Chapter 1

Introduction

Mathematicians want to classify things. However, with partial differential equations (PDEs) they had to stop on a rather unsatisfactory level. The reason for this is that almost all rules of theoretical physics and engineering, and many rules in life sciences and economics, are formulated as ordinary or partial differential equations (ODEs or PDEs). As different as the applications of differential equations are, as different is the behavior of their solutions. Therefore, a mathematical theory which wants to cover all differential equations can only cover the absolute basics. Hence, the books about PDEs necessarily differ strongly by the choice of examples and by the choice of mathematical theory which will be applied to the examples. There are entire books only covering one special important equation. Before we give the goals and objectives of this book we start with a short review of three important examples.

1.1. The three classical linear PDEs

In many courses about PDEs the following three examples, namely the Laplace equation, the heat equation, and the wave equation, play a major, sometimes exclusive, role.

Example 1.1.1. The Laplace equation is an equation for an unknown function $u : \Omega \to \mathbb{R}$ of two or more variables $x = (x_1, \ldots, x_d) \in \Omega \subset \mathbb{R}^d$ in terms of certain of its partial derivatives, namely

$$\Delta u = 0, \tag{1.1}$$

where $\Delta = \partial_{x_1}^2 + \ldots + \partial_{x_d}^2$. This PDE plays an important role in mathematics since the real and imaginary part of an analytic function in the complex plane satisfy the Laplace equation. It also plays a major role in applications. For instance the potential of an irrotational flow of an incompressible fluid such as water, or a stationary temperature field, or the potential of a stationary electric field in the absence of charges in Ω, satisfy this equation.

In order to solve this equation uniquely in a domain Ω, additional conditions are needed. To gain an intuition for the required boundary conditions we consider the factors which should determine a stationary temperature field u in a room $\Omega \subset \mathbb{R}^3$ as sketched from the side in Figure 1.1. The temperature will be determined by the temperature at the walls, the windows, the doors and the heating of the room, mathematically speaking by the conditions at the boundary $\partial\Omega$ of Ω.

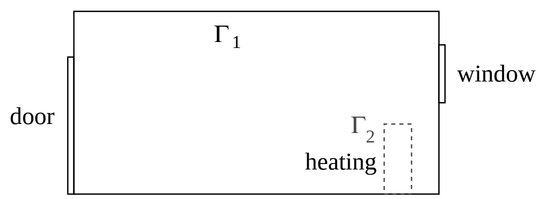

Figure 1.1. Different boundary conditions for the temperature field.

There are mainly two different kinds of boundary conditions. At the heating unit the temperature has a fixed value, while at a window or wall heat will go through the window or wall. Mathematically speaking the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ is split into two parts where in the first part we have so called Dirichlet conditions

$$u|_{\Gamma_1} = g_1,$$

and in the second part we have so called Neumann conditions

$$\partial_n u|_{\Gamma_2} = g_2,$$

with given functions $g_1 : \Gamma_1 \to \mathbb{R}$ and $g_2 : \Gamma_2 \to \mathbb{R}$ and $n : \partial\Omega \to \mathbb{R}^3$ the outer normals.

The Laplace equation is the paradigm of an **elliptic** PDE. It is of second order, i.e., the highest derivative is of order two. There is an extensive theory for elliptic systems, especially for second order elliptic systems. The equilibrium equation of linear elasticity

$$Lu := \mu\Delta u + (\lambda + \mu)\nabla(\nabla \cdot u) \stackrel{!}{=} 0,$$

for the displacement vector $u : \mathbb{R}^3 \supset \Omega \to \mathbb{R}^3$, with constants $\lambda, \mu \in \mathbb{R}$ depending on the material, is also a second order elliptic system. Like the negative Laplace operator $-\Delta$, the linear operator $-L$ defined in this equation is an example of a so called elliptic operator. Due to the important role of elasticity in the construction of cars, bridges, planes, etc., there are well developed numerical schemes such as the finite element method **(FEM)** or the boundary element method **(BEM)**, which are available for computing approximate solutions of such systems. Only in very special cases solutions can be found analytically. ⌋

Example 1.1.2. The heat equation or diffusion equation

$$\partial_t u = \Delta u, \tag{1.2}$$

with $u = u(x,t)$ and $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}$, where t denotes time and x denotes space, describes the evolution of quantities such as heat, chemical concentrations, or the probability distribution of a particle obeying Brownian motion.

It can be derived as follows. Let $V \subset \Omega$ be an arbitrary subset with smooth boundary. The change of the total quantity within V equals the flux through ∂V, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V u \,\mathrm{d}x = -\int_{\partial V} \langle F, n \rangle \,\mathrm{d}S = -\int_V \nabla \cdot F \,\mathrm{d}x$$

with F the flux density, $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^d$, $n : \partial\Omega \to \mathbb{R}^d$ again the outer normal, and where we used the Gauss' integral theorem. Since this relation is true for all sets V, we find

$$\partial_t u = -\nabla \cdot F.$$

Very often the flux density F is proportional to the gradient ∇u of the concentration u, i.e., $F = -a\nabla u$ with a constant $a > 0$. By rescaling time we finally come to the diffusion equation (1.2).

In order to solve this equation uniquely in a domain $\Omega \times \mathbb{R}^+$ additional conditions are needed. As in Example 1.1.1 we need boundary conditions, but also the temperature field at time $t = 0$ has to be known, i.e., we need the initial condition $u|_{t=0} = u_0$ with $u_0 : \Omega \to \mathbb{R}$. Stationary solutions, i.e., time-independent solutions, satisfy the Laplace equation (1.1). The heat equation is the prototype **parabolic** PDE. There is an extensive theory for equations of the form $\partial_t u = Lu$ with an elliptic operator $-L$. ⌋

Example 1.1.3. The linear wave equation

$$\partial_t^2 u = \Delta u, \tag{1.3}$$

$u = u(x,t)$ with $u : \Omega \times \mathbb{R} \to \mathbb{R}$, is a simple model for, e.g., oscillations of a string ($\Omega \subset \mathbb{R}$) or of a membrane ($\Omega \subset \mathbb{R}^2$), or the propagation of light in vacuum. In order to solve this equation uniquely in a domain $\Omega \times (t_0, t_1)$, $t_0 < 0 < t_1$ we again need boundary and initial conditions. Like for scalar second order ODEs we need two initial conditions, namely $u|_{t=0} = u_0$ and $\partial_t u|_{t=0} = u_1$ with $u_0 : \Omega \to \mathbb{R}$ and $u_1 : \Omega \to \mathbb{R}$. The Dirichlet boundary condition $u|_{\partial\Omega}$ corresponds to a membrane which is fixed at the boundary. In this case, the boundary will reflect the waves.

For the wave equation the eigenmodes play a crucial role. An eigenmode is a solution $u(x,t) = \mathrm{e}^{\mathrm{i}\omega t}v(x)$. This yields the eigenvalue problem

$$-\Delta v = \omega^2 v.$$

Such problems play an important role in applications, especially in elasticity theory, where the evolution equations of linear elasticity

$$\partial_t^2 u = \mu\Delta u + (\lambda + \mu)\nabla(\nabla \cdot u)$$

yield to the eigenvalue problem

$$-\mu\Delta v + (\lambda + \mu)\nabla(\nabla \cdot v) = \omega^2 v.$$

If Ω is a bounded set then under suitable boundary conditions there are countably many real eigenvalues $\lambda_n = \omega_n^2$ with $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$. In the construction of cars, bridges, planes, etc., one has to take care that these so called resonant modes are not periodically excited. Hence, there is a big industry using FEM and BEM in order to solve these elliptic eigenvalue problems. The wave equation is the prototype **hyperbolic** PDE. There is an extensive theory for equations of the form $\partial_t^2 u = Lu$ with an elliptic operator $-L$. ⌋

For reasons explained below we will focus on other examples than the three classical ones. The fundamental Examples 1.1.1-1.1.3 cannot be and will not be avoided. However, they will only occur as subproblems which will help to understand the nonlinear problems under consideration.

1.2. Nonlinear PDEs

We now start discussing our main objectives for this book, namely an introduction to nonlinear PDEs from a dynamical systems point of view, with a focus on reduction methods, in particular, the use of amplitude and modulation equations.

Many complications with ODEs or PDEs are due to the fact that **the world is nonlinear.** Ultimately, to solve a PDE means to look for solutions u of an abstract equation $F(u) = 0$. The problem is called linear if for all $\alpha, \beta \in \mathbb{R}$ we have

$$F(\alpha u + \beta v) = \alpha F(u) + \beta F(v).$$

As a consequence, for linear problems we have the superposition of solutions. With u, v being solutions, i.e., $F(u) = 0$ and $F(v) = 0$, also $\alpha u + \beta v$ is a solution, i.e., $F(\alpha u + \beta v) = 0$. Most "real life" problems are nonlinear, i.e., in general

$$F(\alpha u + \beta v) \neq \alpha F(u) + \beta F(v),$$

and therefore a sum of two solutions is no longer a solution of the ODE or PDE. A simple example of a nonlinear function is $F(u) = u^2$. As a consequence, the theory of linear algebra is not available, and the set of solutions in general is more complicated than that for linear problems. In science, for many decades linear problems played a dominating role. Examples 1.1.1 to 1.1.3 are linear. Next we present two famous examples of nonlinear PDEs.

Example 1.2.1. The Navier-Stokes equations

$$\begin{aligned} \partial_t u =& \frac{1}{R}\Delta u - \nabla p - (u \cdot \nabla)u, \\ 0 =& \nabla \cdot u, \end{aligned}$$

describe the evolution of the velocity field $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}^3$ and the pressure field $p : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ of an incompressible fluid, such as water, in a domain $\Omega \subset \mathbb{R}^3$. The Reynolds number R measures the ratio between inertial and viscous forces, and is in some sense proportional to the complexity of the flow. The global existence and uniqueness of smooth solutions of the three-dimensional (3D) Navier-Stokes equations is one of the seven one million dollar or Millennium problems in mathematics presented by the Clay-Foundation in the year 2000. There are a number of reasons for this choice. On the one hand, the Navier-Stokes equations describe the motion of fluids, and the answer to this question would allow us to understand fluids in a much better way. On the other hand, in mathematics the 3D Navier-Stokes equations are interesting PDEs, which so far have resisted all attempts to prove the global existence and uniqueness of solutions. This will be explained in more detail in Chapter 6. ⌋

Example 1.2.2. Maxwell's equations in a medium, for instance a glass fiber, are given by

$$\begin{aligned} \nabla \cdot B =& 0, \\ \nabla \times E + \partial_t B =& 0, \\ \nabla \cdot D =& \rho, \\ \nabla \times H - \partial_t D =& J. \end{aligned}$$

Here $E : \Omega \times \mathbb{R} \to \mathbb{R}^3$ is the electric field, $D = \varepsilon_0 E + P$ is the displacement field, with ε_0 the electric permeability of vacuum, $P : \Omega \times \mathbb{R} \to \mathbb{R}^3$ is the electric polarization of the material, $B : \Omega \times \mathbb{R} \to \mathbb{R}^3$ is the magnetic field, $H = B/\mu_0 - M$ is the magnetizing field, with μ_0 the magnetic permeability of vacuum and $M : \Omega \times \mathbb{R} \to \mathbb{R}^3$ the magnetic polarization of the material, ρ is the charge density, and $J : \Omega \to \mathbb{R}^3$ the charge flow density. Since the first and the third equation above are scalar, while the second and fourth equation are vector valued, so far we have eight equations for the twelve unknowns B_j, E_j, M_j and P_j for $j = 1, 2, 3$. Therefore, these equations have to be closed with constitutive laws $P = P(E, H)$ and $M = M(E, H)$ describing the reaction of the material to the electric and magnetic field. In general, these laws are nonlinear. Moreover, as an additional complication M and P may depend on the past, cf. §11.7. ⌋

The world is instationary, i.e., almost all systems evolve with time. Typical examples are a vibrating beam, the daily change of the weather, or the motion of the planets in the solar system. Hence, from the beginning we will consider **nonlinear time-dependent systems**.

A mathematical concept which is basic to the analytical understanding of all ODEs and PDEs is the concept of **Dynamical Systems**. Until the beginning of the 1960s, Laplace's principle that with the knowledge of all physical rules and the present state of the world, the past and future behavior of the world for all times can be computed, was widely accepted as a relevant philosophical foundation of science. Starting already with the work of H. Poincaré in the 1890s, cf. [**Poi57**], this principle was finally observed to be practically useless at the beginning of the 1960s, for instance by the work of the meteorologist E. Lorenz in 1963 [**Lor63**]. He observed with an analog computer for a three-dimensional model for the weather that the possible time for predictions goes logarithmically with the precision of the initial conditions, i.e., that long-time weather-forecasts are practically impossible. See Figure 1.2 for an illustration of the so called Lorenz attractor and of the sensitivity of solutions w.r.t. the initial conditions.

Certain ODEs and PDEs, or, more general, dynamical systems, can be classified as **chaotic**. The visualization of chaotic dynamical systems was in fashion in the 1980s. Famous examples are the Mandelbrot and the Julia sets. In this book, chaos will not play a central role, but one should keep in mind its existence already in low-dimensional dynamical systems.

1.3. Our choice of equations and the idea of modulation equations

PDEs play an important role in modern engineering. With the help of computer simulations, money can be saved, experiments can be replaced, and data can be gathered which are not available by classical experiments. However, **a numerical simulation of a PDE requires an analytic understanding of the PDE.** The reason for this is again the wide variety of different types of PDEs. Therefore, very often the numerical simulation of a PDE needs an adapted numerical scheme based on an analysis of the PDE. As the example of the crash of the Sleipner oil platform in 1991 shows, a misuse of numerical schemes can cost a lot of money. In the concrete example 700 million dollars [**JR94**].

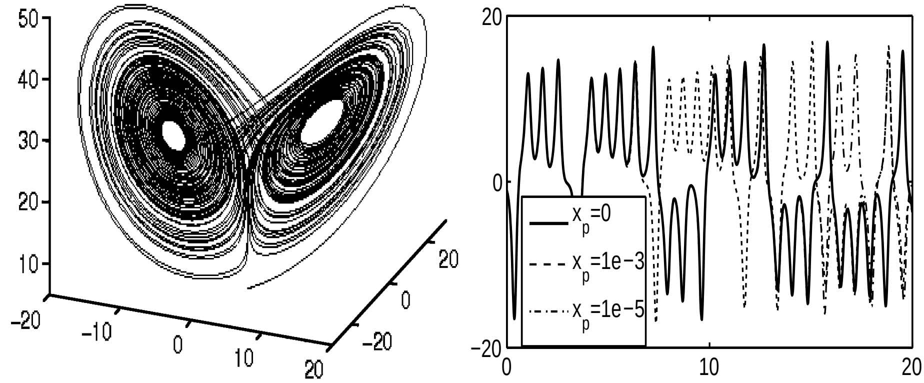

Figure 1.2. Left: Illustration of the attractor of the Lorenz system $\dot{x} = \sigma(y - x)$, $\dot{y} = \rho x - y - xz$, $\dot{z} = -\beta z + xy$ by one orbit in 3D phase space, $\sigma = 10$, $\beta = 8/3$, $\rho = 27$. Right: $x(t)$ for three nearby initial conditions, i.e., $x_1(0)$, $x_2(0) = x_1(0) + 10^{-3}$, and $x_3(0) = x_1(0) + 10^{-5}$, $y(0), z(0)$ always the same. The orbits behave completely different after a certain time, i.e., the orbits to $x_2(0)$ and $x_3(0)$ deviate from the unperturbed one after $t \approx 7$ and $t \approx 16$, respectively. It can be shown that the prediction time goes logarithmically with the precision of the initial conditions.

Moreover, **computers are fast, but never fast enough.** A three-dimensional body $[0,1]^3$ discretized with 100 points in each direction leads to 10^6 variables. A discretization in 1000 points in each direction yields 10^9 variables. Therefore, due to practical reasons one has to decide before what quantities shall be computed. Then the scheme can be adapted to the computation of these quantities.

We are especially interested in problems which cannot be directly studied numerically, i.e., where first **analysis is needed to reduce the dimensionality of the problem.** This is for instance the case in so called spatially extended domains, which means that the wave length of typical solutions is much smaller than the size of the underlying physical domain. In this case often the modeling over an unbounded domain is more reasonable. Then, via a multiple scaling perturbation ansatz simpler models can be derived to describe the phenomena under consideration. These models, called **modulation equations**, belong to the best studied nonlinear PDEs with a status in some scientific areas similar to the three classical linear PDEs from above. Besides the study of these basic nonlinear PDEs from a dynamical systems point of view, one of our main objectives will be the connections between these models and real world problems by going beyond the formal derivation of these modulation equations. This will be called the **justification** of the reduced models.

Example 1.3.1. The digital transport of information in glass fibers is done by sending 0s and 1s through the fiber. In most modern technologies the physical realization of a 1 is an electromagnetic pulsemodulating a carrier wave with a wave length of a few hundred nanometers. There are a number of relevant questions related to the transport of information. For instance:

- Which form is optimal for a pulse to travel a long distance?
- Which distance do two pulses initially need in order to stay separated during the complete journey through the fiber?
- How many kilometers can a pulse travel without an amplifier?
- How do pulses interact if the carrier waves have different frequencies?

There is dispersion in the fiber and thus in general the energy concentrated in a pulse will spread. Moreover, the fiber behaves nonlinearly. Hence, the answers to the above questions are not obvious at all. Numerical simulations, if possible, are much cheaper than experiments. However, suppose that the length of the fiber is $100\text{km} = 10^5\text{m}$. Then, due to the wave length of light of approximately 10^{-7}m, a spatial discretization of Maxwell's equations in the fiber gives at least about 10^{12} points, still neglecting all three-dimensional effects. This number is too big for a direct numerical simulation.

A modulation equation helps. By perturbation analysis the **Nonlinear Schrödinger (NLS) equation**

$$\partial_\tau A = \mathrm{i}\nu_1 \partial_\xi^2 A + \mathrm{i}\nu_2 A|A|^2,$$

with $A(\xi,\tau) \in \mathbb{C}$, $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}$ and coefficients $\nu_1, \nu_2 \in \mathbb{R}$, can be derived, describing the evolution of the envelope A of the pulse alone. On the relevant time scale the dynamics of the envelope of the pulse and the carrier wave which behaves linearly can approximately be separated. The properties of the original system, e.g., the refractive index of the material, and the underlying wave, condense to the coefficients $\nu_j \in \mathbb{R}$. The NLS equation is a **universal** modulation equation which describes slow modulations in time and space of the envelope of a spatially and temporarily oscillating underlying carrier wave in nonlinear dispersive equations.

The spatial discretization can thus be reduced from 10^{12} points to approximately 10^5 or less points, which is quite manageable for numerical schemes. Moreover, a number of problems can be solved analytically for the NLS equation, which is a so called **completely integrable system**. In particular, if $\nu_1\nu_2 > 0$ it has explicit so called **soliton solutions**. These solitons give the optimal form of pulses for the transport of information. These questions will be discussed in detail in Chapter 11. ⌋

Example 1.3.2. At the end of the 20th century a new generation of high speed ferries has caused serious problems, especially those that cross the Channel between England and France and those operating in the Marlborough sound in New Zealand. The waves created by these ferries can propagate without loss of energy over large distances, and thus retain the potential to create enormous havoc when they come ashore, and as a consequence of a fatal accident and other damage there are now speed limits for these ferries [**Ham99**].

Again a modulation equation gives an idea to understand these phenomena. The **Korteweg-de Vries (KdV) equation**

$$\partial_\tau A = \nu_1 \partial_\xi^3 A + \nu_2 A \partial_\xi A,$$

with $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}$, $A(\xi,\tau) \in \mathbb{R}$ and coefficients $\nu_j \in \mathbb{R}$ can be derived with the help of a perturbation ansatz. The KdV equation is a **universal** modulation equation which describes long waves of small amplitude, where the original system condenses to the coefficients $\nu_j \in \mathbb{R}$.

Like the NLS equation, this famous nonlinear equation possesses soliton solutions, very robust solitary waves. These waves interact like particles, i.e., after some nonlinear interaction they reform and look exactly as before the interaction. This observation, made in the middle of the 1960s, that solutions of a PDE show simultaneously the behavior of a particle and a wave, had a big influence on nonlinear science due to the similarity with the particle-wave dualism in quantum mechanics.

For a long time the KdV equation has also been suggested as a model for the description of tsunamis, water waves of only a few meters height, but with a length of up to 100km, i.e., in the ocean they cannot be observed by eye. In the 5000m deep pacific ocean they move with a very high velocity of around 700km/h. If they approach land they become slower and steeper, and cause serious floodings. However, data which is now available from the tsunami at Christmas 2004 in the Indian Ocean seem to indicate that soliton dynamics had played at least for this tsunami no role on the open sea. The validity of the KdV equation will be discussed in Chapter 12. ⌋

Example 1.3.3. Since the 1960s, systems near the onset of a finite wave length instability have been analyzed in detail using modulation equations. These amplitude modulations describe slow changes in time and space of the envelope of the finite wave length pattern close to the first instability. The most famous and generic of such equations is the **Ginzburg-Landau (GL) equation**

$$\partial_\tau A = \nu_2 \partial_\xi^2 A + \nu_0 A + \nu_3 A |A|^2,$$

with $\tau \geq 0$, $\xi \in \mathbb{R}$, $A(\xi,\tau) \in \mathbb{C}$ and coefficients $\nu_j \in \mathbb{C}$. Famous **pattern forming systems** which can be described with the GL equation are reaction-diffusion systems such as the Brusselator, and hydrodynamical stability problems, such as the Couette-Taylor problem, Bénard's problem, or Poiseuille flow. A big part of Part IV, namely Chapter 10, is devoted to the justification of this so called GL approximation for various classes of systems. We explain and prove that the difference of true solutions of the pattern forming systems and the associated GL approximations remains small on the natural time scale of this approximation, and thus prove rigorously that the GL equation makes correct predictions about the dynamics of the original pattern forming systems.

Instead of "modulation equation", in particular the GL equation in the above context is also called the "amplitude equation". Although derived differently, the GL model also plays a crucial role in superconductivity.

A phenomenological model for pattern formation close to the first instability of a spatially homogeneous solution is the **Swift-Hohenberg** equation [**SH77**]

$$\partial_t u = -(1+\partial_x^2)^2 u + \alpha u - u^3, \tag{1.4}$$

with $u = u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$, $t \geq 0$ and control parameter $\alpha \in \mathbb{R}$. This fourth order scalar PDE is probably the simplest example to apply the "Ginzburg-Landau formalism". For small $\alpha =: \varepsilon^2 > 0$, plugging the ansatz

$$u(x,t) = \varepsilon A(\varepsilon x, \varepsilon^2 t) \mathrm{e}^{\mathrm{i}x} + \mathrm{c.c.} \tag{1.5}$$

into (1.4) and sorting w.r.t. powers of ε yields the GL equation

$$\partial_T A = 4\partial_X^2 A + A - 3|A|^2 A \tag{1.6}$$

at order ε^3. ⌋

As already said, the mathematical analysis of the approximation by these three 'generic' modulation equations, namely the KdV, the NLS, and the GL equation, will be one of the mathematical objectives of Part IV of this book. Beside these 'generic' equations there are many more.

Example 1.3.4. The Burgers equation

$$\partial_t u = \partial_x^2 u - \partial_x (u^2),$$

with $t \geq 0$, $x \in \mathbb{R}$, and $u(x,t) \in \mathbb{R}$ arises for instance as a modulation equation for small amplitude long waves on the surface of a viscous liquid running down an inclined plane. It describes this system in case when the trivial solution, the so called Nusselt solution, which possesses a parabolic flow profile and a flat top surface, is spectrally stable. This is the case when the inclination angle θ, which serves as a control parameter in this physical problem, is below a critical value θ_c. This model is used for instance for flood forecasts in rivers.

If the inclination angle is increased, the Nusselt solution becomes unstable via a so called sideband instability. Above the threshold of instability the Kuramoto-Shivashinsky-perturbed KdV equation serves as modulation equation. After some rescaling it has the form

$$\partial_t u = -\partial_x^3 u - \frac{1}{2}\partial_x(u^2) - \varepsilon(\partial_x^2 + \partial_x^4)u,$$

with $t \geq 0$, $x \in \mathbb{R}$, $u = u(x,t) \in \mathbb{R}$, and where $0 < \varepsilon \approx \sqrt{\theta - \theta_c} \ll 1$ is a small parameter. Therefore, complicated dynamics that are present in this equation occur directly at the first instability of the inclined plane problem. The dynamics is dominated by traveling pulse trains consisting of unstable pulses. Time series of the position of the pulses indicate the occurrence of chaotic dynamics. This situation is relevant for cooling units. Again the 3D Navier-Stokes equations for the water flowing down the unit is replaced by a simpler model still containing very complicated dynamics.

Another situation where the Burgers equation arises as a modulation equations are phase or wave number modulations of stable periodic pattern in a pattern forming system, while phase (or wave number) modulations of unstable pattern are generically described by Kuramoto-Shivashinsky type of equations. ⌋

In summary, modulation equations are simpler PDEs, which can be derived by perturbation analysis, and which serve as models for more complicated systems. Hence, modulation equations are a part of mathematical modeling. In Part IV of this book, the derivation and the approximation properties of the above equations will be explained. We will analyze the original system with the help of the modulation equations. We will explain to which extent conclusions based on the modulation equations can be proven to be correct. We will show how mathematics can decide which model of all possible proposed models is the right one. We will explain that modulation equations are universal models, i.e., exactly the same modulation equation describes the same phenomena in completely different physical systems. The much simpler modulation equations itself will be analyzed in Part III of this book.

1.4. Overview

In order to keep the book as an introductory text and as self-contained as possible, in Part I we explain basic dynamical systems concepts for ODEs, such as phase space, fixed points, periodic solutions, attractors, stability and instability, bifurcations and amplitude equations.

In Part II we start to transfer the dynamical systems concepts from finite to infinite dimensions. There are major differences due to the non-equivalence of norms in infinite-dimensional vector spaces and the loss of compactness of closed bounded sets. We explain that PDEs over bounded domains can be considered as dynamical systems with countably many degrees of freedom. As applications we discuss the Chafee-Infante problem and the Navier-Stokes equations.

We have already explained in the previous subsection our choice of equations for the Parts III and IV. In Part III we consider basic model PDEs posed on the real line, namely the Kolmogorov-Petrovsky-Piscounov (KPP) or Fisher equation, the Burgers equation, the Nonlinear Schrödinger (NLS) equation, the Korteweg-deVries (KdV) equation, and the Ginzburg-Landau (GL) equation. We explain fundamental PDE phenomena as diffusion, dispersion, and transport, discuss local and global existence and uniqueness, and construct stationary solutions, or traveling front and pulse solutions, using ODE techniques from Part I. We also give some first results for attractors on unbounded domains and a brief introduction to reaction-diffusion systems.

Part IV is devoted to the analysis of the more complicated systems with the help of the scalar model equations from Part III, which now reappear as modulation equations. Additionally we explain useful concepts such as diffusive stability and spatial dynamics.

At the end of each chapter we collect a number of **exercises**. We in general do not claim any originality for them, and many are taken from the literature, though in some cases we cannot trace back our source. As usual, the exercises are a crucial part of this book.

1.4.1. Grasshopper's Guide. To some extent the four parts of this book are intended to be independent. Moreover, the chapters are kept as self-contained as possible, such that the reader may start to read directly about his or her favorite equation. Therefore, we also give the following guide.

Part I can obviously be read independently of the rest of the book. It is an example-oriented basic course on finite-dimensional dynamical systems which together with Chapters 5 and 6 (and possibly Chapter 13) yields a two semester course about finite- and infinite-dimensional dynamical systems. Chapters 7 and 8 of Part III can subsequently serve as a basis for a seminar.

An alternative one or two semester course is given by §2.2-§2.3 about basic nonlinear ODE dynamics combined with (parts or all of) Part III and some parts of Part IV, for instance the beginning of Chapter 10. Other chapters of Part IV can then serve as a basis for a seminar.

There are other possibilities, for instance Chapter 3 about dissipative ODE dynamics combined with some dissipative PDE dynamics, chosen out of Chapters 5, 7, §8.3, Chapters 9-10, parts of Chapter 13, and Chapter 14. Similarly, Chapter 4 about conservative ODE dynamics could be combined with some conservative PDE dynamics, chosen for instance out of §8.1, §8.2, and Chapters 11 and 12. If the reader is familiar with the contents of Part I and Part II and is interested in an introduction to the mathematical theory of modulation equations, then we recommend to start reading in Part IV and going back to Part III where needed.

Nevertheless, the reader can also work through the book from the beginning to the end.

1.4.2. Recommended literature. Good classical books about PDEs are [**CH89, Joh91, Eva98, Sal08, Vas15**], while [**Str92, SVZZ13, Olv14, Log15a**] give more elementary introductions to PDEs. Books which look at PDEs from a dynamical systems point of view are [**Hen81, Tem97, RR04, Rob01**]. These books cover and extend material similar to that in the first two parts of our book, in particular Part II, while for instance [**SS99b, KP13**] discuss in more detail parts of what is treated in our Part III. For a general background on the functional analytic methods in our book we recommend [**Alt16, Wer00**], but the needed material can be found in most books on functional analysis. For more physically oriented introductions to PDEs see [**Fow97, BK00, TM05, Deb05**], for an overview of developments in the theory of PDEs in the 20th century see [**BB98**], and for an encyclopedic work on PDEs see [**Tay96**]. A "visual approach" to PDE with many motivating pictures is [**Mar07**]. For ODEs we refer for instance to [**Chi06, HSD04, Tes12, Log15b**]. Beginning in Part II, at the end of most Chapters we give an outlook and hints for further reading.

1.4.3. Software. There are many software packages for the numerical solution of ODEs and the graphical presentation of solutions. `Matlab`, `Maple`, and `Mathematica` have built in facilities, and there are various simple to use Java applets available. We strongly encourage the reader to do own experiments with any of these programs.

From the above remarks about the very different types of PDEs it readily follows that there cannot be a general tool for all types of PDEs. However, tools for specific types of PDEs, both commercial and free are widely available. We use some short self-written `matlab` scripts to illustrate some PDE dynamics, mostly for model problems. However, we do not discuss any numerical methods behind these programs and refer to [**Uec09**] and the references therein. For the computation of so called bifurcation diagrams we refer to `AUTO` [**Doe07, Dea16**] and `pde2path` [**UWR14**].

Exercises

1.1. Classify the following PDEs as linear or nonlinear.

a) $\partial_t u = \sum_{i,j=1}^{d} \partial_{x_i}\partial_{x_j}(a^{ij}u) + \sum_{i=1}^{d} \partial_{x_i}(b^i u)$, $a^{ij}, b^i : \mathbb{R}^d \to \mathbb{R}$ smooth functions.

b) $i\partial_t u = \Delta u$. c) $\partial_t V = rV - rS\partial_S V - \frac{1}{2}\sigma^2 S^2 \partial_S^2 V$, $(r, \sigma \in \mathbb{R})$.

d) $\partial_t^2 u = -\partial_x^4 u$. e) $\partial_t u = \Delta(u^\gamma)$, $(\gamma > 0)$.

f) $\partial_t u = \mathrm{div} F(u)$, $F : \mathbb{R} \to \mathbb{R}^d$ a smooth function. g) $\partial_t u = \partial_x^3 u + u\partial_x u$.

1.2. Constant coefficient second order linear partial differential equations in $\mathbb{R}^2$ can be written as

$$Lu = -\sum_{i,j=1,2} a^{ij}\partial_{x_i}\partial_{x_j} u + \sum_{i=1,2} b^i \partial_{x_i} u + c = 0.$$

The operator L is called elliptic if the eigenvalues of the symmetric matrix $A = (a^{ij})$ are strictly positive. It is called hyperbolic if they are nonzero, but have different signs. It is called parabolic if the associated quadratic form ($\partial_x \to \xi$, $\partial_y \to \eta$) defines a parabola. Classify

a) $3\partial_x^2 u + 10\partial_x\partial_y u + 15\partial_y^2 u + 36\partial_x u + 12\partial_y u + 17 = 0$;

b) $3\partial_x^2 u + 4\partial_x u + \partial_y u + 2 = 0$.

1.3. Consider the PDE $\partial_t u = \partial_x u$ for $u = u(x,t)$.

a) Find the general solution for $x \in \mathbb{R}$.

b) Solve the PDE for $x \in (0,1)$ with the initial condition $u(x,0) = 1$ for $x \in (0,1)$ under the boundary condition $u(1,t) = \cos t$.

c) Is it possible to solve the PDE for $x \in (0,1)$ with the initial condition $u(x,0) = 1$ for $x \in (0,1)$ and the boundary condition $u(0,t) = \cos t$?

1.4. Consider a membrane $\Omega = (0,1)^2$ which is fixed at the boundary $\partial\Omega$, i.e., $u|_{\partial\Omega} = 0$.

a) Make an ansatz $u(x,y,t) = v(t)\sin(m\pi x)\sin(n\pi y)$, $(n, m \in \mathbb{N})$ for the solutions of $\partial_t^2 u = \Delta u$. Which equation is satisfied by v?

b) Solve the equation for v with the initial conditions $v(0) = 0$ and $\dot{v}(0) = 1$.

c) Sketch for fixed $m, n \in \mathbb{N}$ the set of $(x,y) \in \Omega$, for which $u(x,y,t) = 0$ for all $t \in \mathbb{R}$.