Preface

In brief, the objective of representation theory is to investigate the different ways in which a given algebraic object—such as an algebra, a group, or a Lie algebra—can act on a vector space. The benefits of such an action are at least twofold: the structure of the acting object gives rise to symmetries of the vector space on which it acts; and, in the other direction, the highly developed machinery of linear algebra can be brought to bear on the acting object itself to help uncover some of its hidden properties. Besides being a subject of great intrinsic beauty, representation theory enjoys the additional benefit of having applications in myriad contexts other than algebra, ranging from number theory, geometry, and combinatorics to probability and statistics [**58**], general physics [**200**], quantum field theory [**212**], the study of molecules in chemistry [**49**], and, more recently, machine learning [**127**].

This book has evolved from my lecture notes for a two-semester graduate course titled *Representation Theory* that I gave at Temple University during the academic years 2012/13 and 2015/16. Some traces of the informality of my original notes and the style of my lectures have remained intact: the text makes rather copious use of pictures and expansively displayed formulae; definitions are not numbered and neither are certain key results, such as Schur's Lemma or Wedderburn's Structure Theorem, which are referred to by name rather than number throughout the book. However, due to the restrictions imposed by having to set forth the material on the page in a linear fashion, the general format of this book does not in fact duplicate my actual lectures and it only locally reflects their content. I will comment more on this below.

The title *A Tour of Representation Theory* (*ToR*) is meant to convey the panoramic view of the subject that I have aimed for.[1] Rather than offering an

[1]The choice of title is also a nod to the Tour de France, and "Tor" in German is "gate" as well as "goal" (scored) and "fool".

in-depth treatment of one particular area, *ToR* gives an introduction to three distinct flavors of representation theory—representations of groups, Lie algebras, and Hopf algebras—and all three are presented as incarnations of algebra representations. The book loops repeatedly through these topics, emphasizing similarities and connections. Group representations, in particular, are revisited frequently after their initial treatment in Part II. For example, Schur-Weyl duality is first discussed in Section 4.7 and later again in Section 8.8; Frobenius-Schur indicators are introduced in §3.6.3 in connection with the Brauer-Fowler Theorem and they are treated in their proper generality in Section 12.5; and Chapter 11, on affine algebraic groups, brings together groups, Lie algebras, and Hopf algebras. This mode of exposition owes much to the "holistic" viewpoint of the monograph [**72**] by Etingof et al., although *ToR* forgoes the delightful historical intermezzos that punctuate [**72**] and it omits quivers in favor of Hopf algebras. Our tour does not venture very far into any of the areas it passes through, but I hope that *ToR* will engender in some readers the desire to pursue the subject and that it will provide a platform for further explorations.

Overview of the Contents. The topics covered in *ToR* and the methods employed are resolutely algebraic. Lie groups, C^*-algebras, and other areas of representation theory requiring analysis are not covered. On the other hand, in keeping with the widely acknowledged truth that algebraic representation theory benefits from a geometric perspective, the discourse involves a modicum of algebraic geometry on occasion and I have also tried my hand at depicting various noncommutative geometric spaces throughout the book. No prior knowledge of algebraic geometry is assumed, however.

Representations of algebras form the unifying thread running through *ToR*. Therefore, Part I is entirely written in the setting of associative algebras. Chapter 1 develops the basic themes of representation theory: irreducibility, complete reducibility, spaces of primitive ideals, characters, . . . ; the chapter establishes notation to be used throughout the remainder of the book; and it furnishes the fundamental general results of representation theory, such as Wedderburn's Structure Theorem. Chapter 2 covers topics that are somewhat more peripheral to the main thrust of *ToR*: projective modules (Section 2.1) and Frobenius algebras (Section 2.2). Readers whose main interest is in groups or Lie algebras may skip this chapter at a first reading. However, Section 2.2 deploys some tools that are indispensable for the discussion of finite-dimensional Hopf algebras in Chapter 12.

Parts II and III are respectively devoted to representations of groups and Lie algebras. To some degree, these two parts can be tackled in any order. However, I have made a deliberate effort at presenting the material on group representations in a palatable manner, offering it as an entryway to representation theory, while the part on Lie algebras is written in a slightly terser style demanding greater mathematical maturity from the reader. Most of Part II deals with *finite-dimensional* representations of *finite* groups, usually over a base field whose characteristic does not divide the order of the group in question. Chapter 3 covers standard territory, with the possible exception of some brief excursions into classical invariant theory (§§3.7.4, 3.8.4). Chapter 4, however, presents the representation theory of the symmetric groups in characteristic 0 via an elegant novel approach devised by Okounkov and Vershik rather than following the route taken by the originators of the theory, Frobenius, Schur, and Young. Much of this chapter elaborates on Chapter 2 of Kleshchev's monograph [**125**], providing full details and some additional background. My presentation of the material on Lie algebras and their representations in Part III owes a large debt to the classics by Dixmier [**63**] and Humphreys [**105**] and also to Fulton and Harris [**83**] as well as the more recent monograph [**69**] by Erdmann and Wildon. The notation and terminology in this part are largely adopted from [**105**] and its Afterword (1994). Departing from tradition, the discussion of the Nullstellensatz and the Dixmier-Mœglin equivalence for enveloping algebras of Lie algebras in Section 5.6 relies on the symmetric ring of quotients rather than the classical ring of quotients; this minimizes the requisite background material from noncommutative ring theory, which is fully provided in Appendix E.

Hopf algebra structures are another recurring theme throughout the book: they are first introduced for the special case of group algebras in Section 3.3; an analogous discussion for enveloping algebras of Lie algebras follows in §5.4.4; and Hopf algebras are finally tackled in full generality in Part IV. While this part of *ToR* is relatively dry in comparison with the rest of the book, the reader familiar with the earlier special cases will be amply prepared and hopefully willing to face up to what may initially seem like a profusion of technicalities. The effort is worthwhile: many facets of the representation theory of groups or Lie algebras, especially those dealing with tensor products of representations, take their most natural form when viewed through the lens of Hopf algebras and, of all parts of *ToR*, it is Part IV that leads closest to the frontier of current research. On the other hand, I believe that students planning to embark on the investigation of Hopf algebras will profit from a grounding in the more classical representation theories of groups and Lie algebras, which is what *ToR* aims to provide.

Prerequisites. The various parts of *ToR* differ rather significantly with regard to their scope and difficulty. However, much of the book was written for a readership having nothing but a first-year graduate algebra course under their belts: the basics of groups, rings, modules, fields, and Galois theory, but not necessarily anything beyond that level. Thus, I had no qualms assuming a solid working knowledge of linear algebra—after all, representation theory is essentially linear algebra with (quite a lot of) extra structure. Appendix B summarizes some formal points of linear algebra, notably the properties of tensor products.

The prospective reader should also be well acquainted with elementary group theory: the isomorphism theorems, Sylow's Theorem, and abelian, nilpotent, and solvable groups. The lead-in to group representations is rather swift; group algebras and group representations are introduced in quick succession and group representation theory is developed in detail from there. On the other hand, no prior knowledge of Lie algebras is expected; the rudiments of Lie algebras are presented in full, albeit at a pace that assumes some familiarity with parallel group-theoretic lines of reasoning.

While no serious use of category theory is made in this book, I have frequently availed myself of the convenient language and unified way of looking at things that categories afford. When introducing new algebraic objects, such as group algebras or enveloping algebras of Lie algebras, I have emphasized their "functorial" properties; this highlights some fundamental similarities of the roles these objects play in representation theory that may otherwise not be apparent. The main players in *ToR* are the category $\mathsf{Vect}_{\Bbbk}$ of vector spaces over a field $\Bbbk$ and the categories $\mathsf{Rep}_{\text{fin}}\,A$ of all finite-dimensional representations of various $\Bbbk$-algebras A. Readers having had no prior exposure to categories and functors may wish to peruse Appendix A before delving into the main body of the text.

Using this Book. *ToR* is intended as a textbook for a graduate course on representation theory, which could immediately follow the standard abstract algebra course, and I hope that the book will also be useful for subsequent reading courses and for readers wishing to learn more about the subject by self-study. Indeed, the more advanced material included in *ToR* places higher demands on its readers than would probably be adequate for an introductory course on representation theory and it is unrealistic to aim for full coverage of the book in a single course, even if it spans two semesters. Thus, a careful selection of topics has to be made by the instructor.

When teaching *Abstract Algebra* over the years, I found that finite groups have tended to be quite popular among my students—starting from minimal prerequisites, one quickly arrives at results of satisfying depth and usefulness. Therefore, I usually start the follow-up course *Representation Theory* by diving right into representations of groups (Part II), covering all of Chapter 3 and some of Chapter 4 in the first semester. Along the way, I add just enough material about algebras from Chapter 1 to explain the general underpinnings, often relegating proofs to reading assignments. In the second semester, I turn to representations of Lie algebras and try to cover as much of Part III as possible. Section 5.6 is generally only presented in a brief "outlook" format and Sections 8.6–8.8 had to be left uncovered so far for lack of time. Instead, in one or two lectures at the end of the second semester of *Representation Theory* or sometimes in a mini-course consisting of four or five lectures in our Algebra Seminar, I try to give the briefest of glimpses into the theory of Hopf algebras and their representations (Part IV).

Alternatively, one could conceivably begin with a quick pass through the representation-theoretic fundamentals of algebras, groups, Lie algebras, and Hopf algebras before spiraling back to cover each or some of these topics in greater depth. Teaching a one-semester course will most likely entail a focus on just one of Parts II, III, or IV depending on the instructor's predilections and the students' background. In order to enable the instructor or reader to pick and choose topics from various parts of the book, I have included numerous cross references and frequent reminders throughout the text.

The exercises vary greatly in difficulty and purpose: some merely serve to unburden various proofs of unsightly routine verifications, while others present substantial results that are not proved but occasionally alluded to in the text. I have written out solutions for the majority of exercises and I'd be happy to make them available to instructors upon request.

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Philadelphia *Martin Lorenz*
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