Preface

The Author List, I: giving credit where credit is due. The first author: Senior grad student in the project. Made the figures.
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Preface to the Second Edition

There are a lot of changes in the second edition. In the first part of the book, starting from Chapter 2, instead of considering real-valued functions, I treat functions $u : I \to Y$, where $I \subseteq \mathbb{R}$ is an interval and Y is a metric space. This change is motivated by the addition of a new chapter, Chapter 8, where I introduce the Bochner integral and study functions mapping time into Banach spaces. This type of functions plays a crucial role in evolution equations.

Another important addition in the first part of the book is Section 7 in Chapter 7, which begins the study of interpolation inequalities for Sobolev functions of one variable. One wants to estimate some appropriate norm of an intermediate derivative in terms of the norms of the function and the highest-order derivative.

Except for Chapter 8, **the first part of the book is meant as a textbook for an advanced undergraduate course or beginning graduate course on real analysis or functions of one variable**. One should simply take Y to be the real line $\mathbb{R}$ so that $\mathcal{H}^1$ reduces to the Lebesgue measure $\mathcal{L}^1$. All the results needed from measure theory are listed in the appendices at the end of the book.

The second part of the book starts with Chapter 9, which went through drastic changes. In the revised version I give an overview of classical results for functions of several variables, which are somehow scattered in the literature. These include Rademacher's and Stepanoff's differentiability theorems, Whitney's extension theorem, and Brouwer's fixed point theorem. The focus of the chapter is now the divergence theorem for Lipschitz domains. While this fundamental result is quoted and used in every book on partial differential equations, it's hard to find a thorough proof in the literature. To introduce the surface integral on the boundary I start by proving the area formula, first in the C^1 case, and then, using Whitney's extension theorem in the Lipschitz case.

In the chapter on distributions, Chapter 10, I added rapidly decreasing functions, tempered distributions, and Fourier transforms. This was long overdue.

The book is structured in such a way that an instructor of a course on Sobolev spaces could actually skip Chapters 9 and 10, which serve mainly as reference chapters and jump to Chapter 11.

Chapters 11, 12, and 13, the first part of Chapter 18 could be used as a textbook on a course on Sobolev spaces. They are mostly self-contained.

One of the main changes in these chapters is that I caved in and decided to include higher order derivatives. The reason why I did not do it in the first edition was because standard operations like the product rule and the chain formula become incredibly messy for higher order derivatives and there are so many multi-indices to take into account that even elementary proofs become unnecessarily complicated. My compromise is to present proofs first in $W^{1,p}(\Omega)$ (first-order derivatives) or in $W^{2,p}(\Omega)$ (second-order derivatives) and only after, when the idea of the proof is clear, to do the general case $W^{m,p}(\Omega)$. I did not always follow this rule, since sometimes there was no significant change in difficulty in treating $W^{m,p}(\Omega)$ rather than $W^{1,p}(\Omega)$.

The advantage in having higher order derivatives is that I can now prove the classical interpolation inequalities of Gagliardo and Nirenberg. These are done in Section 12.5 in Chapter 12 for $\mathbb{R}^N$ and in Section 13.3 in Chapter 13 for uniformly Lipschitz domains. The main novelty with respect to other textbooks is that in the case of uniformly Lipschitz domains I do not assume that functions are in $W^{m,p}(\Omega)$ but only that u is in $L^q(\Omega)$ and the weak derivatives of order m are in $L^p(\Omega)$.

Another new section is Section 12.4 in Chapter 12, where I study superposition in Sobolev spaces.

The last major departure from the first edition is the chapter on Besov spaces, Chapter 17. This chapter was completely rewritten in collaboration with Ian Tice. The main motivation behind the changes was the proof that the trace space of functions in $W^{2,1}(\mathbb{R}^N)$ is given by the Besov space $B^{1,1}(\mathbb{R}^{N-1})$ (see Chapter 18). I am only aware of one complete and simple proof, which is in a recent paper of Mironescu and Russ [**173**]. It makes use of two equivalent norms for $B^{1,1}(\mathbb{R}^{N-1})$, one using second order difference quotients and the other the Littlewood-Paley norm. To introduce the second norm, I went through several different versions of Chapter 17. Eventually, to study Besov spaces I used heavily the K method of real interpolation introduced by Peetre. The interpolation theory needed was added in a new chapter, Chapter 16.

Webpage for mistakes, comments, and exercises: The AMS is hosting a webpage for this book at

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where updates, corrections, and other material may be found.

Preface to the First Edition

There are two ways to introduce Sobolev spaces: The first is through the elegant (and abstract) theory of distributions developed by Laurent Schwartz in the late 1940s; the second is to look at them as the natural development and unfolding of monotone, absolutely continuous, and BV functions[1] of one variable.

To my knowledge, this is one of the first books to follow the second approach. I was more or less forced into it: This book is based on a series of lecture notes that I wrote for the graduate course "Sobolev Spaces", which I taught in the fall of 2006 and then again in the fall of 2008 at Carnegie Mellon University. In 2006, during the first lecture, I found out that half of the students were beginning graduate students with no background in functional analysis (which was offered only in the spring) and very little in measure theory (which, luckily, was offered in the fall). At that point I had two choices: continue with a classical course on Sobolev spaces and thus lose half the class after the second lecture or toss my notes and rethink the entire operation, which is what I ended up doing.

I decided to begin with monotone functions and with the Lebesgue differentiation theorem. To my surprise, none of the students taking the class had actually seen its proof.

I then continued with functions of bounded pointwise variation and absolutely continuous functions. While these are included in most books on real analysis/measure theory, here the perspective and focus are rather different, in view of their applications to Sobolev functions. Just to give an example, most books study these functions when the domain is either the closed interval $[a, b]$ or $\mathbb{R}$. I needed, of course, open intervals (possibly unbounded).

[1]BV functions are functions of bounded variation.

This changed things quite a bit. A lot of the simple characterizations that hold in $[a, b]$ fall apart when working with arbitrary unbounded intervals.

After the first three chapters, in the course I actually jumped to Chapter 7, which relates absolutely continuous functions with Sobolev functions of one variable, and then started with Sobolev functions of several variables. In the book I included three more chapters: Chapter 5 studies curves and arclength. I think it is useful for students to see the relation between rectifiable curves and functions with bounded pointwise variation.

Some classical results on curves that most students in analysis have heard of, but whose proof they have not seen, are included, among them Peano's filling curve and the Jordan curve theorem.

Section 5.4 is more advanced. It relates rectifiable curves with the $\mathcal{H}^1$ Hausdorff measure. Besides Hausdorff measures, it also makes use of the Vitali–Besicovitch covering theorem. All these results are listed in Appendices B and C.

Chapter 6 introduces Lebesgue–Stieltjes measures. The reading of this chapter requires some notions and results from abstract measure theory. Again it departs slightly from modern books on measure theory, which introduce Lebesgue–Stieltjes measures only for right continuous (or left) functions. I needed them for an arbitrary function, increasing or with bounded pointwise variation. Here, I used the monograph of Saks [**201**]. I am not completely satisfied with this chapter: I have the impression that some of the proofs could have been simplified more using the results in the previous chapters. Readers' comments will be welcome

Chapter 4 introduces the notion of decreasing rearrangement. I used some of these results in the second part of the book (for Sobolev and Besov functions). But I also thought that this chapter would be appropriate for the first part. The basic question is how to modify a function that is not monotone into one that is, keeping most of the good properties of the original function. While the first part of the chapter is standard, the results in the last two sections are not covered in detail in classical books on the subject.

As a final comment, the first part of the book could be used for an advanced undergraduate course or beginning graduate course on real analysis or functions of one variable.

The second part of the book starts with one chapter on absolutely continuous transformations from domains of $\mathbb{R}^N$ into $\mathbb{R}^N$. I did not cover this chapter in class, but I do think it is important in the book in view of its ties with the previous chapters and their applications to the change of variables formula for multiple integrals and of the renewed interest in the subject in recent years. I only scratched the surface here.

Chapter 10 introduces briefly the theory of distributions. The book is structured in such a way that an instructor could actually skip it in case the students do not have the necessary background in functional analysis (as was true in my case). However, if the students do have the proper background, then I would definitely recommend including the chapter in a course. It is really important.

Chapter 11 starts (at long last) with Sobolev functions of several variables. Here, I would like to warn the reader about two quite common misconceptions. Believe it or not, if you ask a student what a Sobolev function is, often the answer is "A Sobolev function is a function in L^p whose derivative is in L^p." This makes the Cantor function a Sobolev function :(

I hope that the first part of the book will help students to avoid this danger.

The other common misconception is, in a sense, quite the opposite, namely to think of weak derivatives in a very abstract way not related to the classical derivatives. One of the main points of this book is that weak derivatives of a Sobolev function (but not of a function in BV!) are simply (classical) derivatives of a good representative. Again, I hope that the first part of the volume will help here.

Chapters 11, 12, and 13 cover most of the classical theorems (density, absolute continuity on lines, embeddings, chain rule, change of variables, extensions, duals). This part of the book is more classical, although it contains a few results published in recent years.

Chapter 14 deals with functions of bounded variation of several variables. I covered here only those parts that did not require too much background in measure theory and geometric measure theory. This means that the fundamental results of De Giorgi, Federer, and many others are not included here. Again, I only scratched the surface of functions of bounded variation. My hope is that this volume will help students to build a solid background, which will allow them to read more advanced texts on the subject.

Chapter 17 is dedicated to the theory of Besov spaces. There are essentially three ways to look at these spaces. One of the most successful is to see them as an example/by-product of interpolation theory (see [**7**], [**232**], and [**233**]). Interpolation is very elegant, and its abstract framework can be used to treat quite general situations well beyond Sobolev and Besov spaces.

There are two reasons for why I decided not to use it: First, it would depart from the spirit of the book, which leans more towards measure theory and real analysis and less towards functional analysis. The second reason is that in recent years in calculus of variations there has been an increased interest in nonlocal functionals. I thought it could be useful to present some techniques and tricks for fractional integrals.

The second approach is to use tempered distributions and Fourier theory to introduce Besov spaces. This approach has been particularly successful for its applications to harmonic analysis. Again it is not consistent with the remainder of the book.

This left me with the approach of the Russian school, which relies mostly on the inequalities of Hardy, Hölder, and Young, together with some integral identities. The main references for this chapter are the books of Besov, Il′in, and Nikol′skiĭ [**26**], [**27**].

I spent an entire summer working on this chapter, but I am still not happy with it. In particular, I kept thinking that there should be easier and more elegant proofs of some of the results, but I could not find one.

In Chapter 18 I discuss traces of Sobolev and BV functions. Although in this book I only treat first-order Sobolev spaces, the reason I decided to use Besov spaces over fractional Sobolev spaces (note that in the range of exponents treated in this book these spaces coincide, since their norms are equivalent) is that the traces of functions in $W^{k,1}(\Omega)$ live in the Besov space $B^{k-1,1}(\partial\Omega)$, and thus a unified theory of traces for Sobolev spaces can only be done in the framework of Besov spaces.

Finally, Chapter 15 is devoted to the theory of symmetrization in Sobolev and BV spaces. This part of the theory of Sobolev spaces, which is often missing in classical textbooks, has important applications in sharp embedding constants, in the embedding $N = p$, as well as in partial differential equations.

In Appendices A, B, and C I included essentially all the results from functional analysis and measure theory that I used in the text. I only proved those results that cannot be found in classical textbooks.

What is missing in this book: For didactic purposes, when I started to write this book, I decided to focus on first-order Sobolev spaces. In my original plan I actually meant to write a few chapters on higher-order Sobolev and Besov spaces to be put at the end of the book. Eventually I gave up: It would have taken too much time to do a good job, and the book was already too long.

As a consequence, interpolation inequalities between intermediate derivatives are missing. They are treated extensively in [**7**].

Another important theorem that I considered adding and then abandoned for lack of time was Jones's extension theorem [**122**].

Chapter 14, the chapter on BV functions of several variables, is quite minimal. As I wrote there, I only touched the tip of the iceberg. Good reference books of all the fundamental results that are not included here are [**10**], [**72**], and [**251**].

References: The rule of thumb here is simple: I only quoted papers and books that I actually read at some point (well, there are a few papers in German, and although I do have a copy of them, I only "read" them in a weak sense, since I do not know the language). I believe that misquoting a paper is somewhat worse than not quoting it. Hence, if an important and relevant paper is not listed in the references, very likely it is because I either forgot to add it or was not aware of it. While most authors write books because they are experts in a particular field, I write them because I want to learn a particular topic. I claim no expertise on Sobolev spaces.

Webpage for mistakes, comments, and exercises: In a book of this length and with an author a bit absent-minded, typos and errors are almost inevitable. I will be very grateful to those readers who write to giovanni@andrew.cmu.edu indicating those errors that they have found. The AMS is hosting a webpage for this book at

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where updates, corrections, and other material may be found.

The book contains more than 200 exercises, but they are not equally distributed. There are several on the parts of the book that I actually taught, while other chapters do not have as many. If you have any interesting exercises, I will be happy to post them on the web page.

Giovanni Leoni