Chapter 1

Introduction

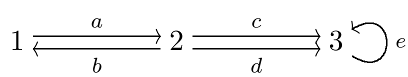
1.1. Basic Definitions and Examples

A **quiver** is a directed graph where multiple edges and loops are allowed. Formally, we define the following.

Definition 1.1.1. A quiver Q is a quadruple $Q = (Q_0, Q_1, h, t)$ where Q_0 is a finite set of **vertices**, Q_1 is a finite set of **arrows**, and h and t are functions $Q_1 \to Q_0$. For an arrow $a \in Q_1$, $h(a)$ and $t(a)$ are called the **head** and the **tail** of a.

In examples, we will often use $1, 2, 3, \dots$ or $x, y, \dots$ for vertices, and $a, b, c, \dots$ for arrows. We will write ha and ta instead of $h(a)$ and $t(a)$. This eases the notation.

Example 1.1.2. For the graph



we have $Q_0 = \{1, 2, 3\}$, $Q_1 = \{a, b, c, d, e\}$, $ta = hb = 1$, $tb = ha = tc = td = 2$ and $hc = hd = he = te = 3$.

We call $Q' = (Q'_0, Q'_1, h', t')$ a **subquiver** of $Q = (Q_0, Q_1, h, t)$ if $Q'_0 \subseteq Q'_0$, $Q'_1 \subseteq Q_1$ and h and t restrict to h' and t' respectively.

Throughout this book, with the exception of Section 8.4 of Chapter 8, we will work over the field $\mathbb{C}$ of complex numbers. The representation theory of quivers can be done over an arbitrary field, but we will stick to $\mathbb{C}$ for simplicity.

Definition 1.1.3. We get a **representation** V of $Q = (Q_0, Q_1, h, t)$ if we attach to every vertex $x \in Q_0$ a finite dimensional $\mathbb{C}$-vector space $V(x)$ and to every arrow $a \in Q_1$ a $\mathbb{C}$-linear map $V(a) : V(ta) \to V(ha)$.

Definition 1.1.4. Suppose that V and W are representations of $Q = (Q_0, Q_1, h, t)$. We get a **morphism** $\phi : V \to W$ if we attach to every vertex $x \in Q_0$ a $\mathbb{C}$-linear map $\phi(x) : V(x) \to W(x)$ such that for every $a \in Q_1$ the diagram

$$\begin{array}{ccc} V(ta) & \xrightarrow{V(a)} & V(ha) \\ {\scriptstyle \phi(ta)}\downarrow & & \downarrow{\scriptstyle \phi(ha)} \\ W(ta) & \xrightarrow[W(a)]{} & W(ha) \end{array}$$

commutes, i.e., $\phi(ha)V(a) = W(a)\phi(ta)$.

The set of all morphisms from V to W is denoted by $\mathrm{Hom}_Q(V, W)$. A morphism $\phi = (\phi(x), x \in Q_0) \in \mathrm{Hom}_Q(V, W)$ can be thought of as an element in $\bigoplus_{x\in Q_0} \mathrm{Hom}(V(x), W(x))$. Consider the $\mathbb{C}$-linear map

$$d_W^V : \bigoplus_{x\in Q_0} \mathrm{Hom}(V(x), W(x)) \longrightarrow \bigoplus_{a\in Q_1} \mathrm{Hom}(V(ta), W(ha)) \tag{1.1}$$

defined by

$$d_W^V(\phi) = \big(\phi(ha)V(a) - W(a)\phi(ta), a \in Q_1\big),$$

where $\phi = (\phi(x), x \in Q_0)$. By definition, the kernel of d_W^V is $\mathrm{Hom}_Q(V, W)$. This shows that $\mathrm{Hom}_Q(V, W)$ is a finite dimensional $\mathbb{C}$-vector space. We will see later that the *cokernel* of d_W^V has an interesting interpretation as well.

Example 1.1.5. Consider the quiver

$$Q : \ 1 \xrightarrow{a} 2 \ .$$

A representation of Q is a linear map $V(a) : V(1) \to V(2)$ where $V(1)$ and $V(2)$ are finite dimensional $\mathbb{C}$-vector spaces. A morphism $\phi : V \to W$ is a pair $(\phi(1), \phi(2))$ such that $\phi(1) : V(1) \to W(1)$ and $\phi(2) : V(2) \to W(2)$ are linear, and $\phi(2)V(a) = W(a)\phi(1)$. The last condition is equivalent to the diagram

$$\begin{array}{ccc} V(1) & \xrightarrow{V(a)} & V(2) \\ {\scriptstyle \phi(1)}\downarrow & & \downarrow{\scriptstyle \phi(2)} \\ W(1) & \xrightarrow[W(a)]{} & W(2) \end{array}$$

being commutative.

Example 1.1.6. Consider the loop quiver

$$Q:\ 1 \circlearrowleft a\ .$$

A representation of Q is a finite dimensional $\mathbb{C}$-vector space $V(1)$ together with an endomorphism $V(a) : V(1) \to V(1)$. An element $\phi(1) \in \mathrm{Hom}(V(1), V(1))$ is a morphism if and only if it commutes with $V(a)$.

The **dimension vector** $\underline{\dim}(V)$ of a representation V is the function $Q_0 \to \mathbb{N}$ defined by $\underline{\dim}(V)(x) = \dim V(x)$ for all $x \in Q_0$. Here $\mathbb{N} = \{0, 1, \dots\}$ is the set of nonnegative integers. The set of dimension vectors is $\mathbb{N}^{Q_0}$, the set of all $\mathbb{N}$-valued functions on Q_0. For a dimension vector α, we define $|\alpha| = \sum_{x \in Q_0} \alpha(x)$.

1.2. The Category of Quiver Representations

We define a category $\mathrm{Rep}(Q)$ where the objects are representations of Q and the morphisms are as defined in the previous section. In this section (and the exercises) we will study this category and investigate the interpretations of monomorphisms, epimorphisms, isomorphisms, kernels, cokernels, products and coproducts in this category.

Suppose that $\phi : V \to W$ is a morphism. In Category Theory, ϕ is an **epimorphism** if for every object Z and every two morphisms $\psi_1, \psi_2 : Z \to W$ with $\psi_1\phi = \psi_2\phi$ we have $\psi_1 = \psi_2$. The morphism ϕ is called a **monomorphism** if for every object Z and every two morphisms $\psi_1, \psi_2 : W \to Z$ with $\phi\psi_1 = \phi\psi_2$ we have $\psi_1 = \psi_2$. We call ϕ an **isomorphism** if there exists a morphism $\psi : W \to V$ such that $\psi\phi$ and $\phi\psi$ are equal to the identity morphisms 1_V and 1_W respectively.

A morphism $\phi : V \to W$ is called injective (respectively surjective) if $\phi(x)$ is injective (respectively surjective) for all $x \in Q_0$. We call ϕ bijective if it is injective and surjective, i.e., if $\phi(x)$ is bijective for all $x \in Q_0$.

Lemma 1.2.1. *Suppose that $\phi : V \to W$ is a morphism of quiver representations.*

(a) *ϕ is injective if and only if ϕ is a monomorphism;*

(b) *ϕ is surjective if and only if ϕ is an epimorphism;*

(c) *ϕ is bijective if and only if ϕ is an isomorphism.*

Proof. The proof is Exercise 1.2.1. □

Definition 1.2.2. A representation W is called a **subrepresentation** of a representation V if $W(x)$ is a subspace of $V(x)$ for all $x \in Q_0$, and $V(a) : V(ta) \to V(ha)$ restricts to $W(a) : W(ta) \to W(ha)$ for all $a \in Q_1$.

Suppose that W is a subrepresentation of V. For every vertex $x \in Q_0$, let $\iota(x) : W(x) \to V(x)$ be the inclusion map. Then $\iota = (\iota(x), x \in Q_0)$ is an injective morphism.

Suppose that $\phi : V \to W$ is a morphism of representations of $Q = (Q_0, Q_1, h, t)$. For every $x \in Q_0$, define $K(x)$ and $C(x)$ to be the kernel and cokernel, respectively, of $\phi(x) : V(x) \to W(x)$. For every $x \in Q_0$, let $\iota(x) : K(x) \to V(x)$ be the inclusion and let $\pi(x) : W(x) \to C(x)$ be the projection. For every arrow $a \in Q_1$, we have

$$(\phi(ha) \circ V(a))(K(ta)) = (W(a) \circ \phi(ta))(K(ta)) = 0,$$

so $V(a)(K(ta)) \subseteq K(ha)$. This means that we can restrict $V(a)$ to a linear map $K(a) : K(ta) \to K(ha)$. So K is a subrepresentation of V. We call the subrepresentation K the **kernel** of ϕ. The inclusion morphism $\iota : K \to V$ is the kernel of $\phi : V \to W$ in the categorical sense (see Exercise 1.2.2).

In a similar fashion, there is a unique way to define $C(a)$ for all $a \in Q_1$ such that the projection $\pi : W \to C$ is a morphism. The quotient representation C is called the **cokernel** of ϕ. The quotient map $\pi : W \to C$ is a cokernel in the categorical sense (see Exercise 1.2.3).

Given two representations V and W, we can form the **direct sum** $V \oplus W$ as follows. For $x \in Q_0$ we define $(V \oplus W)(x) := V(x) \oplus W(x)$, and for every arrow $a \in Q_1$ the map

$$(V \oplus W)(a) : V(ta) \oplus W(ta) \to V(ha) \oplus W(ha)$$

is given by the matrix

$$\begin{pmatrix} V(a) & 0 \\ 0 & W(a) \end{pmatrix}.$$

There are natural inclusions $\iota_1 : V \hookrightarrow V \oplus W$ and $\iota_2 : W \hookrightarrow V \oplus W$ defined by

$$\iota_1(x) = \begin{pmatrix} 1_{V(x)} \\ 0 \end{pmatrix}, \iota_2(x) = \begin{pmatrix} 0 \\ 1_{W(x)} \end{pmatrix}.$$

With these inclusions, $V \oplus W$ is the **coproduct** in the categorical sense (see Exercise 1.2.5). There are also natural projections $\pi_1 : V \oplus W \to V$ and $\pi_2 : V \oplus W \to W$ defined by

$$\pi_1(x) = \begin{pmatrix} 1_{V(x)} & 0 \end{pmatrix}, \pi_2(x) = \begin{pmatrix} 0 & 1_{W(x)} \end{pmatrix}.$$

With these projections, $V \oplus W$ is the **product** in the categorical sense (see Exercise 1.2.4).

One can verify that $\operatorname{Rep}(Q)$ is an abelian category (see Exercise 1.2.6).

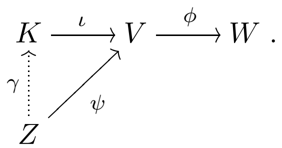
Exercises.

Exercise 1.2.1. *Prove Lemma* 1.2.1.

Exercise 1.2.2. *A morphism $\iota : K \to V$ is called the* **kernel** *of a morphism $\phi : V \to W$ if it has the following properties:*

(1) $\phi\iota = 0$*;*

(2) *if Z is any object and $\psi : Z \to V$ is any morphism with $\phi\psi = 0$, then there exists a unique morphism $\gamma : Z \to K$ such that $\psi = \iota\gamma$.*

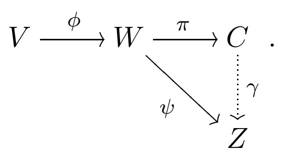


For a morphism $\phi : V \to W$ of quiver representations, let K be the kernel of ϕ, and let $\iota : K \hookrightarrow V$ be the inclusion. Show that $\iota : K \to V$ is a kernel of ϕ in the categorical sense. Also, show that every monomorphism between representations is the kernel of some morphism.

Exercise 1.2.3. *A morphism $\pi : V \to C$ is called the* **cokernel** *of a morphism $\phi : V \to W$ if it has the following properties:*

(1) $\pi\phi = 0$*;*

(2) *if Z is any object and $\psi : W \to Z$ is any morphism with $\psi\phi = 0$, then there exists a unique morphism $\gamma : C \to Z$ such that $\psi = \gamma\pi$.*



For a morphism $\phi : V \to W$ of quiver representations, let $C(x)$ be the cokernel of $\phi(x)$ for all $x \in Q_0$, and define $\pi(x)$ as the projection $W(x) \to C(x)$. Show that there is a unique way to define $C(a)$ for all $a \in Q_1$, such that π is a morphism. Show that $\pi : W \to V$ is a cokernel in the categorical sense. Also, show that every epimorphism between quiver representations is the cokernel of a morphism.

Exercise 1.2.4. *An object P with two morphisms $\pi_1 : P \to V$ and $\pi_2 : P \to W$ is called a product (in the categorical sense) of V and W if it has the following property. For every object Z and every two morphisms $\psi_1 : Z \to V$ and $\psi_2 : Z \to W$ there exist a unique morphism $\gamma : Z \to P$ such that $\psi_1 = \pi_1\gamma$ and $\psi_2 = \pi_2\gamma$. If V and W are two quiver representations, show that $V \oplus W$ together with the projections $\pi_1 : V \oplus W \to V$ and $\pi_2 : V \oplus W \to W$ form a product in the categorical sense.*

Exercise 1.2.5. *An object C with two morphisms $\iota_1 : C \to V$ and $\iota_2 \to W$ is called a coproduct (in the categorical sense) of V and W if it has the following properties: For every object Z and every two morphisms $\psi_1 : Z \to V$ and $\psi_2 : Z \to W$, there exists a unique morphism $\gamma : C \to Z$ such that $\psi_1 = \gamma\iota_1$ and $\psi_2 = \gamma\iota_2$. Show that the $V \oplus W$ together with the inclusions $\iota_1 : V \to V \oplus W$ and $\iota_2 : V \to V \oplus W$ form a coproduct.*

Exercise 1.2.6. *Verify that* $\operatorname{Rep}(Q)$ *is an abelian category.*

1.3. Representation Spaces

In this section we will see that the isomorphism classes of representations of a fixed dimension α have a nice geometric structure. They correspond to orbits in a certain representation $\operatorname{Rep}_\alpha(Q)$ of a certain algebraic group GL_α. This structure allows us to study the classification of quiver representations using (Geometric) Invariant Theory in later chapters. This geometric approach shows that for certain quivers, the classification of quiver representations of a given dimension corresponds to classical problems in linear algebra, such as studying matrices under left and right multiplication by invertible matrices, and studying matrices up to conjugation.

Suppose that $Q = (Q_0, Q_1, h, t)$ and α is a dimension vector. A **framed representation** of dimension α is a representation V for which $V(x) = \mathbb{C}^{\alpha(x)}$ for all $x \in Q_0$. For every $a \in Q_1$, $V(a)$ is an element of $\mathrm{Mat}_{\alpha(ha),\alpha(ta)}$, where $\mathrm{Mat}_{a,b}$ denotes the set of $a \times b$ matrices with complex entries. Define **representation space** of α-dimensional representations by

$$\operatorname{Rep}_\alpha(Q) = \prod_{a \in Q_1} \mathrm{Mat}_{\alpha(ha),\alpha(ta)} \,.$$

So $\operatorname{Rep}_\alpha(Q)$ can be viewed as the set of all framed representations of dimension α. If V is any representation of dimension α, Then we can choose an isomorphism $\phi(x) : V(x) \to \mathbb{C}^{\alpha(x)}$ for all $x \in Q_0$. If we define

$$W(a) = \phi(ha)V(a)\phi(ta)^{-1} \in \mathrm{Mat}_{\alpha(ha),\alpha(ta)},$$

then $W \in \operatorname{Rep}_\alpha(Q)$, and $\phi : V \to W$ is a morphism, because $W(a)\phi(ta) = \phi(ha)V(a)$ for all a. Since $\phi(x)$ is invertible for all $x \in Q_0$, ϕ is an isomorphism. So every representation is isomorphic to a framed representation.

Let GL_a denote the set of all invertible $a \times a$ matrices with complex entries. Define

$$\mathrm{GL}_\alpha = \prod_{x \in Q_0} \mathrm{GL}_{\alpha(x)} \,.$$

Suppose that $\phi = (\phi(x), x \in Q_0) \in \mathrm{GL}_\alpha$. The group GL_α acts on $\operatorname{Rep}_\alpha(Q)$ by

$$(\phi \cdot V)(a) := \phi(ha)V(a)\phi(ta)^{-1}, \quad a \in Q_1. \tag{1.2}$$

If $\phi : V \to W$ is an isomorphism, and $V, W \in \mathrm{Rep}_\alpha(Q)$, then

$$\phi(ha)V(a) = W(a)\phi(ta)$$

and

$$W(a) = \phi(ha)V(a)\phi(ta)^{-1} = (\phi \cdot V)(a)$$

for all $a \in Q_1$. So we have $W = \phi \cdot V$. Conversely, if $W = \phi \cdot V$, it is easy to see that $\phi : V \to W$ is an isomorphism. It follows that two representations $V, W \in \mathrm{Rep}_\alpha(Q)$ are isomorphic if and only if they lie in the same GL_α-orbit.

Corollary 1.3.1. *There is a bijection between the isomorphism classes of α-dimensional representations, and GL_α-orbits in $\mathrm{Rep}_\alpha(Q)$.*

Example 1.3.2. Consider the quiver

$$Q: \; 1 \xrightarrow{a} 2 \; .$$

If $\alpha = (m, n)$, then $\mathrm{Rep}_\alpha(Q) = \mathrm{Mat}_{n,m}$, and $\mathrm{GL}_\alpha = \mathrm{GL}_n \times \mathrm{GL}_m$. The group $\mathrm{GL}_n \times \mathrm{GL}_m$ acts on $\mathrm{Mat}_{n,m}$ by

$$(\phi_1, \phi_2) \cdot A = \phi_1 A \phi_2^{-1}.$$

Example 1.3.3. Consider the quiver

$$Q: \; 1 \circlearrowright a \; .$$

If $\alpha = (n)$, then $\mathrm{Rep}_\alpha(Q) = \mathrm{Mat}_{n,n}$, and $\mathrm{GL}_\alpha = \mathrm{GL}_n$. The action of GL_n on $\mathrm{Mat}_{n,n}$ is given by conjugation:

$$\phi \cdot A = \phi A \phi^{-1}, \quad \phi \in \mathrm{GL}_n, A \in \mathrm{Mat}_{n,n} \, .$$

Remark 1.3.4. Define $H_\alpha \subseteq \mathrm{GL}_\alpha$ as the set of all

$$(\lambda 1_{\alpha(x)}, x \in Q_0)$$

with $\lambda \in \mathbb{C}^\star = \mathbb{C} \setminus \{0\}$. Then H_α acts trivially on $\mathrm{Rep}_\alpha(Q)$.

Exercises.

Exercise 1.3.1. *Consider the quiver*

$$Q: \; a \circlearrowleft 1 \circlearrowright b$$

and let $\alpha = (n)$. What are $\mathrm{Rep}_\alpha(Q)$ and GL_α? How does GL_α act on $\mathrm{Rep}_\alpha(Q)$?

Exercise 1.3.2. *Consider the quiver Q with vertices $\{0, 1, \dots, m\}$ and arrows $a_1, \dots, a_m$, where $a_i : i \to 0$. Suppose that $V_1, V_2, \dots, V_m$ are subspaces of $\mathbb{C}^n$. We can define a representation V of Q by $V(0) = \mathbb{C}^n$, $V(i) = V_i$ for all i, and $V(a_i) : V(i) \to \mathbb{C}^n$ is the inclusion map. Similarly, given subspaces $W_1, \dots, W_m$ of $\mathbb{C}^n$ we can construct a representation W. Show that V and W are isomorphic if and only if there exists an invertible matrix $A \in \mathrm{GL}_n$ such that $A(V_i) = W_i$ for all i.*

1.4. Indecomposable Representations

Indecomposable representations are the building blocks in the representation theory of quivers. Every representation of a quiver is a direct sum of indecomposable representations. In Section 1.7 we will see that such a decomposition is essentially unique. For the problem of classifying matrices up to conjugation, indecomposable representations correspond to Jordan blocks in the Jordan normal form.

A representation V of $Q = (Q_0, Q_1, h, t)$ is called **trivial** if $V(x) = \{0\}$ for all $x \in Q_0$.

Definition 1.4.1. A nontrivial representation V is called **decomposable** if V is isomorphic to $V_1 \oplus V_2$ for some nontrivial representations V_1, V_2 of Q. A nontrivial representation that is not decomposable is called **indecomposable**. A nontrivial representation V is called **irreducible** or **simple** if for every subrepresentation W of V is trivial or equal to V.

For every $x \in Q_0$, we can define a simple representation S_x by $S_x(x) = \mathbb{C}$ and $S_x(y) = 0$ for $x \neq y$. It is clear that every simple representation of Q is indecomposable. Example 1.4.2 below shows that the converse is not true.

Example 1.4.2. As in Example 1.3.2, consider the quiver

$$Q: \ 1 \xrightarrow{a} 2 \ .$$

Suppose that V is an indecomposable representation of Q. If $V(1) = 0$, then $V(2) \neq 0$. If $\dim V(2) > 1$, then we can write $V(2) = W_1 \oplus W_2$ for two nonzero subspaces W_1, W_2 of $V(2)$. Therefore, V is isomorphic to

$$(\, 0 \xrightarrow{0} W_1 \,) \oplus (\, 0 \xrightarrow{0} W_2 \,),$$

so V is decomposable. Contradiction, so we have $\dim V(2) = 1$, and V is isomorphic to

$$0 \longrightarrow \mathbb{C} \ .$$

Similarly, if $V(2) = 0$, then V is isomorphic to

$$\mathbb{C} \longrightarrow 0 \ .$$

Suppose that $V(1)$ and $V(2)$ are both nonzero. Suppose that $V(a)$ is not injective. Let K be the kernel of $V(a)$ and choose a complement L to K in $V(1)$. Then V is isomorphic to the direct sum

$$(K \longrightarrow 0) \oplus (L \longrightarrow V(2)),$$

where the map $L \to V(2)$ is the restriction of $V(a)$. Contradiction, so $V(a)$ is injective. A similar argument shows that $V(a)$ is surjective. So $V(a)$ is an isomorphism. If $\dim V(1) > 1$, then we can write $V(1) = W_1 \oplus W_2$, where

W_1, W_2 are nonzero subspaces of $V(1)$. If we define $Z_i = V(a)(W_i)$, then $V(2) = Z_1 \oplus Z_2$ and V is isomorphic to

$$(W_1 \longrightarrow Z_1) \oplus (W_2 \longrightarrow Z_2),$$

where the maps $W_1 \to Z_1$ and $W_2 \to Z_2$ are restrictions of $V(a)$. Contradiction, so $\dim V(1) = \dim V(2) = 1$, and V is isomorphic to

$$\mathbb{C} \xrightarrow{\ 1\ } \mathbb{C} .$$

Up to isomorphism, the only indecomposable representations of Q are

$$\mathbb{C} \longrightarrow 0 \ , \ 0 \longrightarrow \mathbb{C} \ , \ \mathbb{C} \longrightarrow \mathbb{C} \ ,$$

and these representations are clearly indecomposable. The first two representations are simple, but the last one is not. If we define $W(1) = 0$ and $W(2) = \mathbb{C}$, then W is a proper subrepresentation, because obviously $V(a)(W(1)) = \{0\} \subseteq W(2)$.

Let $Z_{r,s,t}$ be the representation which is a direct summand of r copies of $\mathbb{C} \to \mathbb{C}$, s copies of $\mathbb{C} \to 0$ and t copies of $0 \to \mathbb{C}$. Then $Z_{r,s,t}(a)$ is given by the matrix

$$\begin{pmatrix} 1_r & 0_{r\times s} \\ 0_{t\times r} & 0_{t\times s} \end{pmatrix}, \tag{1.3}$$

where $0_{a\times b}$ denotes the $a \times b$ with zero entries. Our classification of the indecomposable representation implies that every representation V of Q is isomorphic to $Z_{r,s,t}$ for some r, s, t. The integers r, s, t are uniquely determined by V. (This also follows from the Krull-Remak-Schmidt theorem discussed in Section 1.7.)

Of course, this is a well-known result in linear algebra: Any linear transformation L between finite dimensional vector spaces has a matrix of the form (1.3) with respect to some choice of bases. Here r is the rank of the linear map L.

Example 1.4.3. As in Example 1.3.3, consider the quiver

$$Q: \ 1 \circlearrowleft a \ .$$

Suppose that V is a representation of Q. For some choice of basis of $V(1)$, the matrix of $V(a)$ has the block form

$$\begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & J_r \end{pmatrix},$$

where J_i is a $d_i \times d_i$ **Jordan block** for $i = 1, 2, \dots, r$. This is the **Jordan normal form** of $V(a)$. Let V_i be the representation

$$\mathbb{C}^{d_i} \xrightarrow{J_i} \mathbb{C}^{d_i} .$$

Then V is isomorphic to $V_1 \oplus V_2 \oplus \cdots \oplus V_r$. So if the Jordan normal form of $V(a)$ has more than 1 Jordan block, then V is decomposable. Conversely, if the V is decomposable, then the Jordan normal form has more than 1 Jordan block (see Exercise 1.4.1). So a representation V is indecomposable if and only if the Jordan normal form of $V(a)$ consists of a single Jordan block. Moreover, the uniqueness of the Jordan normal form shows that distinct Jordan blocks correspond to nonisomorphic representations. So we have a bijection between isomorphism classes of indecomposable representations and Jordan blocks.

Exercises.

Exercise 1.4.1. *Show that if V is a decomposable representation of the quiver*

$$1 \circlearrowleft a ,$$

then the Jordan normal form of $V(a)$ has more than one Jordan block.

Exercise 1.4.2. *Show that every indecomposable representation of Q is simple if and only if Q has no arrows.*

Exercise 1.4.3. *Find two $n \times n$ matrices A and B such that the $\mathbb{C}$-algebra generated by A and B is $\mathrm{Mat}_{n,n}$. Define a representation of the quiver*

$$b \circlearrowright 1 \circlearrowleft a$$

by $V(1) = \mathbb{C}^n$, $V(a) = A$ and $V(b) = B$. Show that V is simple.

Exercise 1.4.4. *Consider the quiver*

$$Q: \quad \begin{array}{ccc} 1 & \xleftarrow{a} & 2 \\ {\scriptstyle d}\downarrow & & \uparrow{\scriptstyle b} \\ 4 & \xrightarrow[c]{} & 3 \end{array} .$$

For $\lambda \in \mathbb{C}^\star = \mathbb{C} \setminus \{0\}$, define a representation V_λ by $V_\lambda(1) = V_\lambda(2) = V_\lambda(3) = V_\lambda(4) = \mathbb{C}$, $V_\lambda(a) = V_\lambda(b) = V_\lambda(c) = 1_\mathbb{C}$ and $V_\lambda(d) = \lambda \cdot 1_\mathbb{C}$. Show that the $V_\lambda, \lambda \in \mathbb{C}^\star$ are simple and pairwise nonisomorphic.

Show that the only simple representations of Q are S_1, S_2, S_3, S_4 and V_λ, $\lambda \in \mathbb{C}$.

Exercise 1.4.5. *Consider the quiver*

$$1 \longrightarrow 2 \longrightarrow 3 .$$

Show that the indecomposable representations are

$$\mathbb{C} \longrightarrow 0 \longrightarrow 0\,,\; 0 \longrightarrow \mathbb{C} \longrightarrow 0\,,\; 0 \longrightarrow 0 \longrightarrow \mathbb{C}\,,$$

$$\mathbb{C} \longrightarrow \mathbb{C} \longrightarrow 0\,,\; 0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}\,,\; \mathbb{C} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}\,.$$

1.5. The Path Algebra

In this section we will define an associative $\mathbb{C}$-algebra $\mathbb{C}Q$ for every quiver Q. We will show that the category $\operatorname{Rep}(Q)$ of quiver representations is equivalent to the category $\mathbb{C}Q$–mod of finite dimensional left $\mathbb{C}Q$-modules. This allows us to apply techniques for the theory of modules to quiver representations. In Chapter 3 we will study finite dimensional algebras and their modules. As we will see later, it suffices to study finite dimensional algebras that are of the form $\mathbb{C}Q/J$ where J is a certain two-sided ideal in $\mathbb{C}Q$. The role of path algebras in the theory of finite dimensional associative algebras is similar to the role of polynomial rings for commutative algebras.

Definition 1.5.1. A **path** p in a quiver $Q = (Q_0, Q_1, h, t)$ of length $\ell \geq 1$ is a sequence $p = a_\ell a_{\ell-1} \cdots a_1$ of arrows in Q_1 such that $ta_{i+1} = ha_i$ for $i = 1, 2, \dots, \ell - 1$. We define $h(p) = hp = ha_\ell$ and $t(p) = tp = ta_1$. For every $x \in Q_0$, we introduce a **trivial path** e_x of length 0. We define $he_x = te_x = x$.

We think of paths as "going from right to left". A path $p = a_\ell a_{\ell-1} \cdots a_1$ corresponds to the concatenation

$$\circ \xleftarrow{a_\ell} \circ \xleftarrow{a_{\ell-1}} \cdots \xleftarrow{a_2} \circ \xleftarrow{a_1} \circ\,.$$

Suppose that V is a representation of Q. For a path $p = a_\ell a_{\ell-1} \cdots a_1$, we define

$$V(p) = V(a_\ell)V(a_{\ell-1}) \cdots V(a_1).$$

For $x \in Q_0$, we define $V(e_x) := 1_{V(x)} : V(x) \to V(x)$. For all paths p, q we have

$$V(p)V(q) = V(pq) \tag{1.4}$$

whenever $tp = hq$.

Definition 1.5.2. The **path algebra** $\mathbb{C}Q$ is a $\mathbb{C}$-algebra with a basis labeled by all paths in Q. We denote by $\langle p \rangle$ the element of $\mathbb{C}Q$ corresponding to the path p in Q. The multiplication in $\mathbb{C}Q$ is given by

$$\langle p \rangle \cdot \langle q \rangle = \begin{cases} \langle pq \rangle & \text{if } tp = hq, \\ 0 & \text{if } tp \neq hq. \end{cases}$$

Here pq denotes the concatenation of paths, and we use the conventions $pe_x = p$ if $tp = x$, and $e_x p = p$ if $hp = x$.

One easily verifies that the multiplication is associative, because the concatenation of paths is associative. We will often write p instead of $\langle p \rangle$ if there is no danger of confusion.

We have $e_x^2 = e_x$ for all $x \in Q_0$ and $e_x e_y = 0$ if $x \neq y$. So $e_x, x \in Q_0$ form an orthogonal set of idempotents. One easily verifies that $1_{\mathbb{C}Q} := \sum_{x \in Q_0} e_x$ is a 1-element in the associative algebra.

An **oriented cycle** is a nontrivial path p with $hp = tp$. A quiver without oriented cycles is called an **acyclic quiver**.

Lemma 1.5.3. *A quiver Q is acyclic if and only if $\mathbb{C}Q$ is a finite dimensional algebra.*

Proof. If p is an oriented cycle, then $p, p^2, p^3, \ldots$ are distinct paths. So $\mathbb{C}Q$ is infinite dimensional. If Q is acyclic, then $\mathbb{C}Q$ is finite dimensional by Exercise 1.5.2. □

For an associative algebra A, we denote the category of finite dimensional left A-modules by A–mod, and the category of finite dimensional right A-modules by mod–A.

Proposition 1.5.4. *The categories* $\operatorname{Rep}(Q)$ *and* $\mathbb{C}Q$–mod *are equivalent.*

Proof. First we define a function $\mathcal{F} : \mathbb{C}Q\text{–mod} \to \operatorname{Rep}(Q)$. For a $\mathbb{C}Q$-module M we define $\mathcal{F}(M)$ as follows. We define $\mathcal{F}(M)(x) = e_x M$ for all $x \in Q_0$. Let $L_a : M \to M$ be the left-multiplication by a. Then we have

$$L_a(e_{ta}M) = ae_{ta}M = aM = e_{ha}aM \subseteq e_{ha}M.$$

We define $\mathcal{F}(M)(a) : e_{ta}M \to e_{ha}M$ as the restriction of L_a. If $\phi : M \to N$ is a homomorphism of $\mathbb{C}Q$-modules, then we define $\mathcal{F}(\phi) : \mathcal{F}(M) \to \mathcal{F}(N)$ as follows. For every $x \in Q_0$, we have

$$\phi(e_x M) = e_x \phi(M) \subseteq e_x N,$$

so we define $\mathcal{F}(\phi(x)) : e_x M \to e_x N$ as the restriction of ϕ. It is clear that $\mathcal{F}$ is a functor.

Conversely, we define a functor $\mathcal{G} : \operatorname{Rep}(Q) \to$ mod $\mathbb{C}Q$. For a representation V of Q, we define $\mathcal{G}(V) = \bigoplus_{x \in Q_0} V(x)$. If p is a path, and $v \in V(x)$, then we define

$$p \cdot v = \begin{cases} V(p)(v) & \text{if } tp = x, \\ 0 & \text{otherwise.} \end{cases}$$

From (1.4) follows that this gives $\mathcal{G}(V)$ the structure of a $\mathbb{C}Q$-module. If $\phi : V \to W$ is a morphism of representations, then we define $\mathcal{G}(\phi) : \bigoplus_{x \in Q_0} V(x) \to \bigoplus_{x \in Q_0} W(x)$ as the direct sum of all $\phi(x), x \in Q_0$. One can easily verify that $\mathcal{G}$ is a functor.

The compositions $\mathcal{F}\circ\mathcal{G}$ and $\mathcal{G}\circ\mathcal{F}$ are naturally equivalent to the identity. $\square$

We will show that certain tensor algebras are path algebras. This will be useful later in Section 3.3. Suppose that R is an algebra and M is an R-bimodule. We define a **tensor algebra**

$$T_R(M) = \bigoplus_{n=0}^{\infty} M^{\otimes n},$$

where $M^{\otimes 0} = R$, $M^{\otimes 1} = M$ and

$$M^{\otimes n} = \underbrace{M \otimes_R M \otimes_R \cdots \otimes_R M}_{n}$$

for all $n \geq 2$. The tensor algebra has the following universal property proven in Exercise 1.5.1

Proposition 1.5.5. *If A is an algebra, $\lambda_R : R \to A$ is an algebra homomorphism and $\lambda_M : M \to A$ is a R-bimodule homomorphism, then there exists a unique $\mathbb{C}$-algebra homomorphism $\lambda : T_R(M) \to A$ such that the restrictions of λ to R and M are λ_R and λ_M respectively.*

Suppose that $R = \mathbb{C}^n$. Let $Q_0 = \{1, 2, \ldots, n\}$ and define $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ as the i-th basis vector. Then $e_1, e_2, \ldots, e_n$ are **orthogonal idempotents**, i.e., $e_i^2 = e_i$ and $e_ie_j = e_je_i = 0$ for $i \neq j$. Suppose that M is a finite dimensional R-bimodule. We have a direct sum decomposition

$$M = \bigoplus_{i,j\in Q_0} e_jMe_i.$$

An element $a \in M$ is called homogeneous if $a \in e_jMe_i$ for some i and j. In this case, the head of a is $ha = j$ and the tail of a is $ta = i$. Choose a basis Q_1 of M consisting of homogeneous elements. Then $Q = (Q_0, Q_1, h, t)$ is a quiver.

A basis of $M^{\otimes n}$ is given by all paths

$$a_n \otimes a_{n-1} \otimes \cdots \otimes a_1$$

for which $a_na_{n-1}\cdots a_1$ is a path in Q. It is now clear that $T_R(M)$ is isomorphic to the path algebra $\mathbb{C}Q$.

Exercises.

Exercise 1.5.1. *Prove Proposition* 1.5.5

Exercise 1.5.2. *Suppose that Q is an acyclic quiver. Show that one can label the vertices with $1, 2, \ldots, n$ such that $h(a) < t(a)$ for all arrows $a \in Q_1$. Use this to prove that $\mathbb{C}Q$ is finite dimensional.*

Exercise 1.5.3. *Show, using Exercise* 1.5.2*, that* S_x*,* $x \in Q_0$ *are (up to isomorphism) the only simple representations if* Q *is acyclic.*

Exercise 1.5.4. *Suppose that* Q *and* Q' *are acyclic quivers. Show that* $\mathbb{C}Q$ *and* $\mathbb{C}Q'$ *are isomorphic* $\mathbb{C}$*-algebras if and only if* Q *is isomorphic to* Q'*.*

Exercise 1.5.5. *Suppose that* Q *is a quiver and let* $\mathbb{C}\text{–mod}$ *be the category of finite dimensional* $\mathbb{C}$*-vector spaces. Construct a category* $\mathcal{P}_Q$ *where the objects are the vertices in* Q_0*, and* $\operatorname{Hom}_{\mathcal{P}_Q}(y,x)$ *is the set of all paths in* Q *from* x *to* y *for* $x, y \in \mathbb{Q}_0$*. Show that the category* $\operatorname{Rep}(Q)$ *of quiver representations is equivalent to the category* $(\mathbb{C}\text{–mod})^{\mathcal{P}_Q}$ *of all functors* $\mathcal{P}_Q \to \mathbb{C}\text{–mod}$ *(and where morphisms are natural transformations).*

1.6. Duality

Duality is a useful notion in category theory. On the level of algebras, the dual corresponds to the opposite algebra.

Definition 1.6.1. If A is an associative algebra, then its **opposite algebra** A^{op} is defined as follows. As a $\mathbb{C}$-vector space, A^{op} is equal to A, but A^{op} is equipped with a multiplication $\circ$ defined by

$$a \circ b := b \cdot a,$$

(here $\cdot$ is the multiplication for A).

Show that if M is a finite dimensional A-module, then the dual space $M^\star$ is an A^{op}-module as follows: For $u \in A^{\mathrm{op}}$ and $f \in M^\star$ we define $u \cdot f$ by $(u \cdot f)(v) = f(uv)$ for all $v \in M$. If $\phi : M \to N$ is a homomorphism of finite dimensional A-modules, then $\phi^\star : N^\star \to M^\star$ is a homomorphism of A^{op}-modules. This gives a canonical isomorphism of $\mathbb{C}$-vector spaces

$$\operatorname{Hom}_A(M,N) \cong \operatorname{Hom}_{A^{\mathrm{op}}}(N^\star, M^\star).$$

We have defined a contravariant functor $\mathcal{F} : A\text{–mod} \to A^{\mathrm{op}}\text{–mod}$. Similarly, we can define a contravariant functor $\mathcal{F}^{\mathrm{op}} : A^{\mathrm{op}}\text{–mod} \to A\text{–mod}$ and $\mathcal{F}^{\mathrm{op}} \circ \mathcal{F}$ and $\mathcal{F} \circ \mathcal{F}^{\mathrm{op}}$ are naturally equivalent to the indentity functors. This shows that the categories A–mod and A^{op}–mod are dual to each other.

If $Q = (Q_0, Q_1, h, t)$ is a quiver, then we define the **opposite quiver** by $Q^{\mathrm{op}} = (Q_0, Q_1, t, h)$, i.e., the heads and tails of the arrows are interchanged.

Exercises.

Exercise 1.6.1. *Show that there is a canonical isomorphism between* $(\mathbb{C}Q)^{\mathrm{op}}$ *and* $\mathbb{C}(Q^{\mathrm{op}})$*.*

Exercise 1.6.2. *Suppose that* U *is the algebra of upper triangular* $n \times n$ *matrices. Show that* U *and* U^{op} *are isomorphic.*

Exercise 1.6.3. *Let Q be the quiver*

$$\circ \longrightarrow \circ \longleftarrow \circ$$

and let $A = \mathbb{C}Q$. Show that A is not isomorphic to A^{op}.

1.7. The Krull-Remak-Schmidt Theorem

In this section we will prove the Krull-Remak-Schmidt theorem. This result says that every representation of a quiver can be written as a direct sum of indecomposable representations essentially in a unique way. The proof uses the structure of the endomorphism ring $\mathrm{End}_Q(V) = \mathrm{Hom}_Q(V, V)$ of an indecomposable representation V. We will show that such a ring is always local. To see this, we start by studying the generalized eigenspaces of an endomorphism.

Suppose that W is a finite dimensional $\mathbb{C}$-vector space, and $A : W \to W$ is an endomorphism. For $\lambda \in \mathbb{C}$, the **generalized eigenspace** for λ is

$$W_\lambda := \{w \in W \mid \text{for some } N > 0,\ (A - \lambda 1_V)^N(w) = 0\}.$$

It is well-known that W is the direct sum of its generalized eigenspaces:

$$W = \bigoplus_{\lambda \in \mathbb{C}} W_\lambda.$$

A similar decomposition can be done for quiver representations.

Suppose V is a representation of Q, and $\phi : V \to V$ is a morphism of quiver representations. For $\lambda \in \mathbb{C}$ and $x \in Q_0$, let $V(x)_\lambda$ be the generalized eigenspace of $\phi(x) : V(x) \to V(x)$ for eigenvalue λ. Since ϕ is a morphism,

$$\phi(ha)V(a) = V(a)\phi(ta),$$

If $w \in V(ta)_\lambda$, then $(\phi(ta) - \lambda 1)^N w = 0$ for some $N > 0$. From

$$0 = V(a)(\phi(ta) - \lambda 1)^N w = (\phi(ha) - \lambda 1)^N (V(a)w)$$

it follows that $V(a)w \in V(ha)_\lambda$. For every $\lambda \in \mathbb{C}$ we have $V(a)(V(ta)_\lambda) \subseteq V(ha)_\lambda$. So we can define a subrepresentation V_λ of V by $V_\lambda(x) = V(x)_\lambda$ for all $x \in Q_0$ and $\lambda \in \mathbb{C}$. We have an isomorphism

$$V \cong \bigoplus_{\lambda \in \mathbb{C}} V_\lambda. \tag{1.5}$$

Corollary 1.7.1. *If V is an indecomposable representation, and $\phi : V \to V$ is a morphism, then there exists a complex number $\lambda \in \mathbb{C}$ such that $\phi - \lambda 1$ is nilpotent.*

Proof. If V is indecomposable, then $V = V_\lambda$ for some $\lambda \in \mathbb{C}$ by (1.5). □

For a representation V of a quiver Q, the space $\mathrm{Hom}_Q(V, V)$ is a finite dimensional associative $\mathbb{C}$-algebra with 1 (where 1 is the identity).

Corollary 1.7.2. *Suppose that V is an indecomposable representation. For every $\phi \in \mathrm{Hom}_Q(V, V)$, either ϕ is invertible or ϕ is nilpotent. The set of $\mathfrak{m}$ of all noninvertible elements of $\mathrm{Hom}_Q(V, V)$ is the unique maximal (two-sided) ideal of $\mathrm{Hom}_Q(V, V)$ and $\mathrm{Hom}_Q(V, V)/\mathfrak{m} \cong \mathbb{C}$. In particular, $\mathrm{Hom}_Q(V, V)$ is a local ring.*

Proof. If $\phi \in \mathrm{Hom}_Q(V, V)$, then there exists a $\lambda \in \mathbb{C}$ with $\phi - \lambda 1$ nilpotent. If $\lambda = 0$, then ϕ is nilpotent. If $\lambda \neq 0$, then $\phi = (\phi - \lambda 1) + \lambda 1$ is invertible. If $\phi \in \mathfrak{m}$ and $\psi \in \mathrm{Hom}_Q(V, V)$, then ϕ is not invertible, so $\psi\phi$ and $\phi\psi$ are not invertible, and hence in $\mathfrak{m}$. This shows that $\mathfrak{m}$ is an ideal. It is clearly maximal because every element not in $\mathfrak{m}$ is invertible. The composition $\mathbb{C} \to \mathrm{Hom}_Q(V, V) \to \mathrm{Hom}_Q(V, V)/\mathfrak{m}$ is an isomorphism. □

To prepare for the theorem, we need the following technical lemma.

Lemma 1.7.3. *Suppose that V_1, V_2, W_1, W_2 are representations of Q, and*

$$\phi = \begin{pmatrix} \phi_{1,1} & \phi_{1,2} \\ \phi_{2,1} & \phi_{2,2} \end{pmatrix} : V_1 \oplus V_2 \to W_1 \oplus W_2$$

is an isomorphism, where $\phi_{i,j} : V_j \to W_i$. If $\phi_{1,1}$ is an isomorphism, then V_2 is isomorphic to W_2.

Proof. Let

$$\psi = \begin{pmatrix} 1_{W_1} & 0 \\ -\phi_{2,1}\phi_{1,1}^{-1} & 1_{W_2} \end{pmatrix}.$$

Then we have

$$\psi\phi = \begin{pmatrix} \phi_{1,1} & \phi_{1,2} \\ 0 & \phi'_{2,2} \end{pmatrix},$$

where $\phi'_{2,2} = \phi_{2,2} - \phi_{2,1}\phi_{1,1}^{-1}\phi_{1,2}$. Since $\psi\phi$ is invertible, we have that $\phi'_{2,2}$ is an isomorphism. □

Theorem 1.7.4 (Krull-Remak-Schmidt)**.** *Every finite dimensional representation of a quiver Q is isomorphic to a direct sum of indecomposable representations. This decomposition is unique up to isomorphism and permutation of factors. More precisely, if*

$$V_1 \oplus \cdots \oplus V_p \cong W_1 \oplus \cdots \oplus W_r,$$

and $V_1, \ldots, V_p, W_1, \ldots, W_r$ are indecomposable, then $p = r$ and there exists a permutation σ of $\{1, 2, \ldots, p\}$ such that V_i is isomorphic to $W_{\sigma(i)}$ for $i = 1, 2, \ldots, p$.

Proof. We prove by induction on $|\alpha|$ that every representation of dimension α is a direct sum of indecomposable representations. If $|\alpha| = 1$, then every α-dimensional representation is indecomposable. Suppose that V is an

α-dimensional representation with $|\alpha| > 1$. If V is indecomposable, then we are done. Otherwise, $V = W \oplus Z$, where W and Z are nontrivial representations. We have $\alpha = \underline{\dim} V = \underline{\dim} W + \underline{\dim} Z$ and $|\alpha| = |\underline{\dim} W| + |\underline{\dim} Z|$. Since $0 < |\underline{\dim} W|, |\underline{\dim} Z| < |\alpha|$, it follows from the induction hypothesis that W and Z are direct sums of indecomposable representations. Hence V is a direct sum of indecomposable representations. This completes the induction proof.

We now have to show the uniqueness. Suppose that

$$\phi = (\phi_{i,j})_{i,j} : V_1 \oplus \cdots V_p \to W_1 \oplus W_2 \oplus \cdots \oplus W_r$$

is an isomorphism, where $V_1, \ldots, V_p, W_1, \ldots, W_r$ are indecomposable representations, and $\phi_{i,j} \in \operatorname{Hom}_Q(V_j, W_i)$ for all i, j with $1 \leq i \leq r$ and $1 \leq j \leq p$. We show the uniqueness by induction on p. The case $p = 0$ is clear. Define $\psi = \phi^{-1}$ as the inverse of ϕ. Then $\psi = (\psi_{j,i})$, where $\psi_{j,i} \in \operatorname{Hom}_Q(W_i, V_j)$ for all i, j. We have $\sum_i \psi_{p,i}\phi_{i,p} = 1_{V_p}$. Let $\mathfrak{m}$ be the unique maximal ideal of the local ring $\operatorname{Hom}_Q(V_p, V_p)$. Since $1_{V_p} \notin \mathfrak{m}$, we have $\psi_{p,i}\phi_{i,p} \notin \mathfrak{m}$ for some i. After rearranging $W_1, \ldots, W_r$ we may assume that $\psi_{p,r}\phi_{r,p} \notin \mathfrak{m}$. We have that $\psi_{p,r}\phi_{r,p}$ is invertible. It follows that $\phi_{r,p}\psi_{p,r}$ cannot be nilpotent, so it has to be invertible as well. So $\phi_{r,p}$ is an isomorphism. By Lemma 1.7.3, $V_1 \oplus \cdots \oplus V_{p-1} \cong W_1 \oplus \cdots \oplus W_{r-1}$. By the induction hypothesis, we have $p - 1 = r - 1$, so $p = r$. After rearranging $W_1, \ldots, W_{p-1}$ we get that V_i is isomorphic to W_i for $i = 1, 2, \ldots, p$. □

Exercises.

Exercise 1.7.1. *Show that the isomorphism classes of representations of the quiver*

$$1 \xrightarrow{\ a\ } 2 \xrightarrow{\ b\ } 3$$

are uniquely determined by $\dim V(1)$, $\dim V(2)$, $\dim V(3)$, $\operatorname{rank} V(a)$, $\operatorname{rank} V(b)$ *and* $\operatorname{rank} V(b)V(a)$.

1.8. Bibliographical Remarks

The material of this chapter is introductory. We are dealing only with finite dimensional algebras over $\mathbb{C}$. Most of the results are true in greater generality, either over an algebraically closed field of arbitrary characteristic, for finite dimensional algebras over arbitrary field or even for Artin algebras (which can be infinite dimensional). We refer the reader to several books that have been written on the subject. There is a book [**3**] by Auslander, Reiten and Smalø which contains an excellent exposition of the Auslander-Reiten theory. It is based on the use of stable categories and categories of functors. The book [**43**] by Gabriel and Roiter stresses the language of linear algebra and generalizations to the K-linear categories.

Finally, several years ago the three volumes of the book [**38**] by Assem, Simson and Skowroński started appearing. We recommend especially the first volume of this work which contains all the basics in the most general framework.

Chapter 2

Homological Algebra of Quiver Representations

An algebra is semi-simple if all its modules are direct sums of irreducible modules. By Wedderburn's theorem, finite dimensional semi-simple algebras over $\mathbb{C}$ are products of matrix algebras (see Corollary 3.4.13). For example, group algebras of finite groups are semi-simple (over $\mathbb{C}$) which makes the representation theory of finite groups relatively easy. An algebra A is semi-simple precisely when every A-module is projective, i.e., when global dimension of A is zero.

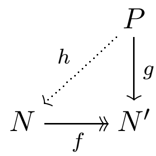
Hereditary algebras are algebras of global dimension at most one. It is natural to study these as the next step. A path of algebras $\mathbb{C}Q$ for quivers Q without oriented cycles is hereditary. To understand the representation theory of hereditary algebras, it suffices to understand the representation theory of path algebras. In this chapter we develop the homological algebra over path algebras, construct explicitly the projective and injective resolutions of all modules. We also give the definition and all the properties of the Ext vector spaces constructing them as classes of extensions and as derived functors. This material can be found in any homological algebra book.

2.1. Projective and Injective Modules

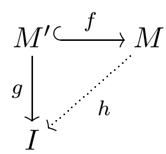
Let A be a $\mathbb{C}$-algebra. We will work in the category of finite dimensional A-modules and all modules are assumed to be finite dimensional. In this section we give the definitions of projective and injective modules, and discuss some basic properties.

Definition 2.1.1. Let A be a $\mathbb{C}$-algebra.

(1) An A-module P is **projective** if the following equivalent conditions are satisfied:
   (a) the functor $\mathrm{Hom}_A(P, -)$ is exact;
   (b) for every epimorphism $f : N \twoheadrightarrow N'$ and every homomorphism $g : P \to N'$ there exist a homomorphism $h : P \to N$ such that $fh = g$.



(2) An A-module I is **injective** if the following equivalent conditions are satisfied:
   (a) the functor $\mathrm{Hom}_A(-, I)$ is exact;
   (b) for every monomorphism $f : M' \hookrightarrow M$ and every homomorphism $g : M' \to I$ there exist a homomorphism $h : M \to I$ such that $hf = g$.



The following lemma is proven in Exercise 2.1.1.

Lemma 2.1.2.

(1) *If P is projective, then every exact sequence*

$$0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$$

of A-modules splits. In particular, M is isomorphic to $N \oplus P$.

(2) *If I is injective, then every exact sequence*

$$0 \longrightarrow I \longrightarrow M \longrightarrow N \longrightarrow 0$$

splits. In particular, M is isomorphic to $I \oplus N$.

The following lemma is proven in Exercise 2.1.2.

Lemma 2.1.3. *An A-module M is projective if and only if $M^\star$ is an injective A^{op}-module.*

For the remainder of this section, assume that A is finite dimensional. For any A-module M we have $\mathrm{Hom}_A(A, M) = M$. So $\mathrm{Hom}_A(A, -)$ is the identity functor, and it is therefore exact. This shows that A is projective. Also, $A = A^{\mathrm{op}}$ is a projective A^{op}-module, so $A^\star$ is an injective A-module.

If P is a projective module and $P = P' \oplus P''$, then P' and P'' are also projective, and a similar statement is true for injective modules (see Exercise 2.1.3).

Corollary 2.1.4. *Every direct summand of A is projective. Every direct summand of $A^\star$ is injective.*

Also the sum of two projective modules is again projective and the sum of two injective modules is again injective (see Exercise 2.1.3). In particular, a free module A^r is projective, and the module $(A^\star)^r$ is injective. Every direct summand of A^r is projective and every direct summand of $(A^\star)^r$ is injective. The converse is also true.

Lemma 2.1.5. *If P is a projective A module, then P is a direct summand of A^r for some positive integer r. If I is an injective A-module, then I is a direct summand of $(A^\star)^r$ for some positive integer r.*

Proof. Suppose that P is generated by $u_1, \ldots, u_r$. Then we define a surjective module homomorphism $\varphi : A^r \twoheadrightarrow P$ defined by $\varphi(a_1, \ldots, a_r) = a_1u_1 + \cdots + a_ru_r$. Let K be the kernel of φ. We have an exact sequence

$$0 \longrightarrow K \longrightarrow A^r \longrightarrow P \longrightarrow 0 \ ,$$

and $A^r \cong K \oplus P$ by Lemma 2.1.2.

If I is injective, then $I^\star$ is a projective A^{op}-module. So $I^\star$ is isomorphic to a direct summand of A^r for some positive integer r (as A^{op}-modules) and I is a direct summand of $(A^\star)^r$. □

Exercises.

Exercise 2.1.1. *Prove Lemma* 2.1.2.

Exercise 2.1.2. *Prove Lemma* 2.1.3.

Exercise 2.1.3. *Suppose that P, P', P'' are A-modules such that $P = P' \oplus P''$. Show that P is projective if and only if both P' and P'' are projective.*

Exercise 2.1.4.

(1) *Suppose that P is a module such that every exact sequence of A-modules*

$$0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$$

splits. Show that P is projective.

(2) *Suppose that I is a module such that every exact sequence of A-modules*

$$0 \longrightarrow I \longrightarrow M \longrightarrow N \longrightarrow 0$$

splits. Show that I is injective.

2.2. Projective and Injective Quiver Representations

In this section we restrict ourselves to acyclic quivers and their path algebras. There is a 1–1 correspondence between indecomposable projective (respectively injective) representations and vertices of the quiver. The projective and injective representation can be explicitly described in terms of paths.

Let $Q = (Q_0, Q_1, h, t)$ be an acyclic quiver. We investigate the structure of projective and injective resolutions in $\operatorname{Rep}(Q)$. For $x \in Q_0$, define $\mathbb{C}Q$-modules by $P_x := \mathbb{C}Qe_x$ and $I_x = (e_x\mathbb{C}Q)^\star$.

Lemma 2.2.1. *We have a direct sum decompositions of $\mathbb{C}Q$-modules.*

$$\mathbb{C}Q = \bigoplus_{x\in Q_0} P_x \text{ and } (\mathbb{C}Q)^\star = \bigoplus_{x\in Q_0} I_x.$$

Proof. If $u \in \mathbb{C}Q$, then we have

$$u = u\cdot 1 = u\cdot \sum_{x\in Q_0} e_x = \sum_{x\in Q_0} ue_x \in \sum_{x\in Q_0} P_x.$$

Suppose that $\sum_{x\in Q_0} a_x = 0$ with $a_x \in P_x$ for all $x \in Q_0$. For $y \in Q_0$ we have $a_x e_y = a_x e_x e_y = 0$ if $x \neq y$, so

$$0 = \Big(\sum_{x\in Q_0} a_x\Big) e_y = a_y e_y = a_y$$

for all $y \in Q_0$. This shows that the sum is direct. The second direct sum decomposition is proven in a similar way. □

Since P_x is a direct summand of $\mathbb{C}Q$, it is projective by Corollary 2.1.4. Note that P_x is a vector space with a basis given by all paths starting at x. We can view P_x now as a representation of the quiver Q. For any vertex $y \in Q_0$ we have that $P_x(y) = e_y\mathbb{C}Qe_x$ is a vector space with a basis given by all paths from x to y. Since I_x is a direct summand of the injective module $(\mathbb{C}Q)^\star$, it is injective as well. The space $I_x^\star = e_x\mathbb{C}Q$ has a basis given by all paths ending at x. The space $I_x(y)^\star = (e_yI_x)^\star = (e_x\mathbb{C}Qe_y)^\star$ has a basis given by all paths from y to x.

Proposition 2.2.2. *We have canonical isomorphisms*

$$\operatorname{Hom}_Q(P_x, V) \cong V(x) \text{ and } \operatorname{Hom}_Q(V, I_x) \cong V(x)^\star.$$

Proof. Define a linear map $\Psi : \mathrm{Hom}_Q(P_x, V) \to V(x)$ by $\Psi(\phi) = \phi(e_x) \in V(x)$. Define a linear map $\Theta : V(x) \to \mathrm{Hom}_Q(P_x, V)$ as follows; for any $v \in V(x)$ there is a unique morphism $\phi_v : P_x \to V$ of quiver representations given by $\phi_v(p) = V(p)v$. Define $\Theta(v) = \phi_v$. It is easy to see that Ψ and Θ are each others inverses.

For the second isomorphism, note that

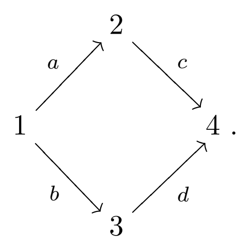
$$\mathrm{Hom}_Q(V, I_x) \cong \mathrm{Hom}_{Q^{\mathrm{op}}}(I_x^\star, V^\star) \cong V^\star(x).$$

□

Proposition 2.2.3. *The indecomposable projective representations are (up to isomorphism) exactly all P_x, $x \in Q_0$. The indecomposable injective representations are (up to isomorphism) exactly all I_x, $x \in Q_0$.*

Proof. Note that P_x is indecomposable for all $x \in Q_0$, because $\mathrm{Hom}_Q(P_x, P_x) \cong \mathbb{C}e_x$ is 1-dimensional. If P is a projective representation, then P is a direct summand of $(\mathbb{C}Q)^r = \bigoplus_{x \in Q_0} P_x^r$. By the Krull-Remak-Schmidt theorem, P has to be isomorphic to P_x for some $x \in Q_0$. The second statement in the proposition follows from duality (see Section 1.6 and Lemma 2.1.3). □

Example 2.2.4. Consider the quiver



The representation P_1 has dimension vector $(1, 1, 1, 2)$. It has a $\mathbb{C}$-basis $\{e_1, a, b, ca, db\}$. The representation P_2 has dimension vector $(0, 1, 0, 1)$ and $\mathbb{C}$-basis $\{e_2, c\}$. Similarly, the representation P_3 has dimension vector $(0, 0, 1, 1)$ and has $\mathbb{C}$-basis $\{e_3, d\}$. Finally, P_4 has dimension vector $(0, 0, 0, 1)$ and its $\mathbb{C}$-basis is $\{e_4\}$. The simple module S_1 has the following resolution:

$$0 \longrightarrow P_2 \oplus P_3 \xrightarrow{(a,b)} P_1 \longrightarrow S_1 \longrightarrow 0.$$

Exercises.

Exercise 2.2.1. *Show that if Q has an oriented cycle, then there are infinitely many distinct isomorphism classes of simple representations. (See Exercise* 1.4.4.*)*

Exercise 2.2.2. *Let S_x be a simple module corresponding to a vertex $x \in Q_0$. There is a natural map $p_x : P_x \to S_x$ sending the generator e_x to the* *element* $1 \in \mathbb{C} = S_x(x)$*. Prove that the kernel of this map is isomorphic to*

$$\bigoplus_{\substack{a \in Q_1 \\ ta = x}} P_{ha}.$$

The resulting complex

$$0 \longrightarrow \bigoplus_{\substack{a \in Q_1 \\ ta = x}} P_{ha} \longrightarrow P_x \longrightarrow S_x \longrightarrow 0$$

gives a projective resolution of length 1 *of a simple object* S_x*.*

Exercise 2.2.3. *Let* $e_x^\star \in I_x(x)$ *be dual to* $e_x \in I_x(x)^\star = \mathbb{C}e_x$*. Prove that for a natural embedding* $q_x : S_x \to I_x$ *defined by sending the generator* $1 \in \mathbb{C} = S_x(x)$ *to the basis element* e_x^* *we have*

$$\text{coker } q_x = \bigoplus_{\substack{a \in Q_1 \\ ha = x}} I_{ta}.$$

The resulting complex

$$0 \longrightarrow S_x \longrightarrow I_x \longrightarrow \bigoplus_{\substack{a \in Q_1 \\ ha = x}} I_{ta} \longrightarrow 0$$

gives an injective resolution of S_x *of length* 1*.*

2.3. The Hereditary Property of Path Algebras

Definition 2.3.1. A category of (left) modules over an algebra A is **hereditary** if a submodule of a projective module is projective.

Exercise 2.2.2 suggests that for acyclic quivers Q, the category $\text{Rep}(Q)$ might be hereditary. We will prove this in this section.

Theorem 2.3.2. *Let* Q *be an acyclic quiver. The category* $\text{Rep}(Q)$ *is hereditary.*

Before giving the proof, we need two propositions.

Proposition 2.3.3. *Assume that there are two exact sequences of representations of* Q*:*

$$0 \longrightarrow P_1 \xrightarrow{d} P_0 \longrightarrow V \longrightarrow 0,$$

$$0 \longrightarrow P_1' \xrightarrow{d'} P_0' \longrightarrow V \longrightarrow 0$$

with P_0, P_1, P_0' *projective. Then* $P_1' \oplus P_0$ *is isomorphic to* $P_1 \oplus P_0'$*. In particular ,*P_1' *is projective as well.*

Proof. Since P'_0 is projective, the identity map on V can be lifted to the map $f_0 : P'_0 \to P_0$ and this map restricts to a map $f_1 : P'_1 \to P_1$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'_1 & \xrightarrow{d'} & P'_0 & \xrightarrow{p'} & V & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle 1_V} & & \\ 0 & \longrightarrow & P_1 & \xrightarrow{d} & P_0 & \xrightarrow{p} & V & \longrightarrow & 0 \end{array}$$

commutative. Consider the complex

$$\mathcal{C}:\ 0 \longrightarrow P'_1 \xrightarrow{g_2} P_1 \oplus P'_0 \xrightarrow{g_1} P_0 \longrightarrow 0,$$

where $g_2 = \begin{pmatrix} -f_1 \\ d' \end{pmatrix}$ and $g_1 = \begin{pmatrix} d & f_0 \end{pmatrix}$. We will show that this complex is exact. Let $H^k(\mathcal{C})$ be the k-th cohomology group of the complex $\mathcal{C}$. The map g_2 is injective because d' is injective, so $H^2(\mathcal{C}) = 0$. Also, g_1 is surjective: if $x_0 \in P_0$, then there exists $x'_0 \in P'_0$ with $p'(x'_0) = pf_0(x'_0) = p(x_0)$. Since $p(x_0 - f_0(x'_0)) = 0$, there exists $x_1 \in P_1$ such that $d(x_1) = x_0 - f_0(x'_0)$. So $x_0 = f_0(x'_0) + d(x_1)$ lies in the image of g_1. Hence $H^0(\mathcal{C}) = 0$. From

$$\begin{aligned} -\dim H^1(\mathcal{C}) &= \dim H^0(\mathcal{C}) - \dim H^1(\mathcal{C}) + \dim H^2(\mathcal{C}) \\ &= \dim P_0 - \dim(P_1 \oplus P'_0) + \dim P_1 \\ &= (\dim P_0 - \dim P_1) - (\dim P'_0 - \dim P'_1) = \dim V - \dim V = 0 \end{aligned}$$

we conclude that $H^1(\mathcal{C}) = 0$. So the complex $\mathcal{C}$ is exact.

Since P_0 is projective our exact sequence splits by Lemma 2.1.2 and we are done. □

For every $\mathbb{C}Q$-module V of Q we define a complex
(2.1)

$$\mathcal{C}(V):\ 0 \longrightarrow \bigoplus_{a \in Q_1} V(ta) \otimes P_{ha} \xrightarrow{d^V} \bigoplus_{x \in Q_0} V(x) \otimes P_x \xrightarrow{f^V} V \longrightarrow 0,$$

where

$$f^V(v \otimes p) = p \cdot v$$

and

$$d^V(v \otimes p) = (a \cdot v) \otimes p - v \otimes pa.$$

It is clear that $\mathcal{C}(V)$ is a complex because

$$f^V(d^V(v \otimes p)) = f^V((a \cdot v) \otimes p - v \otimes pa) = p \cdot (a \cdot v) - (pa) \cdot v = 0.$$

Proposition 2.3.4. *The complex $\mathcal{C}(V)$ (2.1) is exact.*

Proof. Clearly, f^V is onto, because $f^V(v \otimes e_x) = e_x \cdot v = v$ for all $v \in V(x) = e_x V$. We claim that d^V is injective. Suppose that

$$d^V(\sum_{a \in Q_1} v_a \otimes p_a) = 0 \text{ and } \sum_{a \in Q_1} v_a \otimes p_a \neq 0,$$

where $v_a \in V(ta)$ and $p_a \in P_{ha}$ for all $a \in Q_1$. We may assume that $Q_0 = \{1, 2, \ldots, n\}$ and that $h(a) < t(a)$ for all $a \in Q_1$ (see Exercise 1.5.2). Choose $y \in Q_0$ maximal such that there exists an arrow a with $v_a \otimes p_a \neq 0$ and $ta = y$. For any arrow a with $ha = y$ we have $ta > ha = y$ and $v_a \otimes p_a = 0$. The component of $d^V(\sum_{a \in Q_1} v_a \otimes p_a)$ in $V(y) \otimes P_y$ is

$$\sum_{i=1}^{r} -v_{a_i} \otimes p_{a_i} a_i = 0,$$

where $a_1, a_2, \ldots, a_r$ are all arrows with tail y. This is a contradiction because $p_{a_1}a_1, \ldots, p_{a_r}a_r$ are independent, and $v_{a_i} \neq 0$ for all i. From Exercise 2.2.2 it follows that

$$\begin{aligned} &\dim\Big(\bigoplus_{x \in Q_0} V(x) \otimes P_x) - \dim\Big(\bigoplus_{a \in Q_1} V(ta) \otimes P_x\Big) \\ &\quad = \sum_{x \in Q_0} \dim V(x)\Big(\dim P_x - \sum_{\substack{a \in Q_1 \\ ta = x}} \dim P_{ha}\Big) = \sum_{x \in Q_0} \dim V(x) = \dim V. \end{aligned}$$

From this follows that the complex $\mathcal{C}(V)$ is exact in the middle as well. □

The complex $\mathcal{C}(V)$ is called the **canonical projective resolution** of V.

Proof of Theorem 2.3.2. Suppose that P_1' is a subrepresentation of a projective representation P_0', and let $V = P_0'/P_1'$. We have a short exact sequence

$$0 \longrightarrow P_1' \longrightarrow P_0' \longrightarrow V \longrightarrow 0.$$

By Proposition 2.3.4 there exists an exact sequence

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow V \longrightarrow 0,$$

where P_0 and P_1 are projective. So P_1' is projective by Proposition 2.3.3. □

Exercises.

Exercise 2.3.1. *Show for any representation W we have a canonical injective resolution*

$$0 \longrightarrow W \xrightarrow{f_W} \bigoplus_{x \in Q_0} I_x \otimes W(x) \xrightarrow{d_W} \bigoplus_{a \in Q_1} I_{ta} \otimes W(ha) \longrightarrow 0, \tag{2.2}$$

where

$$f_W^\star : \bigoplus_{x \in Q_0} W(x)^\star \otimes I_x^\star \longrightarrow W^\star$$

is defined by

$$f_W^\star(p \otimes w) = p \cdot w$$

and

$$d_W^\star : \bigoplus_{a \in Q_1} W(ha)^\star \otimes I_{ta}^\star \longrightarrow \bigoplus_{x \in Q_0} W(x)^\star \otimes I_x^\star$$

is defined by

$$d_W^\star(w \otimes p) = w \otimes ap - (a \cdot w) \otimes p.$$

2.4. The Extensions Group

Let V and W be two representations of a quiver Q. We introduce the extension group $\mathrm{Ext}_Q(V, W)$. First, we will describe this abelian group as Yoneda extensions. We also describe this group as the derived functor $\mathrm{Ext}_Q^1(V, -)$ of $\mathrm{Hom}_Q(V, -)$ or as the derived functor $\mathrm{Ext}^1(-, W)$ of $\mathrm{Hom}_Q(-, W)$.

An **extension** ξ of V by W is an exact sequence

$$\xi : \ 0 \longrightarrow W \xrightarrow{\ i\ } E \xrightarrow{\ p\ } V \longrightarrow 0\ .$$

The two extensions

$$\xi : \ 0 \longrightarrow W \xrightarrow{\ i\ } E \xrightarrow{\ p\ } V \longrightarrow 0$$

$$\xi' : \ 0 \longrightarrow W \xrightarrow{\ i'\ } E' \xrightarrow{\ p'\ } V \longrightarrow 0\ .$$

are **equivalent** if there exists an isomorphism $f : E \to E'$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \xrightarrow{\ i\ } & E & \xrightarrow{\ p\ } & V & \longrightarrow & 0\ . \\ & & \downarrow{\scriptstyle 1_W} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1_V} & & \\ 0 & \longrightarrow & W & \xrightarrow{\ i'\ } & E' & \xrightarrow{\ p'\ } & V & \longrightarrow & 0 \end{array}$$

commutes.

Definition 2.4.1. We define $\mathrm{Ext}_Q(V, W)$ as the class of equivalence classes of extensions of V by W. This is the **Yoneda extension** group.

Consider the following map:

$$\bigoplus_{x \in Q_0} \mathrm{Hom}(V(x), W(x)) \xrightarrow{\ d_W^V\ } \bigoplus_{a \in Q_1} \mathrm{Hom}(V(ta), W(ha)), \tag{2.3}$$

where

$$d^V_W(\phi(x), x \in Q_0) = (\phi(ha)V(a) - W(a)\phi(ta), a \in Q_1).$$

The following proposition gives us an easy way to understand $\mathrm{Ext}_Q(V, W)$ and to compute it.

Proposition 2.4.2. *The kernel of* d^V_W *is* $\mathrm{Hom}_Q(V, W)$*. The cokernel of* d^V_W *can be identified with* $\mathrm{Ext}_Q(V, W)$*. In particular,* $\mathrm{Ext}_Q(V, W)$ *is a finite dimensional* $\mathbb{C}$*-vector space.*

Proof. It is clear from the definition of $\mathrm{Hom}_Q(V, W)$ that it is the kernel of d^V_W. We define a map

$$\Theta : \bigoplus_{a \in Q_1} \mathrm{Hom}(V(ta), W(ha)) \to \mathrm{Ext}_Q(V, W).$$

Suppose that

$$\psi = (\psi(a), a \in Q_1) \in \bigoplus_{a \in Q_1} \mathrm{Hom}(V(ta), W(ha)).$$

We define $E(x) := V(x) \oplus W(x)$ and $E(a)$ is given in block form by the matrix

$$\begin{pmatrix} V(a) & 0 \\ \psi(a) & W(a) \end{pmatrix}. \tag{2.4}$$

It is clear that the subspaces $\{W(x)\}_{x \in Q_0}$ form a subrepresentation of E and that the factor is isomorphic to V. Thus we have an extension

$$\xi : \ 0 \longrightarrow W \longrightarrow E \longrightarrow V \longrightarrow 0.$$

We define $\Theta(\psi)$ as the equivalence class of ξ.

Suppose that

$$\psi' = (\psi'(a), a \in Q_1) \in \bigoplus_{a \in Q_1} \mathrm{Hom}(V(ta), W(ha))$$

and

$$\xi' : \ 0 \longrightarrow W \longrightarrow E' \longrightarrow V \longrightarrow 0$$

is the extension obtained from ψ'. If ξ and ξ' are equivalent, then there exists an isomorphism $f : E \to E'$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \xrightarrow{i} & E & \xrightarrow{p} & V & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_W} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1_V} & & \\ 0 & \longrightarrow & W & \xrightarrow{i'} & E' & \xrightarrow{p'} & V & \longrightarrow & 0 \end{array} \tag{2.5}$$

commutes. The restriction of $f(x) : V(x) \oplus W(x) \to V(x) \oplus W(x)$ must have the block form

$$f(x) = \begin{pmatrix} 1_V & 0 \\ -\phi(x) & 1_W \end{pmatrix} \tag{2.6}$$

for some $\phi(x) \in \mathrm{Hom}(V(x), W(x))$. Since f is a morphism of quiver representations, we have

$$\begin{pmatrix} 1_{V(ha)} & 0 \\ -\phi(ha) & 1_{W(ha)} \end{pmatrix} \begin{pmatrix} V(a) & 0 \\ \psi(a) & W(a) \end{pmatrix} = \phi(ha)E(a)$$
$$= E'(a)\phi(ta) = \begin{pmatrix} V(a) & 0 \\ \psi'(a) & W(a) \end{pmatrix} \begin{pmatrix} 1_{V(ta)} & 0 \\ -\phi(ta) & 1_{W(ta)} \end{pmatrix}.$$

Looking at the block at position $(2, 1)$ gives us the equation

$$-\phi(ha)V(a) + \psi(a) = \psi'(a) - W(a)\phi(ta)$$

so

$$\psi - \psi' = (\psi(a) - \psi'(a), a \in Q_1) = (\phi(ha)V(a) - W(a)\phi(ta), a \in Q_1) = d_W^V(\phi).$$

Conversely, if $\psi - \psi' = d_W^V(\phi)$ for some $\phi \in \bigoplus_{x \in Q_0} \mathrm{Hom}(V(x), W(x))$, then the diagram (2.5) commutes, where $f : E \to E'$ is the isomorphism defined by (2.6).

Any extension ξ of V by W is equivalent to an extension

$$0 \longrightarrow W \longrightarrow E \longrightarrow V \longrightarrow 0,$$

where $E(x) = V(x) \oplus W(x)$ for all $x \in Q_0$. Then $E(a)$ must be of the form (2.4) for some $\psi = (\psi(a), a \in Q_1)$ and $\Theta(\psi)$ is equivalent to ξ. This shows that Θ is surjective. □

Now, assume that Q is acyclic. Let

$$\mathcal{P}_V : \ 0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow V \longrightarrow 0 \tag{2.7}$$

be a projective resolution of V of length 1, i.e., (2.7) is an exact sequence where P_0 and P_1 are projective. Also, let

$$\mathcal{I}^W : \ 0 \longrightarrow W \longrightarrow I^0 \longrightarrow I^1 \longrightarrow 0 \tag{2.8}$$

be an injective resolution of W of length 1.

We can form the double complex $\mathrm{Hom}_Q(\mathcal{P}_V, \mathcal{I}^W)$. Using the exactness of the functors $\mathrm{Hom}_Q(P_0, -), \mathrm{Hom}_Q(P_1, -), \mathrm{Hom}_Q(-, I^0), \mathrm{Hom}_Q(-, I^1)$, the left-exactness of the functor $\mathrm{Hom}_Q(V, -)$ and the right-exactness of the functor $\mathrm{Hom}_Q(-, W)$, we get the following diagram with exact rows and columns:
(2.9)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_Q(V,W) & \longrightarrow & \mathrm{Hom}_Q(V,I^0) & \longrightarrow & \mathrm{Hom}_Q(V,I^1) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_Q(P_0,W) & \longrightarrow & \mathrm{Hom}_Q(P_0,I^0) & \longrightarrow & \mathrm{Hom}_Q(P_0,I^1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_Q(P_1,W) & \longrightarrow & \mathrm{Hom}_Q(P_1,I^0) & \longrightarrow & \mathrm{Hom}_Q(P_1,I^1) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Lemma 2.4.3. *The cokernel of* $\mathrm{Hom}_Q(V, I^0) \to \mathrm{Hom}_Q(V, I^1)$ *and the kernel of* $\mathrm{Hom}_Q(P_0, W) \to \mathrm{Hom}_Q(P_1, W)$ *are both isomorphic to* $\mathrm{Ext}_Q(V, W)$.

Proof. From the snake lemma applied to the diagram (2.9) it follows that the cokernel of $\mathrm{Hom}_Q(P_0, W) \to \mathrm{Hom}_Q(P_1, W)$ and the cokernel of $\mathrm{Hom}_Q(V, I^0) \to \mathrm{Hom}_Q(V, I^1)$ are isomorphic. If we choose $P_0 = \bigoplus_{x \in Q_0} V(x) \otimes P_x$ and $P_1 = \bigoplus_{a \in Q_1} V(ta) \otimes P_{ha}$ as in (2.1), then the cokernel of $\mathrm{Hom}_Q(P_0, W) \to \mathrm{Hom}_Q(P_1, W)$ is $\mathrm{Ext}_Q(V, W)$ by Exercise 2.4.1. Therefore, the cokernel of $\mathrm{Hom}(V, I^0) \to \mathrm{Hom}(V, I^1)$ is isomorphic to $\mathrm{Ext}_Q(V, W)$. $\square$

In the category of quiver representations, one can define groups $\mathrm{Ext}^i_Q(V, W)$ for all $i \geq 0$ using derived functors of $\mathrm{Hom}_Q(V, -)$ or $\mathrm{Hom}_Q(-, W)$. We have $\mathrm{Ext}^0_Q(V, W) = \mathrm{Hom}_Q(V, W)$. Because of the previous lemma, we have $\mathrm{Ext}^1_Q(V, W) = \mathrm{Ext}_Q(V, W)$. Also, $\mathrm{Ext}^i_Q(V, W) = 0$ for all $i > 1$, because every representation has a projective resolution of length 1.

Exercises.

Exercise 2.4.1. *Show that if we apply the functor* $\mathrm{Hom}_Q(-, W)$ *to* (2.1) *or the functor* $\mathrm{Hom}_Q(V, -)$ *to* (2.2) *then we obtain the map*

$$d^V_W = \mathrm{Hom}_Q(d^V, W) = \mathrm{Hom}_Q(V, d_W)$$

in (2.3).

Exercise 2.4.2. *We have identified* $\mathrm{Ext}_Q(V,W)$ *with the cokernel of* d_W^V *and this gives* $\mathrm{Ext}_Q(V,W)$ *the structure of a finite dimensional* $\mathbb{C}$*-vector space. But the vector space structure of* $\mathrm{Ext}_Q(V,W)$ *can also be directly defined in terms of extensions.*

(1) *Show that the split extensions of* V *by* W *form exactly the equivalence class which corresponds to the* 0 *element in* $\mathrm{Ext}_Q(V,W)$.

(2) *Suppose that*

$$\xi :\ 0 \longrightarrow W \xrightarrow{\ i\ } E \xrightarrow{\ p\ } V \longrightarrow 0$$

is an extension of V *by* W *and denote its equivalence class by* $[\xi]$. *For* $\lambda \in \mathbb{C}$, *define* $E_\lambda := \{(u,v) \in E \oplus V \mid p(u) = \lambda v\}$. *Define* $p_\lambda : E_\lambda \to V$ *as the projection onto* V *and define* $i_\lambda : W \to E_\lambda$ *by* $i_\lambda(w) = (i(w), 0)$. *Show that*

$$\xi_\lambda :\ 0 \longrightarrow W \xrightarrow{\ i_\lambda\ } E_\lambda \xrightarrow{\ p_\lambda\ } V \longrightarrow 0$$

is exact and that $\lambda \cdot [\xi] = [\xi_\lambda]$.

(3) *Suppose that*

$$\xi' :\ 0 \longrightarrow W \xrightarrow{\ i'\ } E' \xrightarrow{\ p'\ } V \longrightarrow 0$$

is another extension of V *by* W. *Let*

$$F = \{(u,u') \in E \oplus E' \mid p(u) = p'(u')\} \subset E \oplus E'$$

be the pull-back in the diagram

$$\begin{array}{ccc} F & \dashrightarrow & E' \\ \vdots & & \downarrow \\ E & \longrightarrow & V \end{array}$$

and let

$$G = \{(i(w), -i'(w)) \mid w \in W\} \subseteq F.$$

Define $E'' = F/G$. *The map* $F \to V$ *contains* G *in the kernel, so it induces a morphism* $p'' : E'' \to W$. *Define* $i'' : W \to E''$ *by* $i''(w) = (i(w), 0) + G \in F/G = E''$. *Show that the sequence*

$$\xi'' :\ 0 \longrightarrow W \xrightarrow{\ i''\ } E'' \xrightarrow{\ p''\ } V \longrightarrow 0$$

is exact and that $[\xi] + [\xi'] = [\xi'']$.

Exercise 2.4.3. *Consider the subsquare*

$$\begin{array}{ccc} \mathrm{Hom}_Q(P_0, I^0) & \xrightarrow{d_0'} & \mathrm{Hom}_Q(P_0, I^1) \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_1} \\ \mathrm{Hom}_Q(P_1, I^0) & \xrightarrow[d_1']{} & \mathrm{hom}_Q(P_1, I^1) \end{array}$$

of (2.9). *We can form the total complex*

$$\mathcal{C}^{\mathrm{tot}}\colon \mathrm{Hom}_Q(P_0, I^0) \longrightarrow \mathrm{Hom}_Q(P_1, I^0) \oplus \mathrm{Hom}_Q(P_0, I^1) \longrightarrow \mathrm{Hom}(P_1, I^1),$$

where the two maps are given by

$$\begin{pmatrix} d_0 \\ -d_0' \end{pmatrix} \text{ and } \begin{pmatrix} d_1' & d_1 \end{pmatrix}.$$

Show that the homology groups are as follows:

$$\begin{aligned} H_0(\mathcal{C}^{\mathrm{tot}}) &= \mathrm{Hom}_Q(V, W), \\ H_1(\mathcal{C}^{\mathrm{tot}}) &= \mathrm{Ext}_Q(V, W), \\ H_2(\mathcal{C}^{\mathrm{tot}}) &= 0. \end{aligned}$$

2.5. The Euler Form

Definition 2.5.1. The Euler characteristic of two representations V, W of Q is

$$\chi(V, W) = \dim \mathrm{Hom}_Q(V, W) - \dim \mathrm{Ext}_Q(V, W).$$

We define the **Euler form** or **Ringel form** on $\mathbb{R}^{Q_0}$ by

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).$$

Note that this bilinear form may not be symmetric (unless Q has no arrows) or skew-symmetric.

Proposition 2.5.2. *If* $\alpha = \underline{\dim}\, V$ *and* $\beta = \underline{\dim}\, W$*, then*

$$\chi(V, W) = \langle \alpha, \beta \rangle.$$

In particular, $\chi(V, W)$ *only depends on the dimension vectors of* V *and* W*.*

Proof. We have an exact sequence

$$0 \to \mathrm{Hom}_Q(V, W) \to \bigoplus_{x \in Q_0} \mathrm{Hom}(V(x), W(x)) \to \bigoplus_{x \in Q_1} \mathrm{Hom}(V(ta), W(ha)) \to \mathrm{Ext}_Q(V, W) \to 0,$$

so

$$\begin{aligned}\dim \operatorname{Hom}_Q(V,W) - \dim \operatorname{Ext}_Q(V,W) \\ = \dim \bigoplus_{x\in Q_0} \operatorname{Hom}(V(x),W(x)) - \dim \bigoplus_{a\in Q_1} \operatorname{Hom}(V(ta),W(ha)) \\ = \sum_{x\in Q_0} \alpha(x)\beta(x) - \sum_{a\in Q_1} \alpha(ta)\beta(ha) = \langle \alpha,\beta\rangle.\end{aligned}$$

□

We can think about the Euler form in terms of matrices.

Definition 2.5.3. Let Q be a quiver with $Q_0 = \{1,2,\dots,n\}$. The **Euler matrix** E is an $n\times n$ matrix with the entries given by the formulas

$$E_{i,j} = \delta_{i,j} - |\{a\in Q_1 \mid ta = i, ha = j\}|,$$

where $\delta_{i,j}$ is the **Kronecker symbol** defined by

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Identifying the dimension vectors with column vectors, we have

$$\langle \alpha,\beta\rangle = \alpha^t E \beta.$$

If Q is an acyclic quiver, then we can relabel the vertices of Q by $1,2,\dots,n$ such that $ha < ta$ for all $a\in Q_1$. In this case, the Euler matrix is lower triangular with 1's on the diagonal.

Example 2.5.4. Consider the quiver

$$1 \longleftarrow 2 \longleftarrow 3 \;.$$

The Euler matrix is

$$E = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Exercises.

Exercise 2.5.1. *Suppose that Q is an acyclic quiver with $Q_0 = \{1,2,\dots,n\}$. Show that $\chi(P_i,S_j) = \delta_{i,j} = \chi(S_i,I_j)$. Show that the columns of $(E^{-1})^t$ are the dimension vectors of the indecomposable projective representations, and the columns of E^{-1} are the dimension vectors of the indecomposable injective representations. Also show that $(E^{-1})_{i,j}$ is exactly the number of paths from i to j.*

Exercise 2.5.2. *Suppose that Q and Q' are acyclic quivers. Use Exercise* 1.5.3 *to show that the categories* $\mathrm{Rep}(Q)$ *and* $\mathrm{Rep}(Q')$ *are equivalent if and only if Q and Q' are isomorphic quivers (i.e., one is obtained from the other by relabeling of the vertices/arrows.)*

2.6. Bibliographical Remarks

The material of this chapter is standard. We choose to develop it from scratch so the book would be accessible to readers not familiar with homological algebra. We only deal with the hereditary case needed to deal with path algebras $\mathbb{C}Q$ for acyclic quivers.

The path algebras $\mathbb{C}Q$ (and even KQ where K is an arbitrary field) are hereditary even when Q is a finite quiver with oriented cycles. In fact, for arbitrary Q and an ideal I in KQ, contained in the square of the ideal generated by the arrows, the category of KQ/I-modules is hereditary if and only if $I = 0$. This is proved in [**43**, Section 8.2]. The general theory of derived functors $\mathrm{Ext}^n(V, W)$ of functors $\mathrm{Hom}(V, W)$ for categories of modules is described for example in a book [**70**] of Rotman. A more sophisticated homological algebra approach, related to derived categories, which will be needed when dealing with cluster categories, can be found in the book [**44**] of Gelfand and Manin.