Chapter 12

Nonlinear Wave Equations

12.1. INTRODUCTION

This final chapter studies the existence (or sometimes nonexistence) of solutions to the initial-value problem for the semilinear wave equation

$$u_{tt} - \Delta u + f(u) = 0. \tag{1}$$

The linear case that $f(u) = m^2 u$ is the *Klein–Gordon* equation. We will also discuss certain mildly quasilinear wave equations having the form

$$u_{tt} - \Delta u + f(Du, u_t, u) = 0, \tag{2}$$

where as usual we write $Du = D_x u$ for the gradient in the x-variables.

We follow the custom of putting the nonlinearity on the left of the equals sign in (1) and (2): this simplifies some later formulas a bit. More complicated quasilinear wave equations, in which the coefficients of the second-order derivatives depend on Du, u_t, u, are beyond the scope of this book.

12.1.1. Conservation of energy. Considering first the semilinear PDE (1), we hereafter set

$$F(z) := \int_0^z f(w)\,dw \qquad (z \in \mathbb{R}).$$

Then $F(0) = 0$ and $F' = f$. We recall from §8.6.2 that the *energy* of a solution u of (1) at time $t \geq 0$ is

$$E(t) := \int_{\mathbb{R}^n} \frac{1}{2}(u_t^2 + |Du|^2) + F(u)\,dx$$

and that this energy is conserved:

THEOREM 1 (Conservation of energy). *Assume that u is a smooth solution of the semilinear wave equation* (1) *and that $u(\cdot,t)$ has compact support in space for each time t. Then*

$$t \mapsto E(t) \quad \textit{is constant.}$$

Proof. We calculate

$$\dot{E}(t) = \int_{\mathbb{R}^n} u_t u_{tt} + Du \cdot Du_t + f(u)u_t \, dx = \int_{\mathbb{R}^n} u_t(u_{tt} - \Delta u + f(u)) \, dx = 0,$$

where $\dot{} = \frac{d}{dt}$. □

The integration by parts in this proof is valid, since u has compact support in space for each time. In many subsequent proofs we will similarly integrate by parts, implicitly relying upon our solution's vanishing for large $|x|$ to justify the computation.

12.1.2. Finite propagation speed. Recalling the domain of dependence calculation for the linear wave equation in §2.4.3, we reintroduce the backwards wave cone:

DEFINITION. Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$ and define the *backwards wave cone* with apex (x_0, t_0):

$$K(x_0,t_0) := \{ (x,t) \mid 0 \le t \le t_0, |x - x_0| \le t_0 - t \}.$$

The curved part of the boundary of $K(x_0,t_0)$ is

$$\Gamma(x_0,t_0) := \{ (x,t) \mid 0 \le t \le t_0, |x - x_0| = t_0 - t \}.$$

THEOREM 2 (Flux estimate for semilinear wave equation). *Assume that u is a smooth solution of the semilinear wave equation* (1).

(i) *For each point $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ we have the identity*

$$\frac{1}{\sqrt{2}} \int_{\Gamma(x_0,t_0)} \frac{1}{2} |u_t \nu - Du|^2 + F(u) \, dS = e(0), \tag{3}$$

where $\nu := \frac{x - x_0}{|x - x_0|}$ and

$$e(t) := \int_{B(x_0,t_0-t)} \frac{1}{2}(u_t^2 + |Du|^2) + F(u) \, dx \quad (0 \le t \le t_0).$$

(ii) *If*

$$F \ge 0$$

and

$$u(\cdot,0), u_t(\cdot,0) \equiv 0 \quad \textit{within } B(x_0,t_0),$$

then $u \equiv 0$ within the cone $K(x_0,t_0)$.

The expression on the left-hand side of (3) is the *energy flux* through the curved surface $\Gamma(x_0, t_0)$.

Proof.

1. We compute that

$$
\begin{aligned}
\dot{e}(t) &= \int_{B(x_0,t_0-t)} u_t u_{tt} + Du \cdot Du_t + f(u)u_t \, dx \\
&\qquad - \int_{\partial B(x_0,t_0-t)} \frac{1}{2}(u_t^2 + |Du|^2) + F(u)\, dS \\
&= \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial \nu} u_t - \frac{1}{2}(u_t^2 + |Du|^2) - F(u)\, dS \\
&= -\int_{\partial B(x_0,t_0-t)} \frac{1}{2}\left|u_t \nu - Du\right|^2 + F(u)\, dS,
\end{aligned} \tag{4}
$$

since

$$
\left|u_t \nu - Du\right|^2 = u_t^2 - 2u_t \frac{\partial u}{\partial \nu} + |Du|^2.
$$

Now integrate in time between 0 and t_0 to derive (3). Notice that the factor $\frac{1}{\sqrt{2}}$ appears when we switch to integration over $\Gamma(x_0, t_0)$, since this surface is tilted at constant angle $\frac{\pi}{4}$ above $B(x_0, t_0) \times \{t = 0\}$.

2. If $u(\cdot, 0), u_t(\cdot, 0) \equiv 0$ on $B(x_0, t_0)$, then $e(0) = 0$ since $F(0) = 0$. As $F \geq 0$, it follows from (4) that $e \equiv 0$. We deduce that $u_t, Du \equiv 0$, and therefore $u \equiv 0$, within the cone $K(x_0, t_0)$. □

Although the mildly quasilinear wave equation (2) does not in general have a conserved energy, we can nevertheless adapt the previous proof to show finite propagation speed.

THEOREM 3 (Domain of dependence)**.** *Assume that*

$$
f(0, 0, 0) = 0
$$

and that u is a smooth solution of the quasilinear wave equation (2)*. If*

$$
u(\cdot, 0), u_t(\cdot, 0) \equiv 0 \quad \textit{within } B(x_0, t_0),
$$

then $u \equiv 0$ *within the cone* $K(x_0, t_0)$*.*

So any disturbance originating outside $B(x_0, t_0)$ does not affect the solution within $K(x_0, t_0)$. Consequently the effects of nonzero initial data propagate with speed at most one.

Proof. Define

$$e(t) := \frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2 + |Du|^2 + u^2\,dx \quad (0 \le t \le t_0).$$

Then

$$\begin{aligned}\dot{e}(t) &= \int_{B(x_0,t_0-t)} u_t u_{tt} + Du \cdot Du_t + u u_t\,dx \\ &\qquad\qquad - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} u_t^2 + |Du|^2 + u^2\,dS \\ &= \int_{B(x_0,t_0-t)} u_t(u_{tt} - \Delta u + u)\,dx \\ &\quad + \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial \nu} u_t\,dS - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} u_t^2 + |Du|^2 + u^2\,dS \\ &\le \int_{B(x_0,t_0-t)} u_t(-f(Du,u_t,u) + u)\,dx.\end{aligned}$$

Since $f(0,0,0) = 0$ and u is smooth,

$$|f(Du,u_t,u)| \le C(|Du| + |u_t| + |u|)$$

for some constant C depending upon $\|Du, u_t, u\|_{L^\infty}$. We conclude that

$$\dot{e}(t) \le C\int_{B(x_0,t_0-t)} u_t^2 + |Du|^2 + u^2\,dx = Ce(t).$$

As $e(0) = 0$, Gronwall's inequality (§B.2) implies $e \equiv 0$. Therefore $u \equiv 0$ within the cone $K(x_0, t_0)$. □

12.2. EXISTENCE OF SOLUTIONS

We devote this section to proving existence theorems for solutions of the mildly quasilinear initial-value problem

$$\text{(1)} \qquad \begin{cases} u_{tt} - \Delta u + f(Du, u_t, u) = 0 & \text{in } \mathbb{R}^n \times (0,T] \\ u = g,\ u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Similarly to §7.2, we say that a function $\mathbf{u} \in L^2(0,T; H^1_{\text{loc}}(\mathbb{R}^n))$, with

$$\mathbf{u}' \in L^2(0,T; L^2_{\text{loc}}(\mathbb{R}^n))$$

and $\mathbf{u}'' \in L^2(0,T; H^{-1}_{\text{loc}}(\mathbb{R}^n))$, is a weak solution of the initial-value problem (1) provided $\mathbf{u}(0) = g, \mathbf{u}'(0) = h$,

$$\mathbf{f} := -f(D\mathbf{u}, \mathbf{u}', \mathbf{u}) \in L^2(0,T; L^2_{\text{loc}}(\mathbb{R}^n)),$$

and

$$\langle \mathbf{u}'', v\rangle + B[\mathbf{u}, v] = (\mathbf{f}, v)$$

for each $v \in H^1(\mathbb{R}^n)$ with compact support and a.e. time $0 \le t \le T$. Here $B[u,v] := \int_{\mathbb{R}^n} Du \cdot Dv\, dx$.

We always assume

$$f(0,0,0) = 0.$$

12.2.1. Lipschitz nonlinearities. We start with a strong assumption on the nonlinearity, namely that

$$f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ is Lipschitz continuous.} \tag{2}$$

THEOREM 1 (Existence and uniqueness)**.**

(i) *Assume* $g \in H^1_{loc}(\mathbb{R}^n)$, $h \in L^2_{loc}(\mathbb{R}^n)$. *Then for each* $T > 0$ *there exists a unique weak solution* $\mathbf{u}$ *of the initial-value problem* (1).

(ii) *If in addition* $g \in H^2_{loc}(\mathbb{R}^n)$, $h \in H^1_{loc}(\mathbb{R}^n)$, *then*

$$\begin{cases} \mathbf{u} \in L^\infty((0,T); H^2_{loc}(\mathbb{R}^n)), \\ \mathbf{u}' \in L^\infty((0,T); H^1_{loc}(\mathbb{R}^n)), \\ \mathbf{u}'' \in L^\infty((0,T); L^2_{loc}(\mathbb{R}^n)). \end{cases}$$

Proof.

1. We will first suppose that $T > 0$ is sufficiently small, as determined below. Given $R >> 1$, let us consider first the initial/boundary-value problem

$$\begin{cases} u_{tt} - \Delta u + f(Du, u_t, u) = 0 & \text{in } B(0,R) \times (0,T] \\ u = 0 & \text{on } \partial B(0,R) \times [0,T] \\ u = g,\ u_t = h & \text{on } B(0,R) \times \{t = 0\}. \end{cases} \tag{3}$$

We temporarily also assume $g \in H^1_0(B(0,R))$.

Introduce the space of functions

$$X := \{\, \mathbf{u} \in L^\infty(0,T; H^1_0(B(0,R))) \mid \mathbf{u}' \in L^\infty(0,T; L^2(B(0,R)))\,\},$$

with the norm

$$\|\mathbf{u}\| := \operatorname*{ess\,sup}_{0 \le t \le T} (\|\mathbf{u}(t)\|_{H^1_0(B(0,R))} + \|\mathbf{u}'(t)\|_{L^2(B(0,R))}).$$

Given $\mathbf{v} \in X$, we hereafter write $\mathbf{u} = A[\mathbf{v}]$ to mean that $\mathbf{u} \in X$ is the unique weak solution of the linear problem

$$\begin{cases} u_{tt} - \Delta u = -f(Dv, v_t, v) & \text{in } B(0,R) \times (0,T) \\ u = 0 & \text{on } \partial B(0,R) \times (0,T) \\ u = g,\ u_t = h & \text{on } B(0,R) \times \{t = 0\}. \end{cases} \tag{4}$$

This weak solution exists according to Theorems 3–5 in §7.2.

Suppose that we are given also $\hat{\mathbf{v}} \in X$, and likewise write $\hat{\mathbf{u}} = A[\hat{\mathbf{v}}]$. Put $\mathbf{w} := \mathbf{u} - \hat{\mathbf{u}}$. Then $\mathbf{w}$ is the unique weak solution of

$$\begin{cases} w_{tt} - \Delta w = f(D\hat{v}, \hat{v}_t, \hat{v}) - f(Dv, v_t, v) & \text{in } B(0,R)\times(0,T) \\ w = 0 & \text{on } \partial B(0,R)\times(0,T) \\ w = 0,\ w_t = 0 & \text{on } B(0,R)\times\{t=0\}. \end{cases}$$

Consequently estimate (50) from §7.2 provides us with the bound

$$\|\mathbf{w}\| \le C\|f(D\hat{\mathbf{v}}, \hat{\mathbf{v}}', \hat{\mathbf{v}}) - f(D\mathbf{v}, \mathbf{v}', \mathbf{v})\|_{L^2(0,T;L^2(B(0,R)))}.$$

In view of the Lipschitz continuity of f, it follows that

$$\|\mathbf{w}\|^2 \le C\int_0^T \int_{B(0,R)} |D\mathbf{v} - D\hat{\mathbf{v}}|^2 + |\mathbf{v}' - \hat{\mathbf{v}}'|^2 + |\mathbf{v} - \hat{\mathbf{v}}|^2\, dxdt \le CT\|\mathbf{v} - \hat{\mathbf{v}}\|^2.$$

Since $\mathbf{w} = \mathbf{u} - \hat{\mathbf{u}} = A[\mathbf{v}] - A[\hat{\mathbf{v}}]$, we deduce that

$$\|A[\mathbf{v}] - A[\hat{\mathbf{v}}]\| \le \frac{1}{2}\|\mathbf{v} - \hat{\mathbf{v}}\|$$

provided $T > 0$ is small enough, depending only upon the Lipschitz constant of f. Banach's Theorem (§9.2.1) now implies the existence of a unique fixed point $\mathbf{u} \in X$, which is the unique weak solution of (3).

2. In particular we may assume that $T < 1$. Let $S = R - 1$. Then the finite propagation speed (Theorem 3 in §12.1) implies that the solution within the cylinder $B(0,S) \times [0,T]$ depends only upon the initial data g, h restricted to $B(0, S+T) \subset\subset B(0,R)$. Consequently our temporary assumption that $g \in H_0^1(B(0,R))$ does not matter, since we can multiply g by a cutoff function vanishing near $\partial B(0,R)$ without affecting the solution within $B(0,S)\times[0,T]$.

Suppose now that we repeat the above construction for another large radius $\hat{R} > R > 1$, to build a weak solution $\hat{\mathbf{u}}$ of (3) (with $\hat{R}$ replacing R). Then owing to uniqueness and finite propagation speed, we have

$$\hat{\mathbf{u}} \equiv \mathbf{u} \quad \text{on } B(0, R-T)\times[0,T].$$

Consequently, we can construct solutions $\mathbf{u}_k$ for a sequence of radii $R_k \to \infty$, and these solutions will exist and agree on any compact subset of $\mathbb{R}^n\times[0,T]$, for sufficiently large k. The common value of these solutions for large k determines our unique weak solution $\mathbf{u}$ of (1) for times $0 \le t \le T$.

We have therefore built a unique solution of (1) on $\mathbb{R}^n \times [0,T]$ provided $T > 0$ is sufficiently small. We then extend the solution to the time intervals $[T, 2T]$, $[2T, 3T]$, etc., to construct a unique weak solution existing for all time.

3. Select $k \in \{1, \dots, n\}$ and let $\tilde{\mathbf{u}} := D_k^h \mathbf{u}$ denote a corresponding difference quotient of $\mathbf{u}$ (§5.8.2). Then $\tilde{\mathbf{u}}$ is the weak solution of

$$\begin{cases} \tilde{u}_{tt} - \Delta \tilde{u} + \mathbf{b} \cdot D\tilde{u} + c\tilde{u} + du_t = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ \tilde{u} = \tilde{g},\ \tilde{u}_t = \tilde{h} & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases}$$

where $\tilde{g} = D_k^h g$, $\tilde{h} = D_k^h h$ and

$$\begin{cases} b^j = \displaystyle\int_0^1 f_{p_j}(sDu(x+he_k,t) + (1-s)Du(x,t), su_t(x+he_k,t) \\ \qquad + (1-s)u_t(x,t), su(x+he_k,t) + (1-s)u(x,t))\, ds, \\ c := \displaystyle\int_0^1 f_z(\cdots)\, ds,\ d := \int_0^1 f_{p_{n+1}}(\cdots)\, ds. \end{cases}$$

As above, we can estimate for large $R > 0$ that

$$\|\tilde{\mathbf{u}}\| = \operatorname*{ess\,sup}_{0 \le t \le T} (\|\tilde{\mathbf{u}}(t)\|_{H^1(B(0,R))} + \|\tilde{\mathbf{u}}'(t)\|_{L^2(B(0,R))}) \le C,$$

the constant C depending only on $\|\tilde{g}\|_{H^1(B(0,2R))} \le C\|g\|_{H^2(B(0,3R))}$ and $\|\tilde{h}\|_{L^2(B(0,2R))} \le C\|h\|_{H^1((0,3R))}$. The above estimates for $k = 1, \dots, n$ show that $\mathbf{u} \in L^\infty((0,T); H^2_{\text{loc}}(\mathbb{R}^n))$ and $\mathbf{u}' \in L^\infty((0,T); H^1_{\text{loc}}(\mathbb{R}^n))$. Finally, we use the PDE $u_{tt} - \Delta u + f(Du, u_t, u) = 0$ to estimate $\int_{B(0,R)} u_{tt}^2\, dx$ and so conclude that $\mathbf{u}'' \in L^\infty((0,T); L^2_{\text{loc}}(\mathbb{R}^n))$. □

12.2.2. Short time existence. Consider again the problem

$$\begin{cases} u_{tt} - \Delta u + f(Du, u_t, u) = 0 & \text{in } \mathbb{R}^n \times (0,T] \\ u = g,\ u_t = h & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{5}$$

We hereafter drop the restrictive assumption (2) that the nonlinearity f be Lipschitz continuous and instead just suppose that f is a given smooth function. Our goal is proving there is a unique solution, existing for at least some short time interval $[0,T]$. We will need more smoothness on the initial data, requiring $g \in H^k(\mathbb{R}^n)$, $h \in H^{k-1}(\mathbb{R}^n)$ for a possibly large integer k (depending on n).

We first introduce some new estimates for the Sobolev space $H^k(\mathbb{R}^n)$:

THEOREM 2 (Sobolev inequalities for H^k)**.** *Suppose that the functions $u_1, \dots, u_m$ belong to $H^k(\mathbb{R}^n)$, where $k > \frac{n}{2}$.*

(i) *If $|\beta_1| + \cdots + |\beta_m| \le k$, then*

$$\|D^{\beta_1}u_1 \cdots D^{\beta_m}u_m\|_{L^2(\mathbb{R}^n)} \le C \prod_{j=1}^m \|u_j\|_{H^k(\mathbb{R}^n)} \tag{6}$$

for a constant $C = C(n,m,k)$.

(ii) *Let* $f : \mathbb{R}^m \to \mathbb{R}$ *be a smooth function satisfying* $f(0) = 0$. *Then* $f(u_1, \dots, u_m) \in H^k(\mathbb{R}^n)$ *and*

$$\|f(u_1, \dots, u_m)\|_{H^k(\mathbb{R}^n)} \leq \Phi(\|u_1\|_{H^k(\mathbb{R}^n)}, \dots, \|u_m\|_{H^k(\mathbb{R}^n)}), \tag{7}$$

where Φ *is a continuous function, nondecreasing in each argument and depending only upon* f, k, n, m.

Proof.

1. We leave the proof of (6) to the reader: see Problem 12.

2. Let $0 < |\alpha| \leq k$. Then $D^\alpha f(u_1, \dots, u_m)$ can be written as a finite sum of terms of the form

$$AD^{\beta_1} u_{j_1} \cdots D^{\beta_l} u_{j_l},$$

where A depends on the partial derivatives of f of order at most k evaluated at $(u_1, \dots, u_m)$, $l \leq k$, $0 \leq |\beta_j| \leq |\alpha|$, and $\beta_1 + \cdots + \beta_l = \alpha$.

Recall from Theorem 6 in §5.6.3 that $k > \frac{n}{2}$ implies the estimate

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{H^k(\mathbb{R}^n)}.$$

Therefore $\|A\|_{L^\infty}$ is bounded by a term depending only upon f and $\|u_j\|_{H^k}$ ($j = 1, \dots, m$). According then to estimate (6),

$$\|AD^{\beta_1} u_{j_1} \cdots D^{\beta_l} u_{j_l}\|_{L^2}$$

is bounded by an expression involving only $\|u_j\|_{H^k}$, for $j = 1, \dots, m$. Since $f(0) = 0$, we can similarly estimate $\|f(u_1, \dots, u_m)\|_{L^2}$. This establishes (7). □

We next demonstrate that if we select the time $T > 0$ sufficiently small, depending upon the initial data g, h, we can find a solution existing on $\mathbb{R}^n \times (0, T]$:

THEOREM 3 (Short time existence). *Assume* $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is smooth,* $f(0, 0, 0) = 0$. *Suppose also* $g \in H^k(\mathbb{R}^n)$, $h \in H^{k-1}(\mathbb{R}^n)$ *for* $k > \frac{n}{2} + 1$.

There exists a time $T > 0$ *such that the initial-value problem*

$$\begin{cases} u_{tt} - \Delta u + f(Du, u_t, u) = 0 & \text{in } \mathbb{R}^n \times (0, T] \\ u = g, \ u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{8}$$

has a unique weak solution $\mathbf{u}$, *with*

$$\mathbf{u} \in L^\infty(0, T; H^k(\mathbb{R}^n)), \ \mathbf{u}' \in L^\infty(0, T; H^{k-1}(\mathbb{R}^n)).$$

The time T of existence provided by the proof depends in a complicated way upon both f and $\|g\|_{H^k(\mathbb{R}^n)}$, $\|h\|_{H^{k-1}(\mathbb{R}^n)}$ and can be very short if $\|g\|_{H^k(\mathbb{R}^n)}$ and $\|h\|_{H^{k-1}(\mathbb{R}^n)}$ are large. We will see in §12.5 that solutions of even the simpler semilinear wave equation need not exist for all time.

Proof.

1. Let

$$X := \{ \mathbf{u} \in L^\infty(0,T;H^1(\mathbb{R}^n)) \mid \mathbf{u}' \in L^\infty(0,T;L^2(\mathbb{R}^n)) \}$$

with the norm

$$\|\mathbf{u}\| := \operatorname*{ess\,sup}_{0\le t\le T}(\|\mathbf{u}(t)\|_{H^1(\mathbb{R}^n)} + \|\mathbf{u}'(t)\|_{L^2(\mathbb{R}^n)}).$$

We introduce also the stronger norm

$$|||\mathbf{u}||| := \operatorname*{ess\,sup}_{0\le t\le T}(\|\mathbf{u}(t)\|_{H^k(\mathbb{R}^n)} + \|\mathbf{u}'(t)\|_{H^{k-1}(\mathbb{R}^n)}).$$

For $\lambda > 0$ define

$$X_\lambda := \{ \mathbf{u} \in X \mid |||\mathbf{u}||| \le \lambda, \mathbf{u}(0) = g, \mathbf{u}'(0) = h \}.$$

If $\mathbf{v} \in X_\lambda$, we write $\mathbf{u} = A[\mathbf{v}]$ to mean that $\mathbf{u}$ solves the linear initial-value problem

$$\begin{cases} u_{tt} - \Delta u = -f(Dv, v_t, v) & \text{in } \mathbb{R}^n \times (0,T] \\ u = g,\ u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{9}$$

Define

$$E_k(t) := \|\mathbf{u}(t)\|^2_{H^k(\mathbb{R}^n)} + \|\mathbf{u}'(t)\|^2_{H^{k-1}(\mathbb{R}^n)}$$

and

$$F_k(t) := \|\mathbf{v}(t)\|^2_{H^k(\mathbb{R}^n)} + \|\mathbf{v}'(t)\|^2_{H^{k-1}(\mathbb{R}^n)}.$$

2. We claim now that we have the estimate

$$E_k(t) \le E_k(0) + C\int_0^t \Psi(F_k(s))\,ds \qquad (0 \le t \le T) \tag{10}$$

for some continuous and monotone function Ψ depending only upon n, k and f.

To prove (10), let $|\alpha| \le k-1$ and apply $D^\alpha = D^\alpha_x$ to the PDE (9), to discover

$$w_{tt} - \Delta w = -D^\alpha(f(Dv, v_t, v))$$

for $w := D^\alpha u$. Therefore we can use estimate (7), with $k-1$ in place of k and with $m = n+2$, to compute

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^n} (w_t^2 + |Dw|^2)\,dx
&= 2\int_{\mathbb{R}^n} w_t w_{tt} + Dw\cdot Dw_t\,dx\\
&= 2\int_{\mathbb{R}^n} w_t(w_{tt} - \Delta w)\,dx\\
&= -2\int_{\mathbb{R}^n} w_t D^\alpha f(Dv, v_t, v)\,dx\\
&\le \int_{\mathbb{R}^n} w_t^2 + |D^\alpha f(Dv, v_t, v)|^2\,dx\\
&\le \int_{\mathbb{R}^n} w_t^2\,dx + C\Phi^2(\|v_{x_1}\|_{H^{k-1}(\mathbb{R}^n)}, \dots, \|v_{x_n}\|_{H^{k-1}(\mathbb{R}^n)}, \|v_t\|_{H^{k-1}(\mathbb{R}^n)}, \|v\|_{H^{k-1}(\mathbb{R}^n)})\\
&\le \int_{\mathbb{R}^n} w_t^2\,dx + \Psi(F_k(t)),
\end{aligned}$$

for some appropriate function Ψ. Apply Gronwall's inequality and then sum the above over all $|\alpha| \le k-1$ to deduce (10).

3. We assert next that if $\lambda > 0$ is large enough and $T > 0$ is small enough, then

$$A : X_\lambda \to X_\lambda.$$

To see this, observe that (10) implies

$$\|\|\mathbf{u}\|\|^2 \le \|g\|^2_{H^k(\mathbb{R}^n)} + \|h\|^2_{H^{k-1}(\mathbb{R}^n)} + CT\Psi(\|\|\mathbf{v}\|\|^2) \tag{11}$$

for some function Ψ. Let

$$\lambda^2 := 2(\|g\|^2_{H^k(\mathbb{R}^n)} + \|h\|^2_{H^{k-1}(\mathbb{R}^n)})$$

and then fix $T > 0$ so small that

$$CT\Psi(\lambda^2) \le \frac{\lambda^2}{2}.$$

Then (11) forces $\|\|A[\mathbf{v}]\|\|^2 \le \lambda^2$, and consequently $A[\mathbf{v}] = \mathbf{u} \in X_\lambda$.

4. Next we claim that if λ is large enough and T is small enough, then

$$\|A[\mathbf{v}] - A[\hat{\mathbf{v}}]\| \le \frac{1}{2}\|\mathbf{v} - \hat{\mathbf{v}}\| \tag{12}$$

for all $\mathbf{v}, \hat{\mathbf{v}} \in X_\lambda$.

To confirm this, let us write $\mathbf{u} = A[\mathbf{v}]$, $\hat{\mathbf{u}} = A[\hat{\mathbf{v}}]$. Put $\mathbf{w} := \mathbf{u} - \hat{\mathbf{u}}$. Then

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n} |Dw|^2 + w_t^2 + w^2\,dx \\ &= 2\int_{\mathbb{R}^n} w_t(f(D\hat{v}, \hat{v}_t, \hat{v}) - f(Dv, v_t, v) + w)\,dx \\ &\leq \int_{\mathbb{R}^n} w_t^2 + w^2\,dx + C\int_{\mathbb{R}^n} |D\hat{v} - Dv|^2 + |\hat{v}_t - v_t|^2 + |\hat{v} - v|^2\,dx,\end{aligned}$$

the constant C depending on $\|D\hat{v}, \hat{v}_t, \hat{v}, Dv, v_t, v\|_{L^\infty}$. This quantity is bounded since the functions $\mathbf{v}, \hat{\mathbf{v}}$ belong to X_λ and $k > \frac{n}{2} + 1$. Invoking Gronwall's inequality, we deduce that

$$\begin{aligned}\max_{0\leq t\leq T}\int |Dw|^2 + |w_t|^2 + w^2\,dx \\ &\leq C\int_0^T\int_{\mathbb{R}^n} |Dv - D\hat{v}|^2 + |v_t - \hat{v}_t|^2 + |v - \hat{v}|^2\,dxdt \\ &\leq CT\|v - \hat{v}\|^2.\end{aligned}$$

We can now select T small enough to ensure (12).

5. Select any $\mathbf{u}_0 \in X_\lambda$. According to the proof of Banach's Theorem from §9.2.1, if we inductively define $\mathbf{u}_{k+1} := A[\mathbf{u}_k]$ for $k = 0, \ldots$, then $\mathbf{u}_k \to \mathbf{u}$ in X and

$$A[\mathbf{u}] = \mathbf{u}.$$

Furthermore, since $|||\mathbf{u}_k||| \leq \lambda$, we have $\mathbf{u} \in X_\lambda$. Uniqueness follows from (12). □

Note carefully the strategy of this proof. We showed that for small $T > 0$ the operator A is a strict contraction in the weaker norm $\|\cdot\|$ and also preserves certain estimates in the stronger norm $|||\cdot|||$. Consequently the iteration scheme from Banach's Theorem provides a sequence that converges in X, to a fixed point that actually lies in the better space X_λ. We did not have to show that A is a strict contraction in the stronger norm.

12.3. SEMILINEAR WAVE EQUATIONS

This section and the next section discuss the initial-value problem for semilinear wave equations:

$$\text{(1)} \qquad \begin{cases} u_{tt} - \Delta u + f(u) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

We will prove much stronger existence theorems than those in §12.2 not only because the nonlinear term $f(u)$ is simpler than $f(u, Du, u_t)$ but also because we have for (1) the conserved energy functional

$$E(t) := \int_{\mathbb{R}^n} \frac{1}{2}(u_t^2 + |Du|^2) + F(u)\,dx,$$

unavailable for the general quasilinear wave equations. Our main goal is discovering when solutions exist for all times $t \geq 0$.

12.3.1. Sign conditions. Our first existence theorem holds for nonlinearities such that $f(z)$ and z have the same sign.

THEOREM 1 (Sign condition on f). *Suppose f is smooth and*

$$zf(z) \geq 0 \qquad (z \in \mathbb{R}). \tag{2}$$

Assume $g \in H^1(\mathbb{R}^n)$, $h \in L^2(\mathbb{R}^n)$, $F(g) \in L^1(\mathbb{R}^n)$.

Then the initial-value problem (1) *has a global weak solution* $\mathbf{u}$ *existing for all times, with*

$$\begin{cases} \mathbf{u} \in L^2_{loc}((0,\infty); L^2(\mathbb{R}^n)) \cap L^\infty((0,\infty); H^1(\mathbb{R}^n)) \\ \mathbf{u}' \in L^\infty((0,\infty); L^2(\mathbb{R}^n)), F(\mathbf{u}) \in L^\infty((0,\infty); L^1(\mathbb{R}^n)). \end{cases}$$

Furthermore, we have the energy inequality

$$E(t) \leq E(0) \qquad \textit{for all times } t \geq 0. \tag{3}$$

Note that we do not assert equality in (3): our solution is not known to be smooth enough for us to calculate rigorously that $\dot{E} \equiv 0$.

Change of notation. Starting with the following proof, we transition away from boldface notation, indicating a mapping of time t into a space of functions of x, and will instead hereafter regard our solution as a function $u = u(x,t)$ of both variables x and t together.

Proof.

1. According to the sign condition (2), $F(z) = \int_0^z f(w)\,dw$ is nondecreasing for $z \geq 0$ and is nonincreasing for $z \leq 0$. Select a sequence of smooth functions $F_k : \mathbb{R} \to \mathbb{R}$ so that

$$F_k \geq 0,\ F_k \to F \text{ pointwise},\ F_k \leq F,\ F_k \equiv F \text{ on } [-k,k]$$

and $f_k := F_k'$ is Lipschitz continuous, with $zf_k(z) \geq 0$ for all $z \in \mathbb{R}$.

We solve the problems

$$\begin{cases} u_{tt}^k - \Delta u^k + f_k(u^k) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u^k = g,\ u_t^k = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{4}$$

Since f_k is Lipschitz continuous, there exists according to Theorem 1 in §12.2 a solution u^k satisfying

$$\begin{cases} u^k,\ Du^k,\ u^k_t \in C([0,\infty);L^2(\mathbb{R}^n)), \\ D^2_x u^k,\ D_x u^k_t,\ u^k_{tt} \in L^\infty_{\text{loc}}((0,\infty);L^2(\mathbb{R}^n)). \end{cases}$$

2. This is enough regularity to allow us to calculate for almost every time t that

$$\dot{E}_k(t) = \frac{d}{dt}\int_{\mathbb{R}^n} \frac{1}{2}((u^k_t)^2 + |Du^k|^2) + F_k(u^k)\,dx = 0.$$

Therefore

$$E_k(t) = E_k(0) = \int_{\mathbb{R}^n} \frac{1}{2}(h^2 + |Dg|^2) + F_k(g)\,dx. \tag{5}$$

Now $F_k(g) \to F(g)$ pointwise as $k \to \infty$, and $0 \le F_k(g) \le F(g)$. Since $F(g) \in L^1$, we can apply the Dominated Convergence Theorem to deduce

$$E^k(0) \to \int_{\mathbb{R}^n} \frac{1}{2}(|Dg|^2 + |h|^2) + F(g)\,dx = E(0).$$

Since $F_k \ge 0$, we also have the bound

$$\max_{0\le t\le T} \|u^k_t, Du^k\|_{L^2(\mathbb{R}^n)} \le C \tag{6}$$

and consequently for each time $T > 0$

$$\max_{0\le t\le T} \|u^k\|_{L^2(\mathbb{R}^n)} \le C. \tag{7}$$

We extract a subsequence (which we reindex and still denote "u^k") such that

$$\begin{cases} u^k \to u \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^n\times(0,\infty)) \text{ and a.e.,} \\ Du^k, u^k_t \rightharpoonup Du, u_t \quad \text{weakly in } L^2_{\text{loc}}(\mathbb{R}^n\times(0,\infty)). \end{cases}$$

3. Next, multiply the PDE (4) by u^k and integrate over $\mathbb{R}^n \times (0,T)$:

$$\int_0^T \int_{\mathbb{R}^n} |Du^k|^2 - (u^k_t)^2 + u^k f_k(u^k)\,dxdt = \int_{\mathbb{R}^n} u^k_t u^k\,dx\Big|_{t=0}^{t=T}.$$

Since $u^k f_k(u^k) \ge 0$, (6) and (7) imply the estimate

$$\int_0^T \int_{\mathbb{R}^n} |u^k f_k(u^k)|\,dxdt \le C. \tag{8}$$

4. We next assert that the functions $\{g^k := f_k(u^k)\}_{k=1}^\infty$ are *uniformly integrable*. This means that for each $\varepsilon > 0$, there exists $\delta > 0$ such that if E is a measurable subset of $\mathbb{R}^n \times (0,T)$ and $|E| \le \delta$, then $\iint_E |g^k|\,dxdt \le \varepsilon$ for all k.

To confirm this, we calculate using (8) that

$$\begin{aligned}\iint_E |g^k|\,dxdt &= \iint_{E\cap\{|u^k|\geq\lambda\}} |f_k(u^k)|\,dxdt + \iint_{E\cap\{|u^k|\leq\lambda\}} |f_k(u^k)|\,dxdt \\ &\leq \frac{1}{\lambda}\iint_E |u^k f_k(u^k)|\,dxdt + |E|\max_{|y|\leq\lambda}|f_k(y)| \\ &\leq \frac{1}{\lambda}C(T) + |E|\max_{|y|\leq\lambda}|f_k(y)|.\end{aligned}$$

Take λ so large that $\frac{C(T)}{\lambda} \leq \frac{\varepsilon}{2}$. Then for all but finitely many k

$$\iint_E |g^k|\,dxdt \leq \frac{\varepsilon}{2} + |E|\max_{|y|\leq\lambda}|f(y)| \leq \varepsilon,$$

provided $|E| \leq \delta$ for $\delta > 0$ sufficiently small.

5. We claim now that

$$f_k(u^k) \to f(u) \text{ in } L^1_{\text{loc}}(\mathbb{R}^n). \tag{9}$$

To see this, again write $g^k := f_k(u^k)$, put $g := f(u)$, and fix $R > 0$, $\varepsilon > 0$. In view of the uniform integrability proved in step 4, we can select $\delta > 0$ so that $|E| \leq \delta$ implies $\int_E |g^k|\,dx \leq \varepsilon$ for $k = 1, \dots$. According to Egoroff's Theorem (§E.2), there exists a measurable set E so that $|E| \leq \delta$ and $g^k \to g$ uniformly on $B(0,R) - E$. Therefore

$$\limsup_{k,l\to\infty}\int_{B(0,R)} |g^k - g^l|\,dx \leq \limsup_{k,l\to\infty}\int_E |g^k - g^l|\,dx \leq 2\varepsilon.$$

This is true for each $\varepsilon > 0$. Consequently $\{g^k\}_{k=1}^\infty$ is a Cauchy sequence in $L^1(B(0,R))$ and so converges to some $\hat{g} \in L^1(B(0,R))$. Since $g^k \to g$ a.e., we deduce $g = \hat{g}$.

6. Because

$$u^k_{tt} - \Delta u^k + f_k(u^k) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty),$$

we deduce upon multiplying by a smooth test function and passing to limits that

$$u_{tt} - \Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty). \tag{10}$$

Furthermore

$$\int_{\mathbb{R}^n} u_t^2 + |Du|^2\,dx \leq \liminf_{k\to\infty}\int_{\mathbb{R}^n} (u^k_t)^2 + |Du^k|^2\,dx;$$

and according to Fatou's Lemma (§E.3),

$$\int_{\mathbb{R}^n} F(u)\,dx \leq \liminf_{k\to\infty}\int_{\mathbb{R}^n} F_k(u^k)\,dx.$$

These last two inequalities together imply $E(t) \leq E(0)$. □

12.3.2. Three space dimensions. Nonlinear wave equations in three space dimensions are physically the most important and turn out to admit useful L^∞ estimates owing to the special form of the solution to the linear nonhomogenous problem provided by the retarded potential formula (44) from §2.4.2.

So let us now look at the initial-value problem

$$\begin{cases} u_{tt} - \Delta u + f(u) = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}^3 \times \{t=0\}. \end{cases} \tag{11}$$

We henceforth always assume that $f : \mathbb{R} \to \mathbb{R}$ is smooth, with $f(0) = 0$, and that $g, h \in C_c^\infty(\mathbb{R}^3)$.

THEOREM 2 (Short time existence and blow-up in L^∞).

(i) *There exists a time* $T > 0$ *and a unique smooth solution* u *of the initial-value problem* (11) *on* $\mathbb{R}^3 \times (0,T)$.

(ii) *If the maximal time* T^* *of existence of this smooth solution is finite, then*

$$\lim_{t \to T^*} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} = \infty. \tag{12}$$

Assertion (12) is important since (unlike Theorem 3 in §12.2.2) it provides a simple criterion for the failure of the solution to exist beyond time T^*. We will see in Theorem 3 below and also in §12.4 that we can sometimes bound the L^∞ norm of solutions and so ensure existence for all time.

Proof.

1. We will look for a solution u having the form

$$u = v + w,$$

where v solves the homogeneous wave equation

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ v = g,\ v_t = h & \text{on } \mathbb{R}^3 \times \{t=0\} \end{cases} \tag{13}$$

and w solves

$$\begin{cases} w_{tt} - \Delta w = -f(u) & \text{in } \mathbb{R}^3 \times (0,\infty) \\ w = w_t = 0 & \text{on } \mathbb{R}^3 \times \{t=0\}. \end{cases}$$

Formula (44) from §2.4.2 lets us write

$$w(x,t) = -\frac{1}{4\pi} \int_{B(x,t)} \frac{f(u(y, t - |y-x|))}{|y-x|}\, dy.$$

Hence our desired solution u must solve the nonlinear integral identity

$$u(x,t) = v(x,t) - \frac{1}{4\pi}\int_{B(x,t)} \frac{f(u^*)}{|x-y|}\,dy \quad (x \in \mathbb{R}^3, t > 0), \tag{14}$$

where for each fixed (x,t) we write

$$u^*(y) := u(y, t - |y-x|) \quad (y \in B(x,t)). \tag{15}$$

2. Introduce the collection of functions

$$X := \{ u \in C([0,T] \times \mathbb{R}^3) \mid u(\cdot, 0) = g,\ \|u - v\|_{L^\infty} \leq 1 \},$$

where v solves the linear, homogeneous problem (13). Since g, h are smooth, so is v. Thus there exists a constant C_1 such that

$$\|u\|_{L^\infty((0,T)\times\mathbb{R}^3)} \leq C_1 \tag{16}$$

for all $u \in X$.

We define the nonlinear mapping $A : X \to C([0,T] \times \mathbb{R}^3)$ by

$$A[u](x,t) := v(x,t) - \frac{1}{4\pi}\int_{B(x,t)} \frac{f(u^*)}{|x-y|}\,dy.$$

Then if u, $\hat{u} \in X$,

$$\begin{aligned}
\|A[u] - A[\hat{u}]\|_{L^\infty((0,T)\times\mathbb{R}^3)} &\leq \sup_{x\in\mathbb{R}^3, 0\leq t\leq T} \left(\frac{1}{4\pi}\int_{B(x,t)} \frac{|f(u^*) - f(\hat{u}^*)|}{|x-y|}\,dy \right) \\
&\leq C \sup_{x\in\mathbb{R}^3, 0\leq t\leq T} \int_{B(x,t)} \frac{|u^* - \hat{u}^*|}{|x-y|}\,dy \\
&\leq C\|u - \hat{u}\|_{L^\infty((0,T)\times\mathbb{R}^3)} \sup_{x\in\mathbb{R}^3, 0\leq t\leq T} \int_{B(x,t)} \frac{dy}{|x-y|} \\
&\leq CT^2 \|u - \hat{u}\|_{L^\infty((0,T)\times\mathbb{R}^3)},
\end{aligned}$$

the constant C in the second line depending upon $\max_{|w|\leq C_1} |f'(w)|$. For the last inequality in this calculation we noted that $\int_{B(0,r)} \frac{dy}{|y|} = Cr^2$.

Fix T so small that A is a strict contraction. Observing also that $A : X \to X$ if T is small, we see from Banach's Theorem (§9.2.1) that A has a unique fixed point u, which consequently solves the integral identities (14).

3. We can apply the same method to estimate the first and even higher derivatives of u. To see this, write $\tilde{u} := D_k^h u$ for a difference quotient, $k = 1, 2, 3$ (§5.8.2). Then

$$\begin{cases} \tilde{u}_{tt} - \Delta\tilde{u} + c\tilde{u} = 0 & \text{in } \mathbb{R}^3 \times (0,T] \\ \tilde{u} = \tilde{g},\ \tilde{u}_t = \tilde{h} & \text{on } \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

where $\tilde{g} := D_k^h g$, $\tilde{h} := D_k^h h$ and $c := \int_0^1 f'(su(x + he_k, t) + (1-s)u(x,t))\,ds$. The function c is bounded, since u is bounded.

As above,

$$\tilde{u}(x,t) = \tilde{v}(x,t) - \frac{1}{4\pi}\int_{B(x,t)} \frac{c^*\tilde{u}^*}{|x-y|}\,dy, \tag{17}$$

where

$$\begin{cases} \tilde{v}_{tt} - \Delta\tilde{v} = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ \tilde{v} = \tilde{g},\ \tilde{v}_t = \tilde{h} & \text{on } \mathbb{R}^3 \times \{t=0\}. \end{cases}$$

In particular, $\tilde{v}$ is smooth, and consequently (17) implies

$$\|\tilde{u}(\cdot,t)\|_{L^\infty} \le C + C\int_0^t \|\tilde{u}(\cdot,s)\|_{L^\infty}\,ds \quad (0 \le t \le T).$$

We invoke Gronwall's inequality (§E.2) next to estimate the L^∞-norms of Du. We can similarly bound u_t. Writing $\tilde{u} = u_{x_k}$ or u_t, we see that $\tilde{u}$ is a weak solution of

$$\tilde{u}_{tt} - \Delta\tilde{u} + f'(u)\tilde{u} = 0.$$

We can now apply our difference quotient argument to this PDE to derive L^∞ bounds on D^2u. By induction we can similarly estimate all the partial derivatives of u in terms of the L^∞-norm of u.

4. Now let T^* denote the maximal time of existence of a smooth solution. If $T^* < \infty$, but (12) fails, we can as above estimate all the derivatives of u on $\mathbb{R}^3 \times [0,T^*)$ and therefore extend the solution beyond this time. □

12.3.3. Subcritical power nonlinearities. We now turn our attention to the nonlinear wave equation in three dimensions with a power-type nonlinearity:

$$\begin{cases} u_{tt} - \Delta u + |u|^{p-1}u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}^3 \times \{t=0\}, \end{cases} \tag{18}$$

the energy for which is

$$E(t) := \int_{\mathbb{R}^3} \frac{1}{2}(u_t^2 + |Du|^2) + \frac{|u|^{p+1}}{p+1}\,dy.$$

We assume $g, h \in C_c^\infty(\mathbb{R}^3)$.

Our goal now is building a global solution for $1 \le p < 5$.

THEOREM 3 (Subcritical existence theorem)**.** *If* $1 < p < 5$, *then there exists a unique global smooth solution of* (18)*.*

Section 12.4 will introduce more sophisticated techniques to handle the critical case $p = 5$.

Proof.

1. In view of Theorem 2, we may assume we have a smooth solution u existing on $\mathbb{R}^3 \times (0,T)$ for some $T > 0$. We need to show that

$$\|u\|_{L^\infty(\mathbb{R}^3\times(0,T))} \leq C$$

for some constant $C = C(T)$.

Let

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ v = g,\ v_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$

so that as in the previous proof

$$u(x,t) = v(x,t) - \frac{1}{4\pi}\int_{B(x,t)} \frac{|u^*|^{p-1}u^*}{|x-y|}\,dy.$$

Recall from (15) our notation that $u^*(y) := u(y, t-|y-x|)$ for $y \in B(x,t)$. Since g, h are smooth, v is bounded on $\mathbb{R}^3 \times [0,T]$. Therefore

$$|u(x,t)| \leq C + I(x,t) \qquad (x \in \mathbb{R}^3, 0 \leq t \leq T) \tag{19}$$

for

$$I(x,t) := \int_{B(x,t)} \frac{|u^*|^p}{|x-y|}\,dy.$$

2. Then

$$I(x,t) \leq \left(\int_{B(x,t)} \frac{|u^*|^2}{|x-y|^2}\,dy\right)^{1/2}\left(\int_{B(x,t)} |u^*|^{2(p-1)}\,dy\right)^{1/2}. \tag{20}$$

Hardy's inequality (Theorem 7 in §5.8.4) implies

$$\int_{B(x,t)} \frac{|u^*|^2}{|y-x|^2}\,dy \leq C\int_{B(x,t)} |Du^*|^2 + \frac{(u^*)^2}{t^2}\,dy. \tag{21}$$

Now

$$Du^* = Du - \nu u_t,$$

for $\nu = \frac{y-x}{|y-x|}$. Consequently the energy flux identity (3) in §12.1.2 implies

$$\int_{B(x,t)} |Du^*|^2\,dy = \frac{1}{\sqrt{2}}\int_{\Gamma(x,t)} |u_t\nu - Du|^2\,dS \leq E(0). \tag{22}$$

Recalling next Poincare's inequality (7) in §5.8.1, we see that

$$\int_{B(x,t)} |u^* - (u^*)_{x,t}|^2\,dy \leq Ct^2\int_{B(x,t)} |Du^*|^2\,dy,$$

for the average $(u^*)_{x,t} := ⨍_{B(x,t)} u^*\,dy$. Hence

$$\frac{1}{t^2}\int_{B(x,t)} (u^*)^2\,dy \leq C\int_{B(x,t)} |Du^*|^2\,dy + Ct|(u^*)_{x,t}|^2. \tag{23}$$

We now compute that

$$
\begin{aligned}
|(u^*)_{x,t}| &\le \frac{C}{t^3}\int_{\Gamma(x,t)} |u|\,dS \le \frac{C}{t^3}\left(\int_{K(x,t)} |u_t|\,dyds + \int_{B(x,t)} |g|\,dy\right)\\
&\le \frac{C}{t^3}\left(\int_{K(x,t)} |u_t|^2\,dyds\right)^{\frac12} |K(x,t)|^{\frac12} + C\|g\|_{L^\infty}\\
&\le \frac{C}{t^{\frac12}} E(0)^{\frac12} + C\|g\|_{L^\infty}.
\end{aligned} \tag{24}
$$

This estimate, combined with (20)-(23), gives

$$
I(x,t) \le C\left(\int_{B(x,t)} |u^*|^{2(p-1)} dy\right)^{1/2}. \tag{25}
$$

3. Suppose now that $1 < p \le 4$. Then $2(p-1) \le 6 = 2^*$, and therefore the Sobolev inequalities (§5.6.3) imply

$$
\|{u^*}^{p-1}\|_{L^2(B(x,t))} \le C\|u^*\|^{p-1}_{H^1(B(x,t))} \le C.
$$

Consequently (19) and (25) yield

$$
|u(x,t)| \le C,
$$

and we have derived the required estimate on $\|u\|_{L^\infty(\mathbb{R}^3\times(0,T))}$.

The next case is $4 < p < 5$. Then $2(p-1) = 2(p-4) + 6 = 2(p-4) + 2^*$, and the foregoing calculations show that

$$
I(x,t) \le C\left(\int_{B(0,t)} |u^*|^{2(p-1)} dy\right)^{1/2} \le C\|u\|^{p-4}_{L^\infty}.
$$

Therefore (19) implies

$$
|u(x,t)| \le C + C\|u\|^{p-4}_{L^\infty} \le C + \frac12\|u\|_{L^\infty},
$$

the last inequality valid since $0 < p-4 < 1$. This again provides a bound on $\|u\|_{L^\infty(\mathbb{R}^3\times(0,T))}$. □

We will see in Problem 15 that the proof can be modified to show for the critical power $p = 5$ that there exists a global smooth solution provided the energy $E(0)$ is small enough. The next section introduces more advanced techniques, to construct a solution even for large energy when $p = 5$.

12.4. CRITICAL POWER NONLINEARITY

We consider now the initial-value problem

$$\begin{cases} u_{tt} - \Delta u + u^5 = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ \qquad u = g,\ u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases} \tag{1}$$

with corresponding energy

$$E(t) := \int_{\mathbb{R}^3} \frac{1}{2}(u_t^2 + |Du|^2) + \frac{u^6}{6}\, dy. \tag{2}$$

We continue to suppose $g,\ h \in C_c^\infty(\mathbb{R}^3)$. Our task is showing that there is a smooth solution, even for this critical power $p = 5$ case for which the methods of the previous section fail.

We will first need more detailed information about the linear wave equation:

THEOREM 1 (Estimates for wave equation in three dimensions)**.** *Let v solve the linear initial-value problem*

$$\begin{cases} v_{tt} - \Delta v = f & \textit{in } \mathbb{R}^3 \times (0, \infty) \\ v = g,\ v_t = h & \textit{on } \mathbb{R}^3 \times \{t = 0\}. \end{cases} \tag{3}$$

For each $T > 0$ we have the estimate

$$\begin{aligned} &\sup_{0\le t\le T} \|v(\cdot,t)\|_{L^6(\mathbb{R}^3)} + \|v\|_{L^4(0,T;L^{12}(\mathbb{R}^3))} \\ &\qquad\qquad \le C(\|Dg\|_{L^2(\mathbb{R}^3)} + \|h\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^1(0,T;L^2(\mathbb{R}^3))}) \end{aligned} \tag{4}$$

for a constant $C = C(T)$.

Proof.

1. Approximating f, g, h by smooth functions if necessary, we may assume v is smooth. We then compute for

$$a(t) := \left(\int_{\mathbb{R}^3} v_t^2 + |Dv|^2\, dx\right)^{\frac{1}{2}}$$

that

$$\begin{aligned} 2\dot{a}(t)a(t) &= 2\int_{\mathbb{R}^3} (v_{tt} - \Delta v)v_t\, dx \\ &= 2\int_{\mathbb{R}^3} f v_t\, dx \le 2a(t)\|f(\cdot,t)\|_{L^2}. \end{aligned}$$

Therefore

$$\sup_{0\le t\le T} \|Dv(\cdot,t)\|_{L^2} \le C(\|Dg\|_{L^2} + \|h\|_{L^2} + \|f\|_{L^1(0,T;L^2)}).$$

Since for $n = 3$ dimensions, $2^* = 6$, the Sobolev inequalities (§5.6.3) provide the stated estimate for $\sup_{0\leq t\leq T} \|v\|_{L^6}$.

2. The proof of the $L^4(0,T;L^{12})$ bound for v is beyond the scope of this textbook. See Sogge [**So**] for details and also Shatah–Struwe [**S-S**]. □

Recalling the energy flux calculation appearing §12.1, we introduce the following

NOTATION. Given $(x_0, T) \in \mathbb{R}^3 \times (0,\infty)$ and $0 < s < T$, write

$$\phi(s) = \phi(s, x_0, T) := \frac{1}{\sqrt{2}} \int_{\Gamma(x_0,t_0)\cap\{s\leq t<T\}} \frac{1}{2}|u_t\nu - Du|^2 + \frac{u^6}{6}\, dS$$

for the energy flux through the curved surface $\Gamma(x_0, T)$ for times between s and T.

THEOREM 2 (L^6-energy flux estimate)**.** *If u is a smooth solution of $u_{tt} - \Delta u + u^5 = 0$ in $\mathbb{R}^3 \times [0, T)$, we have the estimate*

$$\int_{B(x_0,T-s)} u^6\, dx \leq C\phi(s)^{\frac{1}{3}} \tag{5}$$

for each point $x_0 \in \mathbb{R}^3$ and each time $0 \leq s < T$.

Theorem 2 is important since it implies, as we will later see, that the full energy density $\frac{1}{2}(u_t^2 + |Du|^2) + \frac{1}{6}u^6$ cannot "concentrate" near (x_0, T). The proof depends upon a nonlinear variant of the scaling invariance identity introduced for the linear wave equation in §8.6.

Proof.

1. We may assume $x_0 = 0$ and suppose for the time being that $s = 0$. Inspired by Example 4 of §8.6, we multiply the PDE $\Box u + u^5 = 0$ by the multiplier

$$m := (t - T)u_t + x \cdot Du + u.$$

After some rewriting, we derive the Morawetz-type identity

$$p_t - \operatorname{div} \mathbf{q} = -\frac{u^6}{3}, \tag{6}$$

for

$$p := (t - T)\left(\frac{u_t^2 + |Du|^2}{2} + \frac{u^6}{6}\right) + x \cdot Du\, u_t + u\, u_t \tag{7}$$

and

$$\mathbf{q} := ((t - T)u_t + x \cdot Du + u)Du + \left(\frac{u_t^2 - |Du|^2}{2} - \frac{u^6}{6}\right)x. \tag{8}$$

Select a time $0 < \tau < T$. We integrate (6) over the truncated backward wave cone $K(0,T) \cap \{0 \leq t \leq \tau\}$, ending up with three terms corresponding respectively to integrations over the curved side $\Gamma(0,T) \cap \{0 \leq t \leq \tau\}$, the bottom $B(0,T) \times \{t=0\}$ and the top $B(0,T-\tau) \times \{t=\tau\}$. The latter term goes to zero as $\tau \to T$. Thus we discover

$$A - B = -\int_{K(0,T)} \frac{u^6}{3}\,dxdt \leq 0 \tag{9}$$

for

$$A := \frac{1}{\sqrt{2}} \int_{\Gamma(0,T)} p - \mathbf{q} \cdot \nu\, dS, \; B := \int_{B(0,T)} p(\cdot,0)\,dx,$$

where $\nu := \frac{x}{|x|}$.

2. We now claim that

$$A = \frac{1}{\sqrt{2}} \int_{\Gamma(0,T)} (t-T)\left|u_r - u_t + \frac{u}{|x|}\right|^2 dS + \frac{1}{2}\int_{\partial B(0,T)} u^2(\cdot,0)\,dS \tag{10}$$

for $u_r := Du \cdot \frac{x}{|x|}$. To confirm this, we first observe that $|x| = T - t$ on $\Gamma(0,T)$ and then check after a calculation using (7), (8) that

$$p - \mathbf{q}\cdot\nu = -|x|(u_t - u_r)^2 + u(u_t - u_r) \quad \text{on } \Gamma(0,T). \tag{11}$$

We transform the surface integral over the curved surface $\Gamma(0,T)$ to an integral over the ball $B(0,T)$, by putting

$$u^*(y) := u(y, T - |y|). \tag{12}$$

Then $u_r^* = \frac{y}{|y|} \cdot Du^* = u_r - u_t$, and so (11) implies

$$\begin{aligned} A &= -\int_{B(0,T)} |y|(u_r^*)^2 + u^* u_r^*\,dy \\ &= -\int_{B(0,T)} |y|\left(u_r^* + \frac{u^*}{|y|}\right)^2 dy + \int_{B(0,T)} \frac{(u^*)^2}{|y|} + u^* u_r^*\,dy. \end{aligned} \tag{13}$$

Since $\operatorname{div}\left(\frac{y}{|y|}\right) = \frac{2}{|y|}$ in $\mathbb{R}^3$, we can compute

$$\begin{aligned} \int_{B(0,T)} \frac{(u^*)^2}{|y|}\,dy &= \frac{1}{2}\int_{B(0,T)} (u^*)^2 \operatorname{div}\left(\frac{y}{|y|}\right) dy \\ &= -\int_{B(0,T)} u^* u_r^*\,dy + \frac{1}{2}\int_{\partial B(0,T)} (u^*)^2\,dS. \end{aligned}$$

Plugging this identity into (13) and converting back to the original variables gives us the formula (10).

3. Next we assert that

$$B \leq -T \int_{B(0,T)} \frac{u^6}{6}\,dx + \frac{1}{2}\int_{\partial B(0,T)} u^2(\cdot,0)\,dS. \tag{14}$$

To see this, notice first that

$$B = \int_{B(0,T)} (-T)\left(\frac{u_t^2 + |Du|^2}{2} + \frac{u^6}{6}\right) + (x\cdot Du + u)u_t\,dx.$$

Now if $|x| \leq T$, then

$$\begin{aligned}|(x\cdot Du + u)u_t| &\leq \frac{Tu_t^2}{2} + \frac{|x|^2}{2T}\left(Du\cdot\frac{x}{|x|} + \frac{u}{|x|}\right)^2\\ &\leq \frac{Tu_t^2}{2} + \frac{T}{2}\left|Du + \frac{x}{|x|^2}u\right|^2.\end{aligned}$$

Consequently

$$\begin{aligned}B &\leq -T\int_{B(0,T)} \frac{u^6}{6}\,dx + \frac{T}{2}\int_{B(0,T)} \left|Du + \frac{x}{|x|^2}u\right|^2 - |Du|^2\,dx\\ &= -T\int_{B(0,T)} \frac{u^6}{6}\,dx + \frac{T}{2}\int_{B(0,T)} \frac{u^2}{|x|^2} + \frac{2}{|x|}uu_r\,dx.\end{aligned}$$

Since $\operatorname{div}\left(\frac{y}{|y|^2}\right) = \frac{1}{|y|^2}$ in $\mathbb{R}^3$, we calculate as in step 2 that the last integral equals $\frac{1}{2}\int_{\partial B(0,T)} u^2\,dS$. This proves (14).

4. Combining (9), (10) and (14), we deduce that

$$\begin{aligned}T\int_{B(0,t_0)} u^6\,dx &\leq C\int_{\Gamma(0,T)} (T-t)\left|u_r - u_t + \frac{u}{|x|}\right|^2\,dS\\ &\leq CT\phi(0) + C\int_{\Gamma(0,T)} \frac{u^2}{|x|}\,dS \qquad (15)\\ &\leq CT\phi(0) + C\left(\int_{\Gamma(0,T)} u^6\,dS\right)^{1/3}\left(\int_{\Gamma(0,T)} |x|^{-3/2}\,dS\right)^{2/3}.\end{aligned}$$

Since $\int_{\Gamma(0,T)} |x|^{-3/2}\,dS = CT^{3/2}$, we deduce finally that

$$\int_{B(0,T)} u^6\,dx \leq C\phi(0)^{1/3}.$$

This is the inequality (5) for $s = 0$, and the general case that $0 \leq s < T$ follows similarly. □

We present next the major assertion that our critical power wave equation has a smooth solution existing globally in time:

THEOREM 3 (Global existence for u^5 nonlinearity)**.** *Assume that f, g are smooth functions with compact support.*

Then there is a unique smooth solution of the initial-value problem (1) *existing for all time.*

Proof.

1. Assume that $0 < T < \infty$ and that u is a smooth, compactly supported solution existing on $\mathbb{R}^3 \times [0, T)$. We will show that

$$u \in L^\infty(\mathbb{R}^3 \times [0, T)), \tag{16}$$

in which case our results from §12.3 imply that u can be smoothly extended beyond time T.

2. We first assert that if we knew

$$u \in L^4(0, T; L^{12}(\mathbb{R}^3)), \tag{17}$$

then (16) would hold. To see this, differentiate the PDE $\Box u + u^5 = 0$ to find $\Box v + 5u^4 v = 0$, for $v := u_{x_k}$ $(k = 1, 2, 3)$. The linear estimate (4) provides the bound

$$\sup_{0 \le t \le \tau} \|Du\|_{L^6} \le C + C \int_0^\tau \|u^4 Du\|_{L^2}\, dt$$

for $0 \le \tau \le T$.

Now

$$\|u^4 Du\|_{L^2} \le \|Du\|_{L^6} \|u\|_{L^{12}}^4;$$

and consequently

$$\sup_{0 \le t \le \tau} \|Du\|_{L^6} \le C + C \sup_{0 \le t \le \tau} \|Du\|_{L^6} \int_0^\tau \|u\|_{L^{12}}^4\, dt \le C + \frac{1}{2} \sup_{0 \le t \le \tau} \|Du\|_{L^6},$$

provided $\tau > 0$ is so small that

$$\int_0^\tau \|u\|_{L^{12}}^4\, dt \le \delta := \frac{1}{2C}.$$

Since $u \in L^4(0, T; L^{12})$, we can select $\tau = \frac{T}{m} > 0$ such that

$$\int_{k\tau}^{(k+1)\tau} \|u\|_{L^{12}}^4\, dt \le \delta \quad \text{for } k = 1, \dots, m-1.$$

We can then iteratively apply the foregoing argument on the time intervals $[0, \tau], [\tau, 2\tau], \dots, [(m-1)\tau, T)$, eventually to deduce

$$\sup_{0 \le t < T} \|Du\|_{L^6} \le C.$$

Since u has compact support, this implies $\|u\|_{L^\infty((0,T)\times\mathbb{R}^3)} \le C$ and so (16) is valid.

3. Next we show for each point $x_0 \in \mathbb{R}^3$ that

$$\lim_{s\to T} \int_{B(x_0,T-s)} u^6\,dx = 0. \tag{18}$$

Indeed the key estimate (5) provided by Theorem 2 asserts

$$\int_{B(x_0,T-s)} u^6\,dx \le C\phi(s)^{\frac{1}{3}}$$

for the energy flux $\phi(s)$. But according to Theorem 2 in §12.2, $\phi(s) = e(s) - \lim_{r\to 0} e(r)$ for

$$e(t) := \int_{B(x_0,T-t)} \frac{1}{2}(u_t^2 + |Du|^2) + \frac{u^6}{6}\,dx.$$

Since $t \mapsto e(t)$ is nonincreasing, it follows that $\lim_{s\to T}\phi(s) = 0$ and consequently that (18) holds. (Notice in this argument that we do not need to know that $\lim_{r\to 0} e(r) = 0$, although the next step shows this is in fact true.)

4. We assert now that (18) implies

$$\lim_{s\to T} \int_{B(x_0,T-s)} u_t^2 + |Du|^2\,dx = 0 \tag{19}$$

for each point x_0. To confirm this, let us first show that (18) implies

$$u \in L^4(0,T;L^{12}(B(x_0,T-t))). \tag{20}$$

To prove this, we first observe that owing to the linear estimate (4),

$$\|u\|_{L^4(s,\tau,L^{12}(B(x_0,T-t)))} \le C + C\int_s^\tau \|u^5\|_{L^2(B(x_0,T-t))}\,dt \tag{21}$$

for each $s \le \tau < T$. The interpolation inequality $\|u\|_{L^{10}} \le \|u\|_{L^6}^{1/5}\|u\|_{L^{12}}^{4/5}$ (§B.2) implies

$$\|u^5\|_{L^2} = \|u\|_{L^{10}}^5 \le \|u\|_{L^6}\|u\|_{L^{12}}^4.$$

Consequently,

$$\int_s^\tau \|u^5\|_{L^2(B(x_0,T-t))}\,dt \le \sup_{s\le t\le T}\|u\|_{L^6(B(x_0,T-t))}\int_s^\tau \|u\|_{L^{12}(B(x_0,T-t))}^4\,dt.$$

It follows then from (18) and (21) that given any $\epsilon > 0$ we can select a time $0 < s < T$ such that

$$\|u\|_{L^4(s,\tau;L^{12}(B(x_0,T-t)))} \le C + \epsilon\|u\|_{L^4(s,\tau;L^{12}(B(x_0,T-t)))}^4$$

for all $s \le \tau < T$. This expression has the form

$$\phi(\tau) \le C_1 + \epsilon\phi(\tau)^4 \quad (s \le \tau < T)$$

with $\phi(s) = 0$. It follows that

$$\phi(\tau) \le 2C_1 \quad (s \le \tau < T)$$

provided that ϵ is sufficiently small. This proves (20).

But then (20) lets us apply the method of step 2 to $v := u_{x_k}$ ($k = 1, 2, 3$) and $v := u_t$. This reasoning provides the bound

$$\sup_{0\le t<T} \|Du, u_t\|_{L^6(B(x_0,T-t))} \le C,$$

which in turn gives us (19).

5. Next we improve slightly on (20): we claim that for each $x_0 \in \mathbb{R}^3$ there exists $\delta = \delta(x_0) > 0$ such that

$$u \in L^4(0, T; L^{12}(B(x_0, T - t + \delta))). \tag{22}$$

To prove this, notice first that (18) and (19) together imply that given any constant $\epsilon > 0$, we can find $0 < s < T$ for which

$$\int_{B(x_0,T-s)} u_t^2 + |Du|^2 + u^6 \, dx \le \epsilon. \tag{23}$$

Then

$$\int_{B(x_0,T-s+\delta)} u_t^2 + |Du|^2 + u^6 \, dx \le 2\epsilon \tag{24}$$

for some small $\delta > 0$. Consequently, a standard energy calculation shows

$$\sup_{s\le t<T} \int_{B(x_0,T-t+\delta)} u^6 \, dx \le C\epsilon.$$

The techniques introduced above in step 4 then establish (22).

6. In summary so far: for each point x_0, there exists $\delta = \delta(x_0) > 0$ such that u verifies (22). As u has compact support, we can find finitely many points $\{x_k\}_{k=1}^N$ such that $\operatorname{spt} u(\cdot, T) \subset \bigcup_{k=1}^N B(x_k, \delta(x_k))$. It follows that

$$u \in L^4(0, T; L^{12}(\mathbb{R}^3)).$$

Then step 2 implies $u \in L^\infty(\mathbb{R}^3 \times [0, T))$. As noted in step 1, proving this was our goal. □

12.5. NONEXISTENCE OF SOLUTIONS

We next identify some circumstances under which the semilinear nonlinear wave equation $\Box u + f(u) = 0$ does not have a solution existing for all time. Our method will be to introduce appropriate integral quantities depending on the time variable t, for which we can then derive differential inequalities, involving convexity, that lead to contradictions.

12.5.1. Nonexistence for negative energy. We study first the initial-value problem

$$\begin{cases} u_{tt} - \Delta u + f(u) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ \quad = g,\ u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{1}$$

We always assume that g, $h \in C_c^\infty(\mathbb{R}^n)$ and that $f(0) = 0$. Hence a smooth solution, if it exists, will have compact support in space for each time, according to Theorem 3 in §12.1.2.

THEOREM 1 (Nonexistence for negative energy)**.** *Assume that for some constant $\lambda > 2$ we have the inequality*

$$zf(z) \le \lambda F(z) \qquad (z \in \mathbb{R}). \tag{2}$$

Suppose also that the energy is negative:

$$E(0) = \int_{\mathbb{R}^n} \frac{1}{2}(|Dg|^2 + h^2) + F(g)\,dx < 0. \tag{3}$$

Then there cannot exist for all times $t \ge 0$ a smooth solution u of (1)*.*

Therefore the solution constructed in the proof of Theorem 2 in §12.3.2 cannot in general continue for all time.

Proof.

1. Define

$$I(t) := \frac{1}{2}\int_{\mathbb{R}^n} u^2\,dx.$$

Then

$$\begin{aligned} I'' &= \int_{\mathbb{R}^n} u_t^2 + uu_{tt}\,dx = \int_{\mathbb{R}^n} u_t^2 + u(\Delta u - f(u))\,dx \\ &= \int_{\mathbb{R}^n} u_t^2 - |Du|^2 - uf(u)\,dx. \end{aligned} \tag{4}$$

The integration by parts is justified, since u has compact support in space for each time.

According to the conservation of energy, we have

$$E(t) = \int_{\mathbb{R}^n} \frac{1}{2}(u_t^2 + |Du|^2) + F(u)\,dx = E(0)$$

for each time $t > 0$. Add and subtract $(2 + 4\alpha)E(0)$ to (4):

$$\begin{aligned} I'' = (2 + 2\alpha)\int_{\mathbb{R}^n} u_t^2\,dx + 2\alpha\int_{\mathbb{R}^n} |Du|^2\,dx \\ + \int_{\mathbb{R}^n} (2 + 4\alpha)F(u) - uf(u)\,dx - (2 + 4\alpha)E(0). \end{aligned}$$

Select $\alpha > 0$ so that $2 + 4\alpha = \lambda$, where λ is the constant in hypothesis (2). Then the last integral term is nonnegative, and hence

$$I'' \geq (2 + 2\alpha) \int_{\mathbb{R}^n} u_t^2 \, dx - \lambda E(0). \tag{5}$$

Since $I' = \int u u_t \, dx$, inequality (5) implies

$$(1+\alpha)(I')^2 \leq (1+\alpha)\left(\int_{\mathbb{R}^n} u^2 \, dx\right)\left(\int_{\mathbb{R}^n} u_t^2 \, dx\right) \leq I(I'' - \beta), \tag{6}$$

for $\beta := -\lambda E(0) > 0$.

2. Put

$$J := I^{-\alpha}.$$

Then (6) lets us compute

$$J'' = \alpha(\alpha+1) I^{-(\alpha+2)}(I')^2 - \alpha I^{-(\alpha+1)} I'' \leq -\alpha\beta I^{-(\alpha+1)} = -\alpha\beta J^{1+1/\alpha}. \tag{7}$$

This shows that J is a concave function of t. Suppose now that $J'(t_0) < 0$ for some time $t_0 > 0$. Then the concavity of J implies

$$J(t) \leq J(t_0) + (t - t_0) J'(t_0) \qquad (t \geq 0),$$

and this inequality provides the contradiction that $J(t) < 0$ for large times t. Assume instead that $J' \geq 0$ for all $t \geq 0$. Then from (7) it follows that

$$J'' \leq -\alpha\beta J^{1+1/\alpha}(0) =: -\gamma.$$

We have $\gamma > 0$, since our negative energy hypothesis (3) implies $g \not\equiv 0$. Thus

$$J'(t) \leq J'(0) - \gamma t < 0$$

for large t, and we again reach a contradiction. □

12.5.2. Nonexistence for small initial data. Satisfying the negative energy hypothesis (3) in the previous subsection requires that g not be too small. Remarkably, certain semilinear wave equations do not possess solutions existing for all times $t > 0$, even for certain arbitrarily small and smooth initial data g, h.

As an example, consider this initial-value problem in three space dimensions:

$$\begin{cases} u_{tt} - \Delta u - |u|^p = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ u = g, \; u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\}. \end{cases} \tag{8}$$

Take any smooth initial data with

$$\int_{\mathbb{R}^3} g \, dx > 0, \; \int_{\mathbb{R}^3} h \, dx > 0 \tag{9}$$

and $\operatorname{spt} g, \operatorname{spt} h \subset B(0,R)$. We will deduce from these conditions alone that there is no solution, provided the exponent $p > 1$ is small enough.

THEOREM 2 (Nonexistence for small data)**.** *Assume that*

$$1 < p < 1 + \sqrt{2}.$$

Then under the above conditions on g and h, the initial-value problem (8) *does not have a smooth solution u existing for all times* $t \geq 0$.

This statement should be contrasted with the global existence of smooth solutions of $\square u + |u|^{p-1}u = 0$ in $\mathbb{R}^3 \times (0,\infty)$ shown in §12.3 and §12.4 for $1 \leq p \leq 5$.

Proof.

1. We assume to the contrary that u is in fact a solution and derive a contradiction. Since the initial data have support within $B(0,R)$, Theorem 3 from §12.1.2 implies that $u(\cdot,t)$ is supported within the ball $B(0,R+t)$.

Put

$$I(t) = \int_{\mathbb{R}^3} u\,dx.$$

Then

$$I'' = \int_{\mathbb{R}^3} u_{tt}\,dx = \int_{\mathbb{R}^3} |u|^p\,dx.$$

Since $u(\cdot,t)$ vanishes outside the ball $B(0,R+t)$, we have

$$I = \int_{B(0,R+t)} u\,dx \leq \left(\int_{\mathbb{R}^3} |u|^p\,dx\right)^{1/p} |B(0,R+t)|^{1-1/p};$$

and therefore

$$I'' \geq cI^p(1+t)^{-3(p-1)} \tag{10}$$

for a constant $c > 0$.

2. We next introduce the solution v of the linear wave equation

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ v = g,\ v_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\}. \end{cases} \tag{11}$$

According to the retarded potential formula (44) in §2.4.2, we have

$$u(x,t) = v(x,t) + \frac{1}{4\pi}\int_{B(x,t)} \frac{|u|^p(y,t-|y-x|)}{|y-x|}\,dy \geq v(x,t).$$

Now (11) and (9) imply

$$\int_{\mathbb{R}^n} v\,dx = c_1 + c_2 t$$

for constants $c_1, c_2 > 0$. Furthermore, since we are working in $n = 3$ space dimensions, Huygens' principle tells us that v has support within the annular region $A := B(0, t+R) - B(0, t-R)$, which has volume $|A| \leq C(1+t)^2$. Hence

$$c_1 + c_2 t = \int_A v\,dx \leq \int_A u\,dx \leq C\left(\int_{\mathbb{R}^3} |u|^p\,dx\right)^{1/p} (1+t)^{2(1-1/p)}.$$

It follows that

$$I'' = \int_{\mathbb{R}^3} |u|^p\,dx \geq c(1+t)^{2-p}$$

for some constant $c > 0$. Since $I(0), I'(0) > 0$, we deduce that

$$I \geq c(1+t)^{4-p}. \tag{12}$$

3. Let $\varepsilon > 0$ and as follows combine (10) and (12):

$$\begin{aligned} I'' &\geq cI^{1+\varepsilon} I^{p-1-\varepsilon}(1+t)^{-3(p-1)} \\ &\geq cI^{1+\varepsilon}(1+t)^{(4-p)(p-1-\varepsilon)}(1+t)^{-3(p-1)} \\ &= cI^{1+\varepsilon}(1+t)^{-\mu} \end{aligned}$$

for $\mu := (p-1)^2 + \varepsilon(4-p)$. Since $1 < p < 1 + \sqrt{2}$, we can fix $\varepsilon > 0$ so small that

$$0 < \mu < 2. \tag{13}$$

Since $I' > 0$,

$$I'' I' \geq cI' I^{1+\varepsilon}(1+t)^{-\mu};$$

and thus

$$((I')^2)' \geq c(I^{2+\varepsilon}(1+t)^{-\mu})' + c\mu I^{2+\varepsilon}(1+t)^{-\mu+1} \geq c(I^{2+\varepsilon}(1+t)^{-\mu})'.$$

Consequently

$$(I')^2(t) \geq (I')^2(0) + c(I^{2+\varepsilon}(1+t)^{-\mu} - I^{2+\varepsilon}(0)). \tag{14}$$

Now (12) implies $I^{2+\varepsilon}(t)(1+t)^{-\mu} \geq 2I^{2+\varepsilon}(0)$ for large enough times, say, $t \geq t_0$. Therefore we can deduce from (14) that

$$(I')^2 \geq cI^{2+\varepsilon}(1+t)^{-\mu}$$

provided $t \geq t_0$. Then

$$(I^{-\varepsilon/2})' = -\frac{\varepsilon}{2} I^{-\frac{\varepsilon}{2}-1} I' \leq -\frac{c\varepsilon}{2}(1+t)^{-\mu/2}.$$

Integrating, we see that

$$0 \leq I^{-\varepsilon/2}(t) \leq I^{-\varepsilon/2}(t_0) - \frac{c\varepsilon}{2}\int_{t_0}^{t} \frac{ds}{(1+s)^{\mu/2}}.$$

This is a contradiction, since $\mu < 2$ according to (13) and so the integral on the right diverges as $t \to \infty$. □

12.6. PROBLEMS

The following problem set includes questions on both linear and nonlinear wave equations, as well as the related nonlinear Schrödinger equation. All given functions are assumed smooth, unless otherwise stated.

1. Assume u has compact support in space and solves the quasilinear wave equation

$$u_{tt} - \sum_{i=1}^{n} (L_{p_i}(Du))_{x_i} = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

   Determine the appropriate energy $E(t)$ and show $\dot{E} \equiv 0$.

2. Let u solve the *Klein–Gordon equation*

$$(*) \qquad \begin{cases} u_{tt} - \Delta u + m^2 u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

   (a) Show that the energy

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^n} u_t^2 + |Du|^2 + m^2 u^2 \, dx \qquad (t \geq 0)$$

   is constant in time.

   (b) Modify the proof in §4.3.1 showing asymptotic equipartition of energy for the wave equation to prove that

$$\lim_{t\to\infty} \int_{\mathbb{R}^n} |Du|^2 + m^2 u^2 \, dx = E(0).$$

3. Suppose u solves the initial value problem $(*)$ from Problem 2 for the Klein–Gordon equation. Write $\bar{x} = (x, x_{n+1})$ for $x \in \mathbb{R}^n$ and define

$$\bar{u}(\bar{x}, t) := u(x,t)\cos(m x_{n+1}).$$

   (a) Show that $\bar{u}$ solves the wave equation $\Box \bar{u} = 0$ in $\mathbb{R}^{n+1} \times (0,\infty)$.

   (b) Derive a formula for the solution of the initial-value problem for the Klein–Gordon equation when $n = 1$. (This is a variant of the *method of descent*, introduced in §2.4.1.)

4. Assume u solves
$$u_{tt} - \Delta u + du_t = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty),$$
which for $d > 0$ is a damped wave equation. Find a simple exponential term that, when multiplied by u, gives a solution v of
$$v_{tt} - \Delta v + cv = 0$$
for a constant $c < 0$. (This is the opposite of the sign for the Klein–Gordon equation.)

5. Check that for each given $y \in \mathbb{R}^n$, $y \neq 0$, the function $u = e^{i(x\cdot y - \sigma t)}$ solves the Klein–Gordon equation
$$u_{tt} - \Delta u + m^2 u = 0$$
provided $\sigma = (|y|^2 + m^2)^{\frac{1}{2}}$. The phase velocity of this plane wave solution is $\frac{\sigma}{|y|} > 1$. Why does this not contradict the assertions in §12.1 that the speed of propagation for solutions is less than or equal to one?

6. Suppose
$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ u = g, u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$
where g, h have compact support. Show there exists a constant C such that
$$|u(x,t)| \leq C/t \quad (x \in \mathbb{R}^3,\ t > 0).$$
(Hint: Use the representation formula for the solution from §2.4.1.)

7. Let u solve
$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^2 \times (0,\infty) \\ u = g, u_t = h & \text{on } \mathbb{R}^2 \times \{t = 0\}, \end{cases}$$
where g, h have compact support. Show that
$$|u(x,t)| \leq C/t^{\frac{1}{2}} \quad (x \in \mathbb{R}^2,\ t > 0)$$
for some constant C.

8. Suppose u^ε solves the linear wave equation in $n = 2$ space dimensions, with the initial conditions $u^\varepsilon_t = h \equiv 0$ and
$$u^\varepsilon = g^\varepsilon := \begin{cases} e^{-\varepsilon \frac{(r-2)^2}{(r-1)(3-r)}} & \text{if } 1 < r < 3 \\ 0 & \text{otherwise,} \end{cases}$$
where $r = |x|$. Show that although $|g^\varepsilon| \leq 1$, we have
$$\max_{\mathbb{R}^2 \times [0,4]} |u^\varepsilon| \to \infty \qquad \text{as } \varepsilon \to 0.$$

(Hint: Use (26) in §2.4.1 to compute $u^\varepsilon(0,t)$ for $t > 3$.)

9. (Kelvin transform for wave equation) The *hyperbolic Kelvin transform* $\mathcal{K}u = \bar{u}$ of a function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is

$$\bar{u}(x,t) := u(\bar{x},\bar{t})\,\big||\bar{x}|^2 - \bar{t}^2\big|^{\frac{n-1}{2}} = u\left(\frac{x}{|x|^2 - t^2}, \frac{t}{|x|^2 - t^2}\right)\frac{1}{\big||x|^2 - t^2\big|^{\frac{n-1}{2}}},$$

provided $|x|^2 \neq t^2$, where

$$\bar{x} = \frac{x}{|x|^2 - t^2}, \quad \bar{t} = \frac{t}{|x|^2 - t^2}.$$

Show that if $\Box u = 0$, then $\Box \bar{u} = 0$.

(Compare with Problem 11 in Chapter 2.)

10. Assume that u and v solve the system

$$(*) \qquad \begin{cases} (u - v)_t = 2a \sin\left(\frac{u+v}{2}\right) \\ (u + v)_x = \frac{2}{a} \sin\left(\frac{u-v}{2}\right) \end{cases}$$

where $a \neq 0$. Show that both $w := u$ and $w := v$ solve the *sine-Gordon equation* in the form

$$w_{xt} = \sin w.$$

Why is this equivalent to the PDE $\Box w = \sin w$?

11. (Continuation) Alternatively, given a solution v of the sine-Gordon equation, we can try to solve the system $(*)$ to build a second solution u. This procedure is called a *Bäcklund transformation*.

Start with the trivial solution $v \equiv 0$, and use the Bäcklund transformation to compute for each choice of the parameter a another solution u. (Hint: First show u must have the form $f(at + x/a)$. Also show that $(\tan(u/4))_t = a\tan(u/4)$.)

12. Prove the Sobolev-type inequality (6) in §12.2.2.

(Hints: If $u \in H^k$ and β is a multiindex with $|\beta| \leq k$, we have the estimate $\|D^\beta u\|_{L^p} \leq C\|u\|_{H^k}$, where p satisfies (a) $p = \infty$ if $\frac{1}{2} - \frac{k}{n} + \frac{|\beta|}{n} < 0$; (b) $2 \leq p < \infty$ if $\frac{1}{2} - \frac{k}{n} + \frac{|\beta|}{n} = 0$; and (c) $\frac{1}{p} = \frac{1}{2} - \frac{k-|\beta|}{n}$ if $\frac{1}{2} - \frac{k}{n} + \frac{|\beta|}{n} > 0$. Assume that the multiindices $|\beta_1|, \dots, |\beta_r|$ satisfy (a) above, $|\beta_{r+1}|, \dots, |\beta_s|$ satisfy (b), and $|\beta_{s+1}|, \dots, |\beta_m|$ satisfy (c). Estimate $\|D^{\beta_1}u_1 \cdots D^{\beta_m}u_m\|_{L^2}^2$.)

13. A smooth function $\mathbf{u} : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^m$, $\mathbf{u} = (u^1, \dots, u^m)$, is called a *wave map* into the unit sphere $S^{m-1} = \partial B(0,1) \subset \mathbb{R}^m$ provided that everywhere in $\mathbb{R}^n \times [0,\infty)$, $|\mathbf{u}| = 1$ and $\mathbf{u}_{tt} - \Delta\mathbf{u}$ is perpendicular to S^{m-1} at $\mathbf{u}$.

Show that therefore $\mathbf{u}$ solves the system of PDE

$$\mathbf{u}_{tt} - \Delta\mathbf{u} = (|D\mathbf{u}|^2 - |\mathbf{u}_t|^2)\mathbf{u}.$$

14. Prove that if $\mathbf{u}$ is a wave map into the unit sphere, with compact support in space, we have conservation of energy:

$$\frac{d}{dt}\int_{\mathbb{R}^n} |\mathbf{u}_t|^2 + |D\mathbf{u}|^2\,dx = 0.$$

15. (Small energy for $p = 5$) Adapt the proof of Theorem 3 in §12.3.3 to show the existence of a smooth solution of

$$\begin{cases} u_{tt} - \Delta u + u^5 = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ \quad u = g,\ u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$

provided the energy $E(0)$ is sufficiently small.

(Hint: Modify estimate (24) in §12.3.3 by introducing the L^6-norm of g.)

Many techniques developed for semilinear wave equations have counterparts for nonlinear Schrödinger (NLS) equations, to which we devote the remaining exercises.

16. Let u be a complex-valued solution of the nonlinear Schrödinger equation

$$(*) \qquad iu_t + \Delta u = f(|u|^2)u \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

Here $f : \mathbb{R} \to \mathbb{R}$. Demonstrate that if $\xi \in \mathbb{R}^n$, then

$$w(x,t) := e^{\frac{i}{4}(2\xi\cdot x - |\xi|^2 t)} u(x - \xi t, t)$$

also solves the NLS equation. This shows the *Galilean invariance* of solutions.

17. Assume u solves the nonlinear Schrödinger equation $(*)$ from Problem 16 and decays rapidly, along with its derivatives, as $|x| \to \infty$. Derive these identities:

$$\frac{d}{dt}\int_{\mathbb{R}^n} |u|^2\,dx = 0 \quad \text{(conservation of mass)},$$

$$\frac{d}{dt}\int_{\mathbb{R}^n} |Du|^2 + F(|u|^2)\,dx = 0 \quad \text{(conservation of energy)},$$

where $F' = f$, and

$$\frac{d}{dt}\int_{\mathbb{R}^n} \frac{\bar{u}Du - uD\bar{u}}{2i}\,dx = 0 \quad \text{(conservation of momentum)}.$$

Remember that $|u|^2 = u\bar{u}$.

18. (Continuation)
   (a) Under the hypotheses of the previous problem, derive the identity
$$\frac{d^2}{dt^2}\int_{\mathbb{R}^n}|x|^2|u|^2\,dx$$
$$= 8\int_{\mathbb{R}^n}|Du|^2\,dx + 4n\int_{\mathbb{R}^n} f(|u|^2)|u|^2 - F(|u|^2)\,dx.$$
   (b) Use (a) to show that there does not exist a solution of the cubic-NLS equation
$$iu_t + \Delta u + |u|^2 u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty)$$
existing for all times $t \geq 0$, if
$$E(0) = \int_{\mathbb{R}^n}|Du(x,0)|^2 - \frac{|u(x,0)|^4}{2}\,dx < 0$$
and $n \geq 2$. (R. Glassey, J. Math. Physics 18 (1977), 1794–1797)

12.7. REFERENCES

D. Tataru provided me with comments and suggestions for this chapter.

Section 12.2: Shatah–Struwe [**S-S**] was my primary source for this section.

Section 12.3: See Shatah–Struwe [**S-S**] for more. Theorem 1 follows I. Segal (Bull. Soc. Math. France 91 (1963), 129–135) and Theorems 2 and 3 follow K. Jörgens (Math. Z. 77 (1961), 295–308).

Section 12.4: M. Grillakis (Ann. of Math. 132 (1990), 485–509) proved the key estimate in Theorem 2 and the existence Theorem 3. Our presentation of the latter uses simplifications due to Smith and Sogge: consult Sogge [**So**] and Shatah–Struwe [**S-S**].

Section 12.5: See Strauss [**St1**], John [**J3**].

Section 12.6: Problem 8 is from Protter–Weinberger [**P-W**]. Problem 15: J. Rauch observed that small energy implies global existence for $p = 5$. J. Holmer helped me with Problems 16–18.