_{n\ge1}\frac{n^3}{e^{2\pi n}+1} = \frac{1}{240}-\frac{3}{5}\frac{\Gamma(1/4)^8}{2^{13}\pi^6}\;,\ \text{and}$$

$$\sum_{n\ge1}\frac{n^5}{e^{2\pi n}+1} = -\frac{1}{504}+3\frac{\Gamma(1/4)^{12}}{2^{17}\pi^9}\;.$$

For sums of the type $\sum_{n\ge1} n^{k-1}/(e^{2\pi n}\pm1)$ with $k\le0$, see Corollary 11.6.3.

5.15. Show that $\mathfrak{D}^n_{Y,k}(Y) = n!Y^{n+1}$.

5.16. As claimed in the text, show that if the formulas of Theorem 5.8.1(b) are valid for some $A \in \mathrm{SL}_2(\mathbb{Z})$, they are also valid for SA and TA.

5.17. Recall that the group for which the function $\theta_{0,0}(\tau)$ introduced in Chapter 1 is the subgroup Γ_θ of Γ generated by $T^2 = \left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. Show that Γ_θ is the set of $\gamma = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \Gamma$ such that $a-d \equiv b-c \equiv 0 \pmod 2$ and that a system of representatives of $\Gamma_\theta\backslash\Gamma$ is given by $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}1&-1\\1&0\end{smallmatrix}\right)$, so that $[\Gamma:\Gamma_\theta] = 3$.

5.18. From the text, we know that the function

$$h(\tau) = \tau^{-1/2}\theta_{0,0}(\tau)\theta_{0,0}(\tau+1)\theta_{0,0}((\tau-1)/\tau)$$

is modular of weight $3/2$ with multiplier system on Γ.

(i) Show that $h(\tau)/\eta^3(\tau)$ is a modular *function* of weight 0 with trivial multiplier system.

(ii) Show that it has no zeroes or poles, deduce that it is a constant, and compute this constant by considering the Fourier expansions.

(iii) Deduce from Proposition 2.3.13 some formulas satisfied by the function $\eta(\tau)$.

5.19. (i) Show that $j(i) = 1728$, and deduce the following table giving $v_\tau(f)$ for the indicated functions, where τ_0 denotes any element of $\mathfrak{H}$ not equivalent under Γ to i, ρ, or $i\infty$:

$f \quad\diagdown\quad \tau$	ρ	i	$i\infty$	τ_0
$j(\tau)$	3	0	-1	0
$j(\tau)-1728$	0	2	-1	0
$j'(\tau)$	2	1	-1	0

(ii) Deduce that the function

$$f_{a,b,c}(\tau)=\frac{j'(\tau)^a(2i\pi)^{-a}}{j(\tau)^b(j(\tau)-1728)^c}$$

is a modular *form* of weight $2a$ over Γ if and only if $2c\le a$, $3b\le 2a$, and $b+c\ge a$, and give the corresponding conditions for f to be a *cusp* form.

(iii) For instance, show that

$$E_4(\tau)=f_{2,1,1}(\tau)=\frac{j'(\tau)^2(2i\pi)^{-2}}{j(\tau)(j(\tau)-1728)}\ ,$$
$$E_6(\tau)=f_{3,2,1}(\tau)=\frac{j'(\tau)^3(2i\pi)^{-3}}{j(\tau)^2(j(\tau)-1728)}\ ,$$
$$\Delta(\tau)=f_{6,4,3}(\tau)=\frac{j'(\tau)^6(2i\pi)^{-6}}{j(\tau)^4(j(\tau)-1728)^3}\ ,$$

hence that we have

$$\frac{j'(\tau)}{2i\pi}=-\frac{E_{14}(\tau)}{\Delta(\tau)}\ .$$

5.20. (i) Show that $E_2E_4=E_6+3E_4'/2i\pi$, $E_2E_6=E_8+E_6'/i\pi$, $E_2E_8=E_{10}+3E_8'/4i\pi$, $E_2E_{12}=E_{14}+E_{12}'/2i\pi$.

(ii) Using in addition the identities $E_2^2=E_4+(12/2i\pi)E_2'$, $E_4^2=E_8$, $E_4E_6=E_{10}$, and $E_4E_{10}=E_6E_8=E_{14}$ (which we have already proved), compute the expressions

$$\sum_{1\le m\le n-1}\sigma_i(m)\sigma_j(n-m)$$

for the 9 pairs $(i,j)=(1,1)$, $(1,3)$, $(1,5)$, $(1,7)$, $(1,11)$, $(3,3)$, $(3,5)$, $(3,9)$, and $(5,7)$ (note that these are all the pairs (i,j) with i and j odd positive integers with $i\le j$ and such that $k=i+j+2\in\{4,6,8,10,14\}$, which are the only weights k for which E_k generates $M_k(\Gamma)$).

(iii) More generally, compute the expressions

$$\sum_{1\le m\le n-1}m^t\sigma_i(m)\sigma_j(n-m)$$

for the 25 triples (t,i,j) with $t\in\mathbb{Z}_{\ge 0}$ and i, j odd positive integers with $i\le j$ such that $k=2t+i+j+2\in\{4,6,8,10,14\}$.

5.21. (Continuation)

(i) Compute the same expressions as in the above exercise for the 9 analogous triples (t,i,j) such that $2t+i+j+2=12$, in terms involving also Ramanujan's tau function $\tau(n)$.

(ii) In particular, prove the following formulas:

$$\begin{aligned}\tau(n) &= \frac{n}{12}(5\sigma_3(n)+7\sigma_5(n))+70\sum_{m=1}^{n-1}(2n-5m)\sigma_3(m)\sigma_5(n-m)\\ &= \frac{n}{36}(25\sigma_3(n)+11\sigma_9(n))-350\sum_{m=1}^{n-1}m\sigma_3(m)\sigma_5(n-m)\\ &= n^4\sigma_1(n)-24\sum_{m=1}^{n-1}m^2(35m^2-52mn+18n^2)\sigma_1(m)\sigma_1(n-m)\\ &= n^5\sigma_1(n)-120\sum_{m=1}^{n-1}m^2(n-m)(4n-7m)\sigma_1(m)\sigma_1(n-m)\\ &= 50n^4\sigma_3(n)-7n^4(12n-5)\sigma_1(n)-840\sum_{m=1}^{n-1}m^4\sigma_1(m)\sigma_1(n-m)\ .\end{aligned}$$

(iii) The second formula shows that we have the congruence $\tau(n)\equiv n^5\sigma_1(n)\pmod{120}$. Show that if $p\geq 3$ is prime, we have $\tau(p)\equiv p(p+1)\pmod{120}$.

(iv) Show that when n is odd, we have $2\nmid\sigma_1(n)$ if and only if n is a square, and deduce that coefficientwise we have the congruence

$$\Delta(\tau)\equiv\sum_{n\geq 0}q^{(2n+1)^2}\equiv q+q^9+q^{25}+q^{49}+\cdots\pmod 2\ .$$

(v) Show that

$$\frac{7E_4E_8+5E_6^2}{12}=E_{12}-\frac{65520}{691}\Delta\ ,$$

and deduce once again the congruence due to Ramanujan

$$\tau(n)\equiv\sigma_{11}(n)\pmod{691}\ .$$

Note: one can prove other congruences for $\tau(n)$ modulo suitable powers of 2, 3, 5, 7, 23, and 691; see [**SD75**].

5.22. For $k\geq 4$ even, set $u_k=F_k(i)\eta^{2k}(i)$ where we recall that

$$F_k(\tau)=-\frac{B_k}{2k}E_k=\sum_{m\geq 0}\sigma_{k-1}(m)q^m=-\frac{B_k}{2k}+\sum_{n\geq 1}\frac{n^{k-1}q^n}{1-q^n}$$

and $\eta(i)=2^{-1}\pi^{-3/4}\Gamma(1/4)$.

(i) Show that for $k \geq 2$ even, we have the recursions

$$\left(\frac{1}{12} - \frac{1}{(k+1)(k+2)}\right) u_{k+4} = \sum_{\substack{2\leq j\leq k-2 \\ j\equiv 2 \pmod 4}} \binom{k}{j} u_{j+2}u_{k+2-j}$$

with the initial values $u_4 = 1/20$ and $u_6 = 0$.

(ii) Compute u_{4k} and $\sum_{n\geq 1} n^{4k-1}/(e^{2\pi n} - 1)$ for $2 \leq k \leq 5$.

5.23. Let as usual $j(\gamma, \tau) = c\tau + d$, and assume that f is a function satisfying $f(\gamma\tau) = j(\gamma, \tau)^{-k} f(\tau)$ with $k > 0$ (this is not a misprint, we want $j(\gamma, \tau)$ to a negative power).

(i) Show that

$$f^{(m)}(\gamma\tau) = \sum_{i=0}^{m} \binom{m}{i} \frac{(k-i)!}{(k-m)!}(-c)^{m-i} j(\gamma, \tau)^{i+m-k} f^{(i)}(\tau) \ .$$

(ii) Deduce that

$$f^{(k+1)}(\gamma\tau) = j(\gamma, \tau)^{k+2} f(\tau) \ .$$

(iii) More generally, assume that

$$f(\gamma\tau) = j(\gamma, \tau)^{-k} \left(f(\tau) + \sum_{i=0}^{k} \lambda_i j(\gamma, \tau)^i\right)$$

for some constants λ_i (possibly depending on γ but not on τ). Show that the conclusion of (ii) is still valid.

We will study this phenomenon in much more detail in Chapter 11.

5.24. Show that if $y'' + ay' + by = 0$, then $Y = y^2$ satisfies

$$Y''' + 3aY'' + (a' + 2a^2 + 4b)Y' + (2b' + 4ab)Y = 0 \ .$$

5.25. Here is an alternate proof of the formula for $\eta(-1/\tau)$, inspired by proofs of Siegel and Weil, but taken from a preprint of Garrett. Set $F(\tau) = -\sum_{n\geq 1} \log(1 - q^n)$.

(i) Show that

$$F(\tau) = \sum_{n\geq 1} \sum_{m\geq 1} \frac{1}{m} q^{mn} \ .$$

(ii) Prove that the Mellin transform of $F(it)$ is given by

$$\Lambda(F, s) := \int_0^\infty F(it)t^{s-1}\, dt = (2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s+1) \ ,$$

where $\zeta(s)$ is the Riemann zeta function.

(iii) Let $\Lambda(\zeta, s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ be the completed Riemann zeta function and use the duplication formula for the gamma function to show that

$$\Lambda(\zeta, s)\Lambda(\zeta, s + 1) = 2(2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s + 1) = 2\Lambda(F, s).$$

(iv) Deduce from the functional equation for $\zeta(s)$, i.e., $\Lambda(\zeta, s) = \Lambda(\zeta, 1-s)$, that $\Lambda(F, -s) = \Lambda(F, s)$.

(v) Recall the Mellin inversion formula (Proposition 3.1.22) which tells us that for any $c > 2$ (which is the region of absolute convergence of $\zeta(s)\zeta(s+1)$) we have

$$F(it) = \frac{1}{2i\pi} \int_{\Re(s)=c} \Lambda(F, s) t^{-s}\, ds\ .$$

Show that it is possible to shift the line of integration to the left and that only poles are at $s = 1, 0, -1$ and by using $\Lambda(F, -s) = \Lambda(F, s)$ deduce that

$$\begin{aligned} F(it) &= \sum_{n=-1,0,1} \mathrm{Res}_{s=n} \Lambda(F, s) t^{-s} + \frac{1}{2i\pi} \int_{\Re(s)=-c} \Lambda(F, s) t^{-s}\, ds \\ &= \sum_{n=-1,0,1} \mathrm{Res}_{s=n} \Lambda(F, s) t^{-s} + F(i/t)\ . \end{aligned}$$

(vi) Show that $s = 1$ and $s = -1$ are simple poles with residues $\mathrm{Res}_{s=-1} \Lambda(F, s) t^{-s} = -(\pi/12)t$ and $\mathrm{Res}_{s=1} \Lambda(F, s) t^{-s} = (\pi/12)t^{-1}$.

(vii) Show that $s = 0$ is a double pole, and since $\zeta(0) = -1/2$, we have $\Lambda(F, s) = -(1/2)(1/s^2 + a/s + O(1))$ for some constant a. Then deduce that $\mathrm{Res}_{s=0} \Lambda(F, s) t^{-s} = \log(t)/2 - a/2$.

(viii) Conclude that

$$F(it) - F(i/t) = (\pi/12)(t^{-1} - t) + \log(t)/2 - a/2\ ,$$

and observe that since the left-hand side changes sign when $t \mapsto t^{-1}$, it follows that $a = 0$ (this is equivalent to the well-known formula $\zeta'(0) = -\log(2\pi)/2$).

(ix) Finally, note that $\eta(it) = e^{-\pi t/12} e^{-F(it)}$, so we obtain $\eta(i/t) = t^{1/2}\eta(it)$, which implies that $\eta(-1/\tau) = (\tau/i)^{1/2}\eta(\tau)$ by analytic continuation.

5.26. Let $D > 1$ be a fundamental discriminant and define the function η_D as in Exercise 2.24(v) by

$$\eta_D(\tau) = q^{B_2(\chi_D)/4} \prod_{n \ge 1} (1 - q^n)^{\left(\frac{D}{n}\right)}\ .$$

Repeat the proof from the previous exercise and give a formula for $\eta_D(-1/\tau)$.

5.27. Prove the theta series expansion of the 14 modular forms given by Mersmann's theorem (Theorem 5.9.4).

5.28. Using Jacobi's triple and Watson's quintuple product identities (see Proposition 2.1.20 and Exercise 2.30), prove some of these formulas.

5.29. Write a (small) computer program to show that there are 1575 eta products of the form $f(\tau) = \sum_{n\geq 1} a(n)q^n$ with $a(1) = 1$ which are modular on $\Gamma_0(M)$ for some M and some character χ, that 793 of those have integral weight k, and that at most 28 of those can be normalized Hecke eigenforms: 7 in weight 1, 8 in weight 2, 4 in weight 3, 4 in weight 4, 1 in weight 5, 2 in weight 6, 1 in weight 8, and 1 in weight 12.

5.30. Show that $E_8 \equiv E_2 \pmod 7$ (coefficientwise of course), and deduce from the formula for $D_\tau(E_4)$ that we have $\tau(n) \equiv n\sigma_3(n) \pmod 7$ and in particular for p prime $\tau(p) \equiv p^4 + p \pmod 7$.

5.31. In a similar manner show that $\tau(n) \equiv n\sigma_1(n) \pmod 5$, so that for p prime $\tau(p) \equiv p^2 + p \pmod 5$.

5.32. Set

$$\theta_1(\tau) = \sum_{(m,n)\in\mathbb{Z}^2} q^{m^2+mn+6n^2} = 1 + 2q + 2q^4 + 4q^6 + \cdots ,$$

$$\theta_2(\tau) = \sum_{(m,n)\in\mathbb{Z}^2} q^{2m^2+mn+3n^2} = 1 + 2q^2 + 2q^3 + 2q^4 + 2q^6 + \cdots .$$

(i) Show that θ_1 and θ_2 belong to $M_1(\Gamma_0(23), \chi_{-23})$, where as usual $\chi_D(d) = \left(\frac{D}{d}\right)$.

(ii) Show that $\theta_1(0) = \theta_2(0) = -i/\sqrt{23}$, where $\theta_i(0)$ denotes the value of $\theta(\tau)$ at the cusp 0.

(iii) Deduce that

$$D(\tau) = (\theta_1(\tau) - \theta_2(\tau))/2 \in S_1(\Gamma_0(23), \chi_{-23}) .$$

(Note that $S_1(\Gamma_0(N), \chi) = 0$ for all $N < 23$ and any character χ, so D is the "smallest" example of a cusp form of weight 1 on some $\Gamma_0(N)$.)

(iv) The dimension formulas, valid only for weight $k \geq 2$, tell us that $\dim(M_2(\Gamma_0(23))) = 3$ and $\dim(S_2(\Gamma_0(23))) = 2$. Show that this implies that $(\theta_1^2, \theta_1\theta_2, \theta_2^2)$ is a basis of $M_2(\Gamma_0(23))$ and that $(\theta_1^2 - \theta_1\theta_2, \theta_2^2 - \theta_1\theta_2)$ is a basis of $S_2(\Gamma_0(23))$.

(v) Assume by contradiction that $M_1(\Gamma_0(23), \chi_{-23})$ is not spanned by θ_1 and θ_2. By multiplying by θ_i show that we would have θ_1^2/θ_2 and θ_2^2/θ_1 in $M_1(\Gamma_0(23), \chi_{-23})$.

(vi) By looking at the orders of the zeros at all $\tau \in \mathfrak{H}$, deduce that we would have $\theta_2/\theta_1 \in M_0(\Gamma_0(23))$, which is a contradiction. It follows that $M_1(\Gamma_0(23), \chi_{-23})$ has dimension 2 with basis (θ_1, θ_2) and $S_1(\Gamma_0(23), \chi_{-23})$ has dimension 1 and is spanned by D.

(vii) It is easy to see that $M_1(\Gamma_0(23), \chi_{-23}) = S_1(\Gamma_0(23), \chi_{-23}) \oplus \mathbb{C}E_1$ for some Eisenstein series, E_1, of weight 1 (compare Remarks 7.4.2), and by Corollary 8.5.12 it follows that this is in fact given by

$$E_1(\tau) = 1 + \frac{2}{3}\sum_{n\ge1}\left(\sum_{d\mid n}\left(\frac{-23}{d}\right)\right) q^n\ .$$

Show that $E_1 = (\theta_1 + 2\theta_2)/3$ and deduce a formula for the *total* number of representations of a positive integer n by the quadratic forms $x^2 + xy + 6y^2$, $2x^2 + xy + 3y^2$, and $3x^2 - xy + 2y^2$.

(viii) Show that $\eta(\tau)\eta(23\tau) \in S_1(\Gamma_0(23), \chi_{-23})$ and deduce that $D(\tau) = \eta(\tau)\eta(23\tau)$.

(ix) Using the fact that $(1 - q^n)^{23} \equiv 1 - q^{23n} \pmod{23}$ coefficientwise, deduce that (again coefficientwise) $\Delta \equiv D \pmod{23}$, where Δ is the usual discriminant function of weight 12 on Γ.

(x) Deduce that if $\left(\frac{n}{23}\right) = -1$, we have $\tau(n) \equiv 0 \pmod{23}$, where $\tau(n)$ is the Ramanujan tau function.

(xi) If p is a prime such that $\left(\frac{p}{23}\right) = 1$, show that $\tau(p) \equiv 2 \pmod{23}$ if p is of the form $p = x^2 + xy + 6y^2$, and show that $\tau(p) \equiv -1 \pmod{23}$ otherwise.

5.33. (i) Show that we have a coefficientwise congruence $\Delta \equiv f_{11} \pmod{11}$, where

$$f_{11}(\tau) = (\eta(\tau)\eta(11\tau))^2\ ,$$

and show that $f_{11} \in S_2(\Gamma_0(11))$ with trivial character.

(ii) It is easy to show that f_{11} is the modular form associated with the elliptic curve $y^2 + y = x^3 - x^2$. Using Schoof's algorithm for counting points on elliptic curves implemented in many computer algebra systems, compute in this way $\tau(p)$ modulo 11 for a few large values of p.

5.34. Let $p \ge 5$ be a prime number.

(i) For any $s \ge 0$ write $(ps + 1)/24 = u/v$ with $\gcd(u, v) = 1$ and

$$\eta(\tau)\eta^s(p\tau) = \sum_{n\equiv u \pmod{v}} a_s(n)q^{n/v}\ .$$

Show that there are at least $(p-1)/2$ congruence classes of n modulo p such that $a_s(n) = 0$ when $n \equiv u \pmod{v}$.

(ii) Similarly, write $(ps + 3)/24 = u/v$ with $\gcd(u, v) = 1$ and

$$\eta^3(\tau)\eta^s(p\tau) = \sum_{n\equiv u \pmod{v}} b_s(n)q^{n/v}\ .$$

Prove the same result for $b_s(n)$.

5.35. (Hard "exercise") Prove the following conjectures due to the first author, or at least the "if" part:

(i) Write $\eta^{-16}(\tau)\eta^{68}(2\tau)=\sum_{n\ge5}a(n)q^n$. Then $a(n)=0$ if and only if n is a power of 2, i.e., if $n=2^k$ for some $k\ge1$ (see Exercise 13.3 for the "if" part).

(ii) Write

$$\eta^5(\tau)\eta^{11}(2\tau)=\sum_{n\equiv1\ (\mathrm{mod}\ 8),\ n\ge9}b(n)q^{n/8}\ .$$

Then $b(n)=0$ if and only if n is an even power of 31, i.e., if $n=31^{2k}$ for some $k\ge1$.

5.36. With $\Omega=\Omega_i$ with the notation of Corollary 5.10.4, compute the algebraic numbers $E_{4k}(i)/\Omega^{4k}$ for $1\le k\le6$.

5.37. (i) Using Proposition 5.4.4, compute the first 50 canonical Taylor coefficients of Δ around $\tau_0=i$.

(ii) Using this, compute numerically the *periods*

$$r_j(\Delta)=\int_0^{i\infty}\tau^j\Delta(\tau)d\tau$$

for $0\le j\le10$ to 30 decimal digits, after making the evident change of variable $T=(\tau-i)/(\tau+i)$. Note that this computation would be more difficult by using directly the Fourier expansion of Δ at $i\infty$.

(iii) Compute numerically the ratios r_{2j}/r_0 for $0\le j\le5$ and r_{2j+1}/r_1 for $0\le j\le4$. What do you notice? (See Manin's Theorem 11.11.2 for a proof.)

5.38. By Theorem 5.7.3 we can write $E_{12r}=\Delta^rP_r(j)$, where P_r is a rational function.

(i) Show that P_r is in fact a polynomial with rational coefficients, that it has degree r, and that it is monic.

(ii) Show that the sum of the roots of P_r is equal to $720r+24r/B_{12r}$.

(iii) As a special case, using Theorem 5.6.1 and Proposition 5.7.1, show that E_{12} has a single root τ_0 in the standard fundamental domain of $\Gamma\backslash\mathfrak{H}$, that it is on the unit circle, and that $j(\tau_0)=432000/691$.

(iv) F. Rankin and Swinnerton-Dyer have shown in [**RSD**] that for any k, all the roots of the Eisenstein series E_k in the standard fundamental domain lie on the unit circle. Show that this is equivalent to the fact that all the roots of P_r are real and in the interval $[0,1728]$.

Chapter 11

Dirichlet Series, Functional Equations, and Periods

The purpose of this chapter is to discuss the relationship between modular forms and their L-functions. In particular, we want to extend the results of Section 10.7 where we showed that the Euler product expansion of $L(f,s)$ was equivalent to $f \in M_k(\Gamma)$ being a normalized eigenform. In this chapter we will show that if $L(s)$ is *any* Dirichlet series such that the completed L-series $\Lambda(s)$ extends to an entire function which is bounded in vertical strips and satisfies a functional equation, then it is in fact equal to the L-series $L(f,s)$ for some modular form f on the full modular group. Theorems of this form are usually called "converse theorems".

The first result in this direction was obtained by Hamburger [**Ham21**], who showed that the Riemann ζ-function is uniquely characterized by its functional equation and analytical properties. The next important step was taken by Hecke [**Hec36**], who gave a similar characterization of the Dirichlet series for modular forms on the Hecke triangle group, G_λ, which is generated by $\tau \mapsto -1/\tau$ and $\tau \mapsto \tau + \lambda$. Special cases are, of course, $G_1 = \Gamma$, the full modular group, and G_2, which is a subgroup of Γ of index 3 and level 2.

It turns out that finding converse theorems for modular forms on congruence subgroups is much more intricate, and to find the "optimal" converse theorem is still a very active area of research. The first such theorem for congruence subgroups was obtained by Weil [**Wei67**], who obtained a converse theorem requiring that the Dirichlet series satisfies an infinite number of functional equations. Several improvements of this theorem have been made in various aspects (see for example [**CF95**], [**CFOS07**], and [**BK14a**], etc.) but we will not present them here.

We will first state and prove the theorem for the full modular group which is due to Hecke and then also state and prove Weil's theorem before proceeding with applications and the relationships to periods. For additional references we refer the reader to Lang [**Lan95**] and Miyake [**Miy89**].

11.1. Introduction

Let $f = \sum_{n\geq 0} a(n)q^n \in M_k(\Gamma)$ be a modular form. Recall from Section 10.7 that we introduced the Dirichlet series, or L-series,

$$L(f,s) = \sum_{n\geq 1} \frac{a(n)}{n^s} \ .$$

Since we know that $a(n) = O(n^{k-1})$, it follows that this series is absolutely convergent for $\Re(s) > k$. Additionally, if f is a cusp form, then $a(n) = O(n^{k/2-\delta})$ for some $\delta > 0$ and the series is convergent for $\Re(s) > k/2+1-\delta$.

The aim of this chapter is the study of analytic and arithmetic properties of the function $L(f,s)$. In particular, we will show the following:

- It has an analytic continuation to the whole complex plane, with possibly a single pole, which is simple, at $s = k$ with residue equal to $(-1)^{k/2}a(0)(2\pi)^k/(k-1)!$. In particular, if f is a cusp form, then $L(f,s)$ is entire.
- It has a functional equation of the form

$$\Lambda(f,s) = \gamma(s)L(f,s) = (-1)^{k/2}\Lambda(f,k-s),$$

  where $\gamma(s) = (2\pi)^{-s}\Gamma(s)$ is called a *gamma factor*. We will also show that conversely these conditions, together with a necessary regularity condition of $L(f,s)$, imply that f is a modular form.
- The constant coefficient $a(0)$ can be recovered, either from the residue at the pole $s = k$ given above or, equivalently, thanks to the functional equation, by the formula

$$a(0) = -L(f,0) \ .$$

- If $n \in \mathbb{Z}$ and $n < 0$, then $L(f,n) = 0$ (these are called *trivial* zeros).
- If $n \in \mathbb{Z}$ and n is in the "critical strip" $]0,k[$, in other words, if $n = 1,2,\ldots,k-1$, then the "critical value", $L(f,n)$, has interesting arithmetical properties.

The fundamental link between the modular form f and its L-series, or L-function, $L(f,s)$ is the Mellin transform, $\mathcal{M}$, which was introduced in Section 3.1.5. This transform plays a key role in the following preliminary theorem which provides many of the properties we will need. For future reference we formulate it for an arbitrary periodic function of period $\lambda > 0$.

Theorem 11.1.1. *Let $f(\tau) = \sum_{n\geq 0} a(n)q^{n/\lambda}$, where $q^{n/\lambda} = e^{2\pi i(n/\lambda)\tau}$; set*

$$L(f,s) = \sum_{n\geq 1} \frac{a(n)}{n^s} \quad \text{and} \quad \Lambda_{\lambda^2}(f,s) = (2\pi/\lambda)^{-s}\Gamma(s)L(f,s).$$

(a) *If $a(n) = O(n^A)$ for some $A > 0$, then*

$$f(x+iy) = O(y^{-A-1}) \quad \text{as} \quad y \to 0^+,$$

uniformly in $x \in \mathbb{R}$. Conversely, if $f(x+iy) = O(y^{-A-1})$ as $y \to 0^+$, uniformly in $x \in \mathbb{R}$, then $a(n) = O(n^{A+1})$ (note that this is slightly weaker than the expected $O(n^A)$).

(b) *If $\Re(s) > A+1$, then*

$$\Lambda_{\lambda^2}(f,s) = \mathcal{M}(f(it) - a(0))(s) = \int_0^\infty t^{s-1}(f(it) - a(0))dt\ ,$$

and conversely, if $\sigma_0 > A+1$ and $y > 0$, then

$$f(iy) - a(0) = \frac{1}{2\pi i}\int_{\Re(s)=\sigma_0} y^{-s}\Lambda_{\lambda^2}(f,s)\,ds\ .$$

Proof. (a) Since $a(n) = O(n^A)$, the Fourier series for $f(\tau)$ converges for all $\tau \in \mathfrak{H}$ and we immediately obtain the estimate

$$|f(x+iy)| \leq B_1 \sum_{n\geq 0} n^A e^{-2\pi ny/\lambda}$$

for some constant B_1 depending on f. The desired estimate, $f(x+iy) = O(y^{-A-1})$, can now be established by, for instance, comparing the above sum with the integral $\int_0^\infty x^A e^{-2\pi xy/\lambda}dx = \Gamma(A+1)(2\pi y/\lambda)^{-A-1}$ or by an explicit estimate in terms of the function $(1-e^{-2\pi y/\lambda})^{-(A+1)}$ (see Exercise 11.1).

For the converse statement, observe that by definition of the Fourier coefficients, for any $y > 0$ we have

$$a(n) = \frac{1}{\lambda}\int_0^\lambda f(x+iy)e^{-2\pi in(x+iy)/\lambda}\,dx\ .$$

By assumption $|a(n)| \leq By^{-A-1}e^{2\pi ny/\lambda}$, and the estimate $a(n) = O(n^{A+1})$ follows by choosing $y = 1/n$. Note that this proof is essentially the same as the proof of Theorem 9.2.1.

(b) For $\Re(s) > A + 1$ the series $L(f, s)$ converges absolutely; hence

$$\begin{aligned}(2\pi/\lambda)^{-s}\Gamma(s)L(f,s) &= \sum_{n\geq 1}\int_0^\infty a(n)(2\pi n/\lambda)^{-s}t^{s-1}e^{-t}\,dt\\ &= \sum_{n\geq 1}\int_0^\infty a(n)t^{s-1}e^{-2\pi(n/\lambda)t}\,dt.\end{aligned}$$

We may now interchange summation and integration and obtain

$$\Lambda_{\lambda^2}(f,s) = (2\pi/\lambda)^{-s}\Gamma(s)L(f,s) = \int_0^\infty t^{s-1}(f(it) - a(0))\,dt$$

(this is justified by uniform convergence). The second formula follows of course from Mellin's inversion formula, Proposition 3.1.22. Let us, however, give the justification in this simple case: for any $\sigma_0 > 0$ we have

$$f(iy) - a(0) = \sum_{n\geq 1} a(n)e^{-2\pi ny/\lambda} = \frac{1}{2\pi i}\sum_{n\geq 1}\int_{\Re(s)=\sigma_0} a(n)(2\pi ny/\lambda)^{-s}\Gamma(s)\,ds\ .$$

If we assume, in addition, that $\sigma_0 > A + 1$, then the series $\sum_{n\geq 1} a(n)/n^s$ converges absolutely and uniformly for $\Re(s) \geq \sigma_0$ and the second formula follows immediately by interchanging summation and integration. □

11.2. The Main Theorem

We are now able to state the theorem which provides the link between Fourier series which satisfy a "modular" functional equation and Dirichlet series which satisfy a functional equation of the type $s \mapsto k - s$. For convenience we state it in a more general form than we need at the moment.

Definition 11.2.1. We will say that a function is entire and bounded in vertical strips, abbreviated EBV, if it is holomorphic in the whole of $\mathbb{C}$ and bounded in any vertical strip $\sigma_1 \leq \Re(s) \leq \sigma_2$.

We keep the notation of the preceding section and let $\lambda > 0$ be fixed.

Theorem 11.2.2. *Let $f = \sum_{n\geq 0} a(n)q^{n/\lambda}$ and $g = \sum_{n\geq 0} b(n)q^{n/\lambda}$, and assume that $a(n) = O(n^A)$ and $b(n) = O(n^A)$ as $n \to \infty$ for some $A > 0$. Let $k > 0$ and $C \in \mathbb{C}^\times$. Then the following conditions are equivalent:*

(a) *There exists a rational function $R(s)$ such that $\Lambda_{\lambda^2}(f,s) - R(s)$ extends to an EBV function and we have $\Lambda_{\lambda^2}(f, k-s) = C\Lambda_{\lambda^2}(g,s)$.*

(b) *For all* $\tau \in \mathfrak{H}$ *we have the functional equation*

$$f\left(-\frac{1}{\tau}\right) - a(0) = C\left(\frac{\tau}{i}\right)^k (g(\tau) - b(0)) + \sum_{s_0 \text{ pole of } R(s)} \operatorname{Res}_{s=s_0}\left(\left(\frac{\tau}{i}\right)^s R(s)\right) ,$$

where $\operatorname{Res}_{s=s_0}$ *denotes the residue at* $s = s_0$.

Proof. (b) implies (a). If $\Re(s) > A+1$, then Theorem 11.1.1 implies that

$$\Lambda_{\lambda^2}(f,s) = \int_0^\infty t^{s-1}(f(it) - a(0))\,dt$$

and from the Fourier expansion it is clear that $f(it) - a(0) = O(e^{-2\pi t/\lambda})$ as $t \to \infty$. It follows from an immediate estimate that the integral

$$\int_1^\infty t^{s-1}(f(it) - a(0))\,dt$$

converges uniformly in any vertical strip and can be extended to an EBV.

Consider now the integral from 0 to 1. For $\Re(s) > A+1$ we have

$$\int_0^1 t^{s-1}(f(it) - a(0))\,dt = -\frac{a(0)}{s} + \int_0^1 t^{s-1} f(it)\,dt ,$$

and if we replace t by $1/t$ and use the functional equation (b), then

$$\int_0^1 t^{s-1}(f(it) - a(0))\,dt = -\frac{a(0)}{s} + \int_1^\infty t^{1-s} f\left(\frac{i}{t}\right)\frac{dt}{t^2}$$
$$= -\frac{a(0)}{s} + \int_1^\infty t^{-1-s}(Ct^k g(it) + a(0) - Cb(0)t^k + S(t,s))\,dt ,$$

with $S(t,s) = \sum_{s_0 \text{ pole}} \operatorname{Res}_{s=s_0}(R(s)t^s)$. For $\Re(s) > \max(A+1,k)$ we have

$$\int_0^1 t^{s-1}(f(it) - a(0))\,dt = C\int_1^\infty t^{k-1-s}(g(it) - b(0))\,dt + I(s) ,$$

where

$$I(s) = \int_1^\infty t^{-(1+s)} \sum_{s_0 \text{ pole}} \operatorname{Res}_{s=s_0}(R(s)t^s)\,dt .$$

Let σ_1 be strictly greater than all the real parts of the poles of $R(s)$. We want to show that $I(s) = R(s)$ if $\Re(s) > \sigma_1$. By linearity it is sufficient to prove this for $R(s) = (s-a)^{-m}$ with $m \geq 1$. In this case we have

$$\operatorname{Res}_{s=a}((s-a)^{-m}t^s) = t^a \operatorname{Res}_{s=a}\left((s-a)^{-m}\sum_{n\geq 0}(\log(t))^n (s-a)^n/n!\right)$$
$$= t^a \log(t)^{m-1}/(m-1)!$$

and

$$I(s)=\int_1^\infty t^{a-s}\frac{\log(t)^{m-1}}{(m-1)!}\frac{dt}{t}=\frac{1}{(m-1)!}\int_0^\infty e^{-(s-a)u}u^{m-1}du=\frac{1}{(s-a)^m},$$

where we made the change of variables $\log(t)=u$ and used the fact that $\Re(s-a)>0$ (note that this is in fact the Mellin inversion formula again). Thus, for $\Re(s)>\sigma_0$ sufficiently large we have

$$\int_0^1 t^{s-1}(f(it)-a(0))\,dt=C\int_1^\infty t^{k-1-s}(g(it)-b(0))\,dt+R(s)\ .$$

Since $g(it)-b(0)=O(e^{-2\pi t/\lambda})$, the integral on the right-hand side, and hence $\int_0^1 t^{s-1}(f(it)-a(0))\,dt-R(s)$, extends to an EBV function. Furthermore,

$$\Lambda_{\lambda^2}(f,s)=\int_1^\infty t^{s-1}(f(it)-a(0))\,dt+C\int_1^\infty t^{k-1-s}(g(it)-b(0))\,dt+R(s)$$

and (a) now follows since by symmetry we also have

$$\begin{aligned}\Lambda_{\lambda^2}(g,s)=&\int_1^\infty t^{s-1}(g(it)-a(0))\,dt+\frac{1}{C}\int_1^\infty t^{k-1-s}(f(it)-b(0))\,dt\\&+\frac{R(k-s)}{C}.\end{aligned}$$

(a) implies (b). It follows from Theorem 11.1.1 that if $\sigma_0>A+1$, then

$$f(iy)-a(0)=\frac{1}{2\pi i}\int_{\Re(s)=\sigma_0}y^{-s}\Lambda_{\lambda^2}(f,s)\,ds$$

for all $y>0$. By (a) we can shift the line of integration to the left, taking the residues of $R(s)$ into account. If we shift the line until all poles of $R(s)$ are to the right of it, then we obtain

$$f(iy)-a(0)=\sum_{s_0\text{ pole}}\operatorname{Res}_{s=s_0}(R(s)y^{-s})+\frac{1}{2\pi i}\int_{\Re(s)=\sigma_1}y^{-s}\Lambda_{\lambda^2}(f,s)\,ds\ ,$$

where $\sigma_1<-\max_{s_0\text{ pole}}|\Re(s_0)|$. Using the functional equation in (a) gives

$$\begin{aligned}f(iy)-a(0)&=\sum_{s_0}\operatorname{Res}_{s=s_0}(R(s)y^{-s})+\frac{C}{2\pi i}\int_{\Re(s)=\sigma_1}y^{-s}\Lambda_{\lambda^2}(g,k-s)\,ds\\&=\sum_{s_0}\operatorname{Res}_{s=s_0}(R(s)y^{-s})+\frac{C}{2\pi i}\int_{\Re(s)=k-\sigma_1}y^{-(k-s)}\Lambda_{\lambda^2}(g,s)\,ds\\&=\sum_{s_0}\operatorname{Res}_{s=s_0}(R(s)y^{-s})+Cy^{-k}(g(i/y)-b(0))\end{aligned}$$

again by Theorem 11.1.1. If we choose $y=i/\tau$, then (b) follows for purely imaginary $\tau\in\mathfrak{H}$, hence for all $\tau\in\mathfrak{H}$ since both sides are analytic in $\mathfrak{H}$. □

Corollary 11.2.3. *Under the same assumptions as in the theorem, the following two conditions are equivalent:*

(a) *The function*

$$\Lambda_{\lambda^2}(f,s)+\frac{a(0)}{s}+\frac{Cb(0)}{k-s}$$

extends to an EBV on $\mathbb{C}$ *and we have* $\Lambda_{\lambda^2}(f,k-s)=C\Lambda_{\lambda^2}(g,s)$.

(b) *For all* $\tau\in\mathfrak{H}$ *we have* $f(-1/\tau)=C(\tau/i)^k g(\tau)$.

Proof. Simply apply the theorem to $R(s)=-a(0)/s+Cb(0)/(s-k)$. □

Corollary 11.2.4. *Let* $f=\sum_{n\geq 0}a(n)q^n$ *where* $a(n)=O(n^A)$ *for some* $A>0$ *and let* k *be an* even *integer. The following conditions are equivalent:*

(a) *The function*

$$\Lambda(f,s)+a(0)\left(\frac{1}{s}+\frac{(-1)^{k/2}}{k-s}\right)$$

extends to an EBV on $\mathbb{C}$ *and we have* $\Lambda(f,k-s)=(-1)^{k/2}\Lambda(f,s)$.

(b) $f\in M_k(\Gamma)$.

Proof. We apply the preceding corollary with $C=i^k=(-1)^{k/2}$. Since $f(\tau+1)=f(\tau)$ and Γ is generated by $\tau\mapsto\tau+1$ and $\tau\mapsto-1/\tau$, condition (b) of the preceding corollary implies that $f((a\tau+b)/(c\tau+d))=(c\tau+d)^k f(\tau)$ for all $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$. The assumption $a(n)=O(n^A)$ implies that f is holomorphic on $\mathfrak{H}$ and at infinity and hence $f\in M_k(\Gamma)$. The converse is immediate. □

Thanks to the above corollary we now have the basic analytic properties of the *completed* L-series $\Lambda(f,s)=(2\pi)^{-s}\Gamma(s)L(f,s)$ and hence of $L(f,s)$ itself. We summarize these properties in the following:

Proposition 11.2.5. *Let* $f=\sum_{n\geq 0}a(n)q^n\in M_k(\Gamma)$.

(a) *The function* $L(f,s)$ *extends to* $\mathbb{C}$ *into a meromorphic function having at most a single pole, which is simple, at* $s=k$*, with residue*

$$\operatorname{Res}_{s=k}L(f,s)=(-1)^{k/2}\frac{(2\pi)^k}{(k-1)!}a(0)\ .$$

(b) *We have* $L(f,0)=-a(0)$ *and* $L(f,n)=0$ *if* n *is a negative integer.*

(c) *If* $\Lambda(f,s)=(2\pi)^{-s}\Gamma(s)L(f,s)$*, then we have the functional equation*

$$\Lambda(f,k-s)=(-1)^{k/2}\Lambda(f,s)\ .$$

Proof. Thanks to the preceding corollary the function

$$(2\pi)^{-s}\Gamma(s)L(f,s)+a(0)\left(\frac{1}{s}+\frac{(-1)^{k/2}}{k-s}\right)$$

is entire. It follows that $L(f,s)$ is meromorphic in $\mathbb{C}$, with possible poles only at $s=0$ and $s=k$. For $s=0$ we have $(2\pi)^{-s}\Gamma(s)\sim 1/s$ and to compensate the term $a(0)/s$ we must have $L(f,0)=-a(0)$, and in particular there is no pole at $s=0$. For $s=k$ we have $L(f,s)\sim(-1)^{k/2}a(0)(2\pi)^k/((s-k)(k-1)!)$, proving (a). For $n\in\mathbb{Z}_{<0}$ the function $\Gamma(s)$ has a pole; hence $L(f,s)$ must have a zero, proving (b), and (c) is clear from the corollary. ☐

Corollary 11.2.6. *If j is an integer such that $0\le j\le k-2$, then*

$$\frac{(2\pi i)^j}{j!}L(f,k-1-j)=(-1)^{j+1}\frac{(2\pi i)^{k-j-2}}{(k-2-j)!}L(f,j+1)\;.$$

In particular, if $k\equiv 2 \pmod 4$, then $L(f,k/2)=0$.

Proof. This follows from the proposition by specializing to $s=j+1$ after an immediate computation. ☐

Example 11.2.7. If $f=F_k=-B_k/(2k)+\sum_{n\ge1}\sigma_{k-1}(n)q^n$ is an Eisenstein series, we have seen that $L(f,s)=\zeta(s)\zeta(s-k+1)$, so that $L(f,k/2)=\zeta(k/2)\zeta(1-k/2)$, which is indeed equal to zero if $k/2$ is odd and $k\ge 4$.

11.3. Weil's Theorem

The previous results on L-series with functional equations and their connections to modular forms were all stated in the context of the full modular group. In the current section our goal is to extend these results to congruence subgroups and characters; that is, we will consider modular forms in $M_k(\Gamma_0(N),\chi)$. To achieve this we note that the crucial ingredient in the proof of the above results, for instance of Corollary 11.2.4, is that the maps $S:\tau\mapsto-1/\tau$ (or more precisely the operator $|_k S$) and $s\mapsto k-s$ are exchanged under the Mellin transform. Since the transformation S does not belong to $\Gamma_0(N)$ if $N>1$, we cannot hope for the same proof to apply in this case. However, using a similar idea will lead to a connection between a functional equation and the Fricke involution $W_N:\tau\mapsto-1/N\tau$ and it is therefore natural to restrict our results to the spaces $M^{\pm}(\Gamma_0(N),\chi)$ defined in Corollary 10.3.15. This leads to the following immediate result:

Proposition 11.3.1. *Let $f=\sum_{n\ge0}a(n)q^n$, set $L(f,s)=\sum_{n\ge1}a(n)n^{-s}$, and for a positive integer N define*

$$\Lambda_N(f,s)=(2\pi/N^{1/2})^{-s}\Gamma(s)L(f,s)\;.$$

If $f \in M_k^{\varepsilon}(\Gamma_0(N),\chi)$ *where* $\varepsilon = \pm 1$ *and* χ *is real, then* $\Lambda_N(f,s) + a(0)(1/s + \varepsilon/(k-s))$ *extends to an EBV on* $\mathbb{C}$ *and satisfies the functional equation*

$$\Lambda_N(f,k-s) = \varepsilon \Lambda_N(f,s)\ .$$

Conversely, if $\Lambda_N(f,s)$ *satisfies the functional equation above, then*

$$f\big|_k W_N = \varepsilon i^{-k} f$$

(but f *is of course not necessarily in* $M_k(\Gamma_0(N),\chi)$*).*

Proof. The function $f_1(\tau) = f(\tau/N^{1/2})$ satisfies the assumptions of Corollary 11.2.3 with $\lambda = N^{1/2}$, and by definition we have

$$\begin{aligned} f_1(-1/\tau) &= f(-1/(N^{1/2}\tau)) = f(-1/(N(\tau/N^{1/2}))) \\ &= N^{k/2}(\tau/N^{1/2})^k \left(f\big|_k W_N\right)(\tau/N^{1/2}) = \varepsilon(\tau/i)^k f_1(\tau). \end{aligned}$$

With the notation of the corollary, we have $f = g = f_1$ and $C = \varepsilon$ and the result now follows since $\Lambda_N(f,s) = (2\pi/N^{1/2})^{-s}\Gamma(s)L(f,s)$. □

The reader will note that in contrast to Corollary 11.2.4, this proposition is not an if and only if statement: when $N > 1$, it is definitely not the case in general that the functional equation plus the EBV condition implies modularity, only that $f\big|_k W_N = \varepsilon i^{-k} f$. This is simply because the matrices W_N and T do not generate $\Gamma_0^*(N) = \Gamma_0(N) \cup \Gamma_0(N)W_N$ when $N > 1$.

To obtain a converse theorem, we thus need additional analytic conditions. These are provided by the *twists* of the modular form, a notion that we have defined and studied in Definition 10.3.17 and Proposition 10.3.18.

Proposition 11.3.2. *Let* $f = \sum_{n\ge 0} a(n)q^n$*, let* ψ *be a primitive Dirichlet character modulo* $m > 1$*, and define* $L(f_\psi,s) = \sum_{n\ge 1}\psi(n)a(n)n^{-s}$ *and*

$$\Lambda_{Nm^2}(f_\psi,s) = (2\pi/(m^2N)^{1/2})^{-s}\Gamma(s)L(f_\psi,s)\ .$$

If $f \in M_k^{\varepsilon}(\Gamma_0(N),\chi)$ *with* $\varepsilon = \pm 1$ *and* χ *is a real character, then* $\Lambda_{Nm^2}(f_\psi,s)$ *extends to an EBV on* $\mathbb{C}$ *and satisfies the functional equation*

$$\begin{aligned} \Lambda_{Nm^2}(f_\psi,k-s) &= C(\psi)\Lambda_{Nm^2}(f_{\overline{\psi}},s) \quad \textit{with} \\ C(\psi) &= \varepsilon\chi(m)^{-1}\psi(-N)\mathfrak{g}(\psi)/\mathfrak{g}(\overline{\psi}). \end{aligned}$$

Conversely, if $\Lambda_{Nm^2}(f_\psi,s)$ *satisfies the functional equation above, then*

$$f_\psi\big|_k W_{m^2N} = C(\psi)i^{-k}f_{\overline{\psi}}.$$

Proof. Once again we apply Corollary 11.2.3 to the functions

$$f_1(\tau) = f_\psi(\tau/(m^2N)^{1/2}) \quad \text{and} \quad g_1(\tau) = f_{\overline{\psi}}(\tau/(m^2N)^{1/2}) \; ,$$

which satisfy the assumptions of the corollary with $\lambda = (m^2N)^{1/2}$. Hence by Proposition 10.3.18 we have

$$\begin{aligned} f_1(-1/\tau) &= f_\psi(-1/((m^2N)^{1/2}\tau)) = f_\psi(-1/(m^2N(\tau/(m^2N)^{1/2}))) \\ &= (m^2N)^{k/2}(\tau/(m^2N)^{1/2})^k \left(f_\psi\big|_k W_{m^2N}\right)(\tau/(m^2N)^{1/2}) \\ &= C(\psi)(\tau/i)^k g_1(\tau) \; , \end{aligned}$$

where we have set $C(\psi) = \varepsilon\chi(m)^{-1}\psi(-N)\mathfrak{g}(\psi)/\mathfrak{g}(\overline{\psi})$. We therefore deduce by Corollary 11.2.3 that $\Lambda_{Nm^2}(f_\psi, k-s) = C(\psi)\Lambda_{Nm^2}(f_{\overline{\psi}}, s)$ with $C(\psi)$ as stated. In addition, since we assume that $m > 1$, we have $\psi(0) = 0$ and therefore the terms corresponding to poles disappear. □

We will now state and prove the important theorem of Weil [**Wei67**] which we alluded to at the beginning of the chapter. The theorem states that if $\Lambda_{Nm^2}(f_\psi, s)$ extends to an EBV and satisfies the functional equation for *sufficiently many* ψ with the constant $C(\psi)$ as given, then $f \in M_k^\varepsilon(\Gamma_0(N), \chi)$, thus giving a converse theorem similar to that of Corollary 11.2.4.

Theorem 11.3.3 (Weil)**.** *Let $f = \sum_{n\ge0} a(n)q^n$ with $a(n) = O(n^A)$ for some $A > 0$ and define $\Lambda_N(f, s)$ and $\Lambda_{Nm^2}(f_\psi, s)$ as above. Assume that $\Lambda_N(f, s) + a(0)(1/s + \varepsilon/(k-s))$ extends to an EBV and satisfies*

$$\Lambda_N(f, k-s) = \varepsilon\Lambda_N(f, s).$$

Furthermore, assume that for all but a finite number of primes p not dividing N, the function $\Lambda_{Nm^2}(f_\psi, s)$ extends to an EBV and satisfies

$$\Lambda_{Nm^2}(f_\psi, k-s) = C(\psi)\Lambda_{Nm^2}(f_{\overline{\psi}}, s),$$

where $C(\psi)$ is given in Proposition 11.3.2, for all nontrivial characters ψ modulo p. Then $f \in M_k^\varepsilon(\Gamma_0(N), \chi)$. If, in addition, $L(f, s)$ converges absolutely for $s = k - \delta$ for some $\delta > 0$, then f is a cusp form.

Proof. By Propositions 11.3.1 and 11.3.2, we know that $f\big|_k W_N = \varepsilon i^{-k} f$ and $f_\psi\big|_k W_{p^2N} = C(\psi)i^{-k} f_{\overline{\psi}}$ for all nontrivial characters ψ modulo all but a finite number of primes $p \nmid N$. We are going to show that these conditions imply that $f \in M_k(\Gamma_0(N), \chi)$ (hence that $f \in M_k^\varepsilon(\Gamma_0(N), \chi)$). Thus, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. If $b = 0$, we may assume that $a = d = 1$ and

$$\gamma = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = W_N \begin{pmatrix} 1 & -c/N \\ 0 & 1 \end{pmatrix} W_N^{-1} \; ,$$

so that

$$f\big|_k\gamma = \varepsilon i^{-k} f\big|_k \begin{pmatrix} 1 & -c/N \\ 0 & 1 \end{pmatrix} W_N^{-1} = \varepsilon i^{-k} f\big|_k W_N^{-1} = \varepsilon i^{-k} \varepsilon i^k f = f\ .$$

We now assume that $b \neq 0$. Since $N \mid c$ and $ad - bc = 1$, we have $\gcd(a, Nb) = \gcd(d, Nb) = 1$ and by Dirichlet's theorem on primes in arithmetic progression we can find integers s and t such that $p = a + Nbs$ and $q = d + Nbt$ are both odd primes not dividing N, and not in the finite set of primes which was excluded in the assumptions of the theorem. Since $ad \equiv 1 \pmod{N}$, we have $\chi(q) = \chi(d) = \chi^{-1}(a) = \chi^{-1}(p)$. In addition,

$$\gamma' = \begin{pmatrix} 1 & 0 \\ Nt & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & 0 \\ Ns & 1 \end{pmatrix} = \begin{pmatrix} p & b \\ c' & q \end{pmatrix} \in \Gamma_0(N)$$

for some $c' \equiv 0 \pmod{N}$. We now need the following key lemma:

Lemma 11.3.4. *Keep the above assumptions and notation. If p and q are odd primes such that $\chi(q) = \chi^{-1}(p)$, then $f\big|_k\gamma' = \chi(q)f$.*

Since the proof of this lemma is lengthy, we postpone it until after the proof of the theorem. We have shown that $f\big|_k\gamma = f$ when $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$; hence

$$f\big|_k\gamma = f\big|_k \begin{pmatrix} 1 & 0 \\ -Nt & 1 \end{pmatrix} \gamma' \begin{pmatrix} 1 & 0 \\ -Ns & 1 \end{pmatrix} = \chi(q)f = \chi(d)f\ ,$$

proving that f is weakly modular of weight k and character χ on $\Gamma_0(N)$.

It is now immediate to show that f is a modular form: since we assume that $a(n) = O(n^A)$, it follows, for instance, from Lemma 9.3.13 that $f(x + iy) = O(y^{-A-1})$, uniformly in x when $y \to 0$. Now, since f is holomorphic and bounded when $y \to \infty$, it follows from Corollary 5.1.17 that f is also polynomially bounded and hence it is a modular form.

Finally, assume that $L(f, s)$ converges absolutely for $s = k - \delta$ for some $\delta > 0$. By an elementary estimate this implies that $L(f, s)$ converges absolutely for $\Re(s) > k - \delta$ and in particular $\Lambda_N(f, s)$ has no pole at $s = k$. We must therefore have $a(0) = 0$ since, by assumption, $\Lambda_N(f, s) + a(0)(1/s + \varepsilon/(k - s))$ is entire. In addition, choosing for instance $\sigma = k - \delta/2$, we have

$$S_n = \sum_{1 \le j \le n} |a_j| \le n^\sigma \sum_{j \ge 1} |a_j| j^{-\sigma} = O(n^\sigma)$$

and by partial summation

$$\begin{aligned}|f(x+iy)| &\le \sum_{n\ge1}|a(n)|e^{-2\pi ny} = \sum_{n\ge1}(S_n - S_{n-1})e^{-2\pi ny}\\ &= (1-e^{-2\pi y})\sum_{n\ge1}S_n e^{-2\pi ny} = O\Big((1-e^{-2\pi y})\sum_{n\ge1}n^\sigma e^{-2\pi ny}\Big)\\ &= O(y\cdot y^{-\sigma-1}) = O(y^{-\sigma})\end{aligned}$$

by Lemma 9.3.13. It thus follows from Corollary 5.1.17 that f is a cusp form. ☐

Proof of Lemma 11.3.4. It remains to prove Lemma 11.3.4. We closely follow [**Ogg69a**]. For this, we introduce the following notation: if $\alpha = \sum_i n_i[\gamma_i] \in \mathbb{C}[\mathrm{GL}_2^+(\mathbb{R})]$ is a finite formal sum, with $\gamma_i \in \mathrm{GL}_2^+(\mathbb{R})$ and $n_i \in \mathbb{C}$, we set $f|\alpha = \sum_i n_i f\big|_k \gamma_i$ (the weight k being fixed), and we define I_f to be the right ideal of such α satisfying $f|\alpha = 0$. Note that if $\alpha \in I_f$ and $\beta \in \mathbb{C}[\mathrm{GL}_2^+(\mathbb{R})]$, we have $\alpha\beta \in I_f$ (this is why I_f is a right ideal) but in general we do not have $\beta\alpha \in I_f$. In addition, we set $T(b/p) = \left(\begin{smallmatrix}1 & b/p\\ 0 & 1\end{smallmatrix}\right)$. With this notation, by what we have seen in Proposition 10.3.18, we have

$$f_\psi = \frac{\psi(-1)\mathfrak{g}(\psi)}{m} f\Big|\Big(\sum_{x \bmod m}\overline{\psi}(x)T(x/m)\Big)$$

if ψ is a primitive character modulo m, and the assumptions of Weil's theorem are equivalent to $W_N \equiv \varepsilon i^{-k} \pmod{I_f}$ and

$$\mathfrak{g}(\psi)\sum_{x \bmod p}\overline{\psi}(x)T(x/p)W_{p^2N} \equiv C(\psi)i^{-k}\mathfrak{g}(\overline{\psi})\sum_{x \bmod p}\psi(x)T(x/p) \pmod{I_f}\ ,$$

in other words, , since $C(\psi) = \varepsilon\chi(p)^{-1}\psi(-N)\mathfrak{g}(\psi)/\mathfrak{g}(\overline{\psi})$, to

$$\sum_{x \bmod p}\overline{\psi}(x)T(x/p)W_{p^2N} \equiv \varepsilon\chi(p)^{-1}\psi(-N)i^{-k}\sum_{x \bmod p}\psi(x)T(x/p) \pmod{I_f}\ .$$

As we have seen in the proof of Proposition 10.3.18, we have

$$T(x/p)W_{p^2N} = pW_N\begin{pmatrix}p & -y\\ -Nx & (xyN+1)/p\end{pmatrix}T(y/p)\ ,$$

with $y = -(xN)^{-1} \bmod p$, so that

$$\sum_{x \bmod p}\overline{\psi}(x)T(x/p)W_{p^2N} \equiv \psi(-N)\sum_{y \bmod p}\psi(y)W_N\gamma(y)T(y/p) \pmod{I_f}\ ,$$

where $\gamma(y) = \left(\begin{smallmatrix}p & -y\\ -Nx & (xyN+1)/p\end{smallmatrix}\right)$ with $x = -(yN)^{-1} \bmod p$. Note that if $x_2 \equiv x_1 \equiv -(yN)^{-1} \bmod p$, then with evident notation we have $\gamma_2(y)\gamma_1^{-1}(y) = \left(\begin{smallmatrix}1 & 0\\ -(x_2-x_1)N/p & 1\end{smallmatrix}\right)$, so that $\gamma_2(y) \equiv \gamma_1(y) \pmod{I_f}$ since we have already shown that $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) \equiv 1 \pmod{I_f}$. It follows that we may choose any $x \equiv -(yN)^{-1}$ and we therefore have

$$\sum_{x \bmod p} \psi(x)(W_N\gamma(x) - \varepsilon\chi(p)^{-1}i^{-k})T(x/p) \equiv 0 \pmod{I_f}\ ,$$

or equivalently $\sum_{x \bmod p} \psi(x)\lambda(x) \equiv 0 \pmod{I_f}$, where we set

$$\lambda(x) = (1 - \varepsilon\chi(p)i^k W_N\gamma(x))T(x/p)\ .$$

It is clear that if ψ is any character modulo p, trivial or not, then

$$\sum_{x \bmod p} (\overline{\psi}(x_1) - \overline{\psi}(x_2))\psi(x)\lambda(x) \equiv 0 \pmod{I_f}.$$

By summing over all characters ψ modulo p and using the orthogonality of characters, we deduce that $(p-1)(\lambda(x_1) - \lambda(x_2)) \equiv 0 \pmod{I_f}$, proving that if x_1 and x_2 are coprime to p, then $\lambda(x_1) \equiv \lambda(x_2) \pmod{I_f}$.

Since $W_N \equiv \varepsilon i^{-k} \pmod{I_f}$, we have $\lambda(x) \equiv (1 - \chi(p)\gamma(x))T(x/p) \pmod{I_f}$ and hence, if b is coprime to p, then

$$(1 - \chi(p)\gamma(b))T(b/p) \equiv (1 - \chi(p)\gamma(-b))T(-b/p) \pmod{I_f}\ ,$$

and since $T(u+v) = T(u)T(v)$, this is equivalent to

$$1 - \chi(p)\gamma(-b) \equiv (1 - \chi(p)\gamma(b))T(2b/p) \pmod{I_f}\ .$$

Now, as mentioned above, we may choose any $x \equiv -(bN)^{-1} \pmod{p}$ in the definition of $\gamma(b) = \begin{pmatrix} p & -b \\ -Nx & (bxN+1)/p \end{pmatrix}$ and similarly for $\gamma(-b)$. Since $\gamma' = \begin{pmatrix} p & b \\ c' & q \end{pmatrix} \in \Gamma_0(N)$, we also have $\gamma'' = \begin{pmatrix} p & -b \\ -c' & q \end{pmatrix} \in \Gamma_0(N)$, so we choose $x = c'/N$ for $\gamma(b)$, and since $\det(\gamma') = 1$, we have evidently $x \equiv -(bN)^{-1} \pmod{p}$, and thus $\gamma(b) = \gamma''$; similarly, we choose $x = -c'/N$ for $\gamma(-b)$, and we will have $\gamma(-b) = \gamma'$. Thus, the above reads simply

$$1 - \chi(p)\gamma' \equiv (1 - \chi(p)\gamma'')T(2b/p) \pmod{I_f}\ .$$

Now note that $\gamma'^{-1} = \begin{pmatrix} q & -b \\ -c' & p \end{pmatrix} \in \Gamma_0(N)$ and similarly for γ''^{-1}. By using the hypotheses of the theorem, but now for the prime q, we also obtain

$$1 - \chi(q)\gamma''^{-1} \equiv (1 - \chi(q)\gamma'^{-1})T(2b/q) \pmod{I_f}\ .$$

It follows from these congruences and the assumption $\chi(q) = \chi(p)^{-1}$ that

$$\begin{aligned} 1 - \chi(q)\gamma''^{-1} &= -(1 - \chi(p)\gamma'')\chi(q)\gamma''^{-1} \equiv -\chi(q)(1 - \chi(p)\gamma')T(-2b/p)\gamma''^{-1} \\ &= (1 - \chi(q)\gamma'^{-1})\gamma' T(-2b/p)\gamma''^{-1} \pmod{I_f}\ . \end{aligned}$$

Since we also have

$$1 - \chi(q)\gamma''^{-1} \equiv (1 - \chi(q)\gamma'^{-1})T(2b/q) \pmod{I_f}\ ,$$

it follows that $(1-\chi(q)\gamma'^{-1})(1-\mu) \equiv 0 \pmod{I_f}$, where we set

$$\mu = \gamma' T(-2b/p)\gamma''^{-1} T(-2b/q)\ .$$

By using the fact that $pq - bc' = 1$ it is easy to verify that

$$\mu = \begin{pmatrix} 1 & -2b/q \\ 2c'/p & -3+4/pq \end{pmatrix}\ ,$$

so that its characteristic polynomial is $X^2+(2-4/pq)X+1$. The discriminant of this polynomial is $16(1-pq)/(pq)^2$ which is clearly negative since p and q are both greater than 2. Thus, its roots are complex conjugate with modulus 1 and cannot be roots of unity since the only nonreal roots of unity which are of degree 2 over $\mathbb{Q}$ are $\pm i$, $\pm\rho$, and $\pm\overline{\rho}$, which are roots of $X^2 - tX + 1 = 0$ with $t = 0$ and $t = \pm 1$, which is not possible for $t = 4/pq - 2$ since we assume that p and q are both odd primes (and in particular greater than 2).

We now need the following lemma:

Lemma 11.3.5. *Let $\mu = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ be such that its eigenvalues are not real and not roots of unity. If g is a holomorphic function on $\mathfrak{H}$ which satisfies $g(\tau + w) = g(\tau)$ and $g\big|_k\mu = \varepsilon g$ for some constants w, ε, and k with $w > 0$ and $k > 0$, then g is identically zero.*

It is clear that the function $g = f|(1-\chi(q)\gamma'^{-1})$ satisfies the assumptions of the lemma with $\varepsilon = 1$; hence $f\big|_k\gamma' = \chi(q)f$, proving Lemma 11.3.4. □

Proof of Lemma 11.3.5. The assumption that the eigenvalues are not real implies that the map $\tau \mapsto \mu\tau = (a\tau + b)/(c\tau + d)$ has two distinct complex conjugate fixed points: τ_0 and $\overline{\tau_0}$ where we can choose $\tau_0 \in \mathfrak{H}$. Now consider the Cayley transform $\phi(\tau) = (\tau - \tau_0)/(\tau - \overline{\tau_0})$, which is a conformal map from $\mathfrak{H}$ to the open unit disk. The transformation $\mu' = \phi \circ \mu \circ \phi^{-1}$ fixes 0 and ∞ so is necessarily of the form $\mu'(t) = \rho t$ for some $\rho \in \mathbb{C}$, and it is immediate to show that ρ is the square of one of the eigenvalues of μ. If we let $h(\tau) = (\tau - \tau_0)^k$, then since τ_0 is a fixed point of μ, we have

$$h(\mu(\tau)) = \left(\frac{a\tau + b}{c\tau + d} - \frac{a\tau_0 + b}{c\tau_0 + d}\right) = \eta(c\tau + d)^{-k}(\tau - \tau_0)^k = \eta(c\tau + d)^{-k}h(\tau)$$

with $\eta = (c\tau_0 + d)^{-k}$. If we now set $g_1(\tau) = g(\tau)h(\tau)$ and $g_2(t) = g_1(\phi^{-1}(t))$, then $g_1(\mu(\tau)) = \varepsilon\eta g_1(\tau)$ and $g_2(\rho t) = \varepsilon\eta g_2(t)$ since

$$g_2(\mu'(t)) = g_1(\mu(\phi^{-1}(t))) = \varepsilon\eta g_1(\phi^{-1}(t)) = \varepsilon\eta g_2(t)\ .$$

It is clear that h and g_1 are holomorphic on $\mathfrak{H}$ and that g_2 is holomorphic on the open unit disk and can therefore be expanded into a convergent power series $g_2(t) = \sum_{n\ge 0} c_n t^n$. The functional equation $g_2(\rho t) = \varepsilon\eta g_2(t)$ now implies that $\rho^n c_n = \varepsilon\eta c_n$, and therefore either $c_n = 0$ or $\rho^n = \varepsilon\eta$.

If $g \neq 0$ and there do not exist at least two distinct values of n with $c_n \neq 0$, then we would have $g_2(t) = ct^n$ for some nonzero c and some n and working backwards we would have

$$g(\tau) = c(\tau - \tau_0)^{n-k}(\tau - \overline{\tau_0})^{-n} ,$$

which cannot be a periodic function of τ unless $n = k = 0$, which is excluded since $k > 0$. It follows that $\rho^{n_1} = \rho^{n_2}$ for $n_1 \neq n_2$ and hence that $\rho^{n_1 - n_2} = 1$ so that ρ is a root of unity, contradicting the assumption that the eigenvalues are not roots of unity, proving the lemma. □

Remarks 11.3.6.

(a) There exist analogues of Weil's theorem for modular forms on GL_n (ordinary modular forms are on GL_2). In that case, one must twist (tensor product) with modular forms on GL_{n-1}, not only with characters. Thus, in our case, Dirichlet characters can in fact be considered as "modular forms on GL_1".

(b) The standard application of Weil's theorem is to elliptic curves: if one can prove that the L-function of an elliptic curve satisfies the assumptions of the theorem, this shows that the associated q-series is a modular form, in other words, that the curve is modular. As already mentioned, this remarkable result has been shown in general for all elliptic curves over $\mathbb{Q}$ by Wiles and others.

11.4. Application to the Riemann Zeta Function

Although we have already proved the basic properties of the Riemann zeta function in Chapter 1 it is interesting to note that they also follow directly as a special case of the results of the previous section.

Indeed, we have seen that if we set for $\Im(\tau) > 0$

$$\theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2} = 1 + 2\sum_{n\geq 1} q^{n^2/2} = 1 + 2\sum_{n\geq 1} e^{\pi i n^2 \tau} ,$$

then we have the functional equation

$$\theta_3(-1/\tau) = (\tau/i)^{1/2}\theta_3(\tau) .$$

We may therefore apply Corollary 11.2.3 with $\lambda = 2$, $k = 1/2$, $C = 1$, and $f = g = \theta_3$. By definition we have $L(\theta_3, s) = 2\sum_{n\geq 1} 1/(n^2)^s = 2\zeta(2s)$ and hence $\Lambda(\theta_3, s) = 2 \cdot \pi^{-s}\Gamma(s)\zeta(2s)$. It follows that the function $2 \cdot \pi^{-s}\Gamma(s)\zeta(2s) + 1/s + 1/(1/2 - s)$ extends to $\mathbb{C}$ into an EBV function and that

$$2 \cdot \pi^{-(1/2-s)}\Gamma(1/2 - s)\zeta(1 - 2s) = 2 \cdot \pi^{-s}\Gamma(s)\zeta(2s) .$$

It is easy to see that the statements of Theorem 3.3.15(a) and (b) follow immediately from this and we leave the details to the reader.

11.5. Periods and Antiderivatives of Modular Forms

In Chapter 5 we studied successive derivatives, and we will now consider successive *antiderivatives* of modular forms. For now we keep the assumptions of Theorem 11.2.2: let $\lambda > 0$ be fixed, $f = \sum_{n\geq 0} a(n)q^{n/\lambda}$, $g = \sum_{n\geq 0} b(n)q^{n/\lambda}$, $a(n) = O(n^A)$, and $b(n) = O(n^A)$ for some $A > 0$ and $C \neq 0$. However, let $k > 0$ be *integral*, but not necessarily even.

Proposition 11.5.1. *Set*

$$f^*(\tau) = a(0)\frac{\tau^{k-1}}{(k-1)!} + \left(\frac{2\pi i}{\lambda}\right)^{1-k} \sum_{n\geq 1} n^{1-k} a(n) q^{n/\lambda}$$

and similarly for g^* with $a(n)$ replaced by $b(n)$. Then f^* is a $(k-1)$st antiderivative of f; in other words, $(d/d\tau)^{k-1}(f^*) = f$, and if we assume that

$$f(-1/\tau) = C(\tau/i)^k g(\tau),$$

then

$$\begin{aligned} f^*\left(\frac{-1}{\tau}\right) &= (-1)^{k-1} C \left(\frac{\tau}{i}\right)^{2-k} g^*(\tau) \\ &\quad + \left(\frac{2\pi i}{\lambda}\right)^{1-k} \sum_{0\leq j\leq k-2} \frac{(-2\pi i/(\lambda\tau))^j}{j!} L(f, k-1-j) \ . \end{aligned}$$

Proof. The assertion that $(d/d\tau)^{k-1}(f^*) = f$ is immediate by differentiation. Set $f^+(\tau) = f^*(\tau) - a(0)\tau^{k-1}/(k-1)!$ and similarly for g. Then

$$\begin{aligned} \Lambda_{\lambda^2}(f^+, s) &= (2\pi/\lambda)^{-s}\Gamma(s)L(f^+, s) \\ &= (2\pi i/\lambda)^{1-k}(2\pi/\lambda)^{-s}\Gamma(s)\sum_{n\geq 1} n^{1-k}a(n)/n^s \\ &= (2\pi i/\lambda)^{1-k}(2\pi/\lambda)^{-s}\Gamma(s)L(f, s+k-1) \end{aligned}$$

and if we replace s by $2-k-s$, then

$$\Lambda_{\lambda^2}(f^+, 2-k-s) = (2\pi i/\lambda)^{1-k}(2\pi/\lambda)^{s+k-2}\Gamma(2-k-s)L(f, 1-s) \ .$$

Now, since k is a positive integer, it is easy to check that

$$\Gamma(2-k-s)\Gamma(s+k-1) = (-1)^{k-1}\Gamma(s)\Gamma(1-s)$$

and if we substitute this in the expression above, we see that

$$\Lambda_{\lambda^2}(f^+, 2-k-s) = (-i)^{1-k}\Lambda_{\lambda^2}(f, 1-s)\frac{\Gamma(s)}{\Gamma(s+k-1)} \ .$$

In the same way as above, but without changing s, we also have

$$\Lambda_{\lambda^2}(g^+, s) = i^{1-k}\Lambda_{\lambda^2}(g, s+k-1)\frac{\Gamma(s)}{\Gamma(s+k-1)} \ ,$$

and since $\Lambda_{\lambda^2}(f, 1-s) = C\Lambda_{\lambda^2}(g, k-1+s)$ by Corollary 11.2.3, it follows that

$$\Lambda_{\lambda^2}(f^+, 2-k-s) = (-1)^{k-1}C\Lambda_{\lambda^2}(g^+, s)\ .$$

To apply Theorem 11.2.2 we must therefore find a rational function $R(s)$ such that $\Lambda_{\lambda^2}(f^+, s) - R(s)$ is EBV. Now, as we have seen above we have

$$\Lambda_{\lambda^2}(f^+, s) = i^{1-k}\Lambda_{\lambda^2}(f, s+k-1)\frac{\Gamma(s)}{\Gamma(s+k-1)} = \frac{i^{1-k}\Lambda_{\lambda^2}(f, s+k-1)}{s(s+1)\cdots(s+k-2)}$$

and by Corollary 11.2.3 once again, we know that $\Lambda_{\lambda^2}(f, s+k-1)$ has two simple poles, at $s = -(k-1)$ and $s = 1$ with residues $-a(0)$ and $Cb(0)$, respectively. It follows that $\Lambda_{\lambda^2}(f^+, s)$ has only simple poles at $s = -(k-1)$, $s = 1$, and $s = -j$ for $0 \le j \le k-2$ with residues

$$\frac{(-i)^{1-k}(-a(0))}{(k-1)!},\quad \frac{i^{1-k}Cb(0)}{(k-1)!},\quad \text{and}\quad \left(\frac{(-1)^j i^{1-k}\Lambda_{\lambda^2}(f, k-1-j)}{j!(k-2-j)!}\right)_{0\le j\le k-2},$$

respectively, and these are the only poles. Therefore, if we set

$$R(s) = i^{1-k}\left(\frac{1}{(k-1)!}\left(\frac{(-1)^k a(0)}{s+k-1} + \frac{Cb(0)}{s-1}\right)\right.$$
$$\left. + \frac{1}{(k-2)!}\sum_{0\le j\le k-2}\frac{(-1)^j\binom{k-2}{j}\Lambda_{\lambda^2}(f, k-1-j)}{s+j}\right),$$

then the function $\Lambda_{\lambda^2}(f^+, s) - R(s)$ is entire and in fact EBV since

$$\Lambda_{\lambda^2}(f^+, s) = \frac{i^{1-k}\Lambda_{\lambda^2}(f, s+k-1)}{s(s+1)\cdots(s+k-2)}$$

and $\Lambda_{\lambda^2}(f, s)$ is EBV and the functions $1/(s+j)$ tend to 0 when $|t| = |\Im(s)| \to \infty$. We can therefore apply Theorem 11.2.2 to f^+ and g^+ with the function $R(s)$ above, C replaced by $(-1)^{k-1}C$ and k by $2-k$. Since by definition $\Lambda_{\lambda^2}(f, k-1-j) = (2\pi/\lambda)^{j-k+1}(k-2-j)!L(f, k-1-j)$, we have

$$R(s) = i^{1-k}\left(\frac{1}{(k-1)!}\left(\frac{(-1)^k a(0)}{s+k-1} + \frac{Cb(0)}{s-1}\right)\right.$$
$$\left. + \left(\frac{2\pi}{\lambda}\right)^{1-k}\sum_{0\le j\le k-2}\frac{(-2\pi/\lambda)^j}{j!}\frac{L(f, k-1-j)}{s+j}\right),$$

and (b) of Theorem 11.2.2 gives that

$$f^+\left(-\frac{1}{\tau}\right)=(-1)^{k-1}C\left(\frac{\tau}{i}\right)^{2-k}g^+(\tau)$$
$$+\frac{i^{1-k}}{(k-1)!}\left((-1)^k a(0)\left(\frac{\tau}{i}\right)^{1-k}+Cb(0)\frac{\tau}{i}\right)$$
$$+\left(\frac{2\pi i}{\lambda}\right)^{1-k}\sum_{0\le j\le k-2}\frac{(-2\pi/\lambda)^j}{j!}L(f,k-1-j)\left(\frac{\tau}{i}\right)^{-j}\ .$$

Replacing f^+ and g^+ by their respective definitions we obtain immediately

$$f^*\left(\frac{-1}{\tau}\right)=(-1)^{k-1}C\left(\frac{\tau}{i}\right)^{2-k}g^*(\tau)$$
$$+\left(\frac{2\pi i}{\lambda}\right)^{1-k}\sum_{0\le j\le k-2}\frac{(-2\pi i/(\lambda\tau))^j}{j!}L(f,k-1-j)\ ,$$

which proves the proposition. □

Corollary 11.5.2. *If $f\in M_k(\Gamma)$, we have*

$$f^*\left(\frac{-1}{\tau}\right)=\tau^{2-k}f^*(\tau)+(2\pi i)^{1-k}\sum_{0\le j\le k-2}\frac{(-2\pi i/\tau)^j}{j!}L(f,k-1-j)\ .$$

Proof. Simply apply the proposition to $\lambda=1$, $f=g$, and $C=(-1)^{k/2}$. □

It follows from this corollary that the $(k-1)$st antiderivative of a modular form of weight k behaves as a modular form of weight $2-k$, up to a correction factor which is a polynomial of degree $k-2$ in $-1/\tau$. That is, it is a quasi-modular form of weight $2-k$ and depth $k-2$ (see Definition 5.1.7). Our main interest here in these functions is that, up to trivial multiplicative factors, the coefficient of the polynomial in $-1/\tau$ are exactly the "critical values" of $L(f,s)$, in other words, the values $L(f,j)$ for $1\le j\le k-1$, j integral. This will allow us to deduce important relations and arithmetic properties of these values. For now, we reformulate the above result as:

Definition 11.5.3. If $f\in M_k(\Gamma)$, then we define $P_f(X)$, the *period polynomial* associated to f, by

$$P_f(X)=\sum_{0\le j\le k-2}((2\pi i)^j/j!)L(f,k-1-j)X^j.$$

Corollary 11.5.4. *If $f \in M_k(\Gamma)$, then*

$$f^*(-1/\tau) = \tau^{2-k} f^*(\tau) + (2\pi i)^{1-k} P_f(-1/\tau)$$

and the period polynomial P_f satisfies the relations

$$P_f\big|_{2-k} S + P_f = 0 \quad \text{and} \quad P_f\big|_{2-k}(TS)^2 + P_f\big|_{2-k} TS + P_f = 0\ ;$$

in other words,

$$X^{k-2} P_f(-1/X) + P_f(X) = 0 \quad \text{and}$$
$$(X-1)^{k-2} P_f(1/(1-X)) + X^{k-2} P_f((X-1)/X) + P_f = 0\ .$$

Furthermore, if we set

$$\begin{aligned} f^m(\tau) &= f^*(\tau) - (2\pi i)^{1-k} \sideset{}{'}\sum_{0 \le j \le k/2-1} \frac{(2\pi i)^j}{j!} L(f, k-1-j)\tau^j \\ &= f^*(\tau) + (2\pi i)^{1-k} \sideset{}{'}\sum_{0 \le j \le k/2-1} (-1)^j \frac{(2\pi i)^{k-j-2}}{(k-2-j)!} L(f, j+1)\tau^j\ , \end{aligned}$$

where $\sum'$ means that the term with $j = k/2 - 1$ is multiplied by $1/2$, then

$$f^m(-1/\tau) = \tau^{2-k} f^m(\tau).$$

Proof. The expression for $f^*(-1/\tau)$ is simply a reformulation of the preceding corollary, and the first relation satisfied by the period polynomial follows immediately from the functional equation, more precisely from Proposition 11.2.5(c). We delay the proof of the second relation until Lemma 11.8.9.

For the final part, let $f^m(\tau) = f^*(\tau) + (2\pi i)^{1-k} Q(\tau)$, where the polynomial $Q(X) = \sum_{0 \le j \le k/2-1} a_j X^j$ is to be determined so that the functional equation $f^m(-1/\tau) = \tau^{2-k} f^m(\tau)$ holds. By the first part we have

$$\begin{aligned} f^m(-1/\tau) &= f^*(-1/\tau) + (2\pi i)^{1-k} Q(-1/\tau) \\ &= \tau^{2-k}(f^m(\tau) - (2\pi i)^{1-k} Q(\tau)) \\ &\quad + (2\pi i)^{1-k} P_f(-1/\tau) + (2\pi i)^{1-k} Q(-1/\tau)\ ; \end{aligned}$$

hence we need Q to satisfy $-\tau^{2-k} Q(\tau) + Q(-1/\tau) + P_f(-1/\tau) = 0$, in other words, $X^{k-2} Q(-1/X) - Q(X) = P_f(X)$, since $P_f(-1/X) = -X^{2-k} P_f(X)$, which is in fact clearly also a necessary condition for the existence of Q.

If we identify the coefficients in the relation, we obtain the conditions

$$\begin{aligned} &a_j = -((2\pi i)^j/j!) L(f, k-1-j) \quad \text{for } 0 \le j \le k/2-2, \\ &a_j = (-1)^j((2\pi i)^{k-2-j}/(k-2-j)!) L(f, j+1) \quad \text{for } 0 \le j \le k/2-2, \text{ and} \\ &((-1)^{k/2} + 1) a_{k/2-1} = -((2\pi i)^{k/2-1}/(k/2-1)!) L(f, k/2)\ . \end{aligned}$$

Thanks to Corollary 11.2.6 the first two formulas for a_j are equivalent, and the third can always be satisfied since both sides vanish if $k \equiv 2 \pmod 4$. □

Note that we have chosen the polynomial in τ of lowest degree in the definition of the modified antiderivative f^m, but this choice is of course not canonical: since k is even, the vector space of polynomials of degree less than or equal to $k-2$ which satisfy $Q(-1/X) = X^{2-k}Q(X)$ is simply the vector space whose basis is given by $X^{k-2-j} + (-1)^j X^j$ for $0 \le j \le k/2 - 2$ (or $0 \le j \le k/2 - 1$ if $k \equiv 0 \pmod 4$). We can therefore add any linear combination of such polynomials in τ to f^m and keep the same conclusion.

11.6. The Case of Eisenstein Series

We consider the special case where f is an Eisenstein series. The result is then as follows:

Proposition 11.6.1.

(a) *For any even integer $k \ge 4$ set*

$$F_k^*(\tau) = -\frac{B_k}{2 \cdot k!}\tau^{k-1} + (2\pi i)^{1-k} \sum_{n \ge 1} \sigma_{1-k}(n) q^n \ .$$

We then have the functional equation

$$F_k^*\left(\frac{-1}{\tau}\right) = \tau^{2-k} F_k^*(\tau) + \frac{(2\pi i)^{1-k}}{2}\zeta(k-1)(\tau^{2-k} - 1)$$
$$- \frac{1}{2 \cdot k!} \sum_{\substack{1 \le j \le k-3 \\ j \text{ odd}}} \binom{k}{j+1} B_{k-1-j} B_{j+1} \tau^{-j} \ .$$

(b) *For any even integer $k \ge 4$ set*

$$G_k^{(m)}(\tau) = \frac{\zeta(k)}{2\pi i}\tau^{k-1} - \frac{(2\pi i)^{k-1}}{2 \cdot k!} \sideset{}{'}\sum_{\substack{1 \le j \le k/2-1 \\ j \text{ odd}}} \binom{k}{j+1} B_{k-1-j} B_{j+1} \tau^j$$
$$+ \frac{\zeta(k-1)}{2} + \sum_{n \ge 1} \sigma_{1-k}(n) q^n \ .$$

Then $G_k^{(m)}(-1/\tau) = \tau^{2-k} G_k^{(m)}(\tau)$.

Proof. We apply Corollary 11.5.4 to the function

$$f = F_k = -\frac{B_k}{2k} E_k = -\frac{B_k}{2k} + \sum_{n \ge 1} \sigma_{k-1}(n) q^n \ .$$

We have already seen the easy formula $L(F_k, s) = \zeta(s)\zeta(s-k+1)$; hence for $0 \le j \le k-3$ we have $L(F_k, k-1-j) = \zeta(k-1-j)\zeta(-j)$. Thus, using the known values of $\zeta(s)$, we deduce that $L(F_k, k-1-j) = 0$ if j is even and $2 \le j \le k-4$, $L(F_k, k-1-j) = -\zeta(k-1)/2$ if $j = 0$, and if j is odd and $1 \le j \le k-3$, then

$$L(F_k, k-1-j) = \frac{(-1)^{(k-1-j)/2}}{2(j+1)} \frac{(2\pi)^{k-1-j} B_{k-1-j} B_{j+1}}{(k-1-j)!}.$$

For $j = k-2$ we must compute $L(F_k, 1)$. Since $\zeta(s) \sim 1/(s-1)$ and $\zeta(s-k+1) \sim (s-1)\zeta'(2-k)$ when $s \to 1$, it follows that $L(F_k, 1) = \zeta'(2-k)$. By the functional equation of $\zeta(s)$ it now follows that

$$L(F_k, 1) = \frac{(-1)^{k/2-1}}{2}(2\pi)^{2-k}(k-2)!\zeta(k-1) .$$

Thus, applying Corollary 11.5.4 we obtain

$$F_k^*\left(\frac{-1}{\tau}\right) = \tau^{2-k} F_k^*(\tau) + (2\pi i)^{1-k}\Bigg(\frac{\zeta(k-1)}{2}(\tau^{2-k} - 1) + \frac{i}{2} \sum_{\substack{1 \le j \le k-3 \\ j \text{ odd}}} \frac{(-1)^{k/2}(2\pi)^{k-1} B_{k-1-j} B_{j+1}}{(j+1)!(k-1-j)!} \tau^{-j}\Bigg) ,$$

and (a) follows since clearly $\sigma_{1-k}(n) = n^{1-k}\sigma_{k-1}(n)$. (b) follows from (a) together with Corollary 11.5.4. $\square$

Example 11.6.2. In the case $k = 4$ the previous proposition implies that

$$G_4^{(m)}(\tau) = -\frac{\pi^3}{180}\left(\frac{\tau}{i}\right)^3 - \frac{\pi^3}{72}\left(\frac{\tau}{i}\right) + \frac{\zeta(3)}{2} + \sum_{n \ge 1} \sigma_{-3}(n) q^n$$

satisfies the functional equation

$$G_4^{(m)}(-1/\tau) = \tau^{-2} G_4^{(m)}(\tau) .$$

Note that in Proposition 11.6.1, in addition to known rational quantities, there is also the value $\zeta(k-1)$ of the Riemann zeta function at an *odd* positive integer. We can then easily deduce formulas for these numbers by taking special values for τ such as $\tau = i$ or $\tau = \rho$, or after derivation of the formulas. We have, for instance:

Corollary 11.6.3.

(a) *For* $k \equiv 0 \pmod 4$ *we have*

$$\zeta(k-1) = -\frac{2^{k-2}\pi^{k-1}}{k!} \sum_{0 \le n \le k/2} (-1)^n \binom{k}{2n} B_{k-2n} B_{2n} - 2S_k ,$$

with

$$S_k = \sum_{n \ge 1} \sigma_{1-k}(n) e^{-2\pi n} = \sum_{n \ge 1} \frac{1}{n^{k-1}(e^{2\pi n} - 1)} .$$

(b) *For* $k \equiv 2 \pmod 4$ *and* $k \geq 6$ *we have*

$$\zeta(k-1) = \frac{(2\pi)^{k-1}}{(k-2)k!} \sum_{0 \leq n \leq (k-2)/4} (-1)^n (k-4n) \binom{k}{2n} B_{k-2n} B_{2n} - 2 \sum_{n \geq 1} \frac{e^{2\pi n}(1 + 4\pi n/(k-2)) - 1}{n^{k-1}(e^{2\pi n} - 1)^2} \ .$$

Proof. The first formula follows by setting $\tau = i$ in Proposition 11.6.1, and the second by first taking the derivative of the formula of Proposition 11.6.1 and then setting $\tau = i$. The details are left to the reader. □

For example, with $k = 4$ the corollary gives

$$\zeta(3) = \frac{7\pi^3}{180} - 2 \sum_{n \geq 1} \frac{1}{n^3(e^{2\pi n} - 1)} \ .$$

Note the rapid rate of convergence of this series: for instance, the first two terms already give 9 decimals of $\zeta(3)$; see Exercise 11.3 for other examples.

11.7. Transformation under an Arbitrary $\gamma \in \Gamma$

We now want to compute $f^*(\gamma\tau)$ for an arbitrary element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and not only for $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Since the action of integer translations can be trivially computed on the Fourier expansion (in fact $f^*(\tau + b) = f^*(\tau)$ for any $b \in \mathbb{Z}$ when $f \in S_k(\Gamma)$), we may assume that $c \neq 0$, and since the action of γ is the same as that of $-\gamma$, we may assume that $c > 0$. We keep this assumption implicitly in the rest of this section.

Definition 11.7.1. For $f \in M_k(\Gamma)$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $c > 0$ we set

$$f_\gamma(\tau) = f\left(\frac{\tau}{c} - \frac{d}{c}\right) = \sum_{n \geq 0} e^{-2\pi i n d/c} a(n) q^{n/c} \quad \text{and}$$

$$L(f_\gamma, \tau) = \sum_{n \geq 1} e^{-2\pi i n d/c} \frac{a(n)}{n^s} \ .$$

Lemma 11.7.2. *If* γ *is as above, then the following conditions are equivalent:*

(a) $$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \ ,$$

(b) $$f_\gamma\left(\frac{-1}{\tau}\right) = \tau^k f_{\gamma^{-1}}(\tau) \ .$$

Proof. Since $ad - bc = 1$, we check immediately that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -d \\ 0 & c \end{pmatrix}^{-1}$$

and it is clear that

$$f_\gamma = c^{k/2} f\Big|_k \begin{pmatrix} 1 & -d \\ 0 & c \end{pmatrix} \quad \text{and} \quad f_{\gamma^{-1}} = c^{k/2} f\Big|_k \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix} .$$

It follows that $f\big|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = f\big|_k \left(\begin{smallmatrix} 1 & a \\ 0 & c \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & -d \\ 0 & c \end{smallmatrix}\right)^{-1}$ and hence the equality in (a), which can be written $f\big|_k \gamma = f$, is equivalent to $f\big|_k \left(\begin{smallmatrix} 1 & a \\ 0 & c \end{smallmatrix}\right) = f\big|_k \left(\begin{smallmatrix} 1 & -d \\ 0 & c \end{smallmatrix}\right) S$, which is clearly another formulation of the equality in (b). ☐

If we assume that $f \in S_k(\Gamma)$, then $a(n) = O(n^{k/2})$ and in addition condition (a) is satisfied. Since $a(0) = 0$, we deduce from Corollary 11.2.3 that $L(f_\gamma, s)$ is an entire function. We can now state the desired generalization of Corollary 11.5.2.

Proposition 11.7.3. *If $f \in S_k(\Gamma)$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $c > 0$, then*

$$\begin{aligned} f^*(\tau) &= (c\tau + d)^{k-2} f^*\left(\frac{a\tau + b}{c\tau + d}\right) \\ &\quad + (2\pi i)^{1-k} \sum_{0 \le j \le k-2} \frac{(2\pi i)^j}{j!} L(f_\gamma, k-1-j) \left(\tau + \frac{d}{c}\right)^j , \end{aligned}$$

or equivalently

$$\begin{aligned} f^*\left(\frac{a\tau + b}{c\tau + d}\right) &= (c\tau + d)^{2-k} f^*(\tau) \\ &\quad + (2\pi i)^{1-k} \sum_{0 \le j \le k-2} \frac{(-2\pi i/(c(c\tau + d)))^j}{j!} L(f_{\gamma^{-1}}, k-1-j) . \end{aligned}$$

Proof. Thanks to Proposition 11.5.1 with $\lambda = c$ we have

$$\begin{aligned} f_\gamma^*(-1/\tau) &= \tau^{2-k} f_{\gamma^{-1}}^*(\tau) \\ &\quad + (2\pi i/c)^{1-k} \sum_{0 \le j \le k-2} \frac{(-2\pi i/(c\tau))^j}{j!} L(f_\gamma, k-1-j) ; \end{aligned}$$

in other words, by replacing τ with $-1/\tau$ we have

$$f_\gamma^*(\tau) = \tau^{k-2} f_{\gamma^{-1}}^*(-1/\tau) + (2\pi i/c)^{1-k} \sum_{0 \le j \le k-2} \frac{(2\pi i\tau/c)^j}{j!} L(f_\gamma, k-1-j) .$$

Since $(f_\gamma)^*(\tau) = c^{k-1} f^*((\tau - d)/c)$ and $\gamma^{-1} = \left(\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right)$ in $\overline{\Gamma}$, we see that

$$f_{\gamma^{-1}}^*\left(-\frac{1}{\tau}\right) = c^{k-1} f^*\left(\frac{a\tau - 1}{c\tau}\right) = c^{k-1} f^*\left(\frac{a((\tau - d)/c) + b}{c((\tau - d)/c) + d}\right) .$$

By inserting this into the above and changing τ to $c\tau+d$ we obtain

$$f^*(\tau) = (c\tau+d)^{k-2}f^*\left(\frac{a\tau+b}{c\tau+d}\right) + (2\pi i)^{1-k}\sum_{0\le j\le k-2}\frac{(2\pi i)^j}{j!}L(f_\gamma,k-1-j)\left(\tau+\frac{d}{c}\right)^j\ ,$$

proving the first formula. The second follows immediately by changing τ to $(-d\tau+b)/(c\tau-a)$ and then (d,a) to $(-a,-d)$. $\square$

11.8. Eichler Cohomology

We will now define the so-called Eichler cohomology for cusp forms on the full modular group. This cohomology can of course be defined in a more general setting but since our main purpose here is to introduce the key ideas, we prefer to keep the exposition as simple as possible.

To extend the theory in this section to subgroups of the modular group we can proceed in a very similar way except that we need to work with vectors of polynomials; see e.g. [**PP14**]. This is of course in analogy with the fact that modular forms on subgroups can be considered as vector-valued modular forms on Γ (see Section 14.4).

Let $k \geq 4$ be a fixed even integer. For any field K containing $\mathbb{Q}$ (in other words, of characteristic 0) we let $V_k(K)$ denote the K-vector space of polynomials of degree less than or equal to $k-2$ with coefficients in K. Since $\mathbb{Q} \subset K$, there is a natural right group action of Γ on $V_k(K)$ defined for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $P \in V_k(K)$ by

$$(P|\gamma)(\tau) = (c\tau+d)^{k-2}P\left(\frac{a\tau+b}{c\tau+d}\right)\ ,$$

and it is clear that the right-hand side is still an element of $V_k(K)$. The canonical example of an element of $V_k(\mathbb{C})$ is of course the period polynomial P_f of a modular form of weight k.

We now need to define $H^1(\Gamma, V_k(K))$, the first cohomology group of Γ with values in $V_k(K)$.

Definition 11.8.1.

(a) A *cocycle* (or more precisely a 1-*cocycle*) is a map f from Γ to $V_k(K)$ such that for all γ_1 and γ_2 in Γ we have

$$f(\gamma_1\gamma_2) = f(\gamma_1)|\gamma_2 + f(\gamma_2)\ .$$

The set of 1-cocycles is a K-vector space denoted by $Z^1(\Gamma, V_k(K))$.

(b) A *coboundary* (or more precisely a 1-*coboundary*) is a map from Γ to $V_k(K)$ of the form

$$\gamma \mapsto P\big|\gamma - P ,$$

for some fixed $P \in V_k(K)$. The set of 1-coboundaries is a K-vector space denoted by $B^1(\Gamma, V_k(K))$.

(c) We have $B^1(\Gamma, V_k(K)) \subset Z^1(\Gamma, V_k(K))$ and the quotient space

$$H^1(\Gamma, V_k(K)) = Z^1(\Gamma, V_k(K))/B^1(\Gamma, V_k(K))$$

is called the (first) cohomology group of Γ with values in $V_k(K)$.

The inclusion $B^1(\Gamma, V_k(K)) \subset Z^1(\Gamma, V_k(K))$ follows from the identity

$$P\big|\gamma_1\gamma_2 - P = (P\big|\gamma_1 - P)\big|\gamma_2 + (P\big|\gamma_2 - P) .$$

The link between modular forms and the cohomology we just defined is described by the following proposition.

Recall from Section 11.5 that we denote by f^* a specific $(k-1)$st antiderivative of f.

Proposition 11.8.2. *Let $f \in S_k(\Gamma)$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.*

(a) *There exists a polynomial $P_\gamma \in V_k(\mathbb{C})$ such that*

$$f^*(\tau) = (c\tau + d)^{k-2} f^*\left(\frac{a\tau + b}{c\tau + d}\right) - P_\gamma(\tau) ;$$

in other words, $f^*\big|\gamma - f^* = P_\gamma$.

(b) *The maps*

$$\begin{aligned} \phi_{\mathbb{C}}(f) : &\quad \gamma \mapsto P_\gamma \quad \textit{and} \\ \phi_{\mathbb{R}}(f) : &\quad \gamma \mapsto \Re(P_\gamma) \end{aligned}$$

(*the real part is taken coefficientwise*) *are 1-cocycles, and their cohomology class in $H^1(\Gamma, V_k(\mathbb{C}))$ and $H^1(\Gamma, V_k(\mathbb{R}))$, respectively, is independent of the choice of the $(k-1)$st antiderivative f^* of f.*

Proof. (a) By Corollary 11.5.2 the assertion is true for $\gamma = S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Furthermore, it is trivially true for $\gamma = T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ since f is a cusp form and

$$P_T(\tau) = a(0)((\tau + 1)^{k-1} - \tau^{k-1})/(k-1)! = 0.$$

However, if $f^*\big|\gamma_1 - f^* = P_{\gamma_1}$ and $f^*\big|\gamma_2 - f^* = P_{\gamma_2}$, it is clear that

$$f^*\big|\gamma_1\gamma_2 - f^* = (f^*\big|\gamma_1)\big|\gamma_2 - f^* = (f^* + P_{\gamma_1})\big|\gamma_2 - f^* = P_{\gamma_1}\big|\gamma_2 + P_{\gamma_2} .$$

It follows that if P_{γ_1} and P_{γ_2} are in $V_k(\mathbb{C})$, then $P_{\gamma_1\gamma_2} = P_{\gamma_1}\big|\gamma_2 + P_{\gamma_2}$ is also in $V_k(\mathbb{C})$ and (a) follows from the fact that Γ is generated by S and T.

(b) We have just shown that $\phi_{\mathbb{C}}$ is a 1-cocycle. If we replace f^* by some other $(k-1)$st antiderivative of f, it is clear that P_γ is additively modified by a polynomial of the type $Q\big|\gamma - Q$ for some fixed $Q \in V_k(\mathbb{C})$, which by definition is a coboundary, so its class in $H^1(\Gamma, V_k(\mathbb{C}))$ is unchanged.

Since the elements of Γ have real entries, it is clear that $\Re\left(P_{\gamma_1}\big|\gamma_2\right) = (\Re(P_{\gamma_1}))\big|\gamma_2$, so that if $(\gamma \mapsto P_\gamma) \in Z^1(\Gamma, V_k(\mathbb{C}))$, then $(\gamma \mapsto \Re(P_\gamma)) \in Z^1(\Gamma, V_k(\mathbb{R}))$ and similarly for coboundaries, proving (b). □

Definition 11.8.3. Let $(\gamma \mapsto P_\gamma)$ be an element of $Z^1(\Gamma, V_k(K))$. We say that it is a *cuspidal* (or *parabolic*) cocycle if $\deg(P_T) < k-2$, where as usual $T = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. The K-vector space of cuspidal cocycles is denoted $\overline{Z}^1(\Gamma, V_k(K))$.

If $Q \in V_k(K)$, then $(Q\big|T - Q)(\tau) = Q(\tau+1) - Q(\tau)$, which has degree less than or equal to $k-3$; hence a coboundary is always a cuspidal cocycle. In other words, $B^1(\Gamma, V_k(K)) \subseteq \overline{Z}^1(\Gamma, V_k(K))$ and we have the following.

Definition 11.8.4. The first *Eichler cohomology group* of Γ with values in $V_k(K)$ is the K-vector space

$$\overline{H}^1(\Gamma, V_k(K)) = \overline{Z}^1(\Gamma, V_k(K))/B^1(\Gamma, V_k(K))\ .$$

It is clear that $\overline{H}^1(\Gamma, V_k(K)) \subset H^1(\Gamma, V_k(K))$ and the fundamental result of this section is the following, due to Eichler–Shimura:

Theorem 11.8.5 (Eichler–Shimura)**.** *The map $\overline{\phi_{\mathbb{R}}}$, from the space of cusp forms $S_k(\Gamma)$ to the Eichler cohomology group $\overline{H}^1(\Gamma, V_k(\mathbb{R}))$, given by*

$$\overline{\phi_{\mathbb{R}}}(f) = (\gamma \mapsto \Re(f^*\big|\gamma - f^*)) \pmod{B^1(\Gamma, V_k(\mathbb{R}))},$$

is an isomorphism of $\mathbb{R}$-vector spaces.

Proof. (a) Proof of injectivity.

The kernel of $\overline{\phi_{\mathbb{R}}}$ is the set of $f \in S_k(\Gamma)$ such that $(\gamma \mapsto \Re(f^*\big|\gamma - f^*))$ is a 1-coboundary, in other words, such that there exists $P \in V_k(\mathbb{R})$ satisfying

$$\Re(f^*\big|\gamma - f^*) = P\big|\gamma - P \quad \text{for all } \gamma \in \Gamma.$$

Using this for $\gamma = T = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ we deduce that $P(\tau+1) = P(\tau)$ and since P is a polynomial, it follows that P is constant; $P = M$, say. Hence

$$(f^*\big|\gamma - f^*)(\tau) = M((c\tau+d)^{k-2} - 1) + iQ_\gamma(\tau)\ ,$$

where $M \in \mathbb{R}$ and $Q_\gamma \in V_k(\mathbb{R})$. Now by Proposition 11.7.3, for $c > 0$ we have

$$(f^*\big|\gamma - f^*)(\tau) = (2\pi i)^{1-k} \sum_{0\le j\le k-2} \frac{(-2\pi i/c)^j}{j!} L(f_\gamma, k-1-j)(c\tau+d)^{k-2-j}\ .$$

Note that the polynomials $(cX+d)^{k-2-j}$ for $0 \le j \le k-2$ form a basis of $V_k(\mathbb{R})$ and of $V_k(\mathbb{C})$. Since k is even and Q has real coefficients, it follows, by identifying the coefficients of $(c\tau+d)^{k-2-j}$, that the polynomial

$$\sum_{1\le j\le k-3}\frac{(-2\pi i/c)^j}{j!}L(f_\gamma,k-1-j)X^{k-2-j}$$

has real coefficients. By considering the term $j=1$ we see that $L(f_\gamma,k-2)$ is purely imaginary. However, since f is a cusp form, if f is nonzero, then $k\ge 12$, and in particular $k-2>k/2+1$ so that the series $L(f_\gamma,k-2)$ is absolutely convergent. We deduce that if $\gamma\in\Gamma$ with $c>0$, then

$$0=L(f_\gamma,k-2)+\overline{L(f_\gamma,k-2)}=\sum_{n\ge1}e^{-2\pi ind/c}\frac{a(n)}{n^{k-2}}+\sum_{n\ge1}e^{2\pi ind/c}\frac{\overline{a(n)}}{n^{k-2}}\;.$$

Since $\sum_{n\ge1}a(n)/n^{k-2}$ is absolutely convergent, it is clear that the function

$$g(x)=\sum_{n\ge1}e^{2\pi inx}\frac{a(n)}{n^{k-2}}+\sum_{n\ge1}e^{-2\pi inx}\frac{\overline{a(n)}}{n^{k-2}}$$

is a *continuous* function of the real variable x, which vanishes for all rational x by the above, hence by continuity which vanishes for all real x. The series defining $g(x)$ is thus an absolutely convergent Fourier series which vanishes on the real line. It follows that its Fourier coefficients also vanish, in other words, that $a(n)=0$ for all $n\ge1$, and since f is a cusp form, it is clear that $f=0$, which proves the injectivity of $\overline{\phi_{\mathbb{R}}}(f)$.

(b) Proof of surjectivity.

Now that we know that $\overline{\phi_{\mathbb{R}}}(f)$ is injective, to show that it is surjective, hence an isomorphism of $\mathbb{R}$-vector spaces, it is sufficient to show that the $\mathbb{R}$-dimensions of $S_k(\Gamma)$ and of $\overline{H}^1(\Gamma,V_k(\mathbb{R}))$ are equal. Since $\dim_{\mathbb{R}}S_k(\Gamma)=2\dim_{\mathbb{C}}S_k(\Gamma)$, we already know that

$$\dim_{\mathbb{R}}S_k(\Gamma)=\begin{cases}2\left\lfloor\dfrac{k}{12}\right\rfloor & \text{when } k\ge12,\ k\not\equiv2\pmod{12}\;,\\ 2\left\lfloor\dfrac{k}{12}\right\rfloor-2 & \text{when } k\ge12,\ k\equiv2\pmod{12}\;,\\ 0 & \text{when } k<12\;.\end{cases}$$

We must therefore compute $\dim_{\mathbb{R}}\overline{H^1}(\Gamma,V_k(\mathbb{R}))$. This computation is unfortunately rather intricate and we depend on a sequence of lemmas which will be presented in detail after the proof. By using Lemmas 11.8.10, 11.8.11, 11.8.12, and 11.8.13 and Corollary 11.8.15 we deduce that

$$\dim_{\mathbb{R}}\overline{H^1}(\Gamma,V_k(\mathbb{R}))=d_2-d_3-1\;,$$

where

$$d_2=\begin{cases}\lfloor k/2\rfloor & \text{when } k\equiv0\pmod4\;,\\ \lfloor k/2\rfloor-1 & \text{when } k\equiv2\pmod4\end{cases}$$

and

$$d_3 = \begin{cases} \lfloor (k-2)/3 \rfloor & \text{when } k \equiv 0 \pmod{3}\;, \\ \lfloor (k-2)/3 \rfloor + 1 & \text{when } k \not\equiv 0 \pmod{3}\;. \end{cases}$$

When k increases by 12, this dimension thus increases by 2, and for $k = 4$, 6, 8, 10, 12, and 14 we obtain as respective dimensions 0, 0, 0, 0, 2, and 0. It follows that

$$\dim_{\mathbb{R}} \overline{H^1}(\Gamma, V_k(\mathbb{R})) = 2\dim_{\mathbb{C}} S_k(\Gamma) = \dim_{\mathbb{R}} S_k(\Gamma)\;,$$

finishing the proof of the theorem. □

We will now turn to the lemmas which we used in the proof. First we need the following subspaces of cocycles:

$$Z = \{(\gamma \mapsto P_\gamma) \in Z^1(\Gamma, V_k(\mathbb{R})),\ P_T = 0\} \quad \text{and} \quad B = Z \cap B^1(\gamma, V_k(\mathbb{R}))\;.$$

By definition of the cuspidal cocycles, it is clear that Z is a subspace of $\overline{Z^1}(\Gamma, V_k(\mathbb{R}))$ and we have in fact the following:

Lemma 11.8.6. *The canonical projection from the space* $\overline{Z^1}(\Gamma, V_k(\mathbb{R}))$ *to* $\overline{H^1}(\Gamma, V_k(\mathbb{R}))$ *induces an isomorphism:*

$$Z/B \simeq \overline{H^1}(\Gamma, V_k(\mathbb{R}))\;.$$

Proof. It is clear that

$$\begin{aligned}\frac{Z}{B} &= \frac{Z}{Z \cap B^1(\Gamma, V_k(\mathbb{R}))} \\ &\simeq \frac{Z + B^1(\Gamma, V_k(\mathbb{R}))}{B^1(\Gamma, V_k(\mathbb{R}))} \subset \frac{\overline{Z^1}(\Gamma, V_k(\mathbb{R}))}{B^1(\Gamma, V_k(\mathbb{R}))} = \overline{H^1}(\Gamma, V_k(\mathbb{R}))\;.\end{aligned}$$

Conversely, let $(\gamma \mapsto P_\gamma) \in \overline{Z^1}(\Gamma, V_k(\mathbb{R}))$ be given. The map $P \mapsto P(\tau+1) - P(\tau)$ is a linear map from the vector space of polynomials of degree less than or equal to $k-2$ to the vector space of polynomials of degree less than or equal to $k-3$, and its kernel is the 1-dimensional space of constants. It follows that it is *surjective*, in other words, that any polynomial Q of degree less than or equal to $k-3$ is of the form $P(\tau+1) - P(\tau)$ with $\deg(P) \le k-2$. Since $(\gamma \mapsto P_\gamma) \in \overline{Z^1}(\Gamma, V_k(\mathbb{R}))$, by definition $\deg(P_T) \le k-3$, so we deduce the existence of a polynomial $P \in V_k(\mathbb{R})$ such that $P_T(\tau) = P(\tau+1) - P(\tau)$. It is then clear that the cocycle $(\gamma \mapsto (P_\gamma - (P\big|\gamma - P)))$ is equivalent to the initial one modulo the space $B^1(\Gamma, V_k(\mathbb{R}))$ of coboundaries, and by construction it belongs to Z, proving the lemma. □

Lemma 11.8.7. *We have* $\dim_{\mathbb{R}}(B) = 1$.

Proof. If $(\gamma \mapsto (Q\big|\gamma - Q)) \in B$, then $Q(\tau+1) - Q(\tau) = 0$ and hence Q is a constant. Therefore B can be identified with the constant polynomials. □

It follows from the two preceding lemmas that if we want to compute $\dim_{\mathbb{R}}(\overline{H^1}(\Gamma, V_k(\mathbb{R})))$, then we need only compute $\dim_{\mathbb{R}}(Z)$. The crucial Lemma 11.8.9, based on the fact that Γ is generated by S and T with the relations $S^2 = (TS)^3 = I$, is given below.

Definition 11.8.8. We define the following subspaces of $V_k(K)$:

$$W_k^2(K) = \left\{P \in V_k(K) \ : \ P\big|S + P = 0\right\},$$
$$W_k^3(K) = \left\{P \in V_k(K) \ : \ P\big|(TS)^2 + P\big|TS + P = 0\right\},$$

and $W_k(K) = W_k^2(K) \cap W_k^3(K)$; that is,

$$W_k(K) = \left\{P \in V_k(K) \ : \ P\big|(TS)^2 + P\big|TS + P = P\big|S + P = 0\right\}.$$

Lemma 11.8.9. *The map sending $(\gamma \mapsto P_\gamma) \in Z$ to $P_S \in V_k(\mathbb{R})$ is an $\mathbb{R}$-vector space isomorphism from Z to $W_k(\mathbb{R})$.*

Proof. Let $(\gamma \mapsto P_\gamma) \in Z$. Since $S^2 = I$, the cocycle condition gives

$$0 = P_{S^2} = P_S\big|S + P_S\ .$$

Similarly, since $P_T = 0$ and $(TS)^3 = I$, the cocycle condition gives

$$\begin{aligned} 0 &= P_{(TS)^3} = P_{TSTSTS} \\ &= P_T\big|STSTS + P_S\big|TSTS + P_T\big|STS + P_S\big|TS + P_T\big|S + P_S \\ &= P_S\big|TSTS + P_S\big|TS + P_S\ , \end{aligned}$$

proving that $P_S \in W_k(\mathbb{R})$.

If $P_S = 0$, then the cocycle condition implies that $P_\gamma = 0$ for all $\gamma \in \Gamma$ since $P_T = 0$ and Γ is generated by S and T. It is therefore clear that the map $(\gamma \mapsto P_\gamma) \mapsto P_S$ is injective.

Finally, let us show that the map $(\gamma \mapsto P_\gamma) \mapsto P_S$ is surjective. If $P \in W_k(\mathbb{R})$, we consider $(\gamma \mapsto P_\gamma)$ where for $\gamma = T^{n_0}ST^{n_1}\cdots ST^{n_k}$ we set

$$P_\gamma = P_{T^{n_0}ST^{n_1}\cdots ST^{n_k}} = P\big|T^{n_1}S\cdots ST^{n_k} + P\big|T^{n_2}S\cdots ST^{n_k} + \cdots + P\big|T^{n_k}\ ,$$

and the surjectivity follows from the fact that $S^2 = (TS)^3 = I$ generates all relations for Γ, or equivalently, that Γ is the free product of the subgroup of order 2 generated by S with the subgroup of order 3 generated by TS. ☐

The computation of $\dim_{\mathbb{R}}(Z)$ will now be a simple exercise in linear algebra, but we nonetheless provide the details.

Lemma 11.8.10. *With the notation introduced above we have*

$$\begin{aligned} \dim_{\mathbb{R}} \overline{H^1}(\Gamma, V_k(\mathbb{R})) = &\dim_{\mathbb{R}}(W_k^2(\mathbb{R})) + \dim_{\mathbb{R}}(W_k^3(\mathbb{R})) \\ &- \dim_{\mathbb{R}}(W_k^2(\mathbb{R}) + W_k^3(\mathbb{R})) - 1. \end{aligned}$$

Proof. By the preceding lemma we have

$$\begin{aligned}\dim_{\mathbb{R}}(Z) &= \dim_{\mathbb{R}}(W_k^2(\mathbb{R}) \cap W_k^3(\mathbb{R}))\\ &= \dim_{\mathbb{R}}(W_k^2(\mathbb{R})) + \dim_{\mathbb{R}}(W_k^3(\mathbb{R})) - \dim_{\mathbb{R}}(W_k^2(\mathbb{R}) + W_k^3(\mathbb{R}))\end{aligned}$$

and the stated formula now follows from Lemmas 11.8.6 and 11.8.7. □

Lemma 11.8.11. *We have* $W_k^2(\mathbb{R}) + W_k^3(\mathbb{R}) = V_k(\mathbb{R})$ *and in particular*

$$\dim_{\mathbb{R}}(W_k^2(\mathbb{R}) + W_k^3(\mathbb{R})) = k - 1\ .$$

Proof. If $P \in V_k(\mathbb{R})$, then we can write

$$P\big|S = a_0(\tau^{k-2} - 1) + P_0(\tau)\ ,$$

with $\deg(P_0) \leq k - 3$; hence by what we have already seen there exists $Q \in V_k(\mathbb{R})$ such that $P_0(\tau) = Q(\tau + 1) - Q(\tau)$ and hence

$$P\big|S = a_0(\tau^{k-2}-1)+Q(\tau+1)-Q(\tau) = a_0(\tau^{k-2}-1)+Q\big|S-Q-(Q\big|S-Q\big|T)\ .$$

By applying $\big|S$ and using $S^2 = I$ together with the evident facts that $\tau^{k-2} - 1 \in W_k^2(\mathbb{R}) \cap W_k^3(\mathbb{R})$, $Q - Q\big|S \in W_k^2(\mathbb{R})$, and $Q - Q\big|TS \in W_k^3(\mathbb{R})$, we obtain

$$P = -a_0(\tau^{k-2} - 1) + Q - Q\big|S - (Q - Q\big|TS) \in W_k^2(\mathbb{R}) + W_k^3(\mathbb{R})\ .$$ □

Lemma 11.8.12. *The dimension of* $W_k^2(\mathbb{R})$ *over* $\mathbb{R}$ *is given by*

$$\dim_{\mathbb{R}}(W_k^2(\mathbb{R})) = \begin{cases} k/2 & \text{when } k \equiv 0 \pmod 4\ ,\\ k/2 - 1 & \text{when } k \equiv 2 \pmod 4\ .\end{cases}$$

Proof. Let $P(\tau) = \sum_{0\leq j\leq k-2} a_j\tau^j \in W_k^2(\mathbb{R})$. Since k is even, the definition of $W_k^2(\mathbb{R})$ implies that $a_j + (-1)^j a_{k-2-j} = 0$. Hence, if $k \equiv 0 \pmod 4$, then the following set is a basis of $W_k^2(\mathbb{R})$:

$$(\tau^{k-2-j} - (-1)^j\tau^j)_{0\leq j\leq k/2-2} \cup \tau^{k/2-1}\ ,$$

while if $k \equiv 2 \pmod 4$, then a basis of $W_k^2(\mathbb{R})$ is given by the set

$$(\tau^{k-2-j} - (-1)^j\tau^j)_{0\leq j\leq k/2-2}\ ,$$

thus proving the lemma. □

The computation of $\dim_{\mathbb{R}}(W_k^3(\mathbb{R}))$ is slightly more complicated since it involves a three-term relation and not only a two-term relation.

Lemma 11.8.13. *We have* $\dim_{\mathbb{R}}(W_k^3(\mathbb{R})) = k - 1 - \dim_{\mathbb{C}}(E)$ *with*

$$E = \{Q \in V_k(\mathbb{C})\ :\ Q = Q\big|TS\}\ .$$

Proof. Let $\psi : V_k(\mathbb{R}) \to V_k(\mathbb{R})$ be the map given by $Q \mapsto Q - Q\big|TS$. It is clear that $\mathrm{Im}(\psi) \subset W_k^3(\mathbb{R})$, and conversely, if $P \in W_k^3(\mathbb{R})$, then $3P = \psi(2P + P\big|TS) \in \mathrm{Im}(\psi)$. It follows that $W_k^3(\mathbb{R}) = \mathrm{Im}(\psi)$ so that

$$\dim_{\mathbb{R}}(W_k^3(\mathbb{R})) = \dim_{\mathbb{R}}(V_k(\mathbb{R})) - \dim_{\mathbb{R}}(\mathrm{Ker}(\psi)) = k - 1 - \dim_{\mathbb{R}}(\mathrm{Ker}(\psi))\ .$$

Now, it is clear that $V_k(\mathbb{C})$ is canonically isomorphic to $V_k(\mathbb{R}) \times V_k(\mathbb{R})$, hence that $\dim_{\mathbb{R}}(\mathrm{Ker}(\psi))$ is equal to $\dim_{\mathbb{C}}(\mathrm{Ker}(\psi))$, where now ψ is considered as a map from $V_k(\mathbb{C})$ to $V_k(\mathbb{C})$, proving the lemma. ☐

Lemma 11.8.14. *We have the following decomposition of the space E:*

$$E = \bigoplus_{\substack{0 \le j \le k-2 \\ \rho^{k-2-j}\overline{\rho}^j = 1}} (\tau + \rho)^j (\tau + \overline{\rho})^{k-2-j} \mathbb{C}\ .$$

Proof. It is immediate to check that

$$((\tau + \rho)^j (\tau + \overline{\rho})^{k-2-j})_{0 \le j \le k-2}$$

is a basis of $V_k(\mathbb{C})$ and that

$$P(\tau) = \sum_{0 \le j \le k-2} c_j (\tau + \rho)^j (\tau + \overline{\rho})^{k-2-j} = (\tau + \overline{\rho})^{k-2} \sum_{0 \le j \le k-2} c_j \left(\frac{\tau + \rho}{\tau + \overline{\rho}}\right)^j .$$

Now, if we set $\tau = (\rho - \overline{\rho}z)/(z - 1)$, then this is equivalent to

$$\left(\frac{z-1}{\rho - \overline{\rho}}\right)^{k-2} P\left(\frac{\rho - \overline{\rho}z}{z - 1}\right) = \sum_{0 \le j \le k-2} c_j z^j$$

and since the left-hand side is in $V_k(\mathbb{C})$, this proves our claim and also gives the coefficients c_j explicitly. We can therefore write

$$V_k(\mathbb{C}) = \bigoplus_{0 \le j \le k-2} (\tau + \rho)^j (\tau + \overline{\rho})^{k-2-j} \mathbb{C}$$

and it is also immediate to check that the subspace $(\tau + \rho)^j (\tau + \overline{\rho})^{k-2-j}\mathbb{C}$ is the eigenspace corresponding to the eigenvalue $1 - \rho^{k-2-j}\overline{\rho}^j$ of ψ, and since the kernel of ψ corresponds to the eigenvalue 0, the lemma follows. ☐

Corollary 11.8.15. *The dimension of E over $\mathbb{C}$ is given by*

$$\dim_{\mathbb{C}}(E) = \left\lfloor \frac{k-2}{3} \right\rfloor + \begin{cases} 0 & \text{when } k \equiv 0 \pmod{3}\ , \\ 1 & \text{when } k \not\equiv 0 \pmod{3}\ . \end{cases}$$

Proof. By the above lemma, $\dim_{\mathbb{C}}(E) = \dim_{\mathbb{C}}(\mathrm{Ker}(\psi))$ is equal to the number of integers j such that $0 \le j \le k - 2$ and $\rho^{k-2-j}\overline{\rho}^j = 1$, in other words, such that $k - 2 - 2j \equiv 0 \pmod{3}$.

If $k \equiv 0 \pmod{3}$, then $j \equiv 2 \pmod{3}$ and hence $j = 3\ell - 1$ with $1 \le \ell \le k/3 - 1$, which gives a total of $k/3 - 1 = \lfloor (k-2)/3 \rfloor$ values.

If $k \equiv 1 \pmod 3$, then $j \equiv 1 \pmod 3$ and hence $j = 3\ell - 2$ with $1 \le \ell \le (k-1)/3$, which gives a total of $(k-1)/3 = \lfloor (k-2)/3 \rfloor + 1$ values.

If $k \equiv 2 \pmod 3$, then $j \equiv 0 \pmod 3$ and hence $j = 3\ell - 3$ with $1 \le \ell \le (k+1)/3$, which gives a total of $(k+1)/3 = \lfloor (k-2)/3 \rfloor + 1$. □

11.9. Interpretation in Terms of Periods

Definition 11.9.1. Let $f \in S_k(\Gamma)$ be a cusp form and for $j = 0, 1, \ldots, k-2$, let $r_j(f)$, the jth *period* of f, be defined by

$$r_j(f) = \int_0^{i\infty} f(\tau)\tau^j \, d\tau \ .$$

Since f is a cusp form, $f(\tau)$ tends to 0 exponentially fast as $y \to \infty$ and also as $y \to 0$; hence the integral converges absolutely (in fact for any integral value of j).

Lemma 11.9.2. *If $f \in S_k(\Gamma)$, then the jth period is explicitly given by*

$$r_j(f) = (-2\pi i)^{-j-1} j! L(f, j+1) = i^{j+1}\Lambda(f, j+1) \ .$$

Furthermore, if the Fourier coefficients, $a(n)$, of $f = \sum_{n \ge 1} a(n) q^n$ are real, then $r_j(f)$ is real when j is odd, and it is purely imaginary when j is even.

Proof. If we set $\tau = it$, then, by Theorem 11.1.1, we have

$$r_j(f) = i^{j+1} \int_0^{\infty} f(it) t^j \, dt = i^{j+1}\Lambda(f, s+1) \ ,$$

and if, in addition, the Fourier coefficients of f are real, then $f(it)$ is real for $t > 0$ and hence $r_j(f) \in i^{j+1}\mathbb{R}$, proving the lemma. □

It follows immediately from the lemma that the period polynomial can be expressed in terms of the periods in the following way.

Corollary 11.9.3. *If $f \in S_k(\Gamma)$, then*

$$P_f(X) = -\frac{1}{(k-2)!} \sum_{0 \le j \le k-2} \binom{k-2}{j} r_j(f) X^j$$

and if the Fourier coefficients of f are real, then

$$\Re(P_f(X)) = -\frac{1}{(k-2)!} \sum_{\substack{0 \le j \le k-2 \\ j \text{ odd}}} \binom{k-2}{j} r_j(f) X^j \quad \textit{and}$$

$$\Im(P_f(X)) = -\frac{1}{(k-2)!} \sum_{\substack{0 \le j \le k-2 \\ j \text{ even}}} \binom{k-2}{j} r_j(f) X^j.$$

Theorem 11.9.4. *For $0 \le \ell \le k-2$ we have the following relations:*

(a) $$r_\ell(f) + (-1)^\ell r_{k-2-\ell}(f) = 0\ ,$$

(b) $$\sum_{\substack{0\le j\le \ell\\ j\ \text{even}}} \binom{\ell}{j} r_j(f) + \sum_{\substack{\ell+1\le j\le k-2\\ j\ \text{even}}} \binom{k-2-\ell}{j-\ell} r_j(f) = 0\ ,$$

(c) $$\sum_{\substack{0\le j\le \ell\\ j\ \text{odd}}} \binom{\ell}{j} r_j(f) + \sum_{\substack{\ell+1\le j\le k-2\\ j\ \text{odd}}} \binom{k-2-\ell}{j-\ell} r_j(f) = 0\ .$$

Proof. Set $P = f^*\big|S - f^*$. It is then easy to verify that $P \in W_k(\mathbb{C})$ and, additionally, by Corollary 11.5.2 and Lemma 11.9.2 we have

$$\begin{aligned} P(\tau) &= (2\pi i)^{1-k} \sum_{0\le j\le k-2} \frac{(-2\pi i)^j}{j!} \tau^{k-2-j} L(f, k-1-j) \\ &= (2\pi i)^{1-k} \sum_{0\le j\le k-2} \frac{(-2\pi i)^j}{j!} \tau^{k-2-j} r_{k-2-j} \frac{(-2\pi i)^{k-1-j}}{(k-2-j)!} \\ &= -\frac{1}{(k-2)!} \sum_{0\le j\le k-2} \binom{k-2}{j} r_j \tau^j\ , \end{aligned}$$

where we abbreviate $r_j(f)$ as r_j. The first formula of the theorem then follows immediately from the relation $P\big|S + P = 0$.

The second relation, $P\big|TSTS + P\big|TS + P = 0$, gives

$$\tau^{k-2} P\left(\frac{\tau-1}{\tau}\right) + (\tau-1)^{k-2} P\left(-\frac{1}{\tau-1}\right) + P(\tau) = 0\ ,$$

and by changing τ to $\tau+1$ and replacing $\tau^{k-2}P(-1/\tau)$ with $-P(\tau)$ (by the first relation of the lemma), we obtain a three-term relation of the form

$$(\tau+1)^{k-2} P\left(\frac{\tau}{\tau+1}\right) + P(\tau+1) - P(\tau) = 0\ .$$

Now, by identifying the coefficient of τ^ℓ for $0 \le \ell \le k-2$ with 0 we see that

$$\sum_{0\le j\le \ell} \binom{k-2}{j}\binom{k-2-j}{k-2-\ell} r_j + \sum_{\ell+1\le j\le k-2} \binom{k-2}{j}\binom{j}{\ell} r_j = 0\ ,$$

and by using the easily verified identities

$$\binom{k-2}{j}\binom{k-2-j}{k-2-\ell} = \binom{k-2}{\ell}\binom{\ell}{j} \quad \text{and}$$
$$\binom{k-2}{j}\binom{j}{\ell} = \binom{k-2}{\ell}\binom{k-2-\ell}{j-\ell}$$

it follows that this can be written as

$$\sum_{0\le j\le \ell}\binom{\ell}{j}r_j+\sum_{\ell+1\le j\le k-2}\binom{k-2-\ell}{j-\ell}r_j=0\;.$$

Replacing ℓ by $k-2-\ell$, j by $k-2-j$, and using (a) it follows that

$$\sum_{0\le j\le \ell}(-1)^{j+1}\binom{\ell}{j}r_j+\sum_{\ell+1\le j\le k-2}(-1)^{j+1}\binom{k-2-\ell}{j-\ell}r_j=0\;.$$

The last two formulas now follow by adding and subtracting, respectively, these two relations. ☐

Corollary 11.9.5. *For $0\le \ell\le k-2$ set $\widetilde{\ell}=k-2-\ell$. We then have*

$$\sum_{\substack{0\le j\le \ell\\ j\text{ even}}}\binom{\ell}{j}r_j(f)-\sum_{\substack{0\le j\le \widetilde{\ell}-1\\ j\text{ even}}}\binom{\widetilde{\ell}}{j}r_j(f)=0\quad\textit{and}$$

$$\sum_{\substack{0\le j\le \ell\\ j\text{ odd}}}\binom{\ell}{j}r_j(f)+\sum_{\substack{0\le j\le \widetilde{\ell}-1\\ j\text{ odd}}}\binom{\widetilde{\ell}}{j}r_j(f)=0\;.$$

Proof. This follows immediately from the theorem by replacing j with $\widetilde{j}=k-2-j$ in the second sums of (b) and (c) and then using (a). ☐

We have in fact proved more than Theorem 11.9.4:

Definition 11.9.6. For any field K containing $\mathbb{Q}$, let $R_k(K)\subseteq K^{k-1}$ be the subspace of tuples $(r_0,\dots,r_{k-2})$ satisfying the relations of Theorem 11.9.4.

We then have an isomorphism of vector spaces.

Proposition 11.9.7. *The map*

$$(r_0,\dots,r_{k-2})\mapsto -\frac{1}{(k-2)!}\sum_{0\le j\le k-2}\binom{k-2}{j}r_j\tau^j$$

is an isomorphism of K-vector spaces from $R_k(K)$ to the space of polynomials $P\in V_k(K)$ satisfying $P\big|S+P=0$ and $P\big|TSTS+P\big|TS+P=0$.

We can now formulate the Eichler–Shimura isomorphism theorem (Theorem 11.8.5) in terms of periods:

Proposition 11.9.8. *The map $r:S_k(\Gamma)\to R_k(\mathbb{R})$ defined by*

$$f\mapsto(\Re(r_0(f)),\Re(r_1(f)),\dots,\Re(r_{k-2}(f)))$$

is an isomorphism from $S_k(\Gamma)$ to a codimension 1 subspace of $R_k(\mathbb{R})$ not containing the vector $(1,0,0,\dots,0,-1)$.

Proof. Recall $W_k(K)$ defined in Definition 11.8.8. We have the maps

$$S_k(\Gamma) \xrightarrow{\phi_{\mathbb{R}}} Z \overset{\alpha}{\simeq} W_k(\mathbb{R}) \overset{\beta}{\simeq} R_k(\mathbb{R}) ,$$

where the first map, $\phi_{\mathbb{R}}$, is defined by $\phi_{\mathbb{R}}(f) = (\gamma \mapsto \Re(f^*\big|\gamma - f^*))$ and the other maps are given by the isomorphism of Lemma 11.8.9 and the inverse of the isomorphism of Proposition 11.9.7. That is, $\alpha \circ \phi_{\mathbb{R}}(f) = \Re(f^*\big|S - f^*) = \Re(P_f)$ and by Corollary 11.9.3 we see that $\beta \circ \alpha \circ \phi_{\mathbb{R}}(f) = (\Re(r_0(f)), \Re(r_1(f)), \ldots, \Re(r_{k-2}(f))) = r(f)$. To show that r is an isomorphism from $S_k(\Gamma)$ to a codimension 1 subspace of $R_k(\mathbb{R})$ not containing $(1, 0, 0, \ldots, 0, -1)$, which corresponds to the cocycle $\gamma \mapsto -1/(k-2)!(1\big|\gamma-1)$, we must therefore show that this property is true for the map $\phi_{\mathbb{R}} : S_k(\Gamma) \to Z$.

By Lemma 11.8.6 and the Eichler–Shimura isomorphism theorem, we know that the composite map $\overline{\phi_{\mathbb{R}}} : S_k(\Gamma) \longrightarrow Z/B$ is an isomorphism. Additionally, by Lemma 11.8.7 (more precisely by its proof), we know that B is the 1-dimensional subspace of Z generated by the cocycle $(\gamma \mapsto (1\big|\gamma-1))$ and hence the proposition follows. ☐

Definition 11.9.9.

(a) Let $R_k^-(\mathbb{R})$ denote the subspace of $\mathbb{R}^{(k-2)/2}$ consisting of tuples $(r_1, r_3, \ldots, r_{k-3})$ satisfying (a) and (c) of Theorem 11.9.4.

(b) Similarly, let $R_k^+(\mathbb{R})$ be the subspace of $\mathbb{R}^{k/2}$ consisting of tuples $(r_0, r_2, \ldots, r_{k-2})$ satisfying conditions (a) and (b) of Theorem 11.9.4.

(c) Finally, we let $M_k(\Gamma)(\mathbb{R})$ and $S_k(\Gamma)(\mathbb{R})$ be the $\mathbb{R}$-vector space of modular forms and the subspace of cusp forms whose Fourier coefficients are all real.

It is clear that the equality $\mathbb{R}^{(k-2)/2} \oplus \mathbb{R}^{k/2} = \mathbb{R}^{k-1}$ induces the identity

$$R_k(\mathbb{R}) = R_k^-(\mathbb{R}) \oplus R_k^+(\mathbb{R})$$

and there is a corresponding decomposition of the map r from the proposition. More precisely, we have the following:

Proposition 11.9.10. *Let $r^- : S_k(\Gamma) \to R_k^-(\mathbb{R})$ and $r^+ : S_k(\Gamma) \to R_k^+(\mathbb{R})$ be the maps defined by*

$$r^-(f) = (r_1(f), r_3(f), \ldots, r_{k-3}(f)) \quad \text{and}$$
$$r^+(f) = \frac{1}{i}(r_0(f), r_2(f), \ldots, r_{k-2}(f)).$$

Then r^- is an isomorphism from $S_k(\Gamma)(\mathbb{R})$ to $R_k^-(\mathbb{R})$ and r^+ is an isomorphism from $S_k(\Gamma)(\mathbb{R})$ to a codimension 1 subspace of $R_k^+(\mathbb{R})$ not containing $(1, 0, \ldots, 0, -1)$.

Proof. By Lemma 11.9.2 it follows that if $f \in S_k(\Gamma)(\mathbb{R})$, then

$$r(f) = (0, r_1(f), 0, r_3(f), \dots, r_{k-3}(f), 0) \quad \text{and}$$
$$r(f/i) = (1/i)(r_0(f), 0, r_2(f), \dots, 0, r_{k-2}(f))\ ;$$

hence the results follow immediately from Proposition 11.9.8. □

We can use this proposition to obtain some partial results about rationality of (quotients of) periods. For instance:

Proposition 11.9.11. *Assume that $k = 12$.*

(a) *The space $R_k^-(\mathbb{R})$ is 1-dimensional and generated by*

$$(r_1, r_3, r_5, r_7, r_9) = (48, -25, 20, -25, 48)\ .$$

(b) *There exists $\omega^- \in \mathbb{R}$ such that*

$$r_1(\Delta) = r_9(\Delta) = 48\omega^-\ ,$$
$$r_3(\Delta) = r_7(\Delta) = -25\omega^-\ , \quad \textit{and}$$
$$r_5(\Delta) = 20\omega^-\ .$$

(c) *The space $R_k^+(\mathbb{R})$ is 2-dimensional and generated by*

$$(r_0, r_2, r_4, r_6, r_8, r_{10}) = (0, -14, 9, -9, 14, 0)\ ,$$

together with $(1, 0, 0, 0, 0, -1)$.

(d) *There exists $\omega^+ \in \mathbb{R}$ such that*

$$r_2(\Delta) = -r_8(\Delta) = -14\omega^+ i \quad \textit{and}$$
$$r_4(\Delta) = -r_6(\Delta) = 9\omega^+ i\ .$$

Proof. Left as an excellent exercise for the reader (Exercise 11.4). □

It is clear from this example, and in any case from the results that we have obtained, that the rationality problem for periods is not yet solved in a completely satisfactory manner since we have not shown that $r_0(\Delta) = -r_{10}(\Delta)$ is also a rational multiple of ω^+. This is however true, as we will see later, and one of the main purposes of what follows is to prove this for general normalized eigenforms. To achieve this we now need to compute the action of the Hecke operators on periods. Note that the rationality properties that we have proved are essentially equivalent to the modularity of the function. Being an eigenform of Hecke operators is an important additional property that will lead to the complete rationality result. Note that even when $S_k(\Gamma)$ is 1-dimensional, for instance for $k = 12$, this is nontrivial, although evidently in that case all cusp forms must be eigenforms.

11.10. Action of Hecke Operators on Periods

The main goal of this section is to show that the periods $r_j(T(n)f)$, of $T(n)f$, are linear combinations with integer coefficients of the periods $r_i(f)$. For completeness, we will give all the relations explicitly.

We emphasize, however, that the important point is the *existence* of these $\mathbb{Z}$-linear relations, and not necessarily the explicit coefficients. Thus, on first reading, one may skip the explicit technical computations as soon as it is clear that the coefficients are integral.

In this section, we will consider integrals of the form

$$\int_{z_1}^{z_2} f(\tau)\tau^j \, d\tau \ ,$$

where z_1 and z_2 are cusps, in other words, are in $\mathbb{P}_1(\mathbb{Q}) = \mathbb{Q} \cup \{i\infty\}$. It is understood that these integrals are taken along "straight lines" (geodesics) in the upper half-plane, $\mathfrak{H}$, considered with the hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$, in other words, either half-circles or half-lines orthogonal to the real line. It is then easy to see that if f is a cusp form, then these integrals are convergent for all integers j.

Lemma 11.10.1. *If j is an integer, $0 \le j \le k-2$, and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, then*

$$\int_{\gamma(0)}^{\gamma(i\infty)} f(\tau)\tau^j \, d\tau = \int_{b/d}^{a/c} f(\tau)\tau^j \, d\tau = \sum_{0 \le i \le k-2} e_j^i(\gamma) r_i(f) \ ,$$

where $e_j^i(\gamma) \in \mathbb{Z}$ is given explicitly by

$$\begin{aligned} e_j^i(\gamma) &= \sum_{x \in \mathbb{Z}} \binom{j}{x} \binom{k-2-j}{i-x} a^x b^{j-x} c^{i-x} d^{k-2-i-j+x} \\ &= \left(\frac{c}{d}\right)^i \left(\frac{b}{d}\right)^j d^{k-2} \sum_{x \in \mathbb{Z}} \binom{j}{x} \binom{k-2-j}{i-x} \left(\frac{ad}{bc}\right)^x \ , \end{aligned}$$

where we use the convention $\binom{m}{n} = 0$ when $n < 0$ or $n > m$.

Proof. By using the evident change of variable $\tau = \gamma\tau'$ we have

$$\begin{aligned} \int_{\gamma(0)}^{\gamma(i\infty)} f(\tau)\tau^j \, d\tau &= \int_0^{i\infty} (c\tau' + d)^k f(\tau') \left(\frac{a\tau' + b}{c\tau' + d}\right)^j \frac{d\tau'}{(c\tau' + d)^2} \\ &= \int_0^{i\infty} f(\tau)(a\tau + b)^j (c\tau + d)^{k-2-j} \, d\tau \ , \end{aligned}$$

and the lemma follows by the binomial theorem from the expansion of $(a\tau + b)^j$ and $(c\tau + d)^{k-2-j}$. $\square$

In the definition of Hecke operators there are matrices with determinant not equal to 1 and we must therefore generalize the above lemma to these. For this, we need to recall some elementary results on continued fractions.

Let $b/d \in \mathbb{Q}$, where, without loss of generality, we may assume that $\gcd(b,d) = 1$ and $d > 0$. It is well known that b/d has a finite (*simple* or *regular*) continued fraction expansion

$$b/d = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_m}}}} = [a_0, a_1, \dots, a_m]\ ,$$

where $a_i \in \mathbb{Z}_{\geq 1}$ for $i \geq 1$ and $a_0 \in \mathbb{Z}$. Note that the above continued fraction is not unique since if $a_m \geq 2$, we can write $a_m = a_m - 1 + 1/1$, and if $a_m = 1$, we can write $a_{m-1} + 1/a_m = a_{m-1} + 1$. Thus, if $d > 1$, we may always assume that $a_m \geq 2$, and we have

$$b/d = [a_0, a_1, \dots, a_{m-1}, a_m] = [a_0, a_1, \dots, a_{m-1}, a_m - 1, 1]\ .$$

If $d = 1$, we have simply $b/d = [b] = [b-1, 1]$. For the moment we do not need to choose between these two representations. We define the νth *convergent*, b_ν/d_ν, of the continued fraction $[a_0, a_1, \dots, a_m]$ by

$$[a_0, a_1, \dots, a_\nu] = \frac{b_\nu}{d_\nu} \quad \text{with} \quad \gcd(b_\nu, d_\nu) = 1 \quad \text{and} \quad d_\nu > 0$$

and it is well known that b_ν and d_ν satisfy the recursions

$$b_\nu = a_\nu b_{\nu-1} + b_{\nu-2} \quad \text{and} \quad d_\nu = a_\nu d_{\nu-1} + d_{\nu-2}\ ,$$

with the initial conditions

$$b_{-1} = 1\ ,\ b_0 = a_0\ ,\ d_{-1} = 0\ ,\ d_0 = 1\ .$$

This is better expressed in matrix terms: if we set

$$g_\nu = \begin{pmatrix} (-1)^\nu b_{\nu-1} & b_\nu \\ (-1)^\nu d_{\nu-1} & d_\nu \end{pmatrix},$$

then the above relations are equivalent to

$$g_\nu = g_{\nu-1} \begin{pmatrix} 0 & (-1)^{\nu-1} \\ (-1)^\nu & a_\nu \end{pmatrix} \quad \text{with} \quad g_0 = \begin{pmatrix} 1 & a_0 \\ 0 & 1 \end{pmatrix}\ .$$

From this matrix relation the well-known fact that g_ν has determinant 1 follows immediately. In addition, we have clearly $g_\nu(0) = b_\nu/d_\nu$ and $g_\nu(i\infty) = b_{\nu-1}/d_{\nu-1} = g_{\nu-1}(0)$. It follows by assumption that

$$\sum_{0 \leq \nu \leq m} \int_{g_\nu(0)}^{g_\nu(i\infty)} = \int_{b_m/d_m}^{b_{-1}/d_{-1}} = \int_{b/d}^{i\infty}\ .$$

This allows us to prove the following lemma:

Lemma 11.10.2. *Let j be an integer such that $0 \le j \le k-2$, and let $b/d \in \mathbb{Q}$ with $\gcd(b,d) = 1$ and $d > 0$. Then*

$$\int_{b/d}^{i\infty} f(\tau)\tau^j \, d\tau = \sum_{0 \le i \le k-2} c_j^i\left(\frac{b}{d}\right) r_i(f) \, ,$$

where $c_j^i(b/d) \in \mathbb{Z}$ is given by

$$c_j^i\left(\frac{b}{d}\right) = \sum_{0 \le \nu \le m} (-1)^{\nu i} \sum_x \binom{j}{x}\binom{k-2-j}{i-x} b_{\nu-1}^x b_\nu^{j-x} d_{\nu-1}^{i-x} d_\nu^{k-2-i-j+x}$$

and where $b/d = [a_0, \dots, a_m]$, $b_\nu/d_\nu = [a_0, \dots, a_\nu]$, $b_{-1} = 1$, and $d_{-1} = 0$.

Proof. Since g_ν has determinant 1, we can apply Lemma 11.10.1 with $\gamma = g_\nu$, so by the formula given above we have

$$\begin{aligned}\int_{b/d}^{i\infty} f(\tau)\tau^j \, d\tau &= \sum_{0 \le \nu \le m} \int_{g_\nu(0)}^{g_\nu(i\infty)} f(\tau)\tau^j \, d\tau \\ &= \sum_{0 \le \nu \le m} \sum_{0 \le i \le k-2} e_j^i(g_\nu) r_i(f) = \sum_{0 \le i \le k-2} c_j^i(b/d) r_i(f)\end{aligned}$$

with $c_j^i(b/d) \in \mathbb{Z}$, and by Lemma 11.10.1 we have

$$c_j^i(b/d) = \sum_{0 \le \nu \le m} (-1)^{\nu i} \sum_x \binom{j}{x}\binom{k-2-j}{i-x} b_{\nu-1}^x b_\nu^{j-x} d_{\nu-1}^{i-x} d_\nu^{k-2-i-j+x} \, ,$$

proving the lemma. □

It will be convenient to have $b_\nu \ge 0$ and $d_\nu \ge 0$ for all ν. This will be the case if $b/d \ge 0$. Otherwise, we can use the following lemma:

Lemma 11.10.3. *Let j be an integer such that $0 \le j \le k-2$, and let $b/d \in \mathbb{Q}$ with $\gcd(b,d) = 1$ and $d > 0$. Then*

$$\int_{-b/d}^{i\infty} f(\tau)\tau^j \, d\tau = \sum_{0 \le i \le k-2} c'^i_j(b/d) r_i(f) \, ,$$

where $c'^i_j(b/d) \in \mathbb{Z}$ is given by

$$\begin{aligned}c'^i_j(b/d) &= \sum_{0 \le \nu \le m} (-1)^{\nu i + j - i} \sum_x \binom{j}{x}\binom{k-2-j}{i-x} b_{\nu-1}^x b_\nu^{j-x} d_{\nu-1}^{i-x} d_\nu^{k-2-i-j+x} \\ &= (-1)^{j-i} c_j^i(b/d)\end{aligned}$$

and where $b/d = [a_0, \dots, a_m]$, $b_\nu/d_\nu = [a_0, \dots, a_\nu]$, $b_{-1} = 1$, and $d_{-1} = 0$.

Proof. Instead of the matrices g_ν, we use the matrices

$$h_\nu = \begin{pmatrix} (-1)^\nu b_{\nu-1} & -b_\nu \\ (-1)^{\nu-1} d_{\nu-1} & d_\nu \end{pmatrix}$$

which satisfy

$$h_\nu = h_{\nu-1} \begin{pmatrix} 0 & (-1)^\nu \\ (-1)^{\nu-1} & a_\nu \end{pmatrix} \quad \text{with} \quad h_0 = \begin{pmatrix} 1 & -a_0 \\ 0 & 1 \end{pmatrix}$$

and therefore have determinant 1. In addition, $h_\nu(0) = -b_\nu/d_\nu$ and $h_\nu(i\infty) = -b_{\nu-1}/d_{\nu-1} = h_{\nu-1}(0)$; hence

$$\sum_{0 \le \nu \le m} \int_{h_\nu(0)}^{h_\nu(i\infty)} = \int_{-b/d}^{i\infty}$$

and we then conclude as in the preceding lemma. □

We are now able to state and prove the main result of this section.

Theorem 11.10.4. *The periods $r_j(T(n)f)$ are $\mathbb{Z}$-linear combinations of the periods $r_i(f)$, with $i \equiv j \pmod 2$. More precisely, we have*

$$r_j(T(n)f) = \sum_{\substack{0 \le i \le k-2 \\ i \equiv j \pmod 2}} t_j^i(n) r_i(f)\ ,$$

where

$$t_j^i(n) = 2 \sum_{d \mid n} \left(\frac{n}{d}\right)^j \sigma_{k-2-2j}\left(\frac{n}{d}\right) \sum_{0 \le \ell \le j} (-1)^{j-\ell} \binom{j}{\ell} \sum_{\substack{0 \le b \le d/2 \\ \gcd(b,d)=1}}{}' \, b^{j-\ell} d^\ell c_\ell^i \left(\frac{b}{d}\right)$$

(here $\sum'$ means that the terms corresponding to $b = 0$ and $b = d/2$, if they exist, must be multiplied by $1/2$) and

$$c_\ell^i\left(\frac{b}{d}\right) = \sum_{0 \le \nu \le m} (-1)^{\nu i} \sum_x \binom{\ell}{x} \binom{k-2-\ell}{i-x} b_{\nu-1}^x b_\nu^{\ell-x} d_{\nu-1}^{i-x} d_\nu^{k-2-i-\ell+x}$$

and where $b/d = [a_0, \dots, a_m]$, $b_\nu/d_\nu = [a_0, \dots, a_\nu]$, $b_{-1} = 1$, and $d_{-1} = 0$.

Proof. It is easy to see that the definition of $T(n)f$ implies that

$$T(n)f(\tau) = n^{k-1} \sum_{d \mid n} d^{-k} \sum_{b \bmod d} f\left(\frac{n}{d^2}\tau + \frac{b}{d}\right)$$

and with the change of variable $\tau' = (n/d^2)\tau + b/d$ it follows that

$$r_j(T(n)f) = n^{k-1}\sum_{d\mid n} d^{-k} \sum_{b \bmod d} \int_0^{i\infty} f\left(\frac{n\tau}{d^2}+\frac{b}{d}\right)\tau^j\,d\tau$$
$$= n^{k-j-2}\sum_{d\mid n} d^{2j+2-k} \sum_{b \bmod d} \int_{b/d}^{i\infty} f(\tau)\left(\tau-\frac{b}{d}\right)^j\,d\tau\;.$$

If we write $b = b_1\delta$ and $d = d_1\delta$ with $\delta = \gcd(b,d)$, then we have

$$r_j(T(n)f) = n^{k-j-2}\sum_{d_1\mid n} d_1^{2j+2-k}\sigma_{2j+2-k}\left(\frac{n}{d_1}\right)$$
$$\cdot \sum_{\substack{b_1 \bmod d_1\\ \gcd(b_1,d_1)=1}} \int_{b_1/d_1}^{i\infty} f(\tau)\left(\tau-\frac{b_1}{d_1}\right)^j\,d\tau$$

and since $\sigma_a(n/d_1) = \sum_{\delta\mid(n/d_1)}(n/(d_1\delta))^a$, it follows that

$$d_1^{2j+2-k}\sigma_{2j+2-k}(n/d_1) = \sum_{\delta\mid(n/d_1)}(n/\delta)^{2j+2-k} = n^{2j+2-k}\sigma_{k-2-2j}(n/d_1)\;,$$

so that

$$r_j(T(n)f) = n^j\sum_{d\mid n}\sigma_{k-2-2j}\left(\frac{n}{d}\right) \sum_{\substack{b \bmod d\\ \gcd(b,d)=1}} \int_{b/d}^{i\infty} f(\tau)\left(\tau-\frac{b}{d}\right)^j\,d\tau$$
$$= n^j\sum_{d\mid n}\sigma_{k-2-2j}\left(\frac{n}{d}\right)\sum_{0\le\ell\le j}(-1)^{j-\ell}\binom{j}{\ell}$$
$$\cdot \sum_{\substack{b \bmod d\\ \gcd(b,d)=1}} \left(\frac{b}{d}\right)^{j-\ell}\int_{b/d}^{i\infty} f(\tau)\tau^\ell\,d\tau$$
$$= \sum_{d\mid n}\left(\frac{n}{d}\right)^j\sigma_{k-2-2j}\left(\frac{n}{d}\right)\sum_{0\le\ell\le j}(-1)^{j-\ell}\binom{k}{\ell}$$
$$\cdot \sum_{\substack{b \bmod d\\ \gcd(b,d)=1}} d^\ell b^{j-\ell}\int_{b/d}^{i\infty} f(\tau)\tau^\ell\,d\tau\;.$$

It already follows that, as claimed, $r_j(T(n)f)$ is a linear combination with integer coefficients of the $r_i(f)$: indeed, by Lemma 11.10.2 this is the case for each individual integral $\int_{b/d}^{i\infty} f(\tau)\tau^\ell\,d\tau$, and the coefficients which occur in the above formula are integral since $0 \le j \le k-2$ and

$$\left(\frac{n}{d}\right)^j\sigma_{k-2-2j}\left(\frac{n}{d}\right) = \sum_{\delta\delta'=n/d}\delta^{k-2-j}\delta'^j\;.$$

To finish the explicit computation, we choose as system of representatives of b modulo d the set of $\pm b$, where $0 \le b \le d/2$ and where the terms corresponding to the extremities $b = 0$ and $b = d/2$, if they exist, must be multiplied by $1/2$. We will thus let $\sum'$ denote the summation over these values of b. By Lemmas 11.10.2 and 11.10.3 we then have

$$r_j(T(n)f) = \sum_{d\mid n} \left(\frac{n}{d}\right)^j \sigma_{k-2-2j}\left(\frac{n}{d}\right) \sum_{0\le\ell\le j} (-1)^{j-\ell}\binom{j}{\ell} \cdot \sideset{}{'}\sum_{\substack{0\le b\le d/2\\ \gcd(b,d)=1}} b^{j-\ell}d^{\ell}\left(\int_{b/d}^{i\infty} +(-1)^{j-\ell}\int_{-b/d}^{i\infty}\right) f(\tau)\tau^{\ell}\,d\tau\ ;$$

in other words,

$$r_j(T(n)f) = \sum_{d\mid n} \left(\frac{n}{d}\right)^j \sigma_{k-2-2j}\left(\frac{n}{d}\right) \sum_{0\le\ell\le j} (-1)^{j-\ell}\binom{j}{\ell} \cdot \sideset{}{'}\sum_{\substack{0\le b\le d/2\\ \gcd(b,d)=1}} b^{j-\ell}d^{\ell} \sum_{0\le i\le k-2} (1+(-1)^{j-i})c_{\ell}^{i}\left(\frac{b}{d}\right) r_i(f)\ ,$$

proving the theorem. □

Note that even though there is a coefficient of 2 in front of the formula for $t_j^i(n)$, it is not necessarily even since the summation $\sum'$ introduces certain terms with coefficient $1/2$. We will return to this point later.

Corollary 11.10.5. *We have the following symmetrized expression:*

$$r_j(T(n)f) = \sideset{}{^*}\sum_{\substack{0\le i\le (k-2)/2\\ i\equiv j \pmod 2}} t'^{i}_{j}(n) r_i(f)\ ,$$

where $\sum^*$ means that the term corresponding to $i = (k-2)/2$ must be multiplied by $1/2$ and where

$$t'^{i}_{j}(n) = 2\sum_{d\mid n} \left(\frac{n}{d}\right)^j \sigma_{k-2-2j}\left(\frac{n}{d}\right) \sum_{0\le\ell\le j} (-1)^{j-\ell}\binom{j}{\ell} \sideset{}{'}\sum_{\substack{0\le b\le d/2\\ \gcd(b,d)=1}} b^{j-\ell}d^{\ell}c'^{i}_{\ell}\left(\frac{b}{d}\right)$$

and

$$c'^{i}_{\ell}\left(\frac{b}{d}\right) = \sum_{0\le\nu\le m} (-1)^{\nu i}\sum_{x}\binom{\ell}{x}\binom{k-2-\ell}{i-x} \cdot \left(b_{\nu-1}^{x}b_{\nu}^{\ell-x}d_{\nu-1}^{i-x}d_{\nu}^{k-2-\ell-i+x} + (-1)^{i}b_{\nu-1}^{\ell-x}b_{\nu}^{x}d_{\nu-1}^{k-2-\ell-i+x}d_{\nu}^{i-x}\right)\ .$$

In particular,

$$r_0(T(n)f) = \sum_{0\le i\le \lfloor k/4\rfloor -1} {t'}_0^{2i}(n) r_{2i}(f)$$

with

$${t'}_0^{2i}(n) = 2\sum_{d\mid n}\sigma_{k-2}\left(\frac{n}{d}\right)\sideset{}{'}\sum_{\substack{0\le b\le d/2\\ \gcd(b,d)=1}}\sum_{\nu=0}^{m}\binom{k-2}{2i}\left(d_{\nu-1}^{2i}d_{\nu}^{k-2-2i}-d_{\nu-1}^{k-2-2i}d_{\nu}^{2i}\right).$$

Proof. This follows immediately from the theorem by grouping the terms r_i and r_{k-2-i}, using the relation $r_i + (-1)^i r_{k-2-i} = 0$ and after changing x to $\ell - x$ in the inner summation. The special case follows from the fact that for $j = 0$ we must have $\ell = 0$ and hence $x = 0$. □

Lemma 11.10.6. *With notation as above we have* ${t'}_0^0(n) = \sigma_{k-1}(n)$.

Proof. By the above corollary we have

$${t'}_0^0(n) = 2\sum_{d\mid n}\sigma_{k-2}\left(\frac{n}{d}\right)\sideset{}{'}\sum_{\substack{0\le b\le d/2\\ \gcd(b,d)=1}}\sum_{0\le\nu\le m}\left(d_\nu^{k-2}-d_{\nu-1}^{k-2}\right)$$

where the inner sum is telescoping, hence equals $d_m^{k-2} - d_{-1}^{k-2} = d^{k-2}$, and

$$\sideset{}{'}\sum_{\substack{0\le b\le d/2\\ \gcd(b,d)=1}} 1 = \sideset{}{'}\sum_{\substack{d/2\le b\le d\\ \gcd(b,d)=1}} = \frac{1}{2}\sum_{\substack{0\le b<d\\ \gcd(b,d)=1}} 1 = \frac{\phi(d)}{2}\ ,$$

so that

$${t'}_0^0(n) = \sum_{d\mid n}\sigma_{k-2}\left(\frac{n}{d}\right)\phi(d)d^{k-2} = \sum_{d\mid n}\sum_{d'\mid n/d} {d'}^{k-2}d^{k-2}\phi(d)\ .$$

By setting $dd' = D$ we see that

$${t'}_0^0(n) = \sum_{D\mid n} D^{k-2}\sum_{d\mid D}\phi(d)\ .$$

Now, since a cyclic group of order D has exactly $\phi(d)$ elements of order d when $d \mid D$, it follows that $\sum_{d\mid D}\phi(d) = D$ and hence ${t'}_0^0(n) = \sum_{D\mid n} D^{k-1} = \sigma_{k-1}(n)$, proving the lemma. □

11.11. Rationality and Parity Theorems

We are now in a position to prove the complete rationality results, that is, also for $r_0(f)$ and $r_{k-2}(f)$, mentioned above, which are originally due to Y. Manin [**Man73**]. We will in addition also prove *parity* results.

Before considering the general case, let us again consider the special case $f = \Delta$; that is, $k = 12$. Applying Corollary 11.10.5 with, for instance, $n = 2$ and using the fact that

$$T(2)\Delta = \tau(2)\Delta = -24\Delta \ ,$$

we deduce from Lemma 11.10.6 that

$$-24r_0(\Delta) = 2049r_0(\Delta) + \sum_{1\le i\le 2} t'^{2i}_0(2)r_{2i}(\Delta) \ ;$$

hence $r_0(\Delta)/i$ is a linear combination with *rational* coefficients (with denominator dividing $2049 + 24 = 2073 = 3 \cdot 691$) of $r_2(\Delta)/i$ and $r_4(\Delta)/i$, and hence by Proposition 11.9.11, it is a rational multiple of ω^+.

Proposition 11.11.1 (Conclusion of Proposition 11.9.11)**.** *We have*

$$r_0(\Delta) = -r_{10}(\Delta) = \frac{22680}{691}\omega^+ i \ .$$

Proof. Again left as an exercise for the reader. □

The rationality result we mentioned earlier is the following:

Theorem 11.11.2 (Manin)**.** *Let $f \in S_k(\Gamma)$ be a normalized eigenform and let $\mathbb{Q}(f)$ be the number field generated by the Fourier coefficients of f. Then there exist two positive real numbers $\omega^-(f)$ and $\omega^+(f)$ such that*

$$r_j(f) \in \omega^-(f)\mathbb{Q}(f) \quad \textit{if } j \textit{ is odd and } 0 \le j \le k-2 \quad \textit{and}$$
$$\frac{1}{i}r_j(f) \in \omega^+(f)\mathbb{Q}(f) \quad \textit{if } j \textit{ is even and } 0 \le j \le k-2.$$

Proof. If $f = \sum_{n\ge 1} a(n)q^n$, we have seen that the $a(n)$ are $\mathbb{Z}$-linear combinations of the $a(j)$ for $1 \le j \le r-1$, where $r = \dim M_k(\Gamma)$. It follows that $\mathbb{Q}(f) = \mathbb{Q}(a(1), \ldots, a(r-1))$, and by Proposition 10.6.2 we know that $\mathbb{Q}(f)$ is totally real. As before we set

$$r^-(f) = (r_1(f), r_3(f), \ldots, r_{k-3}(f))^t \quad \text{and}$$
$$r^+(f) = \frac{1}{i}(r_0(f), r_2(f), \ldots, r_{k-2}(f))^t \ ,$$

where ^{t} indicates that $r^-(f)$ and $r^+(f)$ are now considered as *column* vectors. By Theorem 11.10.4 there exist two matrices $A^-(n)$ and $A^+(n)$ with integer coefficients, such that $r^\pm(T(n)f) = A^\pm(n)r^\pm(f)$. Since $T(n)f = a(n)f$, it follows that

$$r^\pm(f) \in E_f^\pm = \bigcup_{n\ge 1} \operatorname{Ker}(A^\pm(n) - a(n)I) \ .$$

In addition, by Theorem 11.9.4 we also have $r^\pm(f) \in R_k^\pm(\mathbb{R})$, so in fact

$$r^\pm(f) \in E_f^\pm \cap R_k^\pm(\mathbb{R}) \ .$$

We also need the following lemma which provides a converse statement:

Lemma 11.11.3. *If* $v \in E_f^{\pm} \cap R_k^{\pm}(\mathbb{R})$*, then there exists a cusp form* $g \in S_k(\Gamma)(\mathbb{R})$ *such that* $v = r^{\pm}(g)$*.*

Proof. For the $-$ space, since $v \in R_k^-(\mathbb{R})$, the Eichler–Shimura isomorphism theorem, Theorem 11.8.5, implies that $v = r^-(g)$ for some $g \in S_k(\Gamma)(\mathbb{R})$ and the lemma is therefore immediate in this case. As we have already seen previously, the main difficulty lies in the $+$ space. If $v \in E_f^{\pm} \cap R_k^{\pm}(\mathbb{R})$, then the theorem only asserts that there exist $g \in S_k(\Gamma)(\mathbb{R})$ and $x \in \mathbb{R}$ such that

$$v = r^+(g) + (x, 0, 0, \dots, 0, -x)^t .$$

If we choose the matrix given by Corollary 11.10.5 as $A^+(n)$, then it is clear that the first component of $A^+(n)(x, 0, 0, \dots, 0, -x)^t$ is equal to $x\sigma_{k-1}(n)$. However, this is also equal to the first component of $A^+(n)(v - r^+(g)) = a(n)v - r^+(T(n)g)$, and hence to

$$a(n)\left(\frac{r_0(g)}{i} + x\right) - \frac{r_0(T(n)g)}{i}$$

since $v \in E_f^+$. Since f is a cusp form, we have $a(n) = O(n^{k/2})$ and if we write $g = \sum_{1 \le i \le r-1} \lambda_i f_i$ where the f_i are normalized eigenforms, then

$$r_0(T(n)g) = \sum_{1 \le i \le r-1} \lambda_i r_0(T(n)f_i) = \sum_{1 \le i \le r-1} \lambda_i a_i(n) r_0(f_i) ,$$

where we have set $f_i = \sum_{n \ge 1} a_i(n) q^n$. It follows that we also have the asymptotic estimate $r_0(T(n)g) = O(n^{k/2})$, so that $x\sigma_{k-1}(n) = O(n^{k/2})$. Since $\sigma_{k-1}(n) \ge n^{k-1}$ and $k - 1 > k/2$, this is impossible unless $x = 0$, proving the lemma. □

We now resume our proof of Theorem 11.11.2: if $v \in E_f^{\pm} \cap R_k^{\pm}(\mathbb{R})$, there exists $g \in S_k(\Gamma)(\mathbb{R})$ such that $v = r^{\pm}(g)$. Since $v \in E_f^{\pm}$, we have $A^{\pm}(n)r^{\pm}(g) = a(n)r^{\pm}(g)$ for all n; hence by definition of $A^{\pm}(n)$ we have

$$r^{\pm}(T(n)g - a(n)g) = 0 .$$

However, by Proposition 11.9.10 the maps r^+ and r^- are *injective*, so that for all n we have $T(n)g = a(n)g$, which is equivalent to $g = \lambda f$ for some constant λ since two eigenfunctions for all the $T(n)$ with the same eigenvalues are proportional (the Multiplicity 1 theorem; see Theorem 13.3.9). It follows that $E_f^{\pm} \cap R_k^{\pm}(\mathbb{R})$ has dimension 1, hence that

$$E_f^{\pm} \cap R_k^{\pm}(\mathbb{R}) = r^{\pm}(f)\mathbb{R} .$$

Since $E_f^{\pm}$ and $R_k^{\pm}(\mathbb{R})$ are defined by equations with coefficients in $\mathbb{Q}(f)$, it follows that $E_f^{\pm} \cap R_k^{\pm}(\mathbb{R})$ is a sub-$\mathbb{Q}(f)$-vector space of $r^{\pm}(f)\mathbb{R}$, in other words, that $r^{\pm}(f) \in w^{\pm}\mathbb{Q}(f)^{d^{\pm}}$, where $d^+ = k/2$, $d^- = (k-2)/2$, and $\omega^{\pm} \in \mathbb{R}$, proving the theorem. □

Corollary 11.11.4. *If $f \in S_k(\Gamma)$ is a normalized eigenform, then*

$$\frac{\omega^+(f)\omega^-(f)}{\langle f, f\rangle} \in \mathbb{Q}(f)\ .$$

Proof. By Theorem 10.8.9 we know that if a and b are integers of opposite parity such that $k/2+2 \le b < b+3 \le a \le k-1$, then $\Lambda(f,a)\Lambda(f,b)/\langle f,f\rangle \in \mathbb{Q}(f)$. Thus, if $k \equiv 0 \pmod 4$, we choose for instance $b = k/2+2$ and $a = k-1$, which does satisfy $a \ge b+3$ since $k \ge 12$. If $k \equiv 2 \pmod 4$, we instead choose $b = k/2+2$ and $a = k-2$, which also satisfies $a \ge b+3$ since now $k \ge 16$, and in both cases thanks to the formula $\Lambda(f,a) = i^{-a}r_{a-1}(f)$ we deduce that $\omega^+(f)\omega^-(f)/\langle f,f\rangle \in \mathbb{Q}(f)$. □

We can also deduce many other interesting results from Theorem 11.10.4 and its corollary:

Proposition 11.11.5. *Let $f = \sum_{n\ge1} a(n)q^n \in S_k(\Gamma)$ be a normalized eigenform and assume that $r_0(f) \ne 0$. We then have the explicit formula*

$$a(n) = \sigma_{k-1}(n) + 2\sum_{d\mid n}\sigma_{k-2}\left(\frac{n}{d}\right)\sideset{}{'}\sum_{\substack{0\le b\le d/2\\ \gcd(b,d)=1}}\ \sum_{1\le i\le\lfloor k/4\rfloor-1} c(2i)\frac{r_{2i}(f)}{r_0(f)}\ ,$$

where

$$c(2i) = \sum_{1\le\nu\le m}\binom{k-2}{2i}\left(d_{\nu-1}^{2i}d_\nu^{k-2-2i} - d_{\nu-1}^{k-2-2i}d_\nu^{2i}\right)$$

and $\sum'$ indicates that the term corresponding to $b = d/2$ is multiplied by a factor of $1/2$, $b/d = [a_0, a_1, \dots, a_m]$, and b_ν/d_ν is the νth convergent of b/d.

Note that we only need a finite number of coefficients, $r_{2i}(f)/r_0(f)$, which belong to the field $\mathbb{Q}(f)$ and can be computed once and for all, to compute $a(n)$ for *any* integer n.

Proof. Since $T(n)f = a(n)f$, it follows immediately from Corollary 11.10.5 and Lemma 11.10.6 that, with $c(2i)$ as stated,

$$\begin{aligned}a(n)r_0(f) &= \sigma_{k-1}(n)r_0(f)\\ &\quad + 2\sum_{d\mid n}\sigma_{k-2}\left(\frac{n}{d}\right)\sideset{}{'}\sum_{\substack{0\le b\le d/2\\ \gcd(b,d)=1}}\ \sum_{1\le i\le\lfloor k/4\rfloor-1} c(2i)r_{2i}(f)\ .\end{aligned}$$

□

Corollary 11.11.6. *Under the same assumptions as above we have*

$$a(n) = \sigma_{k-1}(n) + 2\sum_{n=xx'+yy'}\ \sum_{1\le i\le\lfloor k/4\rfloor-1}\frac{r_{2i}(f)}{r_0(f)}\binom{k-2}{2i}\left(y^{2i}x^{k-2-2i}-y^{k-2-2i}x^{2i}\right)\ ,$$

where the sum is taken over all solutions (x, y, x', y') to the equation $n = xx' + yy'$ such that $x > y > 0$ and either $x' > y' > 0$, or $x \mid n$, $x' = n/x$, $y = 0$, and $0 < y/x \le 1/2$, and where the terms with $y/x = 1/2$ must be taken with coefficient $1/2$.

Proof. This easily follows from a theorem due to Heilbronn characterizing all pairs $(d_{\nu-1}, d_\nu)$ occurring in the above proposition, and we omit the details. □

Example 11.11.7. Using the computations made above in Propositions 11.9.11 and 11.11.1 for Δ we obtain

$$\tau(n) = \sigma_{11}(n) + \frac{691}{18}\sum_{n=xx'+yy'}{}' x^2y^2(y^2-x^2)^3\ ,$$

where the summation is taken on quadruples as above. For instance, for $n = 2$ the only possible quadruple is $(x, y, x', y') = (2, 1, 1, 0)$, and it must be counted with coefficient $1/2$; hence we obtain $\tau(2) = 2049 + (691/18)\cdot(-54) = -24$, which is correct.

Note incidentally that this gives an alternate proof of the congruence $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$.

We can also obtain slightly deeper consequences of Corollary 11.10.5.

Lemma 11.11.8. *There exists a matrix $B^-(p)$ with integer entries such that*

$$r^-(T(p)f) = 2B^-(p)r^-(f)\ .$$

Proof. Consider the formula of Corollary 11.10.5 with $n = p$ prime and j odd. Thanks to our conventions, the divisor $d = 1$ in $t'^{i}_{j}(p)$ can give a nonzero contribution only if $b = 0$; hence $\ell = j \ge 1$ and $\nu = 0$, $b_{-1} = 1$, $d_{-1} = 0$, $b_0 = 0$, and $d_0 = 1$, which implies that either $x = i$ (exponent of d_{-1}) *and* $x = \ell$ (exponent of b_0) for the first term, or $x = 0$ (exponent of b_0) and $x = \ell + i + 2 - k$ (exponent of d_{-1}) for the second term, so we have the following two possibilities: $i = j = \ell = x$, contributing $p^j\sigma_{k-2-2j}(p)$, and $i = k-2-j = k-2-\ell$ and $x = 0$, contributing also $p^j\sigma_{k-2-2j}(p)$; hence by our convention on $\sum^*$ the total contribution of the divisor $d = 1$ is always equal to $p^j\sigma_{k-2-2j}(p) = p^j + p^{k-2-j}$. We thus have

$$r_j(T(p)f) = \left(p^j + p^{k-2-j}\right) r_j(f) + \sum^*_{\substack{0\le i\le (k-2)/2 \\ i\equiv 1 \pmod 2}} a_j^i(p) r_i(f)\ ,$$

where

$$a_j^i(p) = 2\sum_{0\le \ell\le j}(-1)^{j-\ell}\binom{j}{\ell}\sum'_{\substack{1\le b\le p/2 \\ \gcd(b,p)=1}} b^{j-\ell}p^{\ell}c'^i_\ell\left(\frac{b}{p}\right)\ .$$

We need to distinguish the case $p > 2$ and $p = 2$.

Case $p > 2$**.** In this case $b = p/2$ does not appear and hence $a_j^i(p)$ is even. Furthermore, if $i = (k-2)/2$, then, changing x to $\ell - x$ shows that

$$\begin{aligned} c'^i_\ell\left(\frac{b}{d}\right) &= 2\sum_{0\le\nu\le m}(-1)^{\nu i}\sum_x\binom{\ell}{x}\binom{k-2-\ell}{i-x} b_{\nu-1}^x b_\nu^{\ell-x} d_{\nu-1}^{i-x} d_\nu^{k-2-\ell-i+x} \\ &= 2c^i_\ell\left(\frac{b}{d}\right)\ ; \end{aligned}$$

hence $a_j^i(p) \equiv 0 \pmod 4$ in this case.

Case $p = 2$**.** We then have

$$a_j^i(2) \equiv c'^i_0\left(\frac{1}{2}\right) \equiv \sum_{0\le\nu\le 1}\binom{k-2}{i}\left(d^i_{\nu-1}d_\nu^{k-2-i} + d_{\nu-1}^{k-2-i}d_\nu^i\right) \pmod 2\ ,$$

and since i is odd, $i > 0$, and $k - 2 - i > 0$, we have

$$d^i_{\nu-1}d_\nu^{k-2-i} + d_{\nu-1}^{k-2-i}d_\nu^i \equiv 0 \pmod 2$$

(this is clear if either $d_{\nu-1}$ or d_ν is even, and also if $d_{\nu-1}$ and d_ν are both odd). We thus have $a_j^i(2) \equiv 0 \pmod 2$. In addition, if $i = (k-2)/2$, we have as before $c'^i_\ell(b/d) = 2c^i_\ell(b/d)$; hence

$$\begin{aligned} \frac{a_j^i(2)}{2} &= \sum_{0\le\ell\le j}(-1)^{j-\ell}\binom{j}{\ell}2^\ell c^i_\ell\left(\frac{1}{2}\right) \equiv c^i_0\left(\frac{1}{2}\right) \\ &\equiv \sum_{0\le\nu\le 1}\binom{k-2}{(k-2)/2}(d_{\nu-1}d_\nu)^{(k-2)/2} \equiv 0 \pmod 2 \end{aligned}$$

since $d_{-1} = 0$, $d_0 = 1$, $d_1 = 2$; hence $d_{-1}d_0 = 0$ and $d_0d_1 = 2$. Since in addition j is odd so that $p^j + p^{k-2-j} \equiv 0 \pmod 2$, the lemma follows. □

The slightly deeper result mentioned above is the following:

Theorem 11.11.9. *For all primes p the eigenvalues of the Hecke operators $T(p)$ acting on $M_k(\Gamma)$ are* twice *totally real algebraic integers* (*in other words, they are "even"*).

Proof. Let $f = \sum_{n\geq 1} a(n)q^n$ be a normalized eigenfunction of all Hecke operators $T(n)$. Since $T(p)f = a(p)f$, by the above lemma we have

$$B^-(p)r^-(f) = \frac{a(p)}{2}r^-(f)\ .$$

Since $f \in S_k(\Gamma)(\mathbb{R})$ and $f \neq 0$, the injectivity of the map r^- (Proposition 11.9.10) implies that $r^-(f) \neq 0$, so that $a(p)/2$ is an eigenvalue of the matrix $B^-(p)$ which has integer entries; hence it is an algebraic *integer*. □

This theorem, conjectured by Serre, was proved simultaneously and independently by Hatada [**Hat77**] and the first author [**Coh77a**] in 1977.

Final Remark 11.11.10. Theorem 11.11.2 does not tell us anything about the arithmetic nature of $\omega^-(f)$, $\omega^+(f)$, or $\omega^+(f)/\omega^-(f)$. It is highly plausible that they are transcendental; on the other hand, recall that we have proved in Corollary 11.11.4 that $\omega^-(f)\omega^+(f) \in \langle f, f\rangle\mathbb{Q}(f)$.

11.12. Rankin–Selberg Theory

Although it is possible to develop the theory in a wider context, in this section we will restrict to the group $\Gamma_0(N)$. For notational simplicity, we set $G = \Gamma_0(N)$ and $\psi(N) = [\Gamma : \Gamma_0(N)] = [\Gamma : G] = N\prod_{p\mid N}(1 + 1/p)$.

Theorem 11.12.1. *Let χ be a character modulo N, let $f = \sum_{n\geq 0} a(n)q^n$ and $g = \sum_{n\geq 0} b(n)q^n$ both belong to $S_k(G,\chi)$, and define the Dirichlet series*

$$S_2(f,g)(s) = \sum_{n\geq 1}\frac{a(n)\overline{b(n)}}{n^s}\ .$$

Then this series converges absolutely for $\Re(s) > k$ and we have

$$\int_{\Gamma_\infty\backslash\mathfrak{H}} f(\tau)\overline{g(\tau)}y^{s+1}\,d\mu = \frac{\Gamma(s)}{(4\pi)^s}S_2(f,g)(s)\ .$$

Proof. A fundamental domain for $\Gamma_\infty\backslash\mathfrak{H}$ is given by the strip $[0,1]\times[0,\infty[$. Assume first that $f = g$ (we will see below that it is easy to reduce to that case). By applying the Parseval–Bessel formula (Proposition 3.1.10) to $F(x) = G(x) = f(x+iy)$, so that $c_n(F) = c_n(G) = a(n)e^{-2\pi ny}$, we see that

$$\int_0^1 |f(x+iy)|^2dx = \sum_{n\geq 1}|a(n)|^2e^{-4\pi ny}\ .$$

Now from the proof of Hecke's bound, Theorem 9.2.1(a), we know that when $y \to 0$, then $f(x+iy) = O(y^{-k/2})$ uniformly in $x \in \mathbb{R}$. Therefore the left-hand side (hence also the right-hand side) of the above formula is $O(y^{-k})$. Thus, if we set $f_n(y) = |a(n)|^2y^{s-1}e^{-4\pi ny}$, then $f_n(y)$ is continuous and since

$a(n)$ is polynomially bounded, the series $\sum_{n\geq 1} f_n(y)$ converges pointwise for all $y>0$. More importantly, for any $0<y\leq 1$ we have

$$\begin{aligned}\left|\sum_{1\leq j\leq n} f_j(y)\right| &\leq \sum_{1\leq j\leq n} |f_j(y)| = y^{\Re(s)-1}\sum_{1\leq j\leq n}|a(j)|^2 e^{-4\pi jy}\\ &\leq y^{\Re(s)-1}\sum_{j\geq 1}|a(j)|^2 e^{-4\pi jy} \leq Ay^{\Re(s)-k-1}\end{aligned}$$

for a suitable constant A. Although a minor point, this is the only place where we use that $f=g$, since otherwise we could not immediately bound the partial sum by the infinite series. If $y\geq 1$, then we trivially have

$$\begin{aligned}\left|\sum_{1\leq j\leq n} f_j(y)\right| &\leq y^{\Re(s)-1}e^{-2\pi y}\sum_{j\geq 1}|a(j)|^2 e^{-2\pi jy}\\ &\leq y^{\Re(s)-1}e^{-2\pi y}\sum_{j\geq 1}|a(j)|^2 e^{-2\pi j} \leq By^{\Re(s)-1}e^{-2\pi y}\end{aligned}$$

for a suitable constant B, again since $a(j)$ is polynomially bounded. Thus, if we take $g(y)=Ay^{\Re(s)-k-1}$ for $0<y\leq 1$ and $g(y)=By^{\Re(s)-1}e^{-2\pi y}$ for $y>1$, the integral $\int_0^\infty g(y)\,dy$ converges for $\Re(s)>k$. This shows that the conditions of Corollary 3.1.5 are satisfied and it follows that we can integrate term by term in y from 0 to ∞ and we deduce that

$$\begin{aligned}\int_0^\infty y^{s-1}\left(\int_0^1 |f(x+iy)|^2\,dx\right)dy &= \sum_{n\geq 0}|a(n)|^2\int_0^\infty y^{s-1}e^{-4\pi ny}\\ &= \frac{\Gamma(s)}{(4\pi)^s}\sum_{n\geq 0}\frac{|a(n)|^2}{n^s}\ ,\end{aligned}$$

where the series on the right-hand side is absolutely convergent. This proves the result when $f=g$.

Since $|a(n)\overline{b(n)}|\leq(|a(n)|^2+|b(n)|^2)/2$, it follows that the Dirichlet series $S_2(f,g)(s)$ is also absolutely convergent for $\Re(s)>k$, and therefore the interchange of summation and integration is justified in this case by the same argument as in the case $f=g$. Alternatively, we easily verify that for any complex numbers a and b we have

$$a\overline{b}=\frac{|a+b|^2-|a-b|^2+i(|a+ib|^2-|a-ib|^2)}{4}$$

and we can simply apply the result obtained above to $f\pm g$ and $f\pm ig$. $\square$

Theorem 11.12.2. *Let χ be a Dirichlet character modulo N and let $f = \sum_{n\geq 0} a(n)q^n$ and $g = \sum_{n\geq 0} b(n)q^n$ both belong to $S_k(G,\chi)$.*

(a) *For $\Re(s) > 1$ we have the formula*

$$\langle E_N(s)f,g\rangle_G = \frac{1}{\psi(N)}\frac{\Gamma(k+s-1)}{(4\pi)^{k+s-1}}\sum_{n\geq 1}\frac{a(n)\overline{b(n)}}{n^{s+k-1}}\ .$$

(b) *The series*

$$S_2(f,g)(s) = \sum_{n\geq 1}\frac{a(n)\overline{b(n)}}{n^s}$$

converges absolutely for $\Re(s) > k$ and can be extended to a meromorphic function in the whole complex plane with a simple pole, at $s = k$, with residue equal to

$$\frac{3}{\pi}\frac{(4\pi)^k}{(k-1)!}\langle f,g\rangle_G$$

and possible poles at the nonreal zeros of $\zeta(2s-2k+2)$. In addition, we have $S_2(f,g)(k-1/2) = 0$.

(c) *It satisfies the functional equation*

$$\mathcal{S}_2(f,g)(2k-1-s) = \mathcal{S}_2(f,g)(s)\ ,$$

where

$$\mathcal{S}_2(f,g)(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+1)\zeta(2s-2k+2)S_2(f,g)(s)\ .$$

Proof. It is easy to see that we can exchange integration and summation and we can then use the usual *unfolding method*:

$$\begin{aligned}
\psi(N)\langle E_N(s)f,g\rangle_G &= \int_{\mathfrak{F}(G)} f(\tau)\overline{g(\tau)}E_N(s)(\tau)y^k\,d\mu \\
&= \sum_{\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash G}\int_{\mathfrak{F}(G)} f(\tau)\overline{g(\tau)}|c\tau+d|^{2k}\Im(\gamma\tau)^{k+s}\,d\mu \\
&= \sum_{\gamma\in\Gamma_\infty\backslash G}\int_{\mathfrak{F}(G)} f(\gamma\tau)\overline{g(\gamma\tau)}\Im(\gamma\tau)^{k+s}\,d\mu \\
&= \sum_{\gamma\in\Gamma_\infty\backslash G}\int_{\gamma(\mathfrak{F}(G))} f(\tau)\overline{g(\tau)}\Im(\tau)^{k+s}\,d\mu \\
&= \int_{\Gamma_\infty\backslash\mathfrak{H}} f(\tau)\overline{g(\tau)}y^{k+s-2}\,dx\,dy.
\end{aligned}$$

It follows from Theorem 11.12.1 that for $\Re(s) > 1$ we have

$$\psi(N)\langle E_N(s)f, g\rangle_G = \frac{\Gamma(k+s-1)}{(4\pi)^{k+s-1}}\sum_{n\geq 1}\frac{a(n)\overline{b(n)}}{n^{s+k-1}}$$

and the series converges absolutely for $\Re(s) > 1$, which proves (a) and the first statement of (b); the other statements of (b) follows from the corresponding properties of $E_N(\tau, s)$ given by Corollary 8.5.9.

(c) Recall that $\mathcal{E}_N(s) = \pi^{-s}\Gamma(s)\zeta(2s)E_N(s)$. Thus, by (a) we have

$$\begin{aligned}\langle \mathcal{E}_N(s)f, g\rangle &= \pi^{-s}(4\pi)^{-(k+s-1)}\Gamma(k+s-1)\Gamma(s)\zeta(2s)S_2(f,g)(k+s-1)\\ &= \pi^{k-1}(2\pi)^{-(2k+2s-2)}\Gamma(k+s-1)\Gamma(s)\zeta(2s)S_2(f,g)(k+s-1)\\ &= \pi^{k-1}\mathcal{S}_2(k+s-1)\ ,\end{aligned}$$

proving the result since $\mathcal{E}_N(s)$ can be analytically continued to the whole complex plane with the functional equation $\mathcal{E}_N(1-s) = \mathcal{E}_N(s)$. ☐

Corollary 11.12.3. *Let $f = \sum_{n\geq 0} a(n)q^n \in S_k(G, \chi)$. The series*

$$\sum_{n\geq 1}\frac{|a(n)|^2}{n^s}$$

converges for $\Re(s) > k$ and can be extended to a meromorphic function in the whole complex plane with a simple pole, at $s = k$, with residue equal to

$$\frac{3}{\pi}\frac{(4\pi)^k}{(k-1)!}\langle f, f\rangle_G\ .$$

Proof. This follows immediately from the theorem. ☐

Remarks 11.12.4.

(a) Observe that Hecke's bound $a(n) = O(n^{k/2})$ (from Theorem 9.2.1) only implies that $S_2(f,g)(s)$ converges for $\Re(s) > k+1$ and the convergence of the series for $\Re(s) > k$ is therefore highly nontrivial. It is clear that this follows from Deligne's theorem (the Ramanujan–Petersson conjecture; see Section 9.2.3) which states that $a(n) = O(n^{(k-1)/2+\varepsilon})$ for all $\varepsilon > 0$ but of course the above result does not prove this. However, we will see in the next subsection that we can still obtain from this a nontrivial upper bound for $a(n)$.

(b) The fact that $\zeta(2s-2k+2)$ appears in the definition of $\mathcal{S}_2$ (and cannot be removed as we will do below) is an indication that the function $S_2(f,g)$ is not the "correct" function to study since we should have only exponential and gamma factors. In the next subsection we will describe the correct function for the case of $f = g$.

11.12.1. Symmetric Square L-Functions. In this subsection, we assume that $f = \sum_{n\ge1} a(n)q^n \in S_k(\Gamma)$ is a normalized eigenform for all Hecke operators. In particular, its L-function has an Euler product

$$L(f,s) = \prod_p \frac{1}{1 - a(p)p^{-s} + p^{k-1-2s}}\ ,$$

which, as we have seen, is equivalent to the fact that the coefficients $a(n)$ are multiplicative and satisfy the second-order linear recursion $a(p^{m+1}) = a(p)a(p^m) - p^{k-1}a(p^{m-1})$. We also know that the coefficients $a(n)$ are totally real algebraic integers and we can write

$$1 - a(p)X + p^{k-1}X^2 = (1-\alpha_p X)(1-\beta_p X)$$

for suitable complex numbers α_p and β_p such that $\alpha_p + \beta_p = a(p)$ and $\alpha_p\beta_p = p^{k-1}$ (by Deligne's theorem α_p and β_p are conjugate nonreal complex numbers of modulus $p^{(k-1)/2}$, but we do not need to know this). Thus,

$$L(f,s) = \prod_p \frac{1}{(1-\alpha_p p^{-s})(1-\beta_p p^{-s})}\ .$$

Since

$$\frac{1}{(1-\alpha X)(1-\beta X)} = \frac{1}{(\alpha-\beta)X}\left(\frac{1}{1-\alpha X} - \frac{1}{1-\beta X}\right)\ ,$$

it is easy to show that

$$a(p^m) = \frac{\alpha_p^{m+1} - \beta_p^{m+1}}{\alpha_p - \beta_p}\ .$$

Definition 11.12.5. With the above notation, we define the *symmetric square L-function* of f by the formula

$$L(\mathrm{Sym}^2(f),s) = \prod_p \frac{1}{(1-\alpha_p^2 p^{-s})(1-\alpha_p\beta_p p^{-s})(1-\beta_p^2 p^{-s})}\ .$$

Lemma 11.12.6. *Keep the above notation and set*

$$A(f,s) = \sum_{n\ge1} \frac{a(n)^2}{n^s} \quad \text{and} \quad B(f,s) = \sum_{n\ge1}\frac{a(n^2)}{n^s}\ .$$

For $\Re(s) > k$ we have

$$A(f,s) = \frac{\zeta(s-(k-1))}{\zeta(2s-2(k-1))} L(\mathrm{Sym}^2(f),s) \quad \text{and}$$
$$B(f,s) = \frac{1}{\zeta(2s-2(k-1))} L(\mathrm{Sym}^2(f),s)\ .$$

In particular, $L(\mathrm{Sym}^2(f),s)=\sum_{n\ge1}A(n)/n^s$ *with*

$$A(n)=\sum_{m^2\mid n}m^{2k-2}a((n/m^2)^2)=\sum_{m\mid n}(-1)^{\Omega(m)}m^{k-1}a(n/m)^2\ ,$$

where $\Omega(m)$ *is the number of prime divisors of* m *counted with multiplicity.*

We note, in passing, that since $a(n)$ is real, we have $a(n)^2=|a(n)|^2$.

Proof. The formulas for $A(f,s)$ and $B(f,s)$ have already been proved in Corollary 10.8.2, and the last formulas follow immediately from the first two, using the elementary identity

$$\frac{\zeta(2s)}{\zeta(s)}=\prod_p(1+p^{-s})^{-1}=\sum_{m\ge1}\frac{(-1)^{\Omega(m)}}{m^s}\ .$$

Note that Corollary 10.8.2 only proved convergence for $\Re(s)>k+1-2\delta$ for some $\delta>0$, but as already mentioned in the proof, Deligne's theorem implies convergence for $\Re(s)>k$. $\square$

Corollary 11.12.7. *Let* $f\in S_k(\Gamma)$ *be a normalized eigenform.*

(a) *For* $\Re(s)>1$ *we have*

$$L(\mathrm{Sym}^2(f),s+k-1)=\frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)}\frac{\zeta(2s)}{\zeta(s)}\langle E(s)f,f\rangle\ .$$

(b) *The function* $L(\mathrm{Sym}^2(f),s)$ *extends to a meromorphic function in the whole complex plane. In addition:*

(i) *We have*

$$L(\mathrm{Sym}^2(f),k)=\frac{\pi}{2}\frac{(4\pi)^k}{(k-1)!}\langle f,f\rangle\ .$$

(ii) *The only possible poles of* $L(\mathrm{Sym}^2(f),s)$ *are the nonreal zeros of* $\zeta(s-k+1)$*, in particular* $k-1<\Re(s)<k$*, and if the Riemann hypothesis is true, then* $\Re(s)=k-1/2$ *(in fact such poles do not exist; see Remarks* 11.12.8 *below).*

(iii) *If we set*

$$\Lambda(\mathrm{Sym}^2(f),s)=\pi^{-3s/2}\Gamma(s/2)\Gamma((s+1)/2)\Gamma((s-k)/2+1)L(\mathrm{Sym}^2(f),s)\ ,$$

we have the functional equation

$$\Lambda(\mathrm{Sym}^2(f),2k-1-s)=\Lambda(\mathrm{Sym}^2(f),s)\ .$$

Proof. (a) From the above lemma we have

$$L(\mathrm{Sym}^2(f),s)=\frac{\zeta(2s-2k+2)}{\zeta(s-k+1)}S_2(f,f)(s)\ ,$$

and hence (a) follows immediately from Theorem 11.12.2(a), as does (b)(i) since $\zeta(s-k+1)$ has a simple pole at $s=k$ with residue 1. In addition, by (a), since $G(s)=\zeta(2s)E(\tau,s)$ is holomorphic except for a simple pole at $s=1$, it follows that $\zeta(s)L(\mathrm{Sym}^2(f),s+k-1)$ has the same property, so that the possible poles of $L(\mathrm{Sym}^2(f),s)$ are the zeros of $\zeta(s-k+1)$. Since

$$L(\mathrm{Sym}^2(f),s)=\frac{(4\pi)^s}{\Gamma(s)}\frac{\zeta(2s-2k+2)}{\zeta(s-k+1)}\langle E(s-k+1)f,f\rangle$$

and since for s tending to $k-1-2m$ with $m\geq 1$ the quantity $\zeta(2s-2k+2)/\zeta(s-k+1)$ does not vanish and since by Corollary 5.2.13 $E(s)$ does not have any real pole except at $s=1$, it follows that only the nontrivial zeros can contribute, proving (b)(ii). Finally, by Theorem 11.12.2 the function

$$\mathcal{S}_2(f,f)(s)=(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+1)\zeta(s-k+1)L(\mathrm{Sym}^2(f),s)$$

satisfies the functional equation $\mathcal{S}_2(f,f)(2k-1-s)=\mathcal{S}_2(f,f)(s)$. As already mentioned, we would like to remove the factor $\zeta(s-k+1)$. Here this is possible: by the duplication formula for the gamma function we have

$$\Gamma(s-k+1)=\pi^{-1/2}2^{s-k}\Gamma((s-k)/2+1)\Gamma((s-k+1)/2)\;,$$

and setting as usual $\Lambda(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$ we have

$$\Gamma(s-k+1)\zeta(s-k+1)=(4\pi)^{(s-k)/2}\Gamma((s-k)/2+1)\Lambda(s-k+1)\;,$$

so that

$$\begin{aligned}&\mathcal{S}_2(f,f)(s)\\&\quad=(2\pi)^{-2s}(4\pi)^{(s-k)/2}\Gamma(s)\Gamma((s-k)/2+1)\Lambda(s-k+1)L(\mathrm{Sym}^2(f),s)\\&\quad=2^{-(s+k)}\pi^{-(3s/2+k/2)}\Gamma(s)\Gamma((s-k)/2+1)\Lambda(s-k+1)L(\mathrm{Sym}^2(f),s)\\&\quad=2^{-(k+1)}\pi^{-(k+1)/2}\Lambda(s-k+1)\Lambda(\mathrm{Sym}^2(f),s)\;,\end{aligned}$$

using again the duplication formula. Since $\Lambda(1-s)=\Lambda(s)$, it follows that $\Lambda(s-k+1)$ is invariant when changing s into $2k-1-s$, proving (iii). □

Remarks 11.12.8.

(a) Since (the denominator of) the Euler factor for $L(\mathrm{Sym}^2(f),s)$ has degree 3 in p^{-s}, it is expected, by the general theory of L-functions, that if $L(\mathrm{Sym}^2(f),s)$ is really an L-function, then the completed function should indeed contain 3 gamma factors of the form $\Gamma(s/2+\mu_i)$ for certain μ_i, as above.

(b) An important result, proved independently by Shimura in 1975 [**Shi75a**] and Zagier in 1976 [**Zag77**], is that $L(\mathrm{Sym}^2(f),s)$ has in fact no poles.

11.12.2. The Rankin Bound for Fourier Coefficients. By using the above theorem and standard methods from analytic number theory it is then possible to obtain nontrivial bounds for the Fourier coefficients of cusp forms. These bounds are not as strong as Deligne's (or even as Weil's using Kloosterman sum bounds) but have the advantage that they are easier to generalize. The result is as follows:

Theorem 11.12.9 (Rankin [**Ran39**]). *Let* $f = \sum_{n\ge0} a(n)q^n \in S_k(\Gamma)$.

(a) *As* $X \to \infty$ *we have*

$$\sum_{1\le n\le X} |a(n)|^2 = \alpha(k)\langle f,f\rangle X^k + O(X^{k-2/5})$$

with $\alpha(k) = (3/\pi)(4\pi)^k/(k-1)!$.

(b) *As* $X \to \infty$ *we have*

$$a(n) = O(X^{k/2-1/5})\ .$$

The proof uses a very technical theorem of Landau [**Lan15**] applied to the function

$$\phi(s) = L_2(f, s+k-1) = \sum_{n\ge1} \frac{b(n)}{n^s} \quad \text{with} \quad b(n) = \frac{|a(n)|^2}{n^{k-1}}$$

(see Corollary 11.12.3) and the standard analytic properties of the symmetric square L-function that we have seen above. Note that the same result is true (with a similar proof) if $f \in S_k(\Gamma_0(N), \chi)$. For more details see [**Ran39**].

Exercises

11.1. Let $A > 0$ and $y > 0$. Show that there exists a constant C independent of A and y such that

$$\sum_{n\ge0} n^A e^{-ny} < C\min((1-e^{-y})^{-(A+1)}, e^{-A}(A/y)^{-(A+1)})$$

(Hint: use Stirling's formula). See also Exercise 9.9.

11.2. Give a direct proof of Theorem 11.9.4 using only the results of Section 11.8, the definition of r_j, and the relations $S^2 = (TS)^3 = I$ in Γ. Hint: show first that

$$\left(\int_0^{i\infty} + \int_{TS(0)}^{TS(i\infty)} + \int_{TSTS(0)}^{TSTS(i\infty)}\right) f(\tau)\tau^j\, d\tau = 0\ .$$

11.3. (i) In [**BBG04**], the authors give the following formulas: for $k \equiv 0 \pmod 4$ we have

$$\zeta(k-1) = -2\sum_{n\geq 1}\frac{1}{n^{k-1}(e^{2\pi n}-1)} + \frac{2}{\pi}\left(\frac{k+3}{4}\zeta(k) - \sum_{1\leq n\leq k/4-1}\zeta(4n)\zeta(k-4n)\right)\ .$$

Show that this is equivalent to the formula of Corollary 11.6.3(a).

(ii) For $k \equiv 2 \pmod 4$ we have

$$\zeta(k-1) = -2\sum_{n\geq 1}\frac{(1+4\pi n/(k-2))e^{2\pi n}-1}{n^{k-1}(e^{2\pi n}-1)^2} + \frac{2}{(k-2)\pi}\Big((k/2)\zeta(k) + \sum_{1\leq n\leq k/2-1}(-1)^n(2n)\zeta(2n)\zeta(k-2n)\Big)\ .$$

Show that this is equivalent to the formula of Corollary 11.6.3(b).

(iii) Still for $k \equiv 2 \pmod 4$, set $N = (k-2)/4$. Prove the following identity:

$$(2-(-4)^{-N})\sum_{n\geq 1}\frac{\coth(n\pi)}{n^{k-1}} - 4^{-2N}\sum_{n\geq 1}\frac{\tanh(n\pi)}{n^{k-1}} = \pi^{k-1}Q_N\ ,$$

where

$$Q_N = -\sum_{0\leq n\leq k/2}((-1)^{n(n+1)/2}(-4)^N 2^n + (-4)^n)\frac{B_{k-2n}B_{2n}}{(k-2n)!(2n)!}\ .$$

(iv) Deduce from this another general identity for $\zeta(k-1)$ when $k \equiv 2 \pmod 4$. For instance, show that

$$\zeta(5) = \frac{\pi^5}{294} + \frac{2}{35}\sum_{n\geq 1}\frac{1}{n^5(e^{2\pi n}+1)} - \frac{72}{35}\sum_{n\geq 1}\frac{1}{n^5(e^{2\pi n}-1)}\ .$$

(v) Show that

$$\sum_{n\geq 1}\frac{1}{n(e^{2\pi n}+1)} + \sum_{n\geq 1}\frac{1}{n(e^{2\pi n}-1)} = \frac{\pi}{6} - \frac{3\log(2)}{4}\ .$$

11.4. Prove Propositions 11.9.11 and 11.11.1

11.5. Extend the parity result in Theorem 11.11.9 and show that if $p \equiv 3 \pmod 4$, the eigenvalues of $T(p)$ are in fact 4 times algebraic integers. More generally, up to what value of a is it true that $p \equiv -1 \pmod{2^a}$ implies that the eigenvalues of $T(p)$ are of the form $2^a\alpha_p$ for some algebraic integer α_p? It is known (because of $\tau(p)$) that this cannot be true for $a \geq 14$.