Preface

"If you want to build a ship, don't herd people together to collect wood and don't assign them tasks and work, but rather teach them to long for the endless immensity of the sea"

ANTOINE DE SAINT-EXUPERY

This is an introductory textbook about nonlinear dynamics of partial differential equations (PDEs), with a focus on problems over unbounded domains and modulation equations. We explain how dynamical systems methods can be used to analyze PDEs in order to get more insight into the real world phenomena behind the equations. Our presentation is example-oriented and the starting point is very often a real world problem. This means that new mathematical tools are developed step by step in order to analyze the equations. They are re-applied and improved in subsequent sections to handle more and more complicated systems. In the end the reader should have learned mathematical tools for the analysis of some important classes of nonlinear PDEs and gained insight into nonlinear dynamics phenomena which may occur in PDEs.

The book is divided into four parts. In order to keep the book as an introductory text and as self-contained as possible, Part I is an introduction into finite-dimensional dynamics, defined by ordinary differential equations (ODEs), including bifurcation theory, attractors, and the basics of Hamiltonian dynamics. In Part II we explain the major differences between finitely and infinitely many dimensions and that in principle a PDE on a bounded domain is isomorphic to a system of countably many ODEs. We give two main applications of this point of view. The first one is the characterization of the attractor for the Allen-Cahn equation on an interval, which is also known as the Chafee-Infante problem. The second one is a very basic introduction to the Navier-Stokes equations, with a focus on periodic boundary conditions.

Genuine PDE phenomena such as transport, diffusion, and dispersion can hardly be understood by the interpretation of PDEs as systems of infinitely many ODEs. In Part III we consider PDEs which are posed on the real line. We start with the linear heat equation, and then turn to nonlinear problems. For famous model equations such as the Kolmogorov-Petrovsky-Piskounov or Fisher equation, the Korteweg-de Vries (KdV) equation, the Nonlinear Schrödinger (NLS) equation, and the Ginzburg-Landau (GL) equation, we discuss the local existence and uniqueness of solutions, special solutions as fronts and pulses, their stability and instability, soliton dynamics, the construction of attractors, and some related results.

The equations from Part III all play an important role in mathematics and have entire monographs devoted to each. Moreover, they have many connections to physics and other fields of applications, where they are often used as simplest possible models for the description of some real world phenomena. In Part IV we explore these connections from a mathematical perspective. The scalar equations from Part III occur as asymptotic effective models, or more specifically as modulation equations, for the more complicated systems from physics considered in Part IV. Examples are pattern forming systems which can be described by the GL equation, light pulses in nonlinear optics which can be described by the NLS equation, or long waves in dispersive systems which can be described by the KdV equation. We discuss how the dynamics of the reduced model equations transfer to the more complicated systems. Thus, in Part IV we give a mathematically rigorous presentation of the formalism of modulation equations in the context of real world applications. While this last part is close to recent research, it is still in textbook style, and often we do not prove the sharpest or most general result possible, but instead refer to the literature for extensions.

All chapters are kept as self-contained as possible, such that the reader can start to read directly about his or her favorite equation. Having a good background in linear ODEs, cf. §2.1, a starting point for our goals and objectives are §2.2-§2.3 about basic nonlinear ODE dynamics combined with Part III. There are other possible combinations, for instance the sections about dissipative dynamics or the sections about conservative dynamics. Nevertheless the reader can also read the book from the beginning to the end. See the Grasshopper's Guide on page 12 for detailed proposals. All chapters contain exercises which we strongly recommend not to skip.

This book grew out of our manuscripts for the lectures and seminars we gave about ODEs and PDEs at the universities of Bayreuth, Karlsruhe, Oldenburg, and Stuttgart. We thank the students who attended our lectures and seminars and urged us to keep the presentation simple and accessible. Moreover, we thank all friends and colleagues with whom we cooperated over the years, mainly on topics from Part IV, in particular, Dirk Blömker, Tom Bridges, Kurt Busch, Martina Chirilus-Bruckner, Christopher Chong, Walter Craig, Markus Daub, Hannes de Witt, Arjen Doelman, Tomas Dohnal, Wolf-Patrick Düll, Wiktor Eckhaus, Jean-Pierre Eckmann, Bernold Fiedler, Thierry Gallay, Dieter Grass, Daniel Grieser, Mark Groves, Tobias Häcker, Mariana Haragus, Ronald Imbihl, Ralf Kaiser, Tasso Kaper, Klaus Kirchgässner, Markus Kunze, David Lannes, Vincent Lescarret, Karsten Matthies, Ian Melbourne, Andreas Melcher, Johannes Müller, Robert Pego, Dmitry Pelinovsky, Jens Rademacher, Björn Sandstede, Arnd Scheel, Zarif Sobirov, Aart van Harten, C. Eugene Wayne, Daniel Wetzel, Peter Wittwer, and Dominik Zimmermann. We thank Stefanie Siegert and the unknown referees for a number of additional proposals to improve the presentation. Especially we thank Alexander Mielke from whom we learned about nonlinear dynamics and PDEs.

Finally we would like to thank Ina Mette from the AMS for her never ending motivation to go on with this book project and many helpful comments to transform our lecture notes into a book.

Guido Schneider and Hannes Uecker
Stuttgart and Oldenburg, February 2017