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Preface

This is the second volume of a graduate course in set theory. Volume I covered the basics of modern set theory and was primarily aimed at beginning graduate students. Volume II is aimed at more advanced graduate students and research mathematicians specializing in fields other than set theory. It contains short but rigorous introductions to various set-theoretic techniques that have found applications outside of set theory. Although we think of Volume II as a natural continuation of Volume I,[1] each volume is sufficiently self-contained to be studied separately.

The main prerequisite for Volume II is a knowledge of basic naive and axiomatic set theory.[2] Moreover, some knowledge of mathematical logic and general topology is indispensible for reading this volume. A minicourse in mathematical logic was given in Chapters 5 and 6 of Volume I, and we include an appendix on general topology at the end of this volume. Our terminology is fairly standard. For the benefit of those readers who learned their basic set theory from a different source than our Volume I we include a short section on somewhat idiosyncratic notations introduced in Volume I. In particular, some of the material on mathematical logic covered in Chapter 5 is briefly reviewed. The book can be used as a text in the classroom as well as for self-study.

We tried to keep the length of the text moderate. This may explain the absence of many a worthy theorem from this book. Our most important criterion for inclusion of an item was frequency of use outside of pure set theory. We want to emphasize that "item" may mean either an important concept (like "equiconsistency with the existence of a measurable cardinal"), a theorem (like Ramsey's Theorem), or a proof technique (like the craft of using Martin's Axiom). Therefore, we occasionally illustrate a technique by proving a somewhat marginal theorem. Of course, the "frequency of use outside set theory" is based on our subjective perceptions.

At the end of most chapters there are "Mathographical Remarks." Their purpose is to show where the material fits in the history and literature of the subject. We hope they will provide some guidance for further reading in set theory. They should not be mistaken for "scholarly remarks" though. We did not make any effort whatsoever to trace the theorems of this book to their origins. However, each of the theorems presented here can also be found in at least one of the more specialized texts reviewed in the "Mathographical Remarks." Therefore, we do not feel guilty of severing chains of historical evidence.

[1] This is the reason why the present volume starts with Chapter 13.

[2] Possible alternatives to our Volume I are such texts as: K. Devlin, *The Joy of Sets. Fundamentals of Contemporary Set Theory,* Springer Verlag, 1993; A. Lévy; *Basic Set Theory,* Springer Verlag, 1979; or J. Roitman, *Introduction to Modern Set Theory,* John Wiley, 1990.

Much of this book is written like a dialogue between the authors and the reader. This is intended to model the practice of creative mathematical thinking, which more often than not takes on the form of an inner dialogue in a mathematician's mind. You will quickly notice that this text contains many question marks. This reflects our conviction that in the mathematical thought process it is at least as important to have a knack for asking the right questions at the right time as it is to know some of the answers.

You will benefit from this format only if you do your part and actively participate in the dialogue. This means in particular: Whenever we pose a rhetorical question, pause for a moment and ponder the question before you read our answer. Sometimes we put a little more pressure on you and call our rhetorical questions EXERCISES. Not all exercises are rhetorical questions that will be answered a few lines later. Often the completion of a proof is left as an exercise. We also may ask you to supply the entire proof of an interesting theorem, or an important example. Nevertheless, we recommend that you attempt the exercises right away, especially all the easier ones. Most of the time it will be easier to digest the ensuing text if you have worked on the exercise, even if you were unable to solve it.

We often make references to solutions of exercises from earlier chapters. Sometimes the new material will make an old and originally quite hard exercise seem trivial, and sometimes a new question can be answered by modifying the solution to a previous problem. Therefore it is a good idea to collect your solutions and even your failed attempts at solutions in a folder where you can look them up later.

The level of difficulty of our exercises varies greatly. To help the reader save time, we rated each exercise according to what we perceive as its level of difficulty. The rating system is the same as used by American movie theatres. Everybody should attempt the exercises rated G (general audience). Beginners are encouraged to also attempt exercises rated PG (parental guidance), but may sometimes want to consult their instructor for a hint. Exercises rated R (restricted) are intended for mature audiences. The X–rated problems must not be attempted by anyone easily offended or discouraged.

In Chapters 17, 18, 19, and 22 we will discuss consequences of statements that are relatively consistent with, but no provable in ZFC: the Continuum Hypothesis (abbreviated CH), Martin's Axiom (abbreviated MA), and the Diamond Principle (abbreviated $\diamondsuit$). We will write "THEOREM n.m (CH)" in order to indicate that Theorem n.m is provable in the theory ZFC + CH rather than ZFC alone.

We are greatly indebted to Mary Anne Swardson of Ohio University for reading the very first draft of this book and generously applying her red pencil to it. Special thanks are due to Ewelina Skoracka-Just for her beautiful typesetting of this volume.