Contents

Preface xi

Chapter 1. A Crash Course in Commutative Algebra 1
§1.1. Basic algebra 1
§1.2. Field extensions 6
§1.3. Modules 8
§1.4. Localization 9
§1.5. Noetherian rings and factorization 10
§1.6. Primary decomposition 13
§1.7. Integral extensions 16
§1.8. Dimension 19
§1.9. Depth 20
§1.10. Normal rings and regular rings 22

Chapter 2. Affine Varieties 27
§2.1. Affine space and algebraic sets 27
§2.2. Regular functions and regular maps of affine algebraic sets 33
§2.3. Finite maps 40
§2.4. Dimension of algebraic sets 42
§2.5. Regular functions and regular maps of quasi-affine varieties 48
§2.6. Rational maps of affine varieties 58

Chapter 3. Projective Varieties 63
§3.1. Standard graded algebras 63

§3.2. Projective varieties 67
§3.3. Grassmann varieties 73
§3.4. Regular functions and regular maps of quasi-projective varieties 74

Chapter 4. Regular and Rational Maps of Quasi-projective Varieties 87
§4.1. Criteria for regular maps 87
§4.2. Linear isomorphisms of projective space 90
§4.3. The Veronese embedding 91
§4.4. Rational maps of quasi-projective varieties 93
§4.5. Projection from a linear subspace 95

Chapter 5. Products 99
§5.1. Tensor products 99
§5.2. Products of varieties 101
§5.3. The Segre embedding 105
§5.4. Graphs of regular and rational maps 106

Chapter 6. The Blow-up of an Ideal 111
§6.1. The blow-up of an ideal in an affine variety 111
§6.2. The blow-up of an ideal in a projective variety 120

Chapter 7. Finite Maps of Quasi-projective Varieties 127
§7.1. Affine and finite maps 127
§7.2. Finite maps 131
§7.3. Construction of the normalization 135

Chapter 8. Dimension of Quasi-projective Algebraic Sets 139
§8.1. Properties of dimension 139
§8.2. The theorem on dimension of fibers 141

Chapter 9. Zariski's Main Theorem 147

Chapter 10. Nonsingularity 153
§10.1. Regular parameters 153
§10.2. Local equations 155
§10.3. The tangent space 156
§10.4. Nonsingularity and the singular locus 159
§10.5. Applications to rational maps 165

§10.6. Factorization of birational regular maps of nonsingular surfaces 168
§10.7. Projective embedding of nonsingular varieties 170
§10.8. Complex manifolds 175

Chapter 11. Sheaves 181
§11.1. Limits 181
§11.2. Presheaves and sheaves 185
§11.3. Some sheaves associated to modules 196
§11.4. Quasi-coherent and coherent sheaves 200
§11.5. Constructions of sheaves from sheaves of modules 204
§11.6. Some theorems about coherent sheaves 209

Chapter 12. Applications to Regular and Rational Maps 221
§12.1. Blow-ups of ideal sheaves 221
§12.2. Resolution of singularities 225
§12.3. Valuations in algebraic geometry 228
§12.4. Factorization of birational maps 232
§12.5. Monomialization of maps 236

Chapter 13. Divisors 239
§13.1. Divisors and the class group 240
§13.2. The sheaf associated to a divisor 242
§13.3. Divisors associated to forms 249
§13.4. Calculation of some class groups 249
§13.5. The class group of a curve 254
§13.6. Divisors, rational maps, and linear systems 259
§13.7. Criteria for closed embeddings 264
§13.8. Invertible sheaves 269
§13.9. Transition functions 271

Chapter 14. Differential Forms and the Canonical Divisor 279
§14.1. Derivations and Kähler differentials 279
§14.2. Differentials on varieties 283
§14.3. n-forms and canonical divisors 286

Chapter 15. Schemes 289
§15.1. Subschemes of varieties, schemes, and Cartier divisors 289
§15.2. Blow-ups of ideals and associated graded rings of ideals 293

§15.3. Abstract algebraic varieties 295
§15.4. Varieties over nonclosed fields 296
§15.5. General schemes 296

Chapter 16. The Degree of a Projective Variety 299

Chapter 17. Cohomology 307
§17.1. Complexes 307
§17.2. Sheaf cohomology 308
§17.3. Čech cohomology 310
§17.4. Applications 312
§17.5. Higher direct images of sheaves 320
§17.6. Local cohomology and regularity 325

Chapter 18. Curves 333
§18.1. The Riemann-Roch inequality 334
§18.2. Serre duality 335
§18.3. The Riemann-Roch theorem 340
§18.4. The Riemann-Roch problem on varieties 343
§18.5. The Hurwitz theorem 345
§18.6. Inseparable maps of curves 348
§18.7. Elliptic curves 351
§18.8. Complex curves 358
§18.9. Abelian varieties and Jacobians of curves 360

Chapter 19. An Introduction to Intersection Theory 365
§19.1. Definition, properties, and some examples of intersection numbers 366
§19.2. Applications to degree and multiplicity 375

Chapter 20. Surfaces 379
§20.1. The Riemann-Roch theorem and the Hodge index theorem on a surface 379
§20.2. Contractions and linear systems 383

Chapter 21. Ramification and Étale Maps 391
§21.1. Norms and Traces 392
§21.2. Integral extensions 393
§21.3. Discriminants and ramification 398

§21.4. Ramification of regular maps of varieties 406
§21.5. Completion 408
§21.6. Zariski's main theorem and Zariski's subspace theorem 413
§21.7. Galois theory of varieties 421
§21.8. Derivations and Kähler differentials redux 424
§21.9. Étale maps and uniformizing parameters 426
§21.10. Purity of the branch locus and the Abhyankar-Jung theorem 433
§21.11. Galois theory of local rings 438
§21.12. A proof of the Abhyankar-Jung theorem 441

Chapter 22. Bertini's Theorems and General Fibers of Maps 451
§22.1. Geometric integrality 452
§22.2. Nonsingularity of the general fiber 454
§22.3. Bertini's second theorem 457
§22.4. Bertini's first theorem 458

Bibliography 469

Index 477