Distributions

Prospective Grad Students, IV: "Can you really live comfortably in this major metropolitan area with that stipend, or will I find myself living out of a closet working part time as a shoe salesman?"

— Jorge Cham, www.phdcomics.com

Throughout this chapter the implicit space is the Euclidean space $\mathbb{R}^N$ and $\Omega \subseteq \mathbb{R}^N$ is an open set, not necessarily bounded.

10.1. The Spaces $\mathcal{D}_K(\Omega)$, $\mathcal{D}(\Omega)$, and $\mathcal{D}'(\Omega)$

In this section, given an open set $\Omega \subseteq \mathbb{R}^N$, we construct a topology on the space $C_c^\infty(\Omega)$. We begin by considering the space of all functions with support in a given compact set $K \subset \Omega$. In this subspace the topology should be consistent with the natural notion of convergence, which is uniform convergence of the functions and of all their partial derivatives of any order.

Consider an open set $\Omega \subseteq \mathbb{R}^N$ and fix a compact set $K \subset \Omega$. Let $\mathcal{D}_K(\Omega)$ be the set of all functions in $C_c^\infty(\Omega)$ whose support is contained in K, that is,

$$\mathcal{D}_K(\Omega) := \{\phi \in C_c^\infty(\Omega) : \operatorname{supp} \phi \subseteq K\}.$$

For each $j \in \mathbb{N}_0$ define the norm $\|\cdot\|_{K,j}$ on $\mathcal{D}_K(\Omega)$ by

$$\|\phi\|_{K,j} := \sup\{|\partial^\alpha \phi(x)| : x \in K, \ \alpha \in \mathbb{N}_0^N \text{ with } |\alpha| \le j\}. \tag{10.1}$$

By Theorem A.27, the family of norms $\{\|\cdot\|_{K,j}\}_j$ turns $\mathcal{D}_K(\Omega)$ into a locally convex space and a base for the topology τ_K is given by all sets of the form

$$\{\phi \in \mathcal{D}_K(\Omega) : \|\phi\|_{K,j_1} < 1/\ell_1, \ldots, \|\phi\|_{K,j_k} < 1/\ell_k\},$$

where $j_1, \ldots, j_k \in \mathbb{N}_0$ and $\ell_1, \ldots, \ell_k \in \mathbb{N}$, $k \in \mathbb{N}$. Taking $j := \max\{j_1, \ldots, j_k\}$ and $\ell := \max\{\ell_1, \ldots, \ell_k\}$, it follows that

$$V_{K,j,\ell} := \{\phi \in \mathcal{D}_K(\Omega) : \|\phi\|_{K,j} < 1/\ell\} \subseteq \{\phi \in \mathcal{D}_K(\Omega) : \|\phi\|_{K,j_1} < 1/\ell_1, \ldots, \|\phi\|_{K,j_k} < 1/\ell_k\}, \tag{10.2}$$

and so it suffices to consider as a local base for the topology τ_K the family of sets $V_{K,j,\ell}$, where $j \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$.

Exercise 10.1. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $K \subset \Omega$ be a compact set. Define

$$d_K(\phi, \psi) := \max_{j \in \mathbb{N}_0} \frac{1}{2^j} \frac{\|\phi - \psi\|_{K,j}}{1 + \|\phi - \psi\|_{K,j}}, \quad \phi, \psi \in \mathcal{D}_K(\Omega). \tag{10.3}$$

(i) Prove that d_K is a metric.
(ii) Prove that the topology τ_K is determined by the metric d_K.
(iii) Prove that $\mathcal{D}_K(\Omega)$ is complete.

To construct a topology on $C_c^\infty(\Omega)$, let $\mathcal{B}_0$ be the collection of all balanced,[1] convex sets $U \subseteq C_c^\infty(\Omega)$ such that

$$U \cap \mathcal{D}_K(\Omega) \in \tau_K \tag{10.4}$$

for every compact set $K \subset \Omega$.

Theorem 10.2. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. The family*

$$\mathcal{B} := \{\phi + V : \phi \in C_c^\infty(\Omega),\ V \in \mathcal{B}_0\}$$

is a base for a locally convex Hausdorff topology τ on $C_c^\infty(\Omega)$ that turns $C_c^\infty(\Omega)$ into a topological vector space.

Proof. Step 1: We first prove that $\mathcal{B}_0$ is nonempty. Indeed, for every $j \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$ define the norms

$$\|\phi\|_j := \sup\{|\partial^\alpha \phi(x)| : x \in \Omega,\ \alpha \in \mathbb{N}_0^N \text{ with } |\alpha| \le j\} \tag{10.5}$$

and

$$V_{j,\ell} := \{\phi \in C_c^\infty(\Omega) : \|\phi\|_j < 1/\ell\}. \tag{10.6}$$

Then $V_{j,\ell}$ is balanced, convex, and

$$V_{j,\ell} \cap \mathcal{D}_K(\Omega) = \{\phi \in \mathcal{D}_K(\Omega) : \|\phi\|_{K,j} < 1/\ell\} = V_{K,j,\ell} \in \tau_K \tag{10.7}$$

by (10.2), and so $V_{j,\ell}$ belongs to $\mathcal{B}_0$.

Step 2: To prove that $\mathcal{B}$ is a base for a topology, it suffices to verify the following two conditions:

(i) for every $\phi \in C_c^\infty(\Omega)$ there exists $U \in \mathcal{B}$ such that $\phi \in U$;

[1] We recall that a subset E of a vector space X is *balanced* if $tx \in E$ for all $x \in E$ and $t \in [-1, 1]$.

(ii) for every $U_1, U_2 \in \mathcal{B}$, with $U_1 \cap U_2 \neq \emptyset$, and for every $\phi \in U_1 \cap U_2$ there exists $U_3 \in \mathcal{B}$ such that $\phi \in U_3$ and $U_3 \subseteq U_1 \cap U_2$.

To prove (i), let $\phi_0 \in C_c^\infty(\Omega)$. Fix $j \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$ and consider the set $V_{j,\ell}$ defined in (10.6). Then $V_{j,\ell} \in \mathcal{B}_0$, and so $U := \phi_0 + V_{j,\ell} \in \mathcal{B}$.

To verify property (ii), let $\phi_1, \phi_2 \in C_c^\infty(\Omega)$ and $V_1, V_2 \in \mathcal{B}_0$ be such that

$$(\phi_1 + V_1) \cap (\phi_2 + V_2) \neq \emptyset$$

and fix $\phi \in (\phi_1 + V_1) \cap (\phi_2 + V_2)$. Since the supports of ϕ_1, ϕ_2, and ϕ are compact sets contained in Ω, we may find a compact set $K \subset \Omega$ such that

$$K \supseteq \operatorname{supp} \phi_1 \cup \operatorname{supp} \phi_2 \cup \operatorname{supp} \phi.$$

Note that if $(\phi - \phi_i)(x) \neq 0$ for some $x \in \Omega$ and for some $i \in \{1, 2\}$, then necessarily $x \in \operatorname{supp} \phi_i \cup \operatorname{supp} \phi$, and so $\phi - \phi_i \in \mathcal{D}_K(\Omega)$, $i = 1, 2$. Since $\phi - \phi_i \in V_i \cap \mathcal{D}_K(\Omega) \in \tau_K$, $i = 1, 2$, using the continuity of scalar multiplication in $\mathcal{D}_K(\Omega)$, we may find $\theta \in (0, 1)$ such that

$$\phi - \phi_i \in (1 - \theta)(V_i \cap \mathcal{D}_K(\Omega)) \subseteq (1 - \theta)V_i$$

for $i = 1, 2$. By the convexity of the sets V_i we have that

$$\phi - \phi_i + \theta V_i \subseteq (1 - \theta)V_i + \theta V_i = V_i$$

for $i = 1, 2$, so that

$$\phi + \theta(V_1 \cap V_2) \subseteq (\phi_1 + V_1) \cap (\phi_2 + V_2).$$

Thus, $\mathcal{B}$ is a base for a topology τ given by all unions of members of $\mathcal{B}$.

Step 3: Next we show that $(C_c^\infty(\Omega), \tau)$ is a topological vector space.

To prove the continuity of scalar multiplication at a point $(t_0, \phi_0) \in \mathbb{R} \times C_c^\infty(\Omega)$, consider an (open) neighborhood $U \in \tau$ of $t_0\phi_0$. Since $\mathcal{B}$ is a base for the topology, we may find $V \in \mathcal{B}_0$ such that $t_0\phi_0 + V \subseteq U$. Let $K := \operatorname{supp} \phi_0$. Then $\phi_0 \in \mathcal{D}_K(\Omega)$ and since $V \cap \mathcal{D}_K(\Omega) \in \tau_K$, by the continuity of scalar multiplication in $\mathcal{D}_K(\Omega)$ we may find $\delta > 0$ so small that

$$\delta\phi_0 \in \tfrac{1}{2}(V \cap \mathcal{D}_K(\Omega)) \subseteq \tfrac{1}{2}V.$$

Let $s := 1/[2(|t_0| + \delta)]$. Then for every $|t - t_0| < \delta$ and $\phi \in \phi_0 + sV$ we have that

$$t\phi - t_0\phi_0 = t(\phi - \phi_0) + (t - t_0)\phi_0 \in tsV + \tfrac{1}{2}V \subseteq \tfrac{1}{2}V + \tfrac{1}{2}V = V,$$

where we have used the fact that V is convex and balanced. Hence, $t\phi \in t_0\phi_0 + V \subseteq U$ for every $|t - t_0| < \delta$ and every $\phi \in \phi_0 + sV$, which proves continuity of scalar multiplication at the point (t_0, ϕ_0).

To prove the continuity of addition at a point $(\phi_1, \phi_2) \in C_c^\infty(\Omega) \times C_c^\infty(\Omega)$, consider a neighborhood $U \in \tau$ of $\phi_1 + \phi_2$. Since $\mathcal{B}$ is a base for the topology, we may find $V \in \mathcal{B}_0$ such that $\phi_1 + \phi_2 + V \subseteq U$. The convexity of $V \in \mathcal{B}_0$ implies that

$$(\phi_1 + \tfrac{1}{2}V) + (\phi_2 + \tfrac{1}{2}V) = \phi_1 + \phi_2 + V.$$

By (10.4) for every compact set $K \subset \Omega$ we have that $V \cap \mathcal{D}_K(\Omega) \in \tau_K$, and since the topology τ_K turns $\mathcal{D}_K(\Omega)$ into a topological vector space, it also follows that $\frac{1}{2}V \cap \mathcal{D}_K(\Omega) \in \tau_K$. Thus, $\frac{1}{2}V \in \mathcal{B}_0$ and, in turn, $\phi_1 + \frac{1}{2}V$, $\phi_2 + \frac{1}{2}V \in \mathcal{B}$.

Step 4: Finally, to prove that $(C_c^\infty(\Omega), \tau)$ is a Hausdorff topological vector space, it suffices to show that singletons are closed. Let $\phi_1, \phi_2 \in C_c^\infty(\Omega)$ be two distinct elements and define

$$V := \{\phi \in C_c^\infty(\Omega) : \sup_{x\in\Omega} |\phi(x)| < \sup_{x\in\Omega} |\phi_1(x) - \phi_2(x)|\}.$$

In view of (10.6) the set V belongs to $\mathcal{B}_0$ and $\phi_1 \notin \phi_2 + V$. Hence, $\{\phi_1\}$ is closed and the proof is completed. □

The space $C_c^\infty(\Omega)$ endowed with the topology τ is denoted $\mathcal{D}(\Omega)$ and its elements are called *testing functions.*

A natural question is why we define τ in such a convoluted way, instead of directly considering the locally convex topology generated by the family of norms $\{\|\cdot\|_j\}_{j\in\mathbb{N}_0}$ defined in (10.5). The problem is that this topology would not be complete.

Exercise 10.3. Take $N = 1$, $\Omega = \mathbb{R}$, and consider a function $\phi \in C_c^\infty(\mathbb{R})$ with support in $[0, 1]$ such that $\phi > 0$ in $(0, 1)$. Prove that the sequence

$$\phi_n(x) := \phi(x-1) + \frac{1}{2}\phi(x-2) + \cdots + \frac{1}{n}\phi(x-n), \quad x \in \mathbb{R},$$

is a Cauchy sequence in the topology generated by the family of norms $\{\|\cdot\|_j\}_{j\in\mathbb{N}_0}$ defined in (10.5), but its limit does not have compact support, and so it does not belong to $C_c^\infty(\mathbb{R})$.

Exercise 10.4. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Let $\{K_n\}_{n\in\mathbb{N}_0}$ be an increasing sequence of compact sets of Ω, with $K_0 := \emptyset$, such that $\operatorname{dist}(K_n, \partial K_{n+1}) > 0$ and $\Omega = \bigcup_{n=1}^\infty K_n$. Given two sequences $\mathbf{m} := \{m_n\}_{n\in\mathbb{N}_0}$ in $\mathbb{N}_0$ and $\mathbf{a} := \{a_n\}_{n\in\mathbb{N}_0}$ in $(0,\infty)$, with $m_n \to \infty$ and $a_n \to 0$, for every $\phi \in \mathcal{D}(\Omega)$ define

$$p_{\mathbf{m},\mathbf{a}}(\phi) := \sup_{n\in\mathbb{N}_0} \sup_{x\in\Omega\setminus K_n} \frac{1}{a_n} \sum_{|\alpha|\le m_n} |\partial^\alpha \phi(x)|.$$

(i) Prove that $p_{\mathbf{m},\mathbf{a}}$ is a seminorm.

(ii) Prove that the family of seminorms $\{p_{\mathbf{m},\mathbf{a}}\}_{\mathbf{m},\mathbf{a}}$, where $\mathbf{m}$ and $\mathbf{a}$ vary among all sequences as above, generates the topology τ defined in Theorem 10.2.

We now show that the topology τ, when restricted to $\mathcal{D}_K(\Omega)$, for some compact set $K \subset \Omega$, does not produce more open sets than the ones in τ_K.

Theorem 10.5. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Then for every compact set $K \subset \Omega$ the topology τ_K coincides with the relative topology of $\mathcal{D}_K(\Omega)$ as a subset of $\mathcal{D}(\Omega)$.*

Proof. Fix a compact set $K \subset \Omega$ and let $U \in \tau$. We claim that $U \cap \mathcal{D}_K(\Omega)$ belongs to τ_K. To see this, it suffices to consider the case in which $U \cap \mathcal{D}_K(\Omega)$ is nonempty. Let $\phi \in U \cap \mathcal{D}_K(\Omega)$. Since $\mathcal{B}$ is a base for τ, there exists $V \in \mathcal{B}_0$ such that $\phi + V \subseteq U$. Hence,

$$\phi + (V \cap \mathcal{D}_K(\Omega)) \subseteq U \cap \mathcal{D}_K(\Omega).$$

Since $\phi \in \mathcal{D}_K(\Omega)$ and $V \cap \mathcal{D}_K(\Omega) \in \tau_K$, we have that $\phi + (V \cap \mathcal{D}_K(\Omega)) \in \tau_K$. This shows that every point of $U \cap \mathcal{D}_K(\Omega)$ is an interior point with respect to τ_K, and so $U \cap \mathcal{D}_K(\Omega) \in \tau_K$.

Conversely, let $U \in \tau_K$. We claim that

$$U = V \cap \mathcal{D}_K(\Omega) \tag{10.8}$$

for some $V \in \tau$. Since the family of sets $V_{K,j,\ell}$, where $j \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$, is a local base for the topology τ_K (see (10.2)), for every $\phi \in U$ we may find $j_\phi \in \mathbb{N}_0$ and $\ell_\phi \in \mathbb{N}$ such that $\phi + V_{K,j_\phi,\ell_\phi} \subseteq U$. Let V_{j_ϕ,ℓ_ϕ} be defined as in (10.6). By Step 1 of the proof of Theorem 10.2,

$$(\phi + V_{j_\phi,\ell_\phi}) \cap \mathcal{D}_K(\Omega) = \phi + V_{K,j_\phi,\ell_\phi} \subseteq U$$

and $\phi + V_{j_\phi,\ell_\phi} \in \mathcal{B}$. In turn, the set $V := \bigcup_{\phi \in U} (\phi + V_{j_\phi,\ell_\phi})$ belongs to τ and (10.8) holds. $\square$

Exercise 10.6. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Prove that for every $x_0 \in \Omega$, for every compact set $K \subset \Omega$, and for every $r > 0$, the set

$$U := \{\phi \in \mathcal{D}_K(\Omega) : |\phi(x_0)| < r\}$$

is open with respect to τ_K.

Next we study topologically bounded sets in $\mathcal{D}(\Omega)$ (see Definition A.20). The following result will be used to study convergence with respect to the topology τ.

Theorem 10.7. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $W \subseteq \mathcal{D}(\Omega)$ be a topologically bounded set. Then there exists a compact set $K \subset \Omega$ such that $W \subseteq \mathcal{D}_K(\Omega)$. Moreover, for every $j \in \mathbb{N}_0$ there exists a constant $M_j > 0$ such that $\|\phi\|_j \leq M_j$ for all $\phi \in W$, where $\|\cdot\|_j$ is defined in* (10.5).

Proof. Assume by contradiction that W is not contained in $\mathcal{D}_K(\Omega)$ for any compact set $K \subset \Omega$. Let $\{K_n\}_n$ be an increasing sequence of compact sets of Ω such that $\operatorname{dist}(K_n, \partial K_{n+1}) > 0$ and $\Omega = \bigcup_{n=1}^\infty K_n$. Then for each $n \in \mathbb{N}$

we may find a function $\phi_n \in W$ and a point $x_n \in K_{n+1} \setminus K_n$ such that $\phi_n(x_n) \neq 0$. Define

$$U := \{\phi \in \mathcal{D}(\Omega) : |\phi(x_n)| < |\phi_n(x_n)|/n \text{ for all } n \in \mathbb{N}\}.$$

Since each compact set $K \subset \Omega$ contains only finitely many x_n, by the previous exercise we have that $U \cap \mathcal{D}_K(\Omega) \in \tau_K$, and so $U \in \tau$. Using the fact that the set W is topologically bounded, we may find $t > 0$ such that $W \subseteq tU$. Consider an integer $n \geq t$. Then $\phi_n(x_n) \neq 0$ and $|\phi_n(x_n)|/t \geq |\phi_n(x_n)|/n$, which implies that $t^{-1}\phi_n \notin U$, or, equivalently, that $\phi_n \notin tU$, which is a contradiction. This shows that $W \subseteq \mathcal{D}_K(\Omega)$ for some compact set $K \subset \Omega$.

To prove the final part of the statement, note that by Theorem 10.5 the set $W = W \cap \mathcal{D}_K(\Omega)$ is bounded with respect to the topology τ_K, and so, by Corollary A.28, for each $j \in \mathbb{N}_0$ the set $\{\|\phi\|_j : \phi \in W\}$ is bounded in $\mathbb{R}$. □

We are now ready to characterize convergence with respect to the topology τ and to prove the completeness of $\mathcal{D}(\Omega)$ (recall Exercise 10.3).

Theorem 10.8. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Then the space $\mathcal{D}(\Omega)$ is complete. Moreover, a sequence $\{\phi_n\}_n$ in $\mathcal{D}(\Omega)$ converges to $\phi \in \mathcal{D}(\Omega)$ with respect to τ if and only if*

(i) *there exists a compact set $K \subset \Omega$ such that the support of every ϕ_n is contained in K,*

(ii) *$\lim_{n\to\infty} \partial^\alpha \phi_n = \partial^\alpha \phi$ uniformly on K for every multi-index α.*

Proof. Let $\{\phi_n\}_n$ be a Cauchy sequence in $\mathcal{D}(\Omega)$. By Proposition A.22, the set $\{\phi_n : n \in \mathbb{N}\}$ is topologically bounded. Hence, we may apply Theorem 10.7 to find a compact set $K \subset \Omega$ such that $\{\phi_n : n \in \mathbb{N}\}$ is contained in $\mathcal{D}_K(\Omega)$. In turn, by Theorem 10.5, we have that $\{\phi_n\}_n$ is a Cauchy sequence in $\mathcal{D}_K(\Omega)$. Hence, for all integers $j \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$ there exists an integer $\overline{n} \in \mathbb{N}$ such that

$$\phi_n - \phi_k \in V_{K,j,\ell} = \{\phi \in \mathcal{D}_K(\Omega) : \|\phi\|_{K,j} < 1/\ell\}$$

for all $k, n \geq \overline{n}$; that is, for every multi-index α, with $|\alpha| \leq j$, we have that

$$\sup_{x\in K} |\partial^\alpha \phi_n(x) - \partial^\alpha \phi_k(x)| \leq 1/\ell.$$

This implies that $\{\partial^\alpha \phi_n\}_n$ is a Cauchy sequence in the space of continuous bounded functions, and so it converges uniformly in Ω to a function ψ_α with support in K. In particular, taking $\alpha = 0$, we have that ϕ_n converges uniformly in Ω to a function ψ_0 with support in K and $\partial^\alpha \psi_0 = \psi_\alpha$ for every multi-index α, with $|\alpha| \leq j$ (why?). Given the arbitrariness of $j \in \mathbb{N}_0$, we conclude that $\psi_0 \in \mathcal{D}_K(\Omega)$ and that the sequence $\{\phi_n\}_n$ converges to $\psi_0 \in \mathcal{D}(\Omega)$ with respect to τ. Thus, $\mathcal{D}(\Omega)$ is complete.

The proof of the second part of the statement is left as an exercise. □

Exercise 10.9. Let $\Omega \subseteq \mathbb{R}^N$ be an open set.

(i) Prove that for every compact $K \subset \Omega$, the space $\mathcal{D}_K(\Omega)$ is closed and has empty interior in $\mathcal{D}(\Omega)$.

(ii) Prove that $\mathcal{D}(\Omega)$ is not metrizable.

In the previous exercise we have proved that $\mathcal{D}(\Omega)$ is not a metrizable space. Despite this fact, we can still prove that linear functionals defined on $\mathcal{D}(\Omega)$ are continuous if and only if they are sequentially continuous. Precisely, the following result holds.

Theorem 10.10. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $T : \mathcal{D}(\Omega) \to \mathbb{R}$ be linear. Then the following properties are equivalent:*

(i) *T is continuous.*

(ii) *T is bounded.*

(iii) *If $\{\phi_n\}_n$ in $\mathcal{D}(\Omega)$ converges to $\phi \in \mathcal{D}(\Omega)$ with respect to τ, then*

$$\lim_{n\to\infty} T(\phi_n) = T(\phi).$$

(iv) *The restriction of T to $\mathcal{D}_K(\Omega)$ is continuous for every compact set $K \subset \Omega$.*

(v) *For every compact set $K \subset \Omega$ there exist an integer $j \in \mathbb{N}_0$ and a constant $c_K > 0$ such that*

$$|T(\phi)| \le c_K \|\phi\|_{K,j} \quad \text{for all } \phi \in \mathcal{D}_K(\Omega). \tag{10.9}$$

Proof. (i)⇒(ii) If T is continuous, then T is bounded by Theorem A.31.

(ii)⇒(iii) Assume that T is bounded and let $\{\phi_n\}_n$ in $\mathcal{D}(\Omega)$ converge to $\phi \in \mathcal{D}(\Omega)$ with respect to τ. Since $\mathcal{D}(\Omega)$ is a topological vector space, by replacing $\{\phi_n\}$ with $\{\phi_n - \phi\}$, without loss of generality, we may assume that $\{\phi_n\}_n$ converges to 0 with respect to τ. By Proposition A.22, the set $\{\phi_n : n \in \mathbb{N}\}$ is topologically bounded. Since T is bounded, it follows that the set $\{T(\phi_n) : n \in \mathbb{N}\}$ is bounded.

By Exercise 10.1 and Theorem 10.8 there is a compact set $K \subset \Omega$ such that $d_K(\phi_n, 0) \to 0$ as $n \to \infty$. Since $\mathcal{D}_K(\Omega)$ is metrizable, it follows by Theorem A.31 that $T(\phi_n) \to 0$.

(iii)⇒(iv) Fix a compact set $K \subset \Omega$ and assume that $\{\phi_n\}$ is in $\mathcal{D}_K(\Omega)$ and such that $d_K(\phi_n, 0) \to 0$ as $n \to \infty$. By Theorem 10.8 we have that $\{\phi_n\}$ converges to ϕ with respect to τ. Hence, by property (iii), $T(\phi_n) \to 0$ as $n \to \infty$. Using Theorem A.31 once more, we get that the restriction of T to $\mathcal{D}_K(\Omega)$ is continuous.

(iv)⇒(i) For every $\varepsilon > 0$ and for every compact set $K \subset \Omega$ the restriction of T to $\mathcal{D}_K(\Omega)$ is continuous at zero, and so $T^{-1}((-\varepsilon,\varepsilon)) \cap \mathcal{D}_K(\Omega) \in \tau_K$. Hence, $T^{-1}((-\varepsilon,\varepsilon)) \in \tau$, which shows that T is continuous at zero and, by linearity, everywhere.

(iv)⇔(v) Assume that (iv) holds and fix a compact set $K \subset \Omega$. Since T restricted to $\mathcal{D}_K(\Omega)$ is continuous at the origin, given $\varepsilon = 1$ there exist $j \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$ such that $V_{K,j,\ell} \subseteq T^{-1}((-1,1))$, that is,

$$|T(\phi)| \le 1$$

for all $\phi \in \mathcal{D}_K(\Omega)$ with $\|\phi\|_{K,j} < 1/\ell$. If $\phi \in \mathcal{D}_K(\Omega)$ and $\phi \neq 0$, then $\|\phi\|_{K,j} \neq 0$ and

$$\left\| \frac{1}{2\ell} \frac{\phi}{\|\phi\|_{K,j}} \right\|_{K,j} < \frac{1}{\ell}.$$

By the linearity of T it follows that $|T(\phi)| \le 2\ell\|\phi\|_{K,j}$, which gives (iv). Conversely, if (v) holds, then by taking ℓ sufficiently large, we have that $|T(\phi)| \le \varepsilon$ for all $\phi \in V_{K,j,\ell}$, which shows the continuity of T restricted to $\mathcal{D}_K(\Omega)$. □

The dual of $\mathcal{D}(\Omega)$ is denoted $\mathcal{D}'(\Omega)$ and its elements are called *distributions*. We often use the duality notation $\langle T, \phi \rangle$ to denote $T(\phi)$. The space $\mathcal{D}'(\Omega)$ is given the weak star topology (see Section A.5 in Appendix A), so that a sequence $\{T_n\}_n$ in $\mathcal{D}'(\Omega)$ converges to $T \in \mathcal{D}'(\Omega)$ if $T_n(\phi) \to T(\phi)$ for every $\phi \in \mathcal{D}(\Omega)$. In this case we say that $\{T_n\}_n$ *converges to T in the sense of distributions.*

Exercise 10.11. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $T \in \mathcal{D}'(\Omega)$.

(i) Prove that if $\phi, \psi \in C^\infty(\Omega)$, then for every multi-index β the *Leibnitz formula*

$$\partial^\alpha(\phi\psi) = \sum_{\alpha \le \beta} c_{\alpha\beta} \partial^{\beta-\alpha}\phi \partial^\alpha \psi$$

holds for some $c_{\alpha\beta} \in \mathbb{R}$.

(ii) Prove that if $\psi \in C^\infty(\Omega)$, then the linear functional $\psi T : \mathcal{D}(\Omega) \to \mathbb{R}$, defined by

$$(\psi T)(\phi) := T(\psi\phi), \quad \phi \in \mathcal{D}(\Omega),$$

is a distribution.

10.2. Order of a Distribution

In this section we define the order of a distribution. We begin by observing that the integer $j \in \mathbb{N}_0$ in (10.9) may change with the compact set $K \subset \Omega$. If the same integer will do for all compact sets $K \subset \Omega$, then the smallest integer $j \in \mathbb{N}_0$ for which (10.9) holds for all compact sets $K \subset \Omega$ is called the *order* of the distribution T. If no such integer exists, then the distribution T is said to have *infinite order*.

Example 10.12. Let $\Omega \subseteq \mathbb{R}^N$ be an open set.

(i) Let λ be a signed Radon measure on Ω. The functional

$$T_\lambda(\phi) := \int_\Omega \phi \, d\lambda, \quad \phi \in \mathcal{D}(\Omega),$$

is a distribution of order zero.

(ii) Fix $x_0 \in \Omega$. The functional δ_{x_0}, defined by $\delta_{x_0}(\phi) := \phi(x_0)$, $\phi \in \mathcal{D}(\Omega)$, is a distribution and is called the *delta Dirac with mass at* x_0. The distribution δ_{x_0} has order zero.

(iii) Let $u \in L^1_{\mathrm{loc}}(\Omega)$. The functional

$$T_u(\phi) := \int_\Omega \phi(x)u(x)\, dx, \quad \phi \in \mathcal{D}(\Omega),$$

is a distribution of order zero.

Actually, it turns out that all distributions of order zero may be identified with measures.

Theorem 10.13. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $T \in \mathcal{D}'(\Omega)$.*

(i) *If T is positive, that is, if $T(\phi) \geq 0$ for all nonnegative functions $\phi \in \mathcal{D}(\Omega)$, then there exists a unique Radon measure $\mu : \mathcal{B}(\Omega) \to [0, \infty]$ such that*

$$T(\phi) = \int_\Omega \phi \, d\mu \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

(ii) *If T has order zero, then there exist two Radon measure $\mu_1, \mu_1 : \mathcal{B}(\Omega) \to [0, \infty]$ such that*

$$T(\phi) = \int_\Omega \phi \, d\mu_1 - \int_\Omega \phi \, d\mu_2 \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

Proof. (i) We claim that T has order zero. Fix a compact set $K \subset \Omega$ and find an open set U such that $K \subset U \Subset \Omega$. Construct a smooth cut-off function $\varphi \in C^\infty(\Omega)$ such that $\varphi \equiv 1$ on K, $\operatorname{supp} \varphi \subset U$, and $0 \leq \varphi \leq 1$ (see Exercise C.23). In particular, $\varphi \in \mathcal{D}(\Omega)$. Since $\varphi \equiv 1$ on K and $\varphi \geq 0$, for every $\phi \in \mathcal{D}_K(\Omega)$ we have that $|\phi(x)| \leq \|\phi\|_{K,0}\varphi(x)$ for all $x \in \Omega$, and so

$$T(\|\phi\|_{K,0}\varphi - \phi) \geq 0, \quad T(\phi + \|\phi\|_{K,0}\varphi) \geq 0;$$

that is, by the linearity of T,

$$|T(\phi)| \leq \|\phi\|_{K,0} T(\varphi) \quad \text{for all } \phi \in \mathcal{D}_K(\Omega),$$

which shows that T has order zero.

Let $\{\Omega_i\}_i$ be an increasing sequence of bounded open sets of Ω such that $\Omega_i \Subset \Omega_{i+1}$ and $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$. We start by showing that for every fixed $i \in \mathbb{N}$ the distribution T can be extended in a unique way as a linear continuous map on $C_c(\Omega_i)$. Since $K_i := \overline{\Omega_i}$ is a compact set contained in Ω and T has order zero, by what we just proved there exists a constant $c_i > 0$ such that

$$|T(\phi)| \le c_i \|\phi\|_{K_i,0} \tag{10.10}$$

for all $\phi \in \mathcal{D}_{K_i}(\Omega)$. If $\phi \in C_c(\Omega_i)$, then $\operatorname{dist}(\operatorname{supp}\phi, \partial\Omega_i) > 0$, and thus if we consider $\phi_n := \varphi_{1/n} * \phi$, where $\varphi_{1/n}$ are standard mollifiers (with $\varepsilon := 1/n$) and $1/n < \operatorname{dist}(\operatorname{supp}\phi, \partial\Omega_i)$, we have that $\{\phi_n\}_n$ is contained in $\mathcal{D}_{K_i}(\Omega)$ and $\phi_n \to \phi$ uniformly on K_i. It follows by (10.10) that

$$|T(\phi_n - \phi_l)| \le c_i \|\phi_n - \phi_l\|_{K_i,0} \to 0$$

as $l, n \to \infty$. Hence, $\{T(\phi_n)\}_n$ is a Cauchy sequence, and therefore it converges to a limit that we denote by $T_i(\phi)$. Moreover, if $\phi \ge 0$, then $\phi_n \ge 0$ also, and so $T_i(\phi) \ge 0$. Note that, again by (10.10), $T_i(\phi)$ is independent of the choice of the approximating sequence $\{\phi_n\}_n$.

By the linearity of T it follows that $T_i : C_c(\Omega_i) \to \mathbb{R}$ is linear and positive, while by (10.10) we have that $|T_i(\phi)| \le c_i \|\phi\|_{C_c(\Omega_i)}$ for all $\phi \in C_c(\Omega_i)$. Since $\Omega_i \subset \Omega_{i+1}$, it follows that $T_{i+1}(\phi) = T_i(\phi)$ for all $\phi \in C_c(\Omega_i)$. Thus $\{T_i\}_i$ defines a unique linear positive extension of T to the union of all $C_c(\Omega_i)$, which coincides with $C_c(\Omega)$. The result now follows from the Riesz representation theorem in $C_c(\Omega)$ for positive linear functionals (see Theorem B.113).

(ii) The second part of the proof of (i) continues to hold in this case. By (10.10) and the fact that $\{\Omega_i\}_i$ covers Ω, we have that the extended functional is locally bounded, and so the result now follows from the Riesz representation theorem in $C_c(\Omega)$ for locally bounded linear functionals (see Theorem B.113). □

10.3. Derivatives of Distributions and Distributions as Derivatives

We now define the notion of a derivative of a distribution.

Definition 10.14. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $T \in \mathcal{D}'(\Omega)$. Given a multi-index $\alpha \in \mathbb{N}_0^N \setminus \{0\}$, we define the αth *derivative* of T as

$$\frac{\partial^{|\alpha|} T}{\partial x^\alpha}(\phi) := (-1)^{|\alpha|}\Big(\frac{\partial^{|\alpha|}\phi}{\partial x^\alpha}\Big), \quad \phi \in \mathcal{D}(\Omega).$$

For $j \in \mathbb{N}$ the symbol $\nabla^j T$ stands for the collection of all αth distributional derivatives of T with $|\alpha| = j$.

For simplicity we often write ∂^α for $\frac{\partial^{|\alpha|}}{\partial x^\alpha}$ and ∂_i^n for $\frac{\partial^n}{\partial x_i^n}$.

Remark 10.15. It can be verified that $\partial^\alpha T$ is still a distribution. Indeed, let $K \subset \Omega$ be a compact set. By Theorem 10.10 there exist an integer $j \in \mathbb{N}_0$ and a constant $c_K > 0$ such that

$$|T(\phi)| \leq c_K \|\phi\|_{K,j}$$

for all $\phi \in \mathcal{D}_K(\Omega)$. It follows that

$$|\partial^\alpha T(\phi)| = |T(\partial^\alpha \phi)| \leq c_K \|\partial^\alpha \phi\|_{K,j} \leq c_K \|\phi\|_{K,j+|\alpha|}$$

for all $\phi \in \mathcal{D}_K(\Omega)$, which shows that $\partial^\alpha T \in \mathcal{D}'(\Omega)$, again by Theorem 10.10.

In particular, if $u \in L^1_{\text{loc}}(\Omega)$ and α is a multi-index, then the αth *weak*, or *distributional*, *derivative* of u is the distribution $\partial^\alpha T_u$.

In a similar way we may define directional derivatives of a distribution (see Section 9.1).

Definition 10.16. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $T \in \mathcal{D}'(\Omega)$. Given a unit vector $\nu \in \mathbb{R}^N \setminus \{0\}$ and $n \in \mathbb{N}$, we define the nth *directional derivative* of T *in the direction* ν as

$$\frac{\partial^n T}{\partial \nu^n}(\phi) := (-1)^n T\Big(\frac{\partial^n \phi}{\partial \nu^n}\Big), \quad \phi \in \mathcal{D}(\Omega).$$

In particular, if $u \in L^1_{\text{loc}}(\Omega)$, $\nu \in \mathbb{R}^N \setminus \{0\}$ and $n \in \mathbb{N}$, then the nth *weak*, or *distributional, directional derivative* of u *in the direction* ν is the distribution $\frac{\partial^n T_u}{\partial \nu^n}$.

Example 10.17. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u \in L^1_{\text{loc}}(\Omega)$. The *Laplacian* of u in the sense of distribution is the distribution $T_{\Delta u} := \sum_{i=1}^N \partial_i^2 T_u$, that is,

$$\begin{aligned} T_{\Delta u}(\phi) &= \sum_{i=1}^N \partial_i^2 T_u(\phi) = \sum_{i=1}^N T_u(\partial_i^2 \phi) \\ &= \sum_{i=1}^N \int_\Omega u \partial_i^2 \phi \, dx = \int_\Omega u \Delta \phi \, dx \end{aligned}$$

for all $\phi \in \mathcal{D}(\Omega)$. So a function $u \in L^1_{\text{loc}}(\Omega)$ is *subharmonic* if "$\Delta u \geq 0$", that is,

$$T_{\Delta u}(\phi) = \int_\Omega u \Delta \phi \, dx \geq 0$$

for all $\phi \in \mathcal{D}(\Omega)$ with $\phi \geq 0$. By Theorem 10.13 there exists a unique (positive) Radon measure $\mu : \mathcal{B}(\Omega) \to [0, \infty]$ such that

$$T_{\Delta u}(\phi) = \int_\Omega u \Delta \phi \, dx = \int_\Omega \phi \, d\mu \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

Thus, with an abuse of notation, we may write $\Delta u = \mu$.

Similarly, for $u \in L^1_{\text{loc}}(\Omega; \mathbb{R}^N)$ one can define the divergence of u in the sense of distributions.

Definition 10.18. Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $u \in L^1_{\text{loc}}(\Omega)$, and let $\alpha \in \mathbb{N}_0^N \setminus \{0\}$ be a multi-index. If there exists a function $v_\alpha \in L^1_{\text{loc}}(\Omega)$ such that

$$T_{v_\alpha}(\phi) = \partial^\alpha T_u(\phi) \quad \text{for all } \phi \in \mathcal{D}(\Omega),$$

then v_α is called the αth *weak,* or *distributional, derivative* of T_u. We write $\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x^\alpha} := v_\alpha$.

Thus, a function $v_\alpha \in L^1_{\text{loc}}(\Omega)$ is the αth weak derivative of $u \in L^1_{\text{loc}}(\Omega)$ if

$$\int_\Omega \phi v_\alpha dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha \phi \, dx \tag{10.11}$$

for all $\phi \in C_c^\infty(\Omega)$. Note that this is consistent with the integration by parts (9.55).

Exercise 10.19. Let $u : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$u(x) := \begin{cases} \cos x & \text{if } -\pi \le x \le 0, \\ 1 - \dfrac{x}{\pi} & \text{if } 0 < x \le \pi, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Calculate the derivative of u in the sense of distributions and find its order.

(ii) Let v be the restriction of u in the interval $(-\pi, \pi)$. Calculate the first and second derivative of v in the sense of distributions in $(-\pi, \pi)$ and find their orders.

(iii) Let $w : \mathbb{R} \to \mathbb{R}$ be differentiable in $\mathbb{R} \setminus \{a\}$ and assume that there exist

$$\lim_{x \to a^-} w(x) \in \mathbb{R}, \qquad \lim_{x \to a^+} w(x) \in \mathbb{R}.$$

Calculate the derivative of w in the sense of distributions. What can you say about its order? Under what conditions can you conclude that it has order zero?

Exercise 10.20. Assume that $u \in L^1((-\infty, \delta) \cup (\delta, \infty))$ for every $\delta > 0$ and define the *principal value integral*

$$\text{PV} \int_{-\infty}^{\infty} u(x) \, dx := \lim_{\delta \to 0^+} \left(\int_\delta^\infty u(x) \, dx + \int_{-\infty}^{-\delta} u(x) \, dx \right)$$

whenever the limit exists. For $\phi \in \mathcal{D}(\mathbb{R})$ define

$$T(\phi) := \int_{-\infty}^{\infty} \phi(x) \log |x| \, dx.$$

Prove that

$$T'(\phi) = \mathrm{PV}\int_{-\infty}^{\infty} \phi(x)\frac{1}{x}\,dx, \quad T''(\phi) = -\,\mathrm{PV}\int_{-\infty}^{\infty} \frac{\phi(x)-\phi(0)}{x^2}\,dx.$$

Exercise 10.21. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $T_n \in \mathcal{D}'(\Omega)$ be such that the limit

$$T(\phi) := \lim_{n\to\infty} T_n(\phi)$$

exists in $\mathbb{R}$ for every $\phi \in \mathcal{D}(\Omega)$.

(i) Prove that $T \in \mathcal{D}'(\Omega)$.

(ii) Prove that for every multi-index α and for every $\phi \in \mathcal{D}(\Omega)$,

$$\partial^\alpha T(\phi) = \lim_{n\to\infty} \partial^\alpha T_n(\phi).$$

In the next few theorems we characterize distributions as weak derivatives of continuous functions.

Theorem 10.22. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $T \in \mathcal{D}'(\Omega)$. Then for every compact set $K \subset \Omega$ there exist a continuous function $u : \Omega \to \mathbb{R}$ and a multi-index α (both depending on K) such that*

$$T(\phi) = (-1)^{|\alpha|}\int_\Omega u\partial^\alpha\phi\,dx \quad \textit{for all } \phi \in \mathcal{D}_K(\Omega).$$

Proof. In what follows, we use the notation (E.2) in Appendix E. Without loss of generality we may assume that $K \subseteq Q := [0,1]^N$. By the mean value theorem, for every $\psi \in \mathcal{D}_Q(\mathbb{R}^N)$, for all $i = 1,\ldots,N$, and for all $x \in Q$ we have

$$\begin{aligned} |\psi(x)| &= |\psi(x_i', x_i) - \psi(x_i', 0)| \\ &= |\partial_i\psi(x_i', t)(x_i - 0)| \le \max_Q |\partial_i\psi|. \end{aligned} \tag{10.12}$$

For $x \in Q$ set

$$Q(x) := \{y \in Q : \, 0 \le y_i \le x_i,\, i = 1,\ldots,N\}.$$

Then for $\psi \in \mathcal{D}_Q(\mathbb{R}^N)$ and $x \in Q$ we have

$$\begin{aligned} \psi(x) &= \psi(0, x_2,\ldots,x_N) + \int_0^{x_1} \partial_1\psi(y_1, x_2,\ldots,x_N)\,dy_1 \\ &= \int_0^{x_1} \partial_1\psi(y_1, x_2,\ldots,x_N)\,dy_1 \\ &= \int_0^{x_1} \left[\partial_1\psi(y_1, 0, x_3,\ldots,x_N) + \int_0^{x_2} \partial^2_{1,2}\psi(y_1,y_2,x_3,\ldots,x_N)\,dy_2\right] dy_1 \\ &= \int_0^{x_1}\int_0^{x_2} \partial^2_{1,2}\psi(y_1,y_2,x_3,\ldots,x_N)\,dy_2 dy_1 = \int_{Q(x)} \partial^\beta\psi(y)\,dy, \end{aligned}$$

where $\beta := (1, \ldots, 1)$, and so

$$\psi(x) = \int_{Q(x)} \partial^\beta \psi(y)\, dy \quad \text{for all } x \in Q. \tag{10.13}$$

Fix an integer $j \in \mathbb{N}_0$ (to be determined later) and let α be a multi-index with $|\alpha| \le j$. By repeated applications of (10.12) we obtain that

$$\max_Q |\partial^\alpha \psi| \le \max_Q |\partial^{j\beta} \psi| \le \int_Q |\partial^{(j+1)\beta} \psi(y)|\, dy, \tag{10.14}$$

where $j\beta = (j, \ldots, j)$ and where in the last inequality we have used (10.13).

By (10.13) the linear operator

$$\begin{aligned} D^\beta : \mathcal{D}_K(\Omega) &\to \mathcal{D}_K(\Omega) \\ \phi &\mapsto \partial^\beta \phi \end{aligned}$$

is one-to-one, and hence so is the linear operator

$$\begin{aligned} L := D^{j\beta} : \mathcal{D}_K(\Omega) &\to \mathcal{D}_K(\Omega) \\ \phi &\mapsto \partial^{(j+1)\beta} \phi, \end{aligned}$$

since $D^{j\beta} = D^\beta \circ \cdots \circ D^\beta$, where the composition is done j times. Let $Y := L(\mathcal{D}_K(\Omega))$ and define the linear functional $T_1 : Y \to \mathbb{R}$ as follows. Given $\psi \in Y$, there exists a unique $\phi \in \mathcal{D}_K(\Omega)$ such that

$$\psi = L(\phi) = \partial^{(j+1)\beta} \phi.$$

Define $T_1(\psi) := T(\phi)$. Since $T \in \mathcal{D}'(\Omega)$, by Theorem 10.10 there exist an integer $j \in \mathbb{N}_0$ and a constant $c_K > 0$ such that for all $\phi \in \mathcal{D}_K(\Omega)$,

$$|T(\phi)| \le c_K \|\phi\|_{K,j} \le c_K \int_K |\partial^{(j+1)\beta} \phi|\, dx,$$

where in the last inequality we have used (10.14). It follows from the definition of T_1 that

$$|T_1(\psi)| \le c_K \int_K |\psi|\, dx$$

for all $\psi \in Y$, and thus we may apply the Hahn–Banach theorem (see Theorem A.32) to extend T_1 as a continuous linear functional defined in $L^1(K)$. By the Riesz representation theorem in $L^1(K)$ (see Theorem B.93) there exists a function $v \in L^\infty(K)$ such that

$$T_1(\psi) = \int_K v\psi\, dx$$

for all $\psi \in L^1(K)$. In particular, if $\phi \in \mathcal{D}_K(\Omega)$, then

$$T(\phi) = T_1(\partial^{(j+1)\beta} \phi) = \int_K v \partial^{(j+1)\beta} \phi\, dx.$$

Extend v by zero outside K and define

$$u(x) := \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_N} v(y)\, dy_N \cdots dy_1.$$

Then u is continuous and by integrating by parts N times we have that for all $\phi \in \mathcal{D}_K(\Omega)$,

$$T(\phi) = (-1)^N \int_K u \partial^{(j+2)\beta} \phi \, dx.$$

To complete the proof, we may define $\alpha := (j+2)\beta$ and, if needed, replace u with $-u$. □

The previous result is local in the sense that the function u and the multi-index α change with K. Next we show that if T has compact support, then the previous local result becomes global.

Definition 10.23 (Support of a distribution)**.** Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $T \in \mathcal{D}'(\Omega)$. If $U \subseteq \Omega$ is open, then we write that $T = 0$ in U if $T(\phi) = 0$ for all $\phi \in \mathcal{D}(U)$. The *support* of T is the complement of V relative to Ω, where V is the union of all open subsets $U \subseteq \Omega$ in which $T = 0$.

The support of T will be written as $\operatorname{supp} T$.

Exercise 10.24. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $T \in \mathcal{D}'(\Omega)$. Show that $T = 0$ in the (possibly empty) open set $\Omega \setminus \operatorname{supp} T$. Prove also that $\partial^\alpha T = 0$ in $\Omega \setminus \operatorname{supp} T$ for every multi-index α.

We now prove a global version of Theorem 10.22 for distributions with compact support.

Theorem 10.25. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $T \in \mathcal{D}'(\Omega)$ be such that* $\operatorname{supp} T$ *is a compact set of* Ω*. Then*

(i) *there exist an integer $j \in \mathbb{N}_0$ and a constant $c > 0$ such that $|T(\phi)| \le c\|\phi\|_j$ for all $\phi \in \mathcal{D}(\Omega)$ (in particular, T has finite order $\ell \le j$), where $\|\cdot\|_j$ is defined in* (10.5),

(ii) *if U is an open set, with* $\operatorname{supp} T \subset U \subseteq \Omega$*, then for each multi-index α, with $\alpha \le \beta := (\ell+2, \dots, \ell+2)$, there exists a function $v_\alpha \in C(\Omega)$, with* $\operatorname{supp} v_\alpha \subset U$*, such that*

$$T(\phi) = \sum_{\alpha \le \beta} \partial^\alpha T_{v_\alpha}(\phi) = \sum_{\alpha} (-1)^{|\alpha|} \int_\Omega v_\alpha \partial^\alpha \phi \, dx \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

Proof. (i) Consider an open set U, with $\operatorname{supp} T \subset U \Subset \Omega$, and (see Exercise C.23) construct a function $\psi \in \mathcal{D}(\Omega)$ such that $\psi = 1$ on U. We claim that $\psi T = T$. Indeed, if $\phi \in \mathcal{D}(\Omega)$, then since $\psi = 1$ on U, we have that $\phi - \psi\phi = 0$ on U, and so $\operatorname{supp}(\phi - \psi\phi) \cap \operatorname{supp} T = \emptyset$, which implies that

$T(\phi - \psi\phi) = 0$, or, equivalently, $T(\phi) = T(\psi\phi) = (\psi T)(\phi)$. Hence, the claim holds.

Since $T \in \mathcal{D}'(\Omega)$, by Theorem 10.10 there exist an integer $j \in \mathbb{N}_0$ and a constant $c_K > 0$ such that $|T(\phi)| \le c_K \|\phi\|_{K,j}$ for all $\phi \in \mathcal{D}_K(\Omega)$, where $K := \operatorname{supp}\psi$. On the other hand, if $\phi \in \mathcal{D}(\Omega)$, then $\psi\phi \in \mathcal{D}_K(\Omega)$, and so, since $\psi T = T$,

$$|T(\phi)| = |T(\psi\phi)| \le c_K \|\psi\phi\|_{K,j}.$$

By the Leibnitz formula (see Exercise 10.11), $\|\psi\phi\|_{K,j} \le c_\psi \|\phi\|_{K,j}$, and so $|T(\phi)| \le c_K c_\psi \|\phi\|_{K,j}$, which shows part (i).

(ii) Consider an open set W, with $\operatorname{supp} T \subset W \Subset U$. By Theorem 10.22 with $\overline{W}$ in place of K there exists a continuous function $u : \Omega \to \mathbb{R}$ such that

$$T(\phi) = (-1)^{|\beta|} \int_\Omega u \partial^\beta \phi \, dx$$

for all $\phi \in \mathcal{D}_{\overline{W}}(\Omega)$, where $\beta := (\ell + 2, \ldots, \ell + 2)$. Consider an open set V, with $\operatorname{supp} T \subset V \Subset W$, and as in (i) construct a function $\psi \in \mathcal{D}(\Omega)$ such that $\psi = 1$ on V and $K := \operatorname{supp}\psi \subset W$. Then, by (i), for all $\phi \in \mathcal{D}(\Omega)$ we have that $\psi\phi \in \mathcal{D}_{\overline{W}}(\Omega)$ and, by the Leibnitz formula (see Exercise 10.11) again,

$$\begin{aligned} T(\phi) = T(\psi\phi) &= (-1)^{|\beta|} \int_\Omega u \partial^\beta (\psi\phi) \, dx \\ &= \sum_{\alpha \le \beta} c_{\alpha\beta} (-1)^{|\beta|} \int_\Omega u \partial^{\beta-\alpha} \psi \partial^\alpha \phi \, dx, \end{aligned}$$

and so it suffices to take $v_\alpha := c_{\alpha\beta} (-1)^{|\beta-\alpha|} u \partial^{\beta-\alpha} \psi$. This concludes the proof. □

Finally, using partitions of unity, we have a similar representation for every distribution.

Theorem 10.26. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $T \in \mathcal{D}'(\Omega)$. Then for each multi-index α there exists a function $v_\alpha \in C(\Omega)$ such that*

(i) *each compact set $K \subset \Omega$ intersects the support of finitely many v_α,*

(ii) *for all $\phi \in \mathcal{D}(\Omega)$,*

$$T(\phi) = \sum_\alpha \partial^\alpha T_{v_\alpha}(\phi) = \sum_\alpha (-1)^{|\alpha|} \int_\Omega v_\alpha \partial^\alpha \phi \, dx.$$

If T has finite order, then only finitely many v_α different from zero are needed.

Proof. (i) Let S be a countable dense set in Ω, e.g., $S := \{x \in \mathbb{Q}^N \cap \Omega\}$, and consider the countable family $\mathcal{F}$ of open cubes

$$\mathcal{F} := \{Q(x,r) : r \in \mathbb{Q},\, 0 < r < 1,\, x \in S,\, Q(x,r) \subseteq \Omega\}.$$

Since $\mathcal{F}$ is countable, we may write $\mathcal{F} = \{Q(x_n, r_n)\}_{n\in\mathbb{N}}$. By the density of S and of the rational numbers we have that $\Omega = \bigcup_{n=1}^{\infty} \overline{Q(x_n, r_n/2)}$. Note that every compact set $K \subset \Omega$ intersects only finitely many cubes $Q(x_n, r_n)$. For each $n \in \mathbb{N}$ construct a function $\phi_n \in \mathcal{D}(\Omega)$ such that $\phi_n = 1$ on $\overline{Q(x_n, r_n/2)}$ and $\operatorname{supp} \phi_n \subset Q(x_n, r_n)$. Use this family to construct a partition of unity $\{\psi_n\}_n$ in $\mathcal{D}(\Omega)$ with $\operatorname{supp} \psi_n \subset Q(x_n, r_n)$ for each $n \in \mathbb{N}$ (see the proof of Theorem C.21). For each $n \in \mathbb{N}$ the distribution $\psi_n T$ has support contained in $Q(x_n, r_n)$, and so by Theorem 10.25 it has finite order ℓ_n and we may find finitely many functions $\{v_{\alpha,n}\}$ in $C(\Omega)$, with $\alpha \le \beta_n := (\ell_n + 2, \dots, \ell_n + 2)$ and $\operatorname{supp} v_{\alpha,n} \subset Q(x_n, r_n)$, such that

$$(\psi_n T)(\phi) = \sum_{\alpha \le \beta_n} (-1)^{|\alpha|} \int_\Omega v_{\alpha,n} \partial^\alpha \phi \, dx$$

for all $\phi \in \mathcal{D}(\Omega)$. For every multi-index α for which $\alpha \le \beta_n$ fails define $v_{\alpha,n} :\equiv 0$. Thus,

$$(\psi_n T)(\phi) = \sum_{\alpha} (-1)^{|\alpha|} \int_\Omega v_{\alpha,n} \partial^\alpha \phi \, dx$$

for all $\phi \in \mathcal{D}(\Omega)$.

Hence, for every multi-index α we may define $v_\alpha := \sum_{n=1}^{\infty} v_{\alpha,n}$. Since each compact set intersects the support of only finitely many $v_{\alpha,n}$, the function v_α is continuous and (i) holds.

To prove (ii), let $\phi \in \mathcal{D}(\Omega)$. Since $\{\psi_n\}_n$ is a partition of unity, we have that $\phi = \sum_{n=1}^{\infty} \psi_n \phi$, where the sum is actually finite since ϕ has compact support. Since the sum is finite and T is linear, we have that

$$\begin{aligned} T(\phi) &= \sum_{n=1}^{\infty} T(\psi_n \phi) = \sum_{n=1}^{\infty} (\psi_n T)(\phi) = \sum_{n=1}^{\infty} \sum_{\alpha} (-1)^{|\alpha|} \int_\Omega v_{\alpha,n} \partial^\alpha \phi \, dx \\ &= \sum_{\alpha} (-1)^{|\alpha|} \int_\Omega \sum_{n=1}^{\infty} v_{\alpha,n} \partial^\alpha \phi \, dx = \sum_{\alpha} (-1)^{|\alpha|} \int_\Omega v_\alpha \partial^\alpha \phi \, dx. \end{aligned}$$

This shows (ii).

Finally, if T has finite order ℓ, then $\psi_n T$ has finite order $\ell_n \le \ell$, and so it suffices to consider only multi-indices $\alpha \le \beta := (\ell + 2, \dots, \ell + 2)$. ☐

Exercise 10.27. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Prove that every continuous linear functional on $C^\infty(\Omega)$ is of the form $\phi \mapsto T(\phi)$, where T is a distribution with compact support.

Exercise 10.28. Let $\Omega = (a_1, b_1) \times \cdots \times (a_N, b_N)$.

(i) Prove that $\phi \in \mathcal{D}(\Omega)$ is such that $\int_\Omega \phi(x)\,dx = 0$ if and only if $\phi = \sum_{i=1}^N \partial_i \phi_i$ for some $\phi_1, \ldots, \phi_N \in \mathcal{D}(\Omega)$. Hint: Use induction on N and look at Step 1 of the proof of Lemma 7.4.

(ii) Prove that if $T \in \mathcal{D}'(\Omega)$ is such that $\frac{\partial T}{\partial x_i} = 0$ for all $i = 1, \ldots, N$, then there exists a constant $c \in \mathbb{R}$ such that $T(\phi) = c \int_\Omega \phi(x)\,dx$ for all $\phi \in \mathcal{D}(\Omega)$, i.e., T is constant.

(iii) Prove that if $\Omega \subseteq \mathbb{R}^N$ is an open connected set and $T \in \mathcal{D}'(\Omega)$ is such that $\partial_i T = 0$ for all $i = 1, \ldots, N$, then T is constant.

10.4. Rapidly Decreasing Functions and Tempered Distributions

In view of the applications to Fourier transforms in this section we consider complex-valued functions. We recall that for a complex number $z = x + iy$, where $x, y \in \mathbb{R}$, the complex conjugate of z is the number $\overline{z} := x - iy$ and the norm of z is

$$\|z\| := \sqrt{x^2 + y^2}.$$

Definition 10.29. The *space of rapidly decreasing functions* $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ is the space of all functions $\phi : \mathbb{R}^N \to \mathbb{C}$ of class C^∞ such that for all multi-indices $\alpha, \beta \in \mathbb{N}_0^N$,

$$\|\phi\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^N} \|x^\alpha \partial^\beta \phi(x)\| < \infty. \tag{10.15}$$

Thus, $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ consists of all functions that, together with all their derivatives, decay to zero faster than any polynomial.

Remark 10.30. The space $C_c^\infty(\mathbb{R}^N;\mathbb{C})$ of all C^∞ functions $\phi : \mathbb{R}^N \to \mathbb{C}$ with compact support is contained in $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$. The function $\phi(x) := e^{-\|x\|^2}$, $x \in \mathbb{R}^N$, is an example of a function in $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ without compact support.

Theorem 10.31. *The space $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ with the the topology induced by the family of seminorms $\|\cdot\|_{\alpha,\beta}$ is a Fréchet space.*

Proof. Since there are countably many seminorms $\|\cdot\|_{\alpha,\beta}$, the space $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ is metrizable, with metric

$$d(\phi,\psi) := \sum_{n=1}^\infty \frac{1}{2^n} \frac{\|\phi - \psi\|_{\alpha_n,\beta_n}}{1 + \|\phi - \psi\|_{\alpha_n,\beta_n}}.$$

It remains to show that it is complete. Let $\{\phi_n\}_n$ be a Cauchy sequence in $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$. Then $\{x^\alpha \partial^\beta \phi_n\}_n$ is a Cauchy sequence in $C_b(\mathbb{R}^N;\mathbb{C})$ for every $\alpha, \beta \in \mathbb{N}_0^+$ and thus it converges uniformly to a function $\psi_{\alpha,\beta}$. Let $\phi := \psi_{0,0}$.

By the fundamental theorem of calculus for $x \in \mathbb{R}^N$, $t \in \mathbb{R}$ and $i = 1, \ldots, N$,

$$\phi_n(x + te_i) = \phi_n(x) + \int_0^t \frac{\partial \phi_n}{\partial x_i}(x + se_i)\, ds.$$

Letting $n \to \infty$ it follows by uniform convergence that

$$\phi(x + te_i) = \phi(x) + \int_0^t \psi_{0,e_i}(x + se_i)\, ds.$$

Hence, there exists $\frac{\partial \phi}{\partial x_i} = \psi_{0,e_i}$. This proves that ϕ is of class C^1. In a similar way we can show that ϕ is of class C^∞ with $\psi_{\alpha,\beta} = x^\alpha \partial^\beta \phi$. Thus, $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$ and since $\|\phi_n - \phi\|_{\alpha,\beta} \to 0$ for all $\alpha, \beta \in \mathbb{N}_0^+$, it follows that $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ is complete. □

Definition 10.32. The dual of $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ is called the *space of tempered distributions* and is denoted $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$.

The following theorem is important for applications. For $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$ and $m, n \in \mathbb{N}_0$ we define

$$\|\phi\|_{m,n} := \sum_{|\alpha|=m} \sum_{|\beta|=n} \|\phi\|_{\alpha,\beta}.$$

Theorem 10.33. *A linear functional $T : \mathcal{S}(\mathbb{R}^N;\mathbb{C}) \to \mathbb{C}$ is continuous if and only if there exist a constant $c > 0$ and $m, n \in \mathbb{N}_0$ such that*

$$\|T(\phi)\| \le c\|\phi\|_{m,n} \quad \text{for every } \phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C}). \tag{10.16}$$

Proof. Exercise. □

In view of the previous theorem, the restriction of a tempered distribution $T : \mathcal{S}(\mathbb{R}^N;\mathbb{C}) \to \mathbb{C}$ to $\mathcal{D}(\mathbb{R}^N;\mathbb{C})$ defines a distribution.

Corollary 10.34. *Given $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$, its restriction to $\mathcal{D}(\mathbb{R}^N;\mathbb{C})$ is a distribution.*

Proof. By the previous theorem there exist a constant $c > 0$ and $m, n \in \mathbb{N}_0$ such that $\|T(\phi)\| \le c\|\phi\|_{m,n}$ for every $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$. Given a compact set $K \subset \mathbb{R}^N$, let $R > 0$ be such that $K \subseteq \overline{B(0,R)}$. If $\phi \in \mathcal{D}_K(\mathbb{R}^N;\mathbb{C})$, then $\sup_{x \in \mathbb{R}^N} \|x^\alpha \partial^\beta \phi(x)\| \le (1 + R^m) \sup_{x \in K} \|\partial^\beta \phi(x)\|$ and so

$$|T(\phi)| \le (1 + R^m)\|\phi\|_{K,n}$$

for all $\phi \in \mathcal{D}_K(\mathbb{R}^N;\mathbb{C})$, where $\|\cdot\|_{K,n}$ is defined in (10.1). It follows from Theorem 10.10 that $T : \mathcal{D}(\mathbb{R}^N;\mathbb{C}) \to \mathbb{C}$ is continuous. □

Next we show that $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ is embedded in L^p for every p.

Theorem 10.35. *For all $1 \le p \le \infty$, $\mathcal{S}(\mathbb{R}^N;\mathbb{C}) \hookrightarrow L^p(\mathbb{R}^N;\mathbb{C}) \hookrightarrow \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$.*

Proof. We only need to consider the case $1 \le p < \infty$. For $p = 1$, write

$$\int_{\mathbb{R}^N} \|\phi(x)\|\, dx = \int_{\mathbb{R}^N} \frac{1 + \|x\|^{N+1}}{1 + \|x\|^{N+1}} \|\phi(x)\|\, dx$$
$$\le c\|\phi\|_{N+1,0} \int_{\mathbb{R}^N} \frac{1}{1 + \|x\|^{N+1}}\, dx.$$

For $1 < p < \infty$ it is enough to observe that

$$\int_{\mathbb{R}^N} \|\phi(x)\|^p\, dx \le \|\phi\|_\infty^{p-1} \int_{\mathbb{R}^N} \|\phi(x)\|\, dx \le c\|\phi\|^p_{N+1,0}.$$

This shows that $\mathcal{S}(\mathbb{R}^N;\mathbb{C}) \hookrightarrow L^p(\mathbb{R}^N;\mathbb{C})$. Given $\psi \in L^p(\mathbb{R}^N;\mathbb{C})$, consider the linear functional $T_\psi : \mathcal{S}(\mathbb{R}^N;\mathbb{C}) \to \mathbb{C}$ defined by

$$T_\psi(\phi) := \int_{\mathbb{R}^N} \phi\psi\, dx, \quad \phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C}). \tag{10.17}$$

Then by Hölder's inequality

$$\|T_\psi(\phi)\| \le \|\phi\|_{L^{p'}} \|\psi\|_{L^p} \le c\|\phi\|_{N+1,0}\|\psi\|_{L^p}.$$

Hence, by (10.16) the functional T_ψ belongs to $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ and the linear mapping $\psi \in L^p(\mathbb{R}^N;\mathbb{C}) \mapsto T_\psi$ is an embedding. □

Remark 10.36. In what follows we identify ψ with T_ψ. Hence, $L^p(\mathbb{R}^N;\mathbb{C})$, and in particular $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$, can be thought as contained in $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$.

Remark 10.37. Note that if $f : \mathbb{R}^N \to \mathbb{C}$ is a polynomial, then

$$T_f(\phi) := \int_{\mathbb{R}^N} \phi f\, dx, \quad \phi \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C}),$$

is well-defined and $\|T_f(\phi)\| \le c_f\|\phi\|_{d+N+1,0}$, where d is the degree of f and c_f is the supremum of the norms of the coefficients of f. By (10.16) the functional T_f belongs to $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$

Example 10.38. Given a measure $\mu : \mathcal{B}(\mathbb{R}^N) \to [0,\infty]$ with the property that

$$\mu(\overline{B(0,r)}) \le c_0(1+r)^k$$

for some $c_0 > 0$, some $k \in \mathbb{N}$, and for all $r > 0$, the linear functional $T_\mu : \mathcal{S}(\mathbb{R}^N;\mathbb{C}) \to \mathbb{C}$ defined by $T_\mu(\phi) := \int_{\mathbb{R}^N} \phi\, d\mu$ is well-defined and continuous.

Indeed, write

$$\begin{aligned}\int_{\mathbb{R}^N} \|\phi\| \, d\mu &= \int_{B(0,1)} \|\phi\| \, d\mu + \sum_{n=2}^{\infty} \int_{\overline{B(0,n)} \setminus B(0,n-1)} \|\phi\| \, d\mu \\ &\leq 2c_0 \|\phi\|_\infty + \sum_{n=2}^{\infty} \int_{\overline{B(0,n)}} \frac{(1+\|x\|)^{2k}}{(1+\|x\|)^{2k}} \|\phi\| \, d\mu \\ &\leq 2c_0 \|\phi\|_\infty + c c_0 \|\phi\|_{0,2k} \sum_{n=1}^{\infty} \frac{(1+n)^k}{(1+n)^{2k}}.\end{aligned}$$

Hence by (10.16), $T_\mu \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$.

Exercise 10.39. Prove that the linear mapping

$$T(\phi) := \mathrm{PV}\Big(\frac{1}{x}\Big)(\phi) := \lim_{\delta \to 0^+} \int_{\mathbb{R} \setminus [-\delta,\delta]} \frac{\phi(x)}{x} \, dx, \quad \phi \in \mathcal{S}(\mathbb{R}),$$

is well-defined and belongs to $\mathcal{S}'(\mathbb{R};\mathbb{C})$.

Exercise 10.40. Let $\psi : \mathbb{R}^N \to \mathbb{C}$ be a function of class C^∞ such that for every multi-index $\alpha \in \mathbb{N}_0^N$ there exist c_α and $n_\alpha \in \mathbb{N}$ such that

$$\|\partial^\alpha \psi(x)\| \leq c_\alpha (1 + \|x\|^2)^{n_\alpha} \tag{10.18}$$

for all $x \in \mathbb{R}^N$.

(i) Prove that if $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$ then $\phi\psi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$.

(ii) Prove that if $h : \mathbb{R}^N \to \mathbb{C}$ is a Lebesgue measurable function such that $h\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$ for all $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$ and the mapping $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C}) \mapsto h\phi$ is continuous, then h must be of class C^∞ and satisfy (10.18).

(iii) Given $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ prove that the linear functional $\psi T : \mathcal{S}(\mathbb{R}^N;\mathbb{C}) \to \mathbb{C}$ defined by $(\psi T)(\phi) := T(\phi\psi)$, $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$, belongs to $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$.

As in Definition 10.18 we can define the notion of a derivative of a tempered distribution.

Definition 10.41. Given $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ and a multi-index $\alpha \in \mathbb{N}_0^N \setminus \{0\}$, we define the αth *derivative* of T as the linear functional $\partial^\alpha T : \mathcal{S}(\mathbb{R}^N;\mathbb{C}) \to \mathbb{C}$ defined by

$$\frac{\partial^{|\alpha|} T}{\partial x^\alpha}(\phi) := (-1)^{|\alpha|} T\Big(\frac{\partial^{|\alpha|}\phi}{\partial x^\alpha}\Big), \quad \phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C}).$$

For simplicity we often write ∂^α for $\frac{\partial^{|\alpha|}}{\partial x^\alpha}$ and ∂_i^n for $\frac{\partial^n}{\partial x_i^n}$.

Theorem 10.42. *For every $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ and every multi-index α, the functional $\partial^\alpha T$ belongs to $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$.*

Proof. Since $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$, by Theorem 10.33 there exist a constant $c > 0$ and $m, n \in \mathbb{N}_0$ such that $\|T(\phi)\| \le c\|\phi\|_{m,n}$ for every $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$. In turn, since for $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$, $\partial^\alpha \phi$ still belongs to $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$, with

$$\|(\partial^\alpha T)(\phi)\| = \|T(\partial^\alpha \phi)\| \le c\|\partial^\alpha \phi\|_{m,n} \le c\|\phi\|_{m,n+|\alpha|},$$

and so, again by Theorem 10.33 it follows that $\partial^\alpha T$ belongs to $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$. □

Exercise 10.43. Prove that if $T \in \mathcal{D}'(\mathbb{R}^N;\mathbb{C})$ has compact support, then $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$.

Exercise 10.44. Prove that $\mathcal{D}(\mathbb{R}^N;\mathbb{C})$ is dense in $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ and that the inclusion

$$\begin{aligned} \mathcal{D}(\mathbb{R}^N;\mathbb{C}) &\to \mathcal{S}(\mathbb{R}^N;\mathbb{C}) \\ T &\mapsto T \end{aligned}$$

is continuous.

Exercise 10.45. Prove that if f is a polynomial, $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$, and $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$, then fT and ϕT belong to $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$.

10.5. Convolutions

Given two measurable functions $\phi : \mathbb{R}^N \to \mathbb{C}$ and $\psi : \mathbb{R}^N \to \mathbb{C}$, the *convolution* of ϕ and ψ is the function $\phi * \psi$ defined by

$$(\phi * \psi)(x) := \int_{\mathbb{R}^N} \phi(x-y)\psi(y)\,dy \tag{10.19}$$

for all $x \in \mathbb{R}^N$ for which the right-hand side is well-defined.

Theorem 10.46. *Given $\phi, \psi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$, the function $\phi * \psi$ belongs to $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$.*

Proof. Fix $x \in \mathbb{R}^N$. For $m \in \mathbb{N}$ with $m > N$, we can write

$$\begin{aligned} \|(\phi * \psi)(x)\| &\le \int_{\mathbb{R}^N} \|\phi(x-y)\|\|\psi(y)\|\,dy \\ &\le c\|\psi\|_{0,m}\|\phi\|_{0,m} \int_{\mathbb{R}^N} \frac{1}{(1+\|y\|)^m}\frac{1}{(1+\|x-y\|)^m}\,dy. \end{aligned}$$

We now split $\mathbb{R}^N$ into the sets $E := \{y \in \mathbb{R}^N : \frac{1}{2}\|x\| \le \|x-y\|\}$ and $\mathbb{R}^N \setminus E$. Then we have

$$\begin{aligned} &\int_E \frac{1}{(1+\|y\|)^m}\frac{1}{(1+\|x-y\|)^m}\,dy \\ &\le \frac{2^m}{(2+\|x\|)^m}\int_{\mathbb{R}^N} \frac{1}{(1+\|y\|)^m}\,dy \le \frac{c}{(2+\|x\|)^m}, \end{aligned}$$

while in $\mathbb{R}^N \setminus E$, $\|y\| \geq \|x\| - \|x-y\| \geq \|x\| - \frac{1}{2}\|x\| = \frac{1}{2}\|x\|$, and so

$$\int_{\mathbb{R}^N\setminus E} \frac{1}{(1+\|y\|)^m}\frac{1}{(1+\|x-y\|)^m}\,dy$$
$$\leq \frac{2^m}{(2+\|x\|)^m}\int_{\mathbb{R}^N} \frac{1}{(1+\|x-y\|)^m}\,dy \leq \frac{c}{(2+\|x\|)^m},$$

where $c = c(m,N) > 0$. Hence,

$$(2+\|x\|)^m\|(\phi*\psi)(x)\| \leq c\|\psi\|_{0,m}\|\phi\|_{0,m}.$$

This shows that $\phi*\psi$ decays to zero faster than any power of $\|x\|$.

On the other hand, by differentiating under the integral sign, for every multi-index α,

$$\frac{\partial^{|\alpha|}(\phi*\psi)}{\partial x^\alpha}(x) = \int_{\mathbb{R}^N}\frac{\partial^{|\alpha|}\phi}{\partial x^\alpha}(x-y)\psi(y)\,dy = \Big(\frac{\partial^{|\alpha|}\phi}{\partial x^\alpha}*\psi\Big)(x),$$

and so by repeating the same calculations above with ϕ replaced by $\frac{\partial^{|\alpha|}\phi}{\partial x^\alpha}$, we get that all derivatives of $\phi*\psi$ decay to zero faster than any power of $\|x\|$, which shows that $\phi*\psi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$. □

Exercise 10.47. Prove that for every ϕ, ψ, $\varphi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$,

$$(\phi*\psi)*\varphi = \phi*(\psi*\varphi).$$

Theorem 10.48 (Young's inequality)**.** *Let $\phi \in L^p(\mathbb{R}^N;\mathbb{C})$, $1 \leq p \leq \infty$, and $\psi \in L^1(\mathbb{R}^N;\mathbb{C})$. Then $(\phi*\psi)(x)$ exists for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$ and*

$$\|\phi*\psi\|_{L^p(\mathbb{R}^N)} \leq \|\phi\|_{L^p(\mathbb{R}^N)}\|\psi\|_{L^1(\mathbb{R}^N)}.$$

Proof. Consider two Borel functions ϕ_0 and ψ_0 such that $\phi_0(x) = \phi(x)$ and $\psi_0(x) = \psi(x)$ for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$. Since the integral in (10.19) is unchanged if we replace ϕ and ψ with ϕ_0 and ψ_0, respectively, in what follows, without loss of generality, we may assume that ϕ and ψ are Borel functions.

Let $h : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{C}$ be the function defined by $h(x,y) := \phi(x-y)$, $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$. Then h is a Borel function, since it is the composition of the Borel function ϕ with the continuous function $\psi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ given by $\psi(x,y) := x-y$. In turn, the function

$$(x,y) \in \mathbb{R}^N \times \mathbb{R}^N \mapsto \phi(x-y)\psi(y)$$

is Borel measurable. We are now in a position to apply Minkowski's inequality for integrals (Corollary B.83) and Tonelli's theorem to conclude that

$$\begin{aligned}\|\phi * \psi\|_{L^p} &= \Big\|\int_{\mathbb{R}^N} \|\phi(\cdot - y)\psi(y)\|\, dy\Big\|_{L^p}\\ &\le \int_{\mathbb{R}^N} \|\phi(\cdot - y)\psi(y)\|_{L^p}\, dy\\ &= \int_{\mathbb{R}^N} \|\psi(y)\|\|\phi(\cdot - y)\|_{L^p}\, dy = \|\phi\|_{L^p}\int_{\mathbb{R}^N} \|\psi(y)\|\, dy,\end{aligned}$$

where in the last equality we have used the fact that the Lebesgue measure is translation invariant. Hence, $\phi * \psi$ belongs to $L^p(\mathbb{R}^N;\mathbb{C})$, and so it is finite $\mathcal{L}^N$-a.e. in $\mathbb{R}^N$. □

The following is the generalized form of the previous inequality.

Theorem 10.49 (Young's inequality, general form)**.** *Let $1 \le p \le q \le \infty$ and let $\phi \in L^p(\mathbb{R}^N;\mathbb{C})$ and $\psi \in L^q(\mathbb{R}^N;\mathbb{C})$. Then $(\phi * \psi)(x)$ exists for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$ and*

$$\|\phi * \psi\|_{L^r(\mathbb{R}^N)} \le \|\phi\|_{L^p(\mathbb{R}^N)}\|\psi\|_{L^q(\mathbb{R}^N)},$$

where

$$1/p + 1/q = 1 + 1/r. \tag{10.20}$$

Proof. If $r = \infty$, then q is the Hölder conjugate exponent of p and the result follows from Hölder's inequality and the translation invariance of the Lebesgue measure, while if $p = 1$, then $r = q$ and the result follows from the previous theorem (with ϕ and ψ interchanged).

In the remaining cases, write

$$\|\phi(x-y)\psi(y)\| = (\|\phi(x-y)\|^p\|\psi(y)\|^q)^{1/r}\|\phi(x-y)\|^{(r-p)/r}\|\psi(y)\|^{(r-q)/r}.$$

Define $p_1 := r$, $p_2 := pr/(r-p)$, $p_3 := qr/(r-q)$. Then $1/p_1 + 1/p_2 + 1/p_3 = 1$, and so by the extended Hölder inequality (see Exercise B.79) and the translation invariance of the Lebesgue measure,

$$\begin{aligned}\|(\phi * \psi)(x)\| &\le \int_{\mathbb{R}^N} \|\phi(x-y)\psi(y)\|\, dy\\ &\le \Big(\int_{\mathbb{R}^N} \|\phi(x-y)\|^p\|\psi(y)\|^q dy\Big)^{1/r}\|\phi\|_{L^p}^{1-p/r}\|\psi\|_{L^q}^{1-q/r}.\end{aligned}$$

Taking the norm in $L^r(\mathbb{R}^N)$ on both sides, we get

$$\|\phi * \psi\|_{L^r} \le \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \|\phi(x-y)\|^p\|\psi(y)\|^q dydx\Big)^{1/r}\|\phi\|_{L^p}^{1-p/r}\|\psi\|_{L^q}^{1-q/r}.$$

Applying the previous theorem (with $p = 1$), we get that the right-hand side of the previous inequality is less than or equal to

$$(\|\|\phi\|^p\|_{L^1}\|\|\psi\|^q\|_{L^1})^{1/r}\|\phi\|_{L^p}^{1-p/r}\|\psi\|_{L^q}^{1-q/r} = \|\phi\|_{L^p}\|\psi\|_{L^q}.$$

This concludes the proof. □

10.6. Convolution of Distributions

In this section we define the convolution of a distribution (or a tempered distribution) T and a function φ. We begin with the case in which $T = T_u$ for some function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, where we recall that $T_u \in \mathcal{D}'(\mathbb{R}^N)$ is defined by

$$T_u(\phi) := \int_{\mathbb{R}^N} \psi(x)\phi(x)\, dx, \quad \phi \in \mathcal{D}(\mathbb{R}^N).$$

By Fubini's theorem

$$\begin{aligned}\int_{\mathbb{R}^N} (u * \varphi)(x)\phi(x)\, dx &= \int_{\mathbb{R}^N} \phi(x) \int_{\mathbb{R}^N} u(x-y)\varphi(y)\, dydx \\ &= \int_{\mathbb{R}^N} u(\xi) \int_{\mathbb{R}^N} \varphi(x-\xi)\phi(x)\, dxd\xi \\ &= \int_{\mathbb{R}^N} u(\xi) \int_{\mathbb{R}^N} \widetilde{\varphi}(\xi - x)\phi(x)\, dxd\xi \\ &= \int_{\mathbb{R}^N} u(\xi)(\widetilde{\varphi} * \phi)(\xi)\, d\xi,\end{aligned}$$

where $\xi := x - y$ and $\widetilde{\varphi}(x) := \varphi(-x)$. Hence, we have shown that $T_{u*\varphi}(\phi) = T_u(\widetilde{\varphi} * \phi)$ for all $\phi \in \mathcal{D}(\mathbb{R}^N)$. Motivated by this formula we define:

Definition 10.50. If $T \in \mathcal{D}'(\mathbb{R}^N)$ and $\varphi \in \mathcal{D}(\mathbb{R}^N)$ the *convolution* of T and φ is the linear functional $T*\varphi : \mathcal{D}(\mathbb{R}^N) \to \mathbb{R}$ defined by $(T*\varphi)(\phi) := T(\widetilde{\varphi}*\phi)$, where

$$\widetilde{\varphi}(x) := \varphi(-x), \quad x \in \mathbb{R}^N. \tag{10.21}$$

It turns out that $T * \varphi$ can be identified with a function. Given $x \in \mathbb{R}^N$ and a function $\varphi : \mathbb{R}^N \to \mathbb{R}$ we define the function

$$\varphi^x(y) := \varphi(x - y), \quad y \in \mathbb{R}^N. \tag{10.22}$$

Exercise 10.51. Let $\phi, \varphi \in \mathcal{D}(\mathbb{R}^N)$. For $h > 0$ define

$$u_h(x) := h^N \sum_{y \in \mathbb{Z}^N} \varphi(x - hy)\phi(hy), \quad x \in \mathbb{R}^N.$$

Prove that $\{u_h\}_h$ converges uniformly to $\varphi * \phi$ as $h \to 0^+$. Hint: Use Riemann sums.

Theorem 10.52. *Let $T \in \mathcal{D}'(\mathbb{R}^N)$ and $\phi, \psi \in \mathcal{D}(\mathbb{R}^N)$. Then $T * \varphi = T_{u_\varphi}$, where u_φ is the function given by $u_\varphi(x) := T(\varphi^x)$, $x \in \mathbb{R}^N$. Moreover,*

(i) $u_\varphi \in C^\infty(\mathbb{R}^N)$,

(ii) $\operatorname{supp} u_\varphi \subseteq \operatorname{supp} T + \operatorname{supp} \varphi$,

(iii) *for every multi-index* $\alpha \in \mathbb{N}_0^N$, $\partial^\alpha u_\varphi = T * \partial^\alpha \varphi = \partial^\alpha T * \varphi$,

(iv) $(T * \varphi) * \psi = T * (\varphi * \psi)$.

Proof. Step 1: Fix $\phi \in \mathcal{D}(\mathbb{R}^N)$. We approximate the function $\widetilde{\varphi} * \phi$ (see (10.21)) with the Riemann sum

$$u_h(x) := h^N \sum_{y \in \mathbb{Z}^N} \widetilde{\varphi}(x - hy)\phi(hy), \quad x \in \mathbb{R}^N,$$

where $h > 0$. Note that for every multi-index α,

$$\frac{\partial^{|\alpha|} u_h}{\partial x^\alpha}(x) = h^N \sum_{y \in \mathbb{Z}^N} \frac{\partial^{|\alpha|} \widetilde{\varphi}}{\partial x^\alpha}(x - hy)\phi(hy), \quad x \in \mathbb{R}^N.$$

Since $\left\{ \frac{\partial^{|\alpha|} u_h}{\partial x^\alpha} \right\}_h$ converges uniformly to $\frac{\partial^{|\alpha|} \widetilde{\varphi}}{\partial x^\alpha} * \phi$ by the previous exercise and

$$\frac{\partial^{|\alpha|} \widetilde{\varphi}}{\partial x^\alpha} * \phi = \frac{\partial^{|\alpha|}}{\partial x^\alpha}(\widetilde{\varphi} * \phi)$$

(why?), by Theorem 10.8 we have that $\{u_h\}_h$ converges to $\widetilde{\varphi} * \phi$ as $h \to 0$ with respect to the topology τ in $\mathcal{D}(\mathbb{R}^N)$.

Hence, by the continuity and linearity of T and by Theorem 10.10 we have that

$$\begin{aligned}(T * \varphi)(\phi) = T(\widetilde{\varphi} * \phi) &= \lim_{h \to 0} T(u_h) = \lim_{h \to 0} h^N \sum_{y \in \mathbb{Z}^N} T(\widetilde{\varphi}(\cdot - hy))\phi(hy) \\ &= \lim_{h \to 0} h^N \sum_{y \in \mathbb{Z}^N} T(\varphi(hy - \cdot))\phi(hy) = \int_{\mathbb{R}^N} T(\varphi^y)\phi(y)\, dy,\end{aligned}$$

where we have used the previous exercise. This shows that $T * \varphi = T_{u_\varphi}$. In what follows we identify $T * \varphi$ with u_φ.

Step 2: If $x_n \to x$ in $\mathbb{R}^N$, then for every $y \in \mathbb{R}^N$,

$$\varphi^{x_n}(y) = \varphi(x_n - y) \to \varphi(x - y) = \varphi^x(y)$$

and conditions (i) and (ii) of Theorem 10.8 are satisfied. Hence, $\{\varphi^{x_n}\}_n$ converges to φ^x with respect to τ, and so by Theorem 10.10,

$$(T * \varphi)(x_n) = T(\varphi^{x_n}) \to T(\varphi^x) = (T * \varphi)(x),$$

which proves that $T * \varphi$ is a continuous function.

To prove (ii), note that if $x \in \mathbb{R}^N$ is such that $\operatorname{supp} \varphi^x \cap \operatorname{supp} T = \emptyset$, then $(T * \varphi)(x) = 0$. Thus,

$$\operatorname{supp}(T * \varphi) \subseteq \{x \in \mathbb{R}^N : \operatorname{supp} \varphi^x \cap \operatorname{supp} T \neq \emptyset\} = \operatorname{supp} T + \operatorname{supp} \varphi.$$

Next we prove (iii). Let e_i be an element of the canonical basis of $\mathbb{R}^N$ and for every $x \in \mathbb{R}^N$ and $h \neq 0$ consider the function

$$\varphi^{x,h,i}(y) := \frac{\varphi(x + he_i - y) - \varphi(x - y)}{h}, \quad y \in \mathbb{R}^N.$$

As $h \to 0$, we have that $\varphi^{x,h,i}(y) \to \frac{\partial \varphi}{\partial x_i}(x-y)$ for all $y \in \mathbb{R}^N$ and conditions (i) and (ii) of Theorem 10.8 are satisfied (why?). Hence, $\{\varphi^{x,h,i}\}_h$ converges to $(\partial_i \varphi)^x$ with respect to τ as $h \to 0$, and so, by the linearity of T and Theorem 10.10,

$$\frac{(T * \varphi)(x + he_i) - (T * \varphi)(x)}{h} = T(\varphi^{x,h,i}) \to T((\partial_i \varphi)^x)$$

as $h \to 0$, which proves that $\partial_i (T * \varphi) = T * \partial_i \varphi$.

Moreover, since for all $x, y \in \mathbb{R}^N$,

$$\Big(\frac{\partial \varphi}{\partial x_i}\Big)^x (y) = \frac{\partial \varphi}{\partial x_i}(x - y) = -\frac{\partial \varphi}{\partial y_i}(x - y) = -\frac{\partial \varphi^x}{\partial y_i}(y),$$

for all $x \in \mathbb{R}^N$ we have

$$\begin{aligned}\Big(\frac{\partial T}{\partial y_i} * \varphi\Big)(x) &= \frac{\partial T}{\partial y_i}(\varphi^x) = -T\Big(\frac{\partial \varphi^x}{\partial y_i}\Big) \\ &= T\Big(-\Big(\frac{\partial \varphi}{\partial y_i}\Big)^x\Big) = \Big(T * \frac{\partial \varphi}{\partial x_i}\Big)(x),\end{aligned}$$

which, together with an induction argument, gives (iii).

Finally, to prove (iv), we define

$$u_h(x) := h^N \sum_{y \in \mathbb{Z}^N} \varphi(x - hy)\psi(hy), \quad x \in \mathbb{R}^N,$$

where $h > 0$. As before we have that $\{u_h\}_h$ converges to $\varphi * \psi$ with respect to τ as $h \to 0$. It follows that for every $x \in \mathbb{R}^N$, $\{(u_h)^x\}_h$ converges to $(\varphi * \psi)^x$ with respect to τ as $h \to 0$. By the linearity of T and by Theorem 10.10 we have that

$$\begin{aligned}(T * (\varphi * \psi))(x) &= T((\varphi * \psi)^x) = \lim_{h \to 0} T((u_h)^x) = \lim_{h \to 0} (T * u_h)(x) \\ &= \lim_{h \to 0} h^N \sum_{y \in \mathbb{Z}^N} (T * \varphi)(x - hy)\psi(hy) \\ &= ((T * \varphi) * \psi)(x),\end{aligned}$$

where we have used the previous exercise. This completes the proof. ☐

In what follows we identify $T * \varphi$ with u_φ. As a consequence of the previous theorem, we can approximate distributions with C^∞ functions.

Theorem 10.53. *Let $T \in \mathcal{D}'(\mathbb{R}^N)$ and let $\{\varphi_\varepsilon\}_\varepsilon$, $\varepsilon > 0$, be a family of standard mollifiers*[2]*. Then $\{T * \varphi_\varepsilon\}_\varepsilon$ converges to T in the sense of distributions as $\varepsilon \to 0^+$; that is,*

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} (T * \varphi_\varepsilon)(x)\phi(x)\,dx = T(\phi)$$

for every $\phi \in \mathcal{D}(\mathbb{R}^N)$.

Proof. By Theorems C.16(i) and C.20 and Theorem 10.8 we have that for every $\phi \in \mathcal{D}(\mathbb{R}^N)$ the sequence $\{\varphi_\varepsilon * \phi\}$ converges to ϕ with respect to τ as $\varepsilon \to 0^+$. Then by Theorem 10.10,

$$\begin{aligned} T(\phi) &= (T * \widetilde{\phi})(0) = T((\widetilde{\phi})^0) = \lim_{\varepsilon \to 0^+} T((\varphi_\varepsilon * \widetilde{\phi})^0) \\ &= \lim_{\varepsilon \to 0^+} (T * (\varphi_\varepsilon * \widetilde{\phi}))(0) = \lim_{\varepsilon \to 0^+} ((T * \varphi_\varepsilon) * \widetilde{\phi})(0) \\ &= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} (T * \varphi_\varepsilon)(y)\widetilde{\phi}(0 - y)\,dy = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} (T * \varphi_\varepsilon)(y)\phi(y)\,dy, \end{aligned}$$

where we have used Theorem 10.52(iv). □

Exercise 10.54. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $T \in \mathcal{D}'(\Omega)$.

(i) Prove that there exists a sequence $\{T_n\}_n$ in $\mathcal{D}'(\Omega)$ such that each T_n has support compactly contained in Ω and $\{T_n\}$ converges to T in the sense of distributions.

(ii) Prove that $C_c^\infty(\Omega)$ is dense in $\mathcal{D}'(\Omega)$ with respect to the weak star topology of $\mathcal{D}'(\Omega)$.

Similarly we can define the convolution of a tempered distribution and a function. We recall that $\widetilde{\varphi}(x) := \varphi(-x)$, $x \in \mathbb{R}^N$ (see (10.21)).

Definition 10.55. If $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ and $\varphi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$ the *convolution* of T and φ is the continuous linear functional $T * \varphi : \mathcal{S}(\mathbb{R}^N;\mathbb{C}) \to \mathbb{C}$ defined by $(T * \varphi)(\phi) := T(\widetilde{\varphi} * \phi)$, $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$.

The proof of the following theorem is similar to the one of Theorem 10.52 and is left as an exercise. We recall that $\varphi^x(y) = \varphi(x - y)$, $y \in \mathbb{R}^N$.

Theorem 10.56. *If $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ and $\varphi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$, then $T * \varphi = T_{u_\varphi}$, where u_φ is the function given by $u_\varphi(x) := T(\varphi^x)$, $x \in \mathbb{R}^N$. Moreover, $u_\varphi \in C^\infty(\mathbb{R}^N;\mathbb{C})$ and for every multi-index α there exist $c_\alpha > 0$ and $n_\alpha \in \mathbb{N}$ such that*

$$\|\partial^\alpha u_\varphi(x)\| \le c_\alpha (1 + \|x\|^2)^{n_\alpha}$$

for all $x \in \mathbb{R}^N$.

[2] See Appendix C.

In what follows we identify $T * \varphi$ with the function u_φ.

Exercise 10.57. Let $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$. Prove that $(T * \varphi) * \psi = (T * \psi) * \varphi$.

10.7. Fourier Transforms

Given $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$, the *Fourier transform of* ϕ is the function

$$\widehat{\phi}(x) = \mathcal{F}(\phi)(x) := \int_{\mathbb{R}^N} e^{-2\pi i x\cdot y}\phi(y)\,dy, \quad x \in \mathbb{R}^N, \tag{10.23}$$

while the *inverse Fourier transform of* ϕ is the function

$$\phi^\vee(x) := \widehat{\phi}(-x) = \int_{\mathbb{R}^N} e^{2\pi i x\cdot y}\phi(y)\,dy, \quad x \in \mathbb{R}^N. \tag{10.24}$$

Since $\mathcal{S}(\mathbb{R}^N;\mathbb{C}) \subset L^1(\mathbb{R}^N;\mathbb{C})$ (see Theorem 10.35), the functions $\widehat{\phi}$ and $\phi^\vee$ are well-defined.

Theorem 10.58. *The Fourier transform $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R}^N)$ into $\mathcal{S}(\mathbb{R}^N)$. Moreover, for every $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$ and for every $\alpha, \beta \in \mathbb{N}_0^N$,*

$$x^\alpha \partial^\beta \widehat{\phi}(x) = \widehat{\psi_{\alpha,\beta}}(x) \quad \textit{for every } x \in \mathbb{R}^N, \tag{10.25}$$

where $\psi_{\alpha,\beta}(x) := \partial^\alpha((-2\pi i x)^\beta \phi(x))$. In particular, $x^\alpha \widehat{\phi}(x) = \widehat{\partial^\alpha \phi}(x)$.

Proof. By differentiating under the integral sign we have that

$$\frac{\partial^{|\beta|}\widehat{\phi}}{\partial x^\beta}(x) = \int_{\mathbb{R}^N} \frac{\partial^{|\beta|}}{\partial x^\beta}(e^{-2\pi i x\cdot y})\phi(y)\,dy = \int_{\mathbb{R}^N} (-2\pi i y)^\beta e^{-2\pi i x\cdot y}\phi(y)\,dy.$$

Hence,

$$\begin{aligned} x^\alpha \frac{\partial^{|\beta|}\widehat{\phi}}{\partial x^\beta}(x) &= \frac{1}{(-2\pi i)^{|\alpha|}} \int_{\mathbb{R}^N} (-2\pi i y)^\beta (-2\pi i x)^\alpha e^{-2\pi i x\cdot y}\phi(y)\,dy \\ &= \frac{1}{(-2\pi i)^{|\alpha|}} \int_{\mathbb{R}^N} (-2\pi i y)^\beta \phi(y) \frac{\partial^{|\alpha|}}{\partial y^\alpha}(e^{-2\pi i x\cdot y})\,dy. \end{aligned}$$

By integrating by parts (see (9.55)) and using the fact that ϕ and its derivatives decay to zero at infinity we get

$$x^\alpha \frac{\partial^{|\beta|}\widehat{\phi}}{\partial x^\beta}(x) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot y} \frac{\partial^{|\alpha|}}{\partial y^\alpha}((-2\pi i y)^\beta \phi(y))\,dy,$$

which proves (10.25). It follows from Leibnitz rule (see Exercise 10.11) that

$$\begin{aligned}\|\widehat{\phi}\|_{\alpha,\beta} &\le (2\pi)^{|\beta|}\int_{\mathbb{R}^N}\left\|\frac{\partial^{|\alpha|}}{\partial y^\alpha}((-y)^\beta\phi(y))\right\|\,dy\\ &= (2\pi)^{|\beta|}\int_{\mathbb{R}^N}\frac{1+\|y\|^{N+1}}{1+\|y\|^{N+1}}\left\|\frac{\partial^{|\alpha|}}{\partial y^\alpha}((-y)^\beta\phi(y))\right\|\,dy\\ &\le c\|\phi\|_{N+1+|\alpha|,|\beta|},\end{aligned}$$

where $c = c(N,\alpha,\beta) > 0$, which shows that $\widehat{\phi} \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$ and that the linear operator $\mathcal{F} : \mathcal{S}(\mathbb{R}^N;\mathbb{C}) \to \mathcal{S}(\mathbb{R}^N;\mathbb{C})$ is continuous. □

Exercise 10.59. Prove that the Fourier transform of the function $\phi(x) := e^{-\pi\|x\|^2}$, $x \in \mathbb{R}^N$, is ϕ.

Exercise 10.60. Prove that the Fourier transform of the function

$$\phi_\varepsilon(x) := e^{2\pi i x\cdot x_0}e^{-\pi\varepsilon^2\|x\|^2}, \quad x \in \mathbb{R}^N,$$

where $\varepsilon > 0$ and $x_0 \in \mathbb{R}^N$, is

$$\widehat{\phi_\varepsilon}(x) = \frac{1}{\varepsilon^N}e^{-\pi\|(x-x_0)/\varepsilon\|^2}, \quad x \in \mathbb{R}^N.$$

Next we prove that $\mathcal{F}$ is invertible with inverse given by $\mathcal{F}^{-1}(\phi) = \phi^\vee$ (see (10.24)).

Proposition 10.61. *For every ϕ, $\psi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$, we have*

$$\int_{\mathbb{R}^N}\phi(x)\widehat{\psi}(x)\,dx = \int_{\mathbb{R}^N}\widehat{\phi}(x)\psi(x)\,dx. \tag{10.26}$$

Proof. By Fubini's theorem

$$\begin{aligned}\int_{\mathbb{R}^N}\phi(x)\widehat{\psi}(x)\,dx &= \int_{\mathbb{R}^N}\phi(x)\int_{\mathbb{R}^N}e^{-2\pi i x\cdot y}\psi(y)\,dydx\\ &= \int_{\mathbb{R}^N}\psi(y)\int_{\mathbb{R}^N}e^{-2\pi i x\cdot y}\phi(x)\,dxdy = \int_{\mathbb{R}^N}\psi(y)\widehat{\phi}(y)\,dy,\end{aligned}$$

which shows (10.26). □

Theorem 10.62 (Fourier inversion theorem)**.** *For every $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$,*

$$(\widehat{\phi})^\vee = \widehat{(\phi^\vee)} = \phi.$$

In particular, the Fourier transform $\mathcal{F}$ is an isomorphism from $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ to $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ with inverse $\mathcal{F}^{-1}$ given by $\mathcal{F}^{-1}(\phi) = \phi^\vee$ for every $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$.

Proof. Fix $x_0 \in \mathbb{R}^N$ and $\varepsilon > 0$ and define $\psi_\varepsilon(x) := e^{2\pi i x\cdot x_0}e^{-\pi\varepsilon^2\|x\|^2}$, $x \in \mathbb{R}^N$. By Exercise 10.60 we have that $\widehat{\psi_\varepsilon}(x) = \frac{1}{\varepsilon^N}e^{-\pi\|(x-x_0)/\varepsilon\|^2}$ and so, taking $\psi = \psi_\varepsilon$ in (10.26), we get

$$\int_{\mathbb{R}^N}\phi(x)\frac{1}{\varepsilon^N}e^{-\pi\|(x-x_0)/\varepsilon\|^2}dx = \int_{\mathbb{R}^N}e^{2\pi i y\cdot x_0}e^{-\pi\varepsilon^2\|y\|^2}\widehat{\phi}(y)\,dy.$$

Note that $\widehat{\psi_\varepsilon}$ is a mollifier. Hence, the left-hand side converges to $\phi(x_0)$ by Exercise C.19 in Appendix C. On the other hand, by the Lebesgue dominated convergence theorem the right-hand side converges to $(\widehat{\phi})^\vee(x_0)$. Hence, $\phi(x_0) = (\widehat{\phi})^\vee(x_0)$, which shows that $(\widehat{\phi})^\vee = \phi$. Similarly we can show that, $\widehat{(\phi^\vee)} = \phi$.

Next observe that if $\widehat{\phi} = 0$, then $\phi = (\widehat{\phi})^\vee = 0^\vee = 0$, and so $\mathcal{F}$ is one-to-one. Since $\widehat{(\phi^\vee)} = \phi$, it follows that $\mathcal{F}$ is onto and that the inverse of $\mathcal{F}$ is $\mathcal{F}^{-1}(\phi) = \phi^\vee$. $\square$

We recall that for a complex number $z = x + iy$, where $x, y \in \mathbb{R}$, the complex conjugate of z is the number $\overline{z} := x - iy$.

Corollary 10.63 (Plancherel). *For every ϕ, $h \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$,*

$$\int_{\mathbb{R}^N} \phi(x)\overline{h(x)}\,dx = \int_{\mathbb{R}^N} \widehat{\phi}(x)\overline{\widehat{h}(x)}\,dx \tag{10.27}$$

and

$$\int_{\mathbb{R}^N} \|\phi(x)\|^2 dx = \int_{\mathbb{R}^N} \|\widehat{\phi}(x)\|^2 dx = \int_{\mathbb{R}^N} \|\phi^\vee(x)\|^2 dx. \tag{10.28}$$

In particular, $\mathcal{F}$ extends uniquely to an isomorphism of $L^2(\mathbb{R}^N;\mathbb{C})$ onto itself.

The identity (10.27) is called *Parseval identity*, while the identity (10.28) is called *Plancherel identity*.

Proof. Let $\psi := \overline{\widehat{h}}$. Then, using the facts that the cosine is even and the sine is odd, we have

$$\begin{aligned}\widehat{\psi}(x) &= \int_{\mathbb{R}^N} e^{-2\pi i y\cdot x}\overline{\widehat{h}(y)}\,dy = \int_{\mathbb{R}^N} \overline{e^{2\pi i y\cdot x}}\overline{\widehat{h}(y)}\,dy \\ &= \int_{\mathbb{R}^N} \overline{e^{2\pi i y\cdot x}\widehat{h}(y)}\,dy = \overline{\int_{\mathbb{R}^N} e^{2\pi i y\cdot x}\widehat{h}(y)\,dy} = \overline{(\widehat{h})^\vee(x)} = \overline{h(x)},\end{aligned}$$

where in the last equality we have used the inversion theorem (Theorem 10.62). Hence, Parseval's identity follows by (10.26). Taking $h = \phi$ and using the fact that $\phi(x)\overline{\phi(x)} = \|\phi(x)\|^2$ gives the first equality in Plancherel's identity. The second equality follows by replacing ϕ with $\phi^\vee$ and using the inversion theorem.

Since $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ is dense in $L^2(\mathbb{R}^N;\mathbb{C})$, if $\{\phi_n\}_n$ is a sequence in $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ converging to ϕ in $L^2(\mathbb{R}^N;\mathbb{C})$, then by Plancherel's identity the sequence $\{\widehat{\phi_n}\}_n$ is a Cauchy sequence in $L^2(\mathbb{R}^N;\mathbb{C})$ and so it converges to a function $\psi \in L^2(\mathbb{R}^N;\mathbb{C})$. Again by Plancherel's identity, the function ψ does not depend on the particular sequence $\{\phi_n\}_n$. We define $\widehat{\phi} := \psi$. Similarly we can extend uniquely the inverse Fourier transform to $L^2(\mathbb{R}^N;\mathbb{C})$ and reasoning as in the last part of the proof of the inversion theorem (Theorem 10.62) we have that the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^N;\mathbb{C}) \to L^2(\mathbb{R}^N;\mathbb{C})$ is an isomorphism with inverse given by the extension of $\mathcal{F}^{-1}$ to $L^2(\mathbb{R}^N;\mathbb{C})$. □

Remark 10.64 (Important)**.** Note that the Fourier transform of a function ϕ in $L^2(\mathbb{R}^N;\mathbb{C})$ is obtained as a limit in $L^2(\mathbb{R}^N;\mathbb{C})$ of functions of the type (10.23), but in general we cannot say that $\widehat{\phi}$ has the form (10.23), since the integral in (10.23) is well-defined for functions in $L^1(\mathbb{R}^N;\mathbb{C})$ but not for functions in $L^2(\mathbb{R}^N;\mathbb{C})$. On the other hand, if $\phi \in L^1(\mathbb{R}^N;\mathbb{C})$, then (10.23) is well-defined. Hence, the Fourier transform of a function in $L^1(\mathbb{R}^N;\mathbb{C})$ is defined pointwise by (10.23), while the Fourier transform of a function in $L^2(\mathbb{R}^N;\mathbb{C})$ is defined as a limit in $L^2(\mathbb{R}^N;\mathbb{C})$.

Theorem 10.65 (Riemann–Lebesgue lemma)**.** $\mathcal{F} : L^1(\mathbb{R}^N;\mathbb{C}) \to C_0(\mathbb{R}^N;\mathbb{C})$ *with*

$$\sup_{x\in\mathbb{R}^N} \|\mathcal{F}(\phi)(x)\| \le \|\phi\|_{L^1(\mathbb{R}^N)} \tag{10.29}$$

In particular, $\lim_{\|x\|\to\infty} \|\widehat{\phi}(x)\| = 0$.

Proof. By (10.23), for every $\phi \in L^1(\mathbb{R}^N;\mathbb{C})$,

$$\|\widehat{\phi}(x)\| \le \|\phi\|_{L^1}$$

for every $x \in \mathbb{R}^N$. Since $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ is dense in $L^1(\mathbb{R}^N;\mathbb{C})$ let $\{\phi_n\}_n$ in $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ converge to ϕ in $L^1(\mathbb{R}^N;\mathbb{C})$. By the previous inequality

$$\sup_{x\in\mathbb{R}^N} \|\widehat{\phi_n}(x) - \widehat{\phi}(x)\| \le \|\phi_n - \phi\|_{L^1}.$$

Hence, the sequence $\{\widehat{\phi_n}\}_n$ converges uniformly to $\widehat{\phi}$. On the other hand, by Theorem 10.58 we have that $\widehat{\phi_n} \in \mathcal{S}(\mathbb{R}^N;\mathbb{C}) \subset C_0(\mathbb{R}^N;\mathbb{C})$ and hence, since $C_0(\mathbb{R}^N;\mathbb{C})$ is a closed under uniform convergence, it follows that $\widehat{\phi} \in C_0(\mathbb{R}^N;\mathbb{C})$. □

Exercise 10.66. Consider the Fourier transform $\mathcal{F} : L^1(\mathbb{R};\mathbb{C}) \to C_0(\mathbb{R};\mathbb{C})$.

(i) Prove that if $f \in L^1(\mathbb{R};\mathbb{C})$ is such that $\widehat{f}$ is odd, then for every $b > 0$,

$$\Big|\int_1^b \frac{\widehat{f}(x)}{x}\,dx\Big| \le c$$

for some constant $c > 0$ independent of b.

(ii) Prove that there exists $g \in C_0(\mathbb{R};\mathbb{C})$ such that g is not the Fourier transform of any function f in $L^1(\mathbb{R};\mathbb{C})$.

(iii) Prove that $\mathcal{F}(L^1(\mathbb{R};\mathbb{C}))$ is dense in $C_0(\mathbb{R};\mathbb{C})$.

Corollary 10.67 (Hausdorff–Young inequality). *Let $1 < p < 2$. Then $\mathcal{F}$ can be extended uniquely to $L^p(\mathbb{R}^N;\mathbb{C})$ and*

$$\|\mathcal{F}(\phi)\|_{L^{p'}(\mathbb{R}^N)} \le \|\phi\|_{L^p(\mathbb{R}^N)} \quad \text{for every } \phi \in L^p(\mathbb{R}^N;\mathbb{C}). \tag{10.30}$$

Proof. By Plancherel's identity and (10.29),

$$\sup_{x\in\mathbb{R}^N} \|\mathcal{F}(\phi)(x)\| \le \|\phi\|_{L^1}, \quad \|\mathcal{F}(\phi)\|_{L^2} = \|\phi\|_{L^2}$$

for every simple function ϕ vanishing at infinity. Hence, $\mathcal{F}$ is of strong type $(1,\infty)$ and $(2,2)$ (see Definition B.95) and so we are in a position to apply the Riesz–Thorin theorem (see Theorem B.96) to obtain that $\mathcal{F}$ is of strong type (p,q) for every $\theta \in (0,1)$, where

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}, \quad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}.$$

Hence, $p = \frac{2}{2-\theta} \in (1,2)$ and $q = p' = 2/\theta$. Moreover, (10.30) holds. □

Exercise 10.68. Let $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$ and let $L : \mathbb{R}^N \to \mathbb{R}^N$ be an orthogonal transformation.

(i) Prove that $(\widehat{\phi \circ L})(x) = \widehat{\phi}(L(x))$ for every $x \in \mathbb{R}^N$.

(ii) Prove that if ϕ is radial, then so is $\widehat{\phi}$.

Exercise 10.69. Prove that if $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$ is real-valued and $\phi(x) = \phi(-x)$ for every $x \in \mathbb{R}^N$, then $\widehat{\phi}$ is real-valued.

Exercise 10.70. Compute the Fourier transform of the function

$$\phi(x) := e^{(-a+ib)\|x\|^2}, \quad x \in \mathbb{R}^N,$$

where $a > 0$ and $b \in \mathbb{R}$.

Theorem 10.71. *For every ϕ, $\psi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$, $\widehat{\phi * \psi} = \widehat{\phi}\widehat{\psi}$.*

Proof. For $x \in \mathbb{R}^N$ by Fubini's theorem we have

$$\begin{aligned}
(\widehat{\phi * \psi})(x) &= \int_{\mathbb{R}^N} e^{-2\pi i x\cdot y}(\phi * \psi)(y)\, dy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-2\pi i x\cdot y}\phi(y-\xi)\psi(\xi)\, d\xi dy \\
&= \int_{\mathbb{R}^N} \psi(\xi) \int_{\mathbb{R}^N} e^{-2\pi i x\cdot y}\phi(y-\xi)\, dy d\xi \\
&= \int_{\mathbb{R}^N} e^{-2\pi i x\cdot \xi}\psi(\xi) \int_{\mathbb{R}^N} e^{-2\pi i x\cdot (y-\xi)}\phi(y-\xi)\, dy d\xi \\
&= \int_{\mathbb{R}^N} e^{-2\pi i x\cdot \xi}\psi(\xi) \int_{\mathbb{R}^N} e^{-2\pi i x\cdot \eta}\phi(\eta)\, d\eta d\xi = \widehat{\psi}(x)\widehat{\phi}(x),
\end{aligned}$$

where we have made the change of variables $\eta := y - \xi$. □

Remark 10.72. The previous theorem continues to hold for $\phi \in L^1(\mathbb{R}^N;\mathbb{C})$ and $\psi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$.

As a corollary of Theorem 10.71 we can show how the support of the Fourier transform relates to the regularity of a function.

Theorem 10.73. *Let $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$ be such that $\operatorname{supp}\widehat{\phi} \subseteq \overline{B(0,R)}$ for some $R > 0$. Then for every $1 \le p \le q \le \infty$ and for every multi-index $\alpha \in \mathbb{N}_0^N$,*

$$\|\partial^\alpha \phi\|_{L^q(\mathbb{R}^N)} \le cR^{|\alpha|+N/p-N/q}\|\phi\|_{L^p(\mathbb{R}^N)}$$

for some constant $c = c(|\alpha|, N, p, q) > 0$. In particular, for $\alpha = 0$,

$$\|\phi\|_{L^q(\mathbb{R}^N)} \le cR^{N/p-N/q}\|\phi\|_{L^p(\mathbb{R}^N)}.$$

On the other hand, if $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$ is such that $\operatorname{supp}\widehat{\phi} \subseteq \overline{B(0,Rr_2)} \setminus B(0,Rr_1)$ for some $R > 0$ and some $0 < r_1 < r_2$, then

$$c^{-1}R^m\|\phi\|_{L^p(\mathbb{R}^N)} \le \sum_{|\alpha|=m} \|\partial^\alpha \phi\|_{L^p(\mathbb{R}^N)} \le cR^m\|\phi\|_{L^p(\mathbb{R}^N)}$$

for some constant $c = c(N, m, p, r_1, r_2) > 0$.

Proof. Using the change of variables $x = Ry$, we can assume $R = 1$.

Step 1: Let $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$ be such that $\operatorname{supp}\widehat{\phi} \subseteq \overline{B(0,1)}$ and consider a function $\varphi \in \mathcal{D}(\mathbb{R}^N)$ such that $\varphi = 1$ in $\overline{B(0,1)}$. Then by Theorem 10.71, $\widehat{\phi} = \varphi\widehat{\phi} = \widehat{\psi}\widehat{\phi} = \widehat{\psi * \phi}$, where $\psi := \varphi^\vee$, which, by Theorem 10.62, implies that $\phi = \psi * \phi$. By differentiating inside the integral it follows from (10.19) that $\partial^\alpha\phi = (\partial^\alpha\psi) * \phi$, and so by Young's inequality (see Theorem 10.49)

$$\|\partial^\alpha\phi\|_{L^q} = \|(\partial^\alpha\psi) * \phi\|_{L^q} \le \|\partial^\alpha\psi\|_{L^r}\|\phi\|_{L^p},$$

where $1/p + 1/r = 1 + 1/q$.

Step 2: Let $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$ be such that $\operatorname{supp}\widehat{\phi} \subseteq \overline{B(0,r_2)} \setminus B(0,r_1)$ and consider a function $\varphi \in \mathcal{D}(\mathbb{R}^N\setminus\{0\})$ such that $\varphi = 1$ in $\overline{B(0,r_2)} \setminus B(0,r_1)$. Writing

$$\|y\|^{2m} = \sum_{|\alpha|=m} c_\alpha (iy)^\alpha(-iy)^\alpha,$$

we have that

$$\widehat{\phi}(y) = \sum_{|\alpha|=m} (iy)^\alpha\widehat{\phi}(y)\psi_\alpha(y),$$

where $\psi_\alpha(y) := \varphi(y)(c_\alpha(-iy)^\alpha/\|y\|^{2m})$. Note that since $\varphi \in \mathcal{D}(\mathbb{R}^N\setminus\{0\})$, we have that $\psi_\alpha \in \mathcal{D}(\mathbb{R}^N\setminus\{0\})$. It follows from Theorems 10.58 and 10.71 that

$$\phi = \sum_{|\alpha|=m} \psi_\alpha^\vee * \partial^\alpha\phi,$$

and so by Young's inequality (see Theorem 10.49)

$$\|\phi\|_{L^p} \le \sum_{|\alpha|=m} \|\psi_\alpha^\vee * \partial^\alpha\phi\|_{L^p} \le \sum_{|\alpha|=m} \|\psi_\alpha^\vee\|_{L^1}\|\partial^\alpha\phi\|_{L^p} \le c\sum_{|\alpha|=m} \|\partial^\alpha\phi\|_{L^p}.$$

The other inequality follows from Step 1. □

Exercise 10.74. For $n, m \in \mathbb{N}$ let $f_n = \chi_{[-n,n]}$ and $g_{n,m} := \frac{1}{m} f_n * f_m$.

(i) Compute the Fourier transform of f_n.

(ii) Compute the Fourier transform of $g_{n,m}$.

(iii) Calculate the Lebesgue integral $\int_{\mathbb{R}} \frac{\sin^2 x}{x^2}\,dx$.

(iv) Compute the improper Riemann integral $\int_{\mathbb{R}} \frac{\sin x}{x}\,dx$.

We can also define the Fourier transform of tempered distributions. Given $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$, the *Fourier transform* $\widehat{T}$ *of* T is the tempered distribution given by

$$\widehat{T}(\phi) := T(\widehat{\phi}), \quad \phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C}). \tag{10.31}$$

Given $\psi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$, consider the linear functional $T : \mathcal{S}(\mathbb{R}^N;\mathbb{C}) \to \mathbb{C}$ defined by (10.17). By (10.26), for every $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$, we have

$$T_{\widehat{\psi}}(\phi) = \int_{\mathbb{R}^N} \phi(x)\widehat{\psi}(x)\,dx = \int_{\mathbb{R}^N} \widehat{\phi}(x)\psi(x)\,dx = T_\psi(\widehat{\phi}) = \widehat{T_\psi}(\phi). \tag{10.32}$$

Since we are identifying ψ with T_ψ in $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$, this shows that the Fourier transform defined on $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ extends the Fourier transform defined in $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$.

Similarly, given $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$, the *inverse Fourier transform of* T is the tempered distribution given by

$$T^\vee(\phi) := T(\phi^\vee), \quad \phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C}). \tag{10.33}$$

Exercise 10.75. Let $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$.

(i) Prove that $\widehat{T} \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$.

(ii) Prove that if $\{T_n\}_n$ is a sequence in $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ such that $T_n \overset{*}{\rightharpoonup} T$ in $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$, then $\widehat{T_n} \overset{*}{\rightharpoonup} \widehat{T}$.

(iii) Prove that Theorem 10.62 continues to hold in $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$.

(iv) Prove that $\mathcal{F} : \mathcal{S}'(\mathbb{R}^N;\mathbb{C}) \to \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ is an isomorphism with inverse $\mathcal{F}^{-1}$ given by $\mathcal{F}^{-1}(T) = T^\vee$ for every $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$.

In view of Theorem 10.35 and the previous exercise, for every function $\psi \in L^p(\mathbb{R}^N)$ with $p > 2$, the Fourier transform $\widehat{\psi}$ of ψ is the Fourier transform $\widehat{T_\psi}$ of the tempered distribution T_ψ defined in (10.17). Hence, $\widehat{\psi}$ belongs to $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ but in general $\widehat{\phi}$ cannot be identified with a function. A simple example is given by $\psi = 1 \in L^\infty(\mathbb{R}^N)$. In this case

$$T_1(\phi) = \int_{\mathbb{R}^N} \phi 1\,dx, \quad \phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C}), \tag{10.34}$$

and so by Theorem 10.62,

$$\widehat{1}(\phi) = \widehat{T_1}(\phi) = \int_{\mathbb{R}^N} \widehat{\phi}(x)\,dx = \int_{\mathbb{R}^N} e^{2\pi i 0\cdot y}\widehat{\phi}(x)\,dx = (\widehat{\phi})^{\vee}(0) = \phi(0),$$

which shows that $\widehat{1}$ is δ_0.

Exercise 10.76. Given $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$, prove that the following properties are equivalent:

(i) $\operatorname{supp}\widehat{T} = \{0\}$,

(ii) $T = T_f$ for some polynomial $f : \mathbb{R}^N \to \mathbb{C}$,

(iii) $T = \sum_{|\alpha|\le n} a_\alpha \partial^\alpha \delta_0$ for some $n \in \mathbb{N}$ and some $a_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{N}_0^N$, $0 \le |\alpha| \le n$.

Exercise 10.77. Prove that Theorem 10.71 continues to hold, that is, that if $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ and $\varphi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$, then $\widehat{T * \varphi} = \widehat{\varphi}\widehat{T}$.

10.8. Littlewood–Paley Decomposition

Let $\omega_0 \in C_c^\infty(\mathbb{R}^N)$ be a nonnegative function such that

$$\operatorname{supp}\omega_0 = \overline{B(0,1)}, \quad \omega_0 = 1 \text{ in } B(0,1/2), \tag{10.35}$$

and define

$$\varphi_0(x) := \omega_0(x/2) - \omega_0(x), \quad x \in \mathbb{R}^N. \tag{10.36}$$

Given $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ and $k \in \mathbb{Z}$ we define the tempered distributions

$$S_k(T) := (\omega_0(2^{-k}\cdot)\widehat{T})^{\vee}, \tag{10.37}$$

$$\dot{\Lambda}_k(T) := (\varphi_0(2^{-k}\cdot)\widehat{T})^{\vee}. \tag{10.38}$$

The tempered distribution $\dot{\Lambda}_k(T)$ is called the kth *dyadic block of the Littlewood–Paley decomposition of* T.

Observe that by Exercise 10.40, (10.31), (10.33) and (10.37), for every $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$,

$$\begin{aligned} S_k(T)(\phi) &= (\omega_0(2^{-k}\cdot)\widehat{T})^{\vee}(\phi) \\ &= T(\mathcal{F}(\omega_0(2^{-k}\cdot)\mathcal{F}^{-1}(\phi))) = T(S_k(\phi)), \end{aligned} \tag{10.39}$$

where the last equality follows from Theorems 10.62 and 10.71, and where $S_k(\phi) := (\omega_0(2^{-k}\cdot)\widehat{\phi})^{\vee}$. Similarly, for every $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$,

$$\dot{\Lambda}_k(T)(\phi) = T(\dot{\Lambda}_k(\phi)), \tag{10.40}$$

where $\dot{\Lambda}_k(\phi) := (\varphi_0(2^{-k}\cdot)\widehat{\phi})^{\vee}$.

Theorem 10.78 (Littlewood–Paley decomposition)**.** *Let* $T \in \mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ *and let* $\omega_0 \in C_c^\infty(\mathbb{R}^N)$ *be a nonnegative function satisfying* (10.35)*. Then for every* $l \in \mathbb{Z}$*,*

$$T = S_l(T) + \sum_{k=l}^{\infty} \dot{\Lambda}_k(T). \tag{10.41}$$

Moreover, if

$$S_k(T) \overset{*}{\rightharpoonup} 0 \quad \text{as } k \to -\infty \text{ in } \mathcal{S}'(\mathbb{R}^N;\mathbb{C}), \tag{10.42}$$

then

$$T = \sum_{k=-\infty}^{\infty} \dot{\Lambda}_k(T). \tag{10.43}$$

The identity (10.43) is called the *homogeneous Littlewood–Paley decomposition of* T.

Proof. By (10.36), (10.37), and (10.38) for $n \in \mathbb{N}$ we have

$$S_l(T) + \sum_{k=l}^{n} \dot{\Lambda}_k(T) = S_{n+1}(T)$$

and so, by (10.39), for every $\phi \in \mathcal{S}(\mathbb{R}^N;\mathbb{C})$,

$$S_{n+1}(T)(\phi) = T(S_{n+1}(\phi)).$$

We leave as an exercise to check that since $\omega_0(2^{-n-1}x) = 1$ for $x \in B(0, 2^n)$ by (10.35), we have that $\omega_0(2^{-n-1}\cdot)\widehat{\phi} \to \widehat{\phi}$ in $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ as $n \to \infty$. In turn, since $\mathcal{F}$ is an isomorphism from $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ to $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$ (see Theorem 10.62), it follows that $S_{n+1}(\phi) \to \phi$ in $\mathcal{S}(\mathbb{R}^N;\mathbb{C})$, and so

$$S_l(T)(\phi) + \sum_{k=l}^{n} \dot{\Lambda}_k(T)(\phi) = T(S_{n+1}(\phi)) \to T(\phi)$$

as $n \to \infty$, which shows (10.41).

If (10.42) holds, then (10.43) follows by letting $l \to -\infty$ in (10.41). □

Remark 10.79. The previous theorem continues to hold if instead of (10.35) we take $\omega_0 \in C_c^\infty(\mathbb{R}^N)$ to be a nonnegative function such that

$$\operatorname{supp} \omega_0 = \overline{B(0,r)}, \quad \omega_0 = 1 \text{ in } B(0, r/2)$$

for some $r > 0$.

The Littlewood–Paley decomposition (10.43) fails without (10.42). Indeed, by taking the functional T_1 corresponding to the constant function 1 (see (10.34)), we have that $\dot{\Lambda}_k(T_1) = 0$ for all k, so (10.43) cannot hold.