Preface

Combinatorics is not a field, it's an attitude.
Anon

A *combinatorial reciprocity theorem* relates two classes of combinatorial objects via their counting functions: consider a class $\mathcal{X}$ of combinatorial objects and let $f(n)$ be the function that counts the number of objects in $\mathcal{X}$ of size n, where *size* refers to some specific quantity that is naturally associated with the objects in $\mathcal{X}$. Similar to canonization, it requires two miracles for a combinatorial reciprocity to occur:

1. the function $f(n)$ is the restriction of some reasonable function (e.g., a polynomial) to the positive integers, and
2. the evaluation $f(-n)$ is an integer of the same sign $\sigma = \pm 1$ for all $n \in \mathbb{Z}_{>0}$.

In this situation it is only human to ask if $\sigma\, f(-n)$ has a combinatorial meaning, that is, if there is a natural class $\mathcal{X}^\circ$ of combinatorial objects such that $\sigma\, f(-n)$ counts the objects of $\mathcal{X}^\circ$ of size n (where *size* again refers to some specific quantity naturally associated to $\mathcal{X}^\circ$). Combinatorial reciprocity theorems are among the most charming results in mathematics and, in contrast to canonization, can be found all over enumerative combinatorics and beyond.

As a first example we consider the class of maps $[k] \to \mathbb{Z}_{>0}$ from the finite set $[k] := \{1, 2, \dots, k\}$ into the positive integers, and so $f(n) = n^k$ counts the number of maps with codomain $[n]$. Thus $f(n)$ is the restriction of a polynomial and $(-1)^k f(-n) = n^k$ satisfies our second requirement above. This relates the number of maps $[k] \to [n]$ to itself. This relation is a genuine combinatorial reciprocity but the impression one is left with is that of being underwhelmed rather than charmed. Later in the book it will become clear that this example is not boring at all, but for now let's try again.

The term *combinatorial reciprocity theorem* was coined by Richard Stanley in his 1974 paper [**162**] of the same title, in which he developed a firm foundation of the subject. Stanley starts with an appealing reciprocity that he attributes to John Riordan: For a set S and $d \in \mathbb{Z}_{\geq 0}$, the collection of **$d$-subsets**[1] of S is

$$\binom{S}{d} := \{A \subseteq S : |A| = d\} .$$

For d fixed, the number of d-subsets of S depends only on the cardinality $|S|$, and the number of d-subsets of an n-set is

$$f(n) = \binom{n}{d} = \frac{1}{d!}\, n(n-1)\cdots(n-d+2)(n-d+1)\,, \tag{0.0.1}$$

which is the restriction of a polynomial in n of degree d. From the factorization we can read off that $(-1)^d f(-n)$ is a positive integer for every $n > 0$. More precisely,

$$(-1)^d f(-n) = \frac{1}{d!}\, n(n+1)\cdots(n+d-2)(n+d-1) = \binom{n+d-1}{d},$$

which is the number of *d-multisubsets* of an n-set, that is, the number of picking d elements from $[n]$ with repetition but without regard to the order in which the elements are picked. Now this is a combinatorial reciprocity! In formulas it reads

$$(-1)^d \binom{-n}{d} = \binom{n+d-1}{d}. \tag{0.0.2}$$

This is enticing in more than one way. The identity presents an intriguing connection between subsets and multisubsets via their counting functions, and its formal justification is completely within the realms of an undergraduate class in combinatorics. Equation (0.0.2) can be found in Riordan's book [**143**] on combinatorial analysis without further comment and, charmingly, Stanley states that his paper [**162**] can be considered as "further comment". That further comment is necessary is apparent from the fact that the formal proof above falls short of explaining why these two sorts of objects are related by a combinatorial reciprocity. In particular, comparing coefficients in (0.0.2) cannot be the method of choice for establishing more general reciprocity relations.

In this book we develop tools and techniques for handling combinatorial reciprocities. However, our own perspective is firmly rooted in *geometric* combinatorics and, thus, our emphasis is on the geometric nature of the

[1] All our definitions will look like that: incorporated into the text but bold-faced and so hopefully clearly visible.

combinatorial reciprocities. That is, for every class of combinatorial objects we associate a geometric object (such as a polytope or a polyhedral complex) in such a way that combinatorial features, including counting functions and reciprocity, are reflected in the geometry. In short, this book can be seen as *further comment with pictures*. At any rate, our text was written with the intention to give a comprehensive introduction to contemporary enumerative geometric combinatorics.

A Quick Tour. The book naturally comes in two parts with a special role played by the first chapter: Chapter 1 introduces four combinatorial reciprocity theorems that we set out to establish in the course of the book. Chapters 2–4 are for-the-most-part-independent introductions to three major themes of combinatorics: partially ordered sets, polyhedra, and generating functions. Chapters 5–7 treat more sophisticated topics in geometric combinatorics and are meant to be digested in order. Here is what to expect.

Chapter 1 sets the rhythm. We introduce four functions to count colorings and flows on graphs, order-preserving functions on partially ordered sets, and lattice points in dilations of lattice polygons. The definitions in this chapter are kept somewhat informal, to provide an easy entry into the themes of the later chapters. In all four cases we state a surprising combinatorial reciprocity and we point to some of the relations and connections between these examples, which will make repeated appearances later on. All in all, this chapter is a source of examples and motivation. You should revisit it from time to time to see how the various ways to view these objects shape your perspective.

Chapter 2 gives an introduction to partially ordered sets (*posets*, for short). Relating posets by means of order-preserving maps gives rise to the order polynomials from Chapter 1. One of the highlights here is a purely combinatorial proof of the reciprocity surrounding order polynomials (and only later will we see that there was geometry behind it). This gives us an opportunity to introduce important machinery, including Möbius inversion, zeta polynomials, and Eulerian posets in a hands-on and nonstandard form.

Geometry enters (quite literally) the picture in Chapter 3, in which we introduce convex polyhedra. Polyhedra are wonderful objects to study in their own right, as we hope to convey here, and much of their combinatorial structure comes in poset-theoretic terms. Our main motivation, however, is to develop a language that enables us to give the objects from Chapters 1 and 2 a geometric incarnation. The main player in Chapter 3 is the Euler characteristic, which is a powerful tool to obtain combinatorial truths from geometry. Two applications of the Euler characteristic, which we will witness in this chapter, are Zaslavsky's theorem for hyperplane arrangements and the Brianchon–Gram relation for polytopes.

Chapter 4 sets up the main algebraic machinery for our book: (rational) generating functions. We start gently with natural examples of compositions and partitions, and combinatorial reciprocity theorems appear almost instantly and just as naturally. The second half of Chapter 4 connects the world of generating functions with that of polyhedra and cones, where we develop Ehrhart and Hilbert series from first principles, including Stanley's reciprocity theorem for rational simplicial cones, which is at the heart of this book. This connection, in turn, allows us to view the first half of Chapter 4 from a new, geometric, perspective.

Chapter 5 is devoted to decomposing polyhedra into simple pieces. In particular, organizing the various pieces automatically suggests to view triangulations and, more generally, subdivisions as posets. Together with the technologies developed in the first part of the book, this culminates in a proof of our main combinatorial reciprocity theorems for polytopes and cones. The theory of subdividing polyhedra is worthy of study in its own right and we only glimpse at it by studying various ways to subdivide polytopes in a geometric, algorithmic, and, of course, combinatorial fashion. A powerful tool is that of half-open decompositions that quite remarkably help us to see some deep combinatorics in a clear way.

In Chapter 6 we give general posets life in Euclidean space as polyhedral cones. The theory of order cones allows us to utilize Chapters 2–5, often in surprisingly interconnected ways, to study posets using geometric means and, at the same time, interesting arithmetic objects derived from posets. Just as interesting are applications of this theory, which include permutation statistics, order polytopes, P-partitions, and their combinatorial reciprocity theorems.

Chapter 7 finishes the framework that was started in Chapter 1: we develop a unifying geometric approach to certain families of combinatorial polynomials. The last missing piece of the puzzle is formed by hyperplane arrangements, which constitute the main players of Chapter 7. They open a window to certain families of graph polynomials, including chromatic and flow polynomials, and we prove combinatorial reciprocity theorems for both. Hyperplane arrangements also naturally connect to two important families of polytopes, namely, alcoved polytopes and zonotopes.

The prerequisites for this book are minimal: undergraduate knowledge of linear algebra and combinatorics should suffice. The numerous exercises throughout the text are designed so that the book could easily be used for a graduate class in combinatorics or discrete geometry. The exercises that are needed for the main body of the text are marked by ⬠.

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