Chapter 6

The Berkovich Projective Line

In his thesis [**114**], Rivera-Letelier pioneered the use of the Berkovich projective line $\mathbb{P}^1_{\text{an}}$ in the study of non-archimedean dynamics. Roughly speaking, $\mathbb{P}^1_{\text{an}}$ is a topological space containing $\mathbb{P}^1(\mathbb{C}_v)$ but which has the advantages of being compact, Hausdorff, and path-connected.

In the interest of streamlining our presentation, we defer many of the proofs of results in this chapter to Chapter 15. Nevertheless, readers new to Berkovich's theory might find even this lighter version to be rather demanding. It may help to keep in mind throughout that the principal conclusions are the following.

(a) The Berkovich projective line $\mathbb{P}^1_{\text{an}}$ is a path-connected, Hausdorff compactification of $\mathbb{P}^1(\mathbb{C}_v)$.

(b) The points of $\mathbb{P}^1(\mathbb{C}_v)$ in $\mathbb{P}^1_{\text{an}}$ are said to be of Type I.

(c) Every closed subdisk $\overline{D}(a,r) \subseteq \mathbb{C}_v$ corresponds to a point in $\mathbb{P}^1_{\text{an}}$ that we denote $\zeta(a,r)$. This point is said to be of Type II if $r \in |\mathbb{C}_v^\times|$, or of Type III otherwise.

(d) All other points in $\mathbb{P}^1_{\text{an}}$ correspond to decreasing sequences of disks $D_1 \supsetneq D_2 \supsetneq \cdots$ with empty intersection. Such points are said to be of Type IV.

See Figure 6.3, which is a sketch of the Berkovich disk $\overline{D}_{\text{an}}(a,r)$, and Figure 6.1, which illustrates a path between a Type I point $0 \in \mathbb{C}_v$ and a Type II point $\zeta(x,r)$. Try to keep all these goals in mind as you work your way through the details of this chapter.

Berkovich spaces were introduced by Vladimir Berkovich in the late 1980s as an improvement on the theory of rigid analytic spaces. Related seminorm constructions appeared previously in [**51**], but Berkovich's theory seems to be the proper setting for non-archimedean dynamics. For deeper background on Berkovich spaces, we refer the reader to Berkovich's monograph [**26**]. The first two chapters of [**8**] provide more accessible introduction, given that they focus on the projective line rather than arbitrary varieties as in [**26**]. Much of our presentation here is influenced by [**8**]. Finally, Rivera-Letelier's papers [**114, 115, 117**] construct the Berkovich projective line in a different way, motivated more directly by the dynamical applications; see Appendix B.4.

6.1. Seminorms as Berkovich points

Definition 6.1. Let A be a ring with unity. A *multiplicative seminorm* on A is a function $\|\cdot\| : A \to [0,\infty)$ such that

(a) $\|0\| = 0$ and $\|1\| = 1$;

(b) for all $f, g \in A$, $\|fg\| = \|f\| \cdot \|g\|$; and

(c) for all $f, g \in A$, $\|f+g\| \leq \|f\| + \|g\|$.

If $\|f+g\| \leq \max\{\|f\|, \|g\|\}$ for all $f, g \in A$, we say $\|\cdot\|$ is *non-archimedean.*

We say another multiplicative seminorm $\|\cdot\|_\zeta$ is *bounded* with respect to $\|\cdot\|$, and that $\|\cdot\|$ *bounds* $\|\cdot\|_\zeta$, if there is a constant $C > 0$ such that

$$\|f\|_\zeta \leq C\|f\| \quad \text{for all } f \in A.$$

Notably absent from Definition 6.1 is the requirement that $\|f\| = 0$ implies $f = 0$, whence the terminology *semi*norm.

We write most of our seminorms with a subscript, as we did with $\|\cdot\|_\zeta$ above. In that case, we often write simply ζ to indicate the seminorm $\|\cdot\|_\zeta$.

The standard constructions of the Berkovich projective line and affine line start with the Berkovich unit disk, which is the set of multiplicative seminorms on the power series ring $\overline{\mathcal{A}}(0,1)$ that are bounded by the supnorm on $\overline{D}(0,1)$. However, since we do not need the full theory of Berkovich spaces, we begin with the following simpler but equivalent definition of the Berkovich affine line.

Definition 6.2. The *Berkovich affine line*, denoted $\mathbb{A}^1_{\mathrm{an}}$, is the set of multiplicative seminorms $\zeta = \|\cdot\|_\zeta$ on the polynomial ring $\mathbb{C}_v[z]$ that extend the absolute value on $\mathbb{C}_v$. We equip $\mathbb{A}^1_{\mathrm{an}}$ with the *Gel'fand topology*, which is the weakest topology such that for every $f \in \mathbb{C}_v[z]$, the function mapping $\mathbb{A}^1_{\mathrm{an}}$ to $\mathbb{R}$ by $\zeta \mapsto \|f\|_\zeta$ is continuous.

The above definition of the Gel'fand topology means that for any open subset V of $\mathbb{R}$ and any $f \in \mathbb{C}_v[z]$, the set

$$\{\zeta \in \mathbb{A}^1_{\mathrm{an}} : \|f\|_\zeta \in V\}$$

is an open subset of $\mathbb{A}^1_{\mathrm{an}}$. More precisely, the Gel'fand topology is the topology generated by such sets. Later, in Theorem 6.17, we present an alternative description of the Gel'fand topology that may provide a better intuition for what open sets in $\mathbb{P}^1_{\mathrm{an}}$ can be. Until then, simply understanding Berkovich *points* will occupy much of our attention through the end of Section 6.3.

As noted above, we abuse notation and use the symbols ζ and $\|\cdot\|_\zeta$ interchangeably. Generally, however, we write ζ when we want to emphasize that we are talking about a Berkovich point, and we write $\|\cdot\|_\zeta$ when we want to emphasize that we are talking about a seminorm.

6.1.1. The four types of Berkovich points. Before developing the theoretical properties of $\mathbb{A}^1_{\mathrm{an}}$, we give some examples of its elements. We will formalize the following categorization of Berkovich points in Definition 6.8.

First, for any point $x \in \mathbb{C}_v$, define a function $\|\cdot\|_x : \mathbb{C}_v[z] \to [0,\infty)$ by

$$\|f\|_x := |f(x)|.$$

The reader may check that $\|\cdot\|_x$ is indeed a multiplicative seminorm extending $|\cdot|$, although it is important to note that it is *not* a norm, since $\|z - x\|_x = 0$; see Exercise 6.1. We will see in Theorem 6.21 that the map $x \mapsto \|\cdot\|_x$ is an embedding of $\mathbb{A}^1(\mathbb{C}_v)$ into $\mathbb{A}^1_{\mathrm{an}}$. Berkovich points of the form $\|\cdot\|_x$ are called *Type I points*.

Next, for any (rational or irrational) closed disk $\overline{D}(a,r) \subseteq \mathbb{C}_v$, define $\zeta(a,r)$ to be

$$\|f\|_{\zeta(a,r)} := \sup\{|f(x)| : x \in \overline{D}(a,r)\},$$

which is the sup-norm of Definition 3.6. By Proposition 3.7, it is a multiplicative norm on $\overline{\mathcal{A}}(a,s)$ for any $s > r$. Hence, $\zeta(a,r)$ is a multiplicative norm on $\mathbb{C}_v[z] \subseteq \overline{\mathcal{A}}(a,s)$ extending $|\cdot|$. That is, $\zeta(a,r)$ is a point in $\mathbb{A}^1_{\mathrm{an}}$. Such points are called *Type II points* if $r \in |\mathbb{C}_v^\times|$, or *Type III points* otherwise.

There is also a fourth type of Berkovich point, at least in the case that $\mathbb{C}_v$ is not spherically complete; see Definition 2.6. Consider a decreasing sequence of closed disks $D_1 \supsetneq D_2 \supsetneq \cdots$ with empty intersection. Denoting the sup-norm on $\mathbb{C}_v[z]$ associated to D_n by ζ_n, define a corresponding multiplicative norm $\|\cdot\| : \mathbb{C}_v[z] \to [0,\infty)$ by

$$\|f\| := \lim_{n\to\infty} \|f\|_{\zeta_n}.$$

We leave it to the reader to check, in Exercise 6.2, that this limit converges, that $\|\cdot\|$ is a multiplicative norm extending $|\cdot|$, and that any equivalent sequence of closed disks $\{E_n\}$ gives the same norm. Here, we say that two

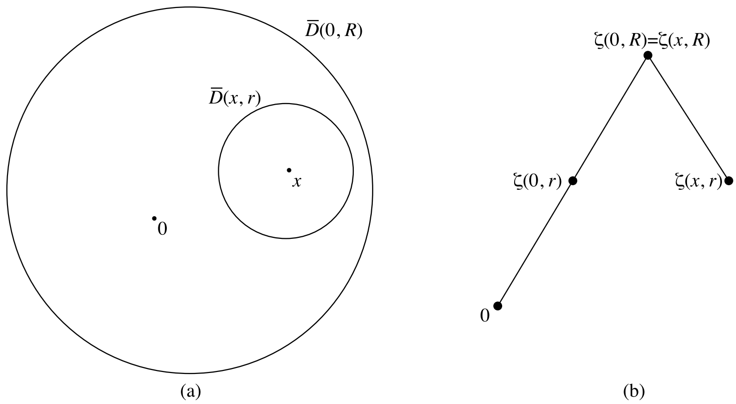

Figure 6.1. The path from 0 to $\zeta(x,r)$ in $\mathbb{A}^1_{\rm an}$

decreasing sequences $\{D_n\}$, $\{E_n\}$ of closed disks with empty intersection are *equivalent* if for every $N \geq 1$, there is some $M \geq N$ such that $E_M \subseteq D_N$ and $D_M \subseteq E_N$. A point $\zeta \in \mathbb{A}^1_{\rm an}$ is said to be of *Type IV* if it is associated in this way with an equivalence class of decreasing sequences of disks with empty intersection.

Even though we have not discussed the topology on $\mathbb{A}^1_{\rm an}$ carefully, we can already begin to understand why the space is path-connected. For example, given $x \in \mathbb{C}_v$ and $r > 0$ such that $0 < r < |x|$, we can construct a path from the Type I point 0 to the Type II or III point $\zeta(x,r)$ as follows.

Set $R = |x|$, and consider the arrangement of the points 0 and x and the disks $\overline{D}(x,r)$ and $\overline{D}(0,R)$ in Figure 6.1(a). To draw the path in Berkovich space, start at the Type I point 0, which we can informally think of as a disk of radius $s = 0$. Increase the radius s through a path consisting of Type II and III points of the form $\zeta(0,s)$ until $s = R$. Then $\zeta(0,R) = \zeta(x,R)$, because $\overline{D}(0,R) = \overline{D}(x,R)$. Thus, we may now consider Type II and III points $\zeta(x,s)$ corresponding to disks $\overline{D}(x,s)$ centered at s. By decreasing the radius s, we eventually arrive at the desired point $\zeta(x,r)$, corresponding to the disk $\overline{D}(x,r)$; see Figure 6.1(b).

6.1.2. Diameter and absolute value.

Definition 6.3. Let $\zeta \in \mathbb{A}^1_{\rm an}$. The *absolute value* of ζ is

$$|\zeta| := \|z\|_\zeta,$$

and the *diameter* of ζ is

$$\operatorname{diam}(\zeta) := \inf\left\{\|z-a\|_\zeta : a \in \mathbb{C}_v\right\}.$$

Clearly, the absolute value of a Type I point x coincides with the usual definition of $|x|$. On the other hand, the absolute value of a Type II or III point $\zeta(a,r)$ is $|\zeta(a,r)| = \max\{|a|, r\}$. If ζ is a Type IV point approximated by a decreasing sequence $\overline{D}(a_1,r_1) \supsetneq \overline{D}(a_2,r_2) \supsetneq \cdots$ of disks with empty intersection, then $|\zeta| = |\zeta(a_n,r_n)|$ for any n large enough that $0 \notin \overline{D}(a_n,r_n)$.

If $\zeta = \zeta(a,r)$ is the point of Type II or III corresponding to the disk $\overline{D}(a,r)$, then

$$\operatorname{diam}(\zeta) = r = \operatorname{diam}\left(\overline{D}(a,r)\right),$$

where the second use of diam above is the metric space diameter of the disk, justifying our use of the term "diameter" in this context. The diameter of a Type I point $x \in \mathbb{C}_v$ is 0, which is the same as the diameter of the set $\{x\}$. For a Type IV point ζ defined by a decreasing sequence $D_1 \supsetneq D_2 \supsetneq \cdots$, the reader can check that

$$\operatorname{diam}(\zeta) = \lim_{n\to\infty} \operatorname{diam}(D_n).$$

(See Exercise 6.3.) In particular, by Lemma 2.7(b), Type IV points have strictly positive diameter. Unlike points of the other three types, however, there is no corresponding subset of $\mathbb{C}_v$ whose diameter is $\operatorname{diam}(\zeta)$.

6.2. Disks in the Berkovich affine line

Just as $\mathbb{A}^1_{\text{an}}$ is a Berkovich space version of $\mathbb{A}^1(\mathbb{C}_v)$, we also have Berkovich versions of disks, as follows.

Definition 6.4. Let $a \in \mathbb{C}_v$, and let $r > 0$.

(a) The *closed Berkovich disk* $\overline{D}_{\text{an}}(a,r)$ is

$$\overline{D}_{\text{an}}(a,r) := \left\{\zeta \in \mathbb{A}^1_{\text{an}} : \|z-a\|_\zeta \le r\right\}.$$

(b) The *open Berkovich disk* $D_{\text{an}}(a,r)$ is

$$D_{\text{an}}(a,r) := \left\{\zeta \in \mathbb{A}^1_{\text{an}} : \|z-a\|_\zeta < r\right\}.$$

Open and closed disks both inherit the Gel'fand topology from $\mathbb{A}^1_{\text{an}}$. If $r \in |\mathbb{C}_v^\times|$, we say the Berkovich disks $\overline{D}_{\text{an}}(a,r)$ and $D_{\text{an}}(a,r)$ are *rational*; otherwise, they are *irrational*.

The names "open" and "closed" in Definition 6.4 provide some intuition about the Gel'fand topology, via the following proposition.

Proposition 6.5. *Let $a \in \mathbb{C}_v$ and $r > 0$. Then*

$$D_{\text{an}}(a,r) \textit{ is open, and } \overline{D}_{\text{an}}(a,r) \textit{ is closed in } \mathbb{A}^1_{\text{an}}.$$

Proof. Let $f(z) = z-a \in \mathbb{C}_v[z]$, and let $U := (-\infty, r) \subseteq \mathbb{R}$. By definition of the Gel'fand topology, the map $F : \mathbb{A}^1_{\text{an}} \to \mathbb{R}$ by $F(\zeta) := \|f\|_\zeta$ is continuous, and hence

$$D_{\text{an}}(a,r) = \left\{\zeta \in \mathbb{A}^1_{\text{an}} : \|z-a\|_\zeta < r\right\} = F^{-1}(U)$$

is an open subset of $\mathbb{A}^1_{\mathrm{an}}$.

Similarly, $V := (-\infty, r] \subseteq \mathbb{R}$ is closed in $\mathbb{R}$, and hence

$$\overline{D}_{\mathrm{an}}(a,r) = \left\{\zeta \in \mathbb{A}^1_{\mathrm{an}} : \|z-a\|_\zeta \le r\right\} = F^{-1}(V)$$

is closed in $\mathbb{A}^1_{\mathrm{an}}$. □

It is immediate from Definition 6.4 that $\overline{D}_{\mathrm{an}}(a,r)$ contains all Type I points of the form $\|\cdot\|_x$ for $x \in \overline{D}(a,r)$, that $D_{\mathrm{an}}(a,r)$ contains all Type I points of the form $\|\cdot\|_x$ for $x \in D(a,r)$, and that

$$D_{\mathrm{an}}(a,r) = \bigcup_{0<s<r} \overline{D}_{\mathrm{an}}(a,s).$$

We leave it to the reader to check that

$$\mathbb{A}^1_{\mathrm{an}} = \bigcup_{s>0} \overline{D}_{\mathrm{an}}(a,s) = \bigcup_{s>0} D_{\mathrm{an}}(a,s). \tag{6.1}$$

(See Exercise 6.4.) It is also easy to check that

- $\overline{D}(a,r) \subseteq \overline{D}(b,s)$ if and only if $\overline{D}_{\mathrm{an}}(a,r) \subseteq \overline{D}_{\mathrm{an}}(b,s)$,
- $D(a,r) \subseteq D(b,s)$ if and only if $D_{\mathrm{an}}(a,r) \subseteq D_{\mathrm{an}}(b,s)$,

and similarly if we replace the inclusions by equalities. In particular, the Berkovich disk $\overline{D}_{\mathrm{an}}(a,r)$ or $D_{\mathrm{an}}(a,r)$ is independent of the choice of center a of the corresponding closed disk $\overline{D}(a,r) \subseteq \mathbb{C}_v$ or, respectively, the corresponding open disk $D(a,r) \subseteq \mathbb{C}_v$.

Thus, there is a one-to-one correspondence between closed Berkovich disks and closed disks in $\mathbb{C}_v$, and between open Berkovich disks and open disks in $\mathbb{C}_v$. However, the reader should note that an important subtlety arises for irrational disks. Recall that if $r \notin |\mathbb{C}_v^\times|$, then the irrational closed and open disks $\overline{D}(a,r)$ and $D(a,r)$ in $\mathbb{C}_v$ coincide. However, the corresponding Berkovich disks $\overline{D}_{\mathrm{an}}(a,r)$ and $D_{\mathrm{an}}(a,r)$ differ, solely because the former contains the Type III point $\zeta(a,r)$ but the latter does not.

Of course, if $r \in |\mathbb{C}_v^\times|$, then the rational closed disk $\overline{D}_{\mathrm{an}}(a,r)$ contains *many* points not in $D_{\mathrm{an}}(a,r)$, including all the Type I points in the closed annulus $\{|z-a| = r\} := \overline{D}(a,r) \smallsetminus D(a,r)$. In fact, we have the following result.

Proposition 6.6. *Let $a \in \mathbb{C}_v$, and let $r > 0$. Then $\overline{D}_{\mathrm{an}}(a,r) \smallsetminus \{\zeta(a,r)\}$ is the disjoint union*

$$\overline{D}_{\mathrm{an}}(a,r) \smallsetminus \{\zeta(a,r)\} = \coprod_{c \in \mathcal{C}} D_{\mathrm{an}}(c,r),$$

where $\mathcal{C}$ is a set consisting of one representative from each open disk $D(c,r)$ of radius r contained in $\overline{D}(a,r)$.

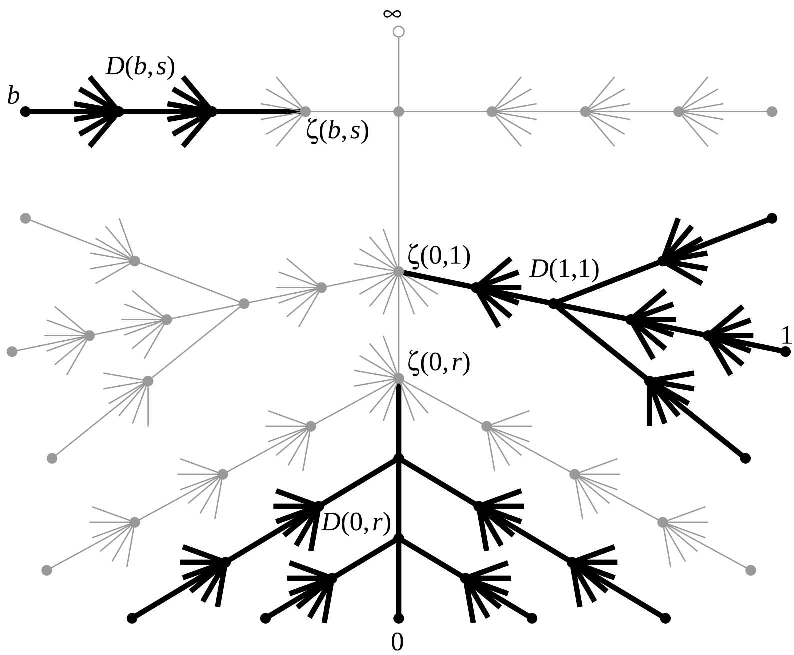

Figure 6.2. $\mathbb{A}^1_{\mathrm{an}}$ with three rational open disks highlighted

Proof. See Section 15.2. □

Figure 6.2 illustrates $\mathbb{A}^1_{\mathrm{an}}$, with three rational open disks highlighted: $D_{\mathrm{an}}(0,r)$ for some $0 < r < 1$; $D_{\mathrm{an}}(1,1)$; and $D_{\mathrm{an}}(b,s)$ for some $|b| > 1$ and $0 < s < |b|$. Note that each open disk has a unique boundary point (in this case, $\zeta(0,r)$, $\zeta(1,1) = \zeta(0,1)$, and $\zeta(b,s)$, respectively), and that the rational open disk consists of precisely one of the infinitely many branches emanating from that boundary point.

Remark 6.7. Proposition 6.6 implicitly distinguishes between points of Types II and III, as follows. Any closed Berkovich disk $\overline{D}_{\mathrm{an}}(a,r)$ has a corresponding point $\zeta(a,r)$ of Type II or III that we say *bounds* the disk. If $r \in |\mathbb{C}_v^\times|$, then there are infinitely many open disks $D(c,r)$ contained in $\overline{D}(a,r)$, corresponding to the infinitely many elements of the residue field. On the other hand, if $r \notin |\mathbb{C}_v^\times|$, then $D(a,r) = \overline{D}(a,r)$. Thus, a closed Berkovich disk bounded by a Type II point can be partitioned into infinitely many open subdisks of the same radius (along with the bounding point), whereas if the bounding point is Type III, there is only one open disk in the partition.

6.3. Berkovich's classification

We now formally define the four types of Berkovich points we described in Section 6.1.1. Recall our notation of the seminorms $\|\cdot\|_x$ and $\|\cdot\|_{\zeta(a,r)}$, given by $\|f\|_x := |f(x)|$ and

$$\|f\|_{\zeta(a,r)} := \sup\big\{|f(x)| : x \in D(a,r)\big\} = \sup\big\{|f(x)| : x \in \overline{D}(a,r)\big\}.$$

Definition 6.8. We say that a Berkovich point $\zeta \in \mathbb{A}^1_{\text{an}}$ is of

Type I if $\|\cdot\|_\zeta = \|\cdot\|_x$ for some $x \in \mathbb{C}_v$;

Type II if $\|\cdot\|_\zeta = \|\cdot\|_{\zeta(a,r)}$ for some rational closed disk $\overline{D}(a,r) \subseteq \mathbb{C}_v$;

Type III if $\|\cdot\|_\zeta = \|\cdot\|_{\zeta(a,r)}$ for some irrational disk $\overline{D}(a,r) \subseteq \mathbb{C}_v$; or

Type IV if $\|\cdot\|_\zeta = \lim_{n\to\infty} \|\cdot\|_{\zeta_n}$, where the norms $\{\zeta_n\}$ correspond to a decreasing sequence of closed disks in $\mathbb{C}_v$ with empty intersection.

Of the four types, note that only the Type I points fail to be norms.

Theorem 6.9 (Berkovich's classification)**.** *Let $\zeta \in \mathbb{A}^1_{\text{an}}$. Then ζ is of exactly one of the four types in Definition* 6.8*. Moreover, ζ determines the corresponding geometric data up to equivalence, in the sense that*

(a) *for points of Type I, $\|\cdot\|_x = \|\cdot\|_y$ if and only if $x = y$;*

(b) *for points of Type II or III, $\zeta(a,r) = \zeta(b,s)$ if and only if $\overline{D}(a,r) = \overline{D}(b,s)$;*

(c) *for points of Type IV, $\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} \xi_n$ if and only if the sequences $\{D_n\}$ and $\{E_n\}$ of closed disks corresponding to $\{\zeta_n\}$ and $\{\xi_n\}$ are equivalent.*

Proof. See Section 15.2. □

Berkovich proved Theorem 6.9 in [**26**, Section 1.4.4]; another presentation appears in [**8**, Theorem 1.2].

With Theorem 6.9 in hand to help us understand what Berkovich points are, we have the language needed to describe the structure of Berkovich disks, at least informally. Intuitively, the closed Berkovich disk $\overline{D}_{\text{an}}(a,r)$ looks like a tree branching downward from the top-most point $\zeta(a,r)$, with infinitely many branches at every Type II point, as described by Proposition 6.6 and Remark 6.7. More precisely, the branches at a Type II point $\zeta(b,s)$ correspond to the infinitely many open subdisks $D(c,s)$ of $\overline{D}(b,s)$ of the same radius, as well as (if $s < r$) one more branch extending upward, corresponding to increasing the radius. On the other hand, by the same proposition and remark, the Type III points are interior points with no branching.

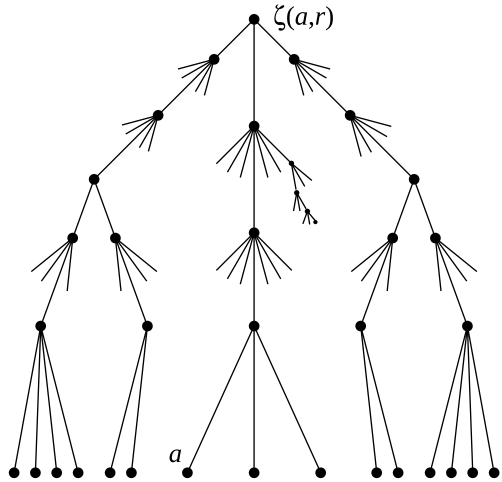

Figure 6.3. The closed Berkovich disk $\overline{D}_{\mathrm{an}}(a, r)$

The limbs of the tree end at the Type I and Type IV points. Figure 6.3 illustrates the tree, with Berkovich points of larger diameter appearing higher up in the figure. Only a few of the (Type II) branch points appear in Figure 6.3; in reality, such points would be dense in each line segment, with an infinite number of branches at each one. The tips of the branches appearing at the bottom of the tree are Type I points. Figure 6.3 also attempts to illustrate one Type IV point, which appears as a small chain of branches near the middle of the tree; the full Berkovich disk has such points scattered throughout. The Type IV point in question would be at the end of an infinite chain of such branches, but still with positive diameter and hence strictly above the bottom of the figure. Figure 6.3 also contains a diagram of the open Berkovich disk $D_{\mathrm{an}}(a, r)$, which is the single branch emanating from the point $\zeta(a, r)$ and containing the Type I point a.

Warning 6.10. The Gel'fand topology is weaker than the tree image might suggest at first. For example, the Type I points (at the tips of the branches) are dense in the full space, as we will see in Theorem 6.25. In particular, any open set containing a Type II point ζ must in fact contain all but finitely many of the branches emanating from ζ; see Proposition 6.19 and Lemma 15.6.

6.4. The Berkovich projective line

Just as the classical projective line in algebraic geometry can be defined by gluing two copies of the affine line $\mathbb{A}^1$ via the map $z \mapsto 1/z$, the Berkovich projective line can be defined by a similar gluing. First, however, we need

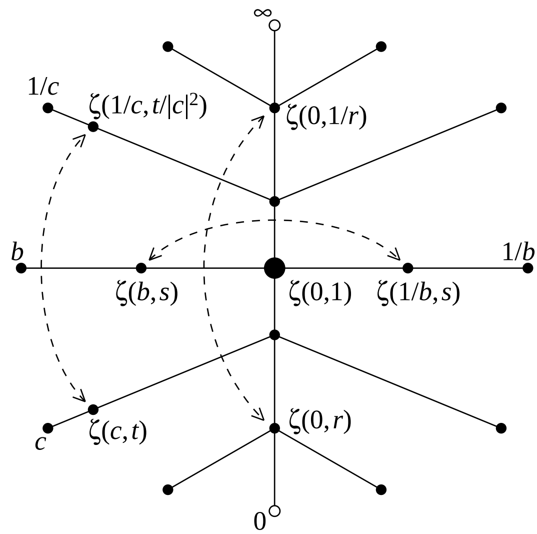

Figure 6.4. The map $\zeta \mapsto 1/\zeta$ on $\mathbb{A}^1_{\mathrm{an}} \setminus \{0\}$

to define the map $\zeta \mapsto 1/\zeta$, which we do as follows. This map is a function from $\mathbb{A}^1_{\mathrm{an}} \setminus \{0\}$ to itself, taking the seminorm $\|\cdot\|_\zeta$ on $\mathbb{C}_v[z]$ to the seminorm $\|\cdot\|_{1/\zeta}$ given by

$$\|f\|_{1/\zeta} := \frac{\|z^{\deg(f)} f(1/z)\|_\zeta}{\|z\|_\zeta^{\deg(f)}} \quad \text{for } f \in \mathbb{C}_v[z]. \tag{6.2}$$

We leave it to the reader to check that

- $\mathbb{A}^1_{\mathrm{an}} \setminus \{0\}$ is an open subset of $\mathbb{A}^1_{\mathrm{an}}$;
- $1/\zeta$, as defined in (6.2), is indeed an element of $\mathbb{A}^1_{\mathrm{an}} \setminus \{0\}$;
- the map $\zeta \mapsto 1/\zeta$ is a homeomorphism of $\mathbb{A}^1_{\mathrm{an}} \setminus \{0\}$; and
- the map $\zeta \mapsto 1/\zeta$ preserves Berkovich types (I–IV).

See Exercises 6.5, 6.6, and 6.7. Figure 6.4 shows how $\zeta \mapsto 1/\zeta$ acts on some points in $\mathbb{A}^1_{\mathrm{an}} \setminus \{0\}$, as described in Exercise 6.7.

Having verified the above properties of this map, we are now prepared to define $\mathbb{P}^1_{\mathrm{an}}$.

Definition 6.11. The Berkovich projective line $\mathbb{P}^1_{\mathrm{an}}$ is the topological space formed by gluing two copies of $\mathbb{A}^1_{\mathrm{an}}$ along their subsets $\mathbb{A}^1_{\mathrm{an}} \setminus \{0\}$ via the map $\zeta \mapsto 1/\zeta$ of equation (6.2).

As is common for the classical projective line $\mathbb{P}^1$, we often write

$$\mathbb{P}^1_{\mathrm{an}} = \mathbb{A}^1_{\mathrm{an}} \cup \{\infty\},$$

where ∞ is the point 0 on the second copy of $\mathbb{A}^1_{\mathrm{an}}$. Note that $\mathbb{A}^1_{\mathrm{an}}$ is an open subset of $\mathbb{P}^1_{\mathrm{an}}$, by the first bullet point above.

We now extend Definition 6.3 of the diameter and absolute value to every point $\zeta \in \mathbb{P}^1_{\mathrm{an}}$, by

$$|\zeta| := \begin{cases} |\zeta| & \text{if } \zeta \in \mathbb{A}^1_{\mathrm{an}}, \\ \infty & \text{if } \zeta = \infty, \end{cases} \quad \text{and} \quad \mathrm{diam}(\zeta) := \begin{cases} \mathrm{diam}(\zeta) & \text{if } \zeta \in \mathbb{A}^1_{\mathrm{an}}, \\ 0 & \text{if } \zeta = \infty. \end{cases}$$

In particular, a point $\zeta \in \mathbb{P}^1_{\mathrm{an}}$ is of Type I if and only if its diameter is zero.

Warning 6.12. The absolute value function $|\cdot| : \mathbb{P}^1_{\mathrm{an}} \to \mathbb{R} \cup \{\infty\}$ is continuous with respect to the Gel'fand topology, but the diameter function $\mathrm{diam} : \mathbb{P}^1_{\mathrm{an}} \to \mathbb{R}$ is *not*. The problem is not just the point at ∞; the diam function fails to be continuous on $\mathbb{A}^1_{\mathrm{an}}$ and on every affinoid. However, diam is upper semicontinuous on $\mathbb{A}^1_{\mathrm{an}}$; see Exercise 6.9.

Unlike the classical projective line from algebraic geometry, however, $\mathbb{P}^1_{\mathrm{an}}$ can also be constructed by gluing two closed unit *disks* via $\zeta \mapsto 1/\zeta$. The intuitive reason why this is possible in the non-archimedean setting is that the set $\{x \in \mathbb{P}^1(\mathbb{C}_v) : |x| = 1\}$ is an *open* subset of $\mathbb{P}^1(\mathbb{C}_v)$, unlike the unit circle $\{x \in \mathbb{P}^1(\mathbb{C}) : |x| = 1\}$ in the Riemann sphere.

Proposition 6.13. *The Berkovich projective line $\mathbb{P}^1_{\mathrm{an}}$ is homeomorphic to the space formed by gluing two copies of the Berkovich disk $\overline{D}_{\mathrm{an}}(0,1)$ along the Berkovich annulus*

$$\overline{D}_{\mathrm{an}}(0,1) \smallsetminus D_{\mathrm{an}}(0,1) = \big\{\zeta \in \mathbb{A}^1_{\mathrm{an}} : |\zeta| = 1\big\},$$

via the identification $\zeta \mapsto 1/\zeta$ of equation (6.2).

Proof. See Section 15.3. Incidentally, the function $\zeta \mapsto 1/\zeta$ maps this annulus $\{\zeta : |\zeta| = 1\}$ bijectively onto itself; see Exercise 6.10. ☐

Proposition 6.13 allows us to picture $\mathbb{P}^1_{\mathrm{an}}$ as $\overline{D}_{\mathrm{an}}(0,1)$ with an extra copy of the open tree $D_{\mathrm{an}}(0,1)$ attached to the top of the point $\zeta(0,1)$. The new top portion contains all points ζ of $\mathbb{P}^1_{\mathrm{an}}$ with $|\zeta| > 1$, including ∞. That is, it contains all Type I points $x \in \mathbb{P}^1(\mathbb{C}_v)$ with $|x| > 1$, all Type II and III points $\zeta(a,r)$ with $|a| > 1$ or $r > 1$, and all Type IV points corresponding to decreasing sequences of disks outside $\overline{D}(0,1)$. For a rough idea of the space, see Figure 6.5, which highlights the four Type I points $0, 1, \infty$, and a, for some $a \in \mathbb{C}_v$ with $0 < |a| < 1$. It also highlights several Type II points, including $\zeta(0,1)$ and $\zeta(0,|a|)$. As is the case with $\mathbb{A}^1_{\mathrm{an}}$, the space looks like a tree with infinite branching at a dense set of points (the Type II points) along every edge.

The infinite branching at each Type II point, as well as the lack of branching at Type III points, can be summed up more formally by the

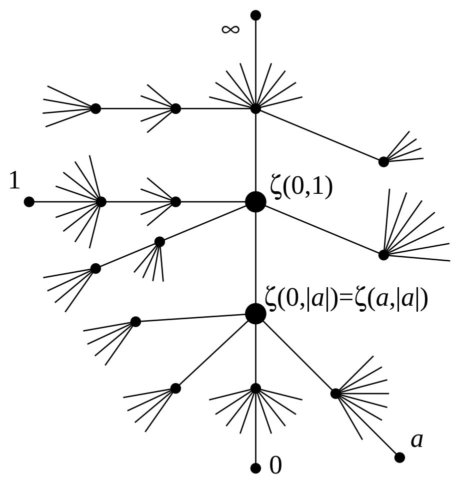

Figure 6.5. The Berkovich projective line $\mathbb{P}^1_{\text{an}}$

following extension of Proposition 6.6. To see the contrast between points of Type II and III, recall again that if $\zeta(a,r)$ is of Type III, then there is only one open disk of the form $D(c,r)$ contained in $\overline{D}(a,r)$; but if $\zeta(a,r)$ is of Type II, there are infinitely many.

Corollary 6.14. *Let $a \in \mathbb{C}_v$, and let $r > 0$. The set $\mathbb{P}^1_{\text{an}} \smallsetminus \{\zeta(a,r)\}$ formed by deleting the Type II or III point $\zeta(a,r)$ from the Berkovich projective line is the disjoint union*

$$\mathbb{P}^1_{\text{an}} \smallsetminus \{\zeta(a,r)\} = W_\infty \amalg \coprod_{c\in\mathcal{C}} D_{\text{an}}(c,r),$$

where $\mathcal{C}$ is a set consisting of one representative from each open disk $D(c,r)$ of radius r contained in $\overline{D}(a,r)$, and

$$W_\infty := \mathbb{P}^1_{\text{an}} \smallsetminus \overline{D}_{\text{an}}(a,r).$$

Proof. This is immediate from Proposition 6.6. □

Remark 6.15. The Type II point $\zeta(0,1)$ corresponding to the disk $\overline{D}(0,1)$ is not qualitatively different from any other Type II point. For example, we will see in Remark 7.11 that any Type II point may be moved to $\zeta(0,1)$ by a change of coordinates. Still, it is common to fix a choice of coordinate on $\mathbb{P}^1_{\text{an}}$, just as it is common to fix a choice of coordinate on the classical projective line. In that case, the point $\zeta(0,1)$ is often called the *Gauss point* (for example, in [**8**]) or the *canonical point* (for example, in [**114, 115, 117**]).

6.5. Disks and affinoids in $\mathbb{P}^1_{\mathrm{an}}$

We saw in Proposition 6.5 that open disks in $\mathbb{A}^1_{\mathrm{an}}$ are topologically open, and similarly for closed disks. That observation leads us to define the following Berkovich analogues of $\mathbb{P}^1(\mathbb{C}_v)$-disks and affinoids.

Definition 6.16.

(a) A *closed Berkovich disk* in $\mathbb{P}^1_{\mathrm{an}}$ is either a closed Berkovich disk $\overline{D}_{\mathrm{an}}(a,r)$ or the complement $\mathbb{P}^1_{\mathrm{an}} \smallsetminus D_{\mathrm{an}}(a,r)$ of an open Berkovich disk.

An *open Berkovich disk* in $\mathbb{P}^1_{\mathrm{an}}$ is either an open Berkovich disk $D_{\mathrm{an}}(a,r)$ or the complement $\mathbb{P}^1_{\mathrm{an}} \smallsetminus \overline{D}_{\mathrm{an}}(a,r)$ of a closed Berkovich disk.

If $r \in |\mathbb{C}_v^\times|$, we say the Berkovich disk is *rational*; otherwise, we say it is *irrational*.

(b) A *connected Berkovich affinoid* is a nonempty intersection of finitely many $\mathbb{P}^1_{\mathrm{an}}$-disks.

If $D_1, \ldots, D_n$ are closed (respectively, open, rational, irrational) Berkovich disks, we say that the connected Berkovich affinoid

$$D_1 \cap \cdots \cap D_n$$

is also *closed* (respectively, *open*, *rational*, *irrational*).

(c) A *Berkovich affinoid* is a nonempty finite union of connected Berkovich affinoids. If all of the components in the union are closed (respectively, open, rational, irrational) connected Berkovich affinoids, we say the resulting Berkovich affinoid is also *closed* (respectively, *open*, *rational*, *irrational*).

Figure 6.6 shows $\mathbb{P}^1_{\mathrm{an}} \smallsetminus (D_{\mathrm{an}}(1,1) \cup D_{\mathrm{an}}(b,s) \cup D_{\mathrm{an}}(0,r))$, which is a closed connected Berkovich affinoid obtained by removing three open disks from $\mathbb{P}^1_{\mathrm{an}}$; in fact, the disks removed are precisely the ones shown earlier in Figure 6.2. On the other hand, Figure 6.7 shows an open connected Berkovich affinoid formed by removing two closed disks from $\mathbb{P}^1_{\mathrm{an}}$. (*Note*: The radii r and s are different in the latter figure.) One of the two disks removed contains ∞, and thus the affinoid in question is the (Berkovich) annulus $U = D_{\mathrm{an}}(0,s) \smallsetminus \overline{D}_{\mathrm{an}}(a,r)$. Indeed, it is appropriate to call U an annulus, because it can also be written as

$$U = D_{\mathrm{an}}(a,s) \smallsetminus \overline{D}_{\mathrm{an}}(a,r) = \big\{\zeta \in \mathbb{P}^1_{\mathrm{an}} : r < \|z-a\|_\zeta < s\big\}.$$

Note that according to Definition 6.16, the full set $\mathbb{P}^1_{\mathrm{an}}$ is considered a connected Berkovich affinoid, since it is the intersection of zero disks. In addition, as noted above, all open Berkovich disks are open subsets of $\mathbb{P}^1_{\mathrm{an}}$, and closed Berkovich disks are closed. Hence, open Berkovich affinoids are

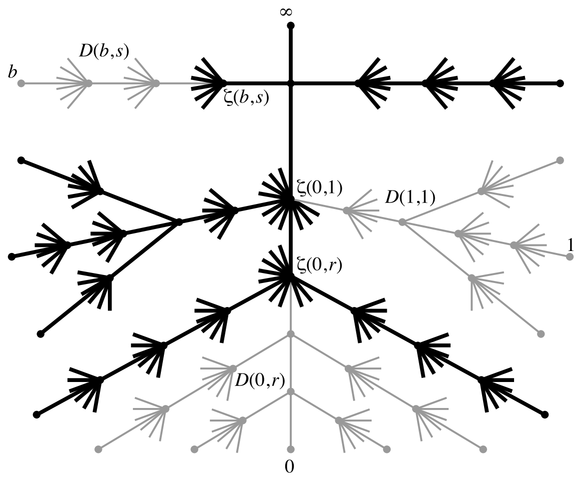

Figure 6.6. The closed connected Berkovich affinoid $\mathbb{P}^1_{\mathrm{an}} \setminus (D_{\mathrm{an}}(1,1) \cup D_{\mathrm{an}}(b,s) \cup D_{\mathrm{an}}(0,r))$

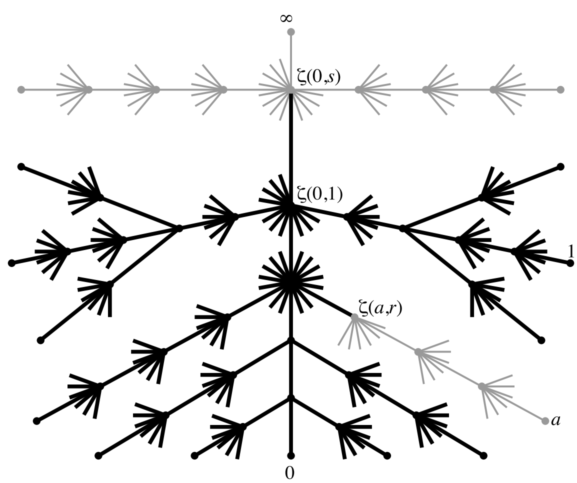

Figure 6.7. The open connected Berkovich affinoid $D_{\mathrm{an}}(0,s) \setminus \overline{D}_{\mathrm{an}}(a,r)$

open subsets, and closed Berkovich affinoids are closed. In addition, as we will see in Theorem 6.32, connected Berkovich affinoids are connected in the topological sense. For the moment, however, the following result shows that Berkovich affinoids provide us with a very useful characterization of the Gel'fand topology. Recall that a *basis* $\mathcal{B}$ for a topology $\mathcal{U}$ is a subset of $\mathcal{U}$ such that every element of $\mathcal{U}$ is a union of elements of $\mathcal{B}$.

Theorem 6.17. *The set of open connected Berkovich affinoids in* $\mathbb{P}^1_{\mathrm{an}}$ *forms a basis for the Gel'fand topology.*

Proof. See Section 15.3. □

Warning 6.18.

(a) The set of open Berkovich *disks* does *not* form a basis for the Gel'fand topology; see Exercise 6.12.

(b) There is a strong parallel between the $\mathbb{P}^1(\mathbb{C}_v)$ and the $\mathbb{P}^1_{\text{an}}$ versions of affinoids, including disks; but this parallel has its limitations. In particular, unlike affinoids in $\mathbb{P}^1(\mathbb{C}_v)$, closed Berkovich affinoids are *not* topologically open, and open Berkovich affinoids are *not* topologically closed. (That's good, because we will show in Theorem 6.32 that $\mathbb{P}^1_{\text{an}}$ is path-connected.)

(c) The closure of a rational open Berkovich disk is *not* the corresponding closed disk. Instead, the closure of the rational open disk $D_{\text{an}}(a,r)$ is $D_{\text{an}}(a,r) \cup \{\zeta(a,r)\}$. Since $r \in |\mathbb{C}_v^\times|$, the closure is *not* the closed disk $\overline{D}_{\text{an}}(a,r)$, which also contains infinitely many other open disks $D_{\text{an}}(c,r)$, for $c \in \{|z-a| = r\}$; see Proposition 6.6.

In light of Theorem 6.17, it is important to understand open connected Berkovich affinoids.

Proposition 6.19. *Let $U \subseteq \mathbb{P}^1_{\text{an}}$ be an open connected Berkovich affinoid.*

(a) *There is a finite set of pairwise disjoint closed Berkovich disks $\{V_1, \ldots, V_n\}$ such that*

$$U = \mathbb{P}^1_{\text{an}} \smallsetminus \big(V_1 \cup \cdots \cup V_n\big).$$

(b) *Let $\zeta = \zeta(a,r) \in U$ be a Type II point. Then U contains all but finitely many of the branches emanating from ζ. That is, U contains all but finitely many of the disks W_∞ and $D_{\text{an}}(c,r)$ for $c \in \overline{D}(a,r)$ from Corollary* 6.14.

In addition, for all *of the branches W emanating from ζ, we have $U \cap W \neq \varnothing$.*

Proof. This is left to the reader; see Exercise 6.11. □

Proposition 6.19(b) says that an open connected Berkovich affinoid U is obtained from the tree of Figure 6.5 by choosing a point ζ in the tree, making cuts at finitely many other points of Type II or III, and letting U be the portion still connected to ζ via some path through the tree. Note that when the cuts are made, the point ξ of the cut itself is removed, along with any branches off ξ, besides the one containing ζ.

Example 6.20. Choose $a \in \mathbb{C}_v$ with $0 < |a| < 1$, as in Figure 6.5. Let

$$U := \{|a| < |\zeta| < 1\} = D_{\text{an}}(0,1) \smallsetminus \overline{D}_{\text{an}}(0,|a|),$$

which is an open Berkovich annulus. (That is, it is the open connected Berkovich affinoid formed by intersecting the two open disks $D_{\text{an}}(0,1)$ and $\mathbb{P}^1_{\text{an}} \smallsetminus \overline{D}_{\text{an}}(0,|a|)$.) Let $\zeta = \zeta(0,|a|^{1/2}) \in U$, which lies on the line segment between $\zeta(0,|a|)$ and $\zeta(0,1)$ in Figure 6.5. Then U is formed by cutting the tree at $\zeta(0,|a|)$ and $\zeta(0,1)$, and only keeping the portion that is still connected to our chosen point ζ.

Thus, U appears in Figure 6.5 as the interval from $\zeta(0,|a|)$ to $\zeta(0,1)$, *not* including the two endpoints. Of course, there are infinitely branches (not shown in Figure 6.5) emanating from each of the infinitely many Type II points in that interval. However, $\zeta(0,1)$ and all of its branches have been removed, with the exception of the one pointing downward toward 0. (This does not contradict Proposition 6.19(b), because $\zeta(0,1) \notin U$.) Similarly, $\zeta(0,|a|)$ has also been removed along with all of its branches other than the one pointing upward toward ∞.

On the other hand, $\zeta = \zeta(0,|a|^{1/2})$ is a Type II point in U, and we now confirm that it satisfies the conclusions of Proposition 6.19(b). Although U does *not* contain the branch $W_\infty := \mathbb{P}^1_{\text{an}} \smallsetminus \overline{D}_{\text{an}}(0,|a|^{1/2})$, it *does* intersect it, for example, at $\zeta(0,|a|^{1/4})$ and at any Type I point x with $|a|^{1/2} < |x| < 1$. Similarly, U does not contain the branch $D_{\text{an}}(0,|a|^{-1/2})$, but it intersects it at points of absolute value strictly between $|a|$ and $|a|^{1/2}$. Every other branch off ζ is an open Berkovich disk of the form $V = D_{\text{an}}(c,|a|^{1/2})$ with $|c| = |a|^{1/2}$, and we have $V \subseteq U$, as predicted by Proposition 6.19(b).

The topological boundary of a Berkovich disk is a single point of Type II or III. Similarly, the boundary of a Berkovich affinoid is a finite set of points of Types II and III; see Exercise 6.19. For instance, in Example 6.20, the boundary of the annulus $U = \{|a| < |\zeta| < 1\}$ consists of the two points $\zeta(0,|a|)$ and $\zeta(0,1)$. The line segment between these two points is called the *skeleton* of U; see Exercise 6.29 for the definition of skeleta of arbitrary connected affinoids.

Among its many uses, Theorem 6.17 allows us to prove the next two results, which together say that $\mathbb{P}^1_{\text{an}}$ is a Hausdorff compactification of $\mathbb{P}^1(\mathbb{C}_v)$. In particular, in light of the following theorem, we hereafter consider $\mathbb{P}^1(\mathbb{C}_v)$ to be simply the set of Type I points in $\mathbb{P}^1_{\text{an}}$.

Theorem 6.21. *The function $i : \mathbb{P}^1(\mathbb{C}_v) \to \mathbb{P}^1_{\text{an}}$ given by*

$$i(x) = \begin{cases} \|\cdot\|_x & \text{if } x \in \mathbb{C}_v, \\ \infty & \text{if } x = \infty, \end{cases}$$

is an embedding of topological spaces. Moreover, the image $i(\mathbb{P}^1(\mathbb{C}_v))$ is precisely the set of Type I points of $\mathbb{P}^1_{\text{an}}$.

Proof. See Section 15.4. □

Theorem 6.22. *The Berkovich projective line* $\mathbb{P}^1_{\text{an}}$ *is compact and Hausdorff.*

Proof. See Section 15.4. □

As a compact Hausdorff space, $\mathbb{P}^1_{\text{an}}$ is also normal, meaning that any two disjoint closed subsets can be separated by open sets. Incidentally, it follows fairly quickly from the definitions that any two disjoint closed Berkovich affinoids can be separated by open Berkovich affinoids; see Exercise 6.20.

Remark 6.23. The reader can easily check the following relationship between Berkovich disks and their classical counterparts in $\mathbb{P}^1(\mathbb{C}_v)$; see Exercise 6.15. For any $a \in \mathbb{C}_v$ and any $r > 0$,

$$\begin{aligned} \overline{D}_{\text{an}}(a,r) \cap \mathbb{P}^1(\mathbb{C}_v) &= \overline{D}(a,r), \\ D_{\text{an}}(a,r) \cap \mathbb{P}^1(\mathbb{C}_v) &= D(a,r), \end{aligned} \tag{6.3}$$

and similarly for the disks $\mathbb{P}^1_{\text{an}} \smallsetminus \overline{D}_{\text{an}}(a,r)$ and $\mathbb{P}^1_{\text{an}} \smallsetminus D_{\text{an}}(a,r)$. Since every disk in $\mathbb{P}^1_{\text{an}}$ or $\mathbb{P}^1(\mathbb{C}_v)$ is of this form, every Berkovich disk gives us a classical disk, and every classical disk comes from a Berkovich disk.

This relationship is almost a one-to-one correspondence between Berkovich disks and classical disks, but not quite. The same subtlety we noted about irrational disks in Section 6.2 (just before Proposition 6.6) is again the issue. That is, if $r \notin |\mathbb{C}_v^\times|$, then

$$\overline{D}_{\text{an}}(a,r) \cap \mathbb{P}^1(\mathbb{C}_v) = D_{\text{an}}(a,r) \cap \mathbb{P}^1(\mathbb{C}_v) = \overline{D}(a,r) = D(a,r),$$

even though $\overline{D}_{\text{an}}(a,r) \supsetneq D_{\text{an}}(a,r)$. The same thing happens for the disks $\mathbb{P}^1_{\text{an}} \smallsetminus \overline{D}_{\text{an}}(a,r)$ and $\mathbb{P}^1_{\text{an}} \smallsetminus D_{\text{an}}(a,r)$. Fortunately, this hiccup for irrational disks is the only obstacle to the correspondence being one-to-one; see Exercise 6.15(b).

Similarly, if $W \subseteq \mathbb{P}^1_{\text{an}}$ is a connected Berkovich affinoid, then its set of Type I points $W \cap \mathbb{P}^1(\mathbb{C}_v)$ is a classical connected affinoid, as can be easily deduced by writing W as an intersection of disks and applying equation (6.3) to each disk. Of course, if some of the disks are irrational, then the hiccup above makes things a little more complicated. Fortunately, if we specify in advance that the disks involved are either all open or all closed, then the ambiguity vanishes; see, for example, Exercise 6.28. Thus, we now set the convention that when switching back and forth between Berkovich affinoids and their classical analogues, then unless otherwise specified, **we consider each irrational disk in $\mathbb{C}_v$ to be open**.

Remark 6.24. The reader can easily check that a point $\zeta \in \mathbb{A}^1_{\text{an}}$ belongs to a given rational Berkovich disk $D \subseteq \mathbb{A}^1_{\text{an}}$ if and only if the geometric data corresponding to ζ is contained in the associated classical disk $D_{\text{I}} := D \cap \mathbb{C}_v$;

see Exercise 6.16. That is,

- if $\zeta = x \in \mathbb{C}_v$ is of Type I, we need $x \in D_{\mathrm{I}}$;
- if $\zeta = \zeta(b, s)$ is of Type II or III, we need $\overline{D}(b, s) \subseteq D_{\mathrm{I}}$; and
- if ζ is of Type IV, we need all but finitely many of the disks approximating ζ to lie in D_{I}.

The condition for ζ to belong to an irrational Berkovich disk $D \subseteq \mathbb{A}^1_{\mathrm{an}}$ is the same, with the one exception that if $D = D_{\mathrm{an}}(a, r)$ is an irrational *open* Berkovich disk, then $\zeta(a, r) \notin D$ even though $\overline{D}(a, r) \subseteq D \cap \mathbb{C}_v$.

For a Berkovich affinoid W, the situation is a bit more complicated. If $\zeta \in \mathbb{P}^1_{\mathrm{an}}$ is of Type I or IV, then it turns out that again, $\zeta \in W$ if and only if the geometric data corresponding to ζ is contained in the associated classical affinoid $W_{\mathrm{I}} := W \cap \mathbb{P}^1(\mathbb{C}_v)$; see Exercise 6.17. However, if ζ is of Type II or III, it is entirely possible that $\zeta \in W$ but the disk corresponding to ζ does *not* sit inside W. For example, if $0 < r < s < t$ and W is the annulus $W = \{r < |\zeta| < t\}$, then $\zeta(0, s) \in W$, even though $\overline{D}(0, s) \not\subseteq W$.

Nevertheless, it is easy to check whether a given point ζ lies in W, simply by checking whether ζ lies in each Berkovich disk defining W.

The following result states that each of the four types of points forms a dense subset of a Berkovich disk. We leave the proof as an exercise.

Theorem 6.25.

(a) *The set of Type I points in $\mathbb{P}^1_{\mathrm{an}}$ forms a dense subset of $\mathbb{P}^1_{\mathrm{an}}$.*

(b) *The set of Type II points in $\mathbb{P}^1_{\mathrm{an}}$ forms a dense subset of $\mathbb{P}^1_{\mathrm{an}}$.*

(c) *If $|\mathbb{C}_v^\times| \neq (0, \infty)$, then the set of Type III points in $\mathbb{P}^1_{\mathrm{an}}$ forms a dense subset of $\mathbb{P}^1_{\mathrm{an}}$.*

(d) *If $\mathbb{C}_v$ is not spherically complete, then the set of Type IV points in $\mathbb{P}^1_{\mathrm{an}}$ forms a dense subset of $\mathbb{P}^1_{\mathrm{an}}$.*

Proof. See Exercise 15.7. □

6.6. Paths and path-connectedness

In this section we discuss a key property of connected Berkovich affinoids, including disks and $\mathbb{P}^1_{\mathrm{an}}$ itself, that we have stated only informally so far: these spaces are path-connected.

Definition 6.26. Let $\zeta, \xi \in \mathbb{A}^1_{\mathrm{an}}$. We say that ζ *lies above* ξ, and we write $\xi \preceq \zeta$, if $\|\cdot\|_\xi$ is bounded by $\|\cdot\|_\zeta$; that is, if

$$\|f\|_\xi \leq \|f\|_\zeta \quad \text{for all } f \in \mathbb{C}_v[z].$$

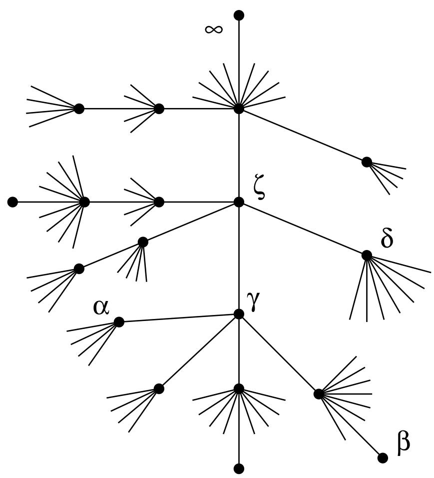

Figure 6.8. Lying above: $\alpha, \beta \preceq \gamma$ and $\alpha, \beta, \gamma, \delta \preceq \zeta$.

Lying above a seminorm is thus exactly the same as bounding it. We only use the different notation because "bounding" places an emphasis on the ring and the seminorm, where as "lying above" places the emphasis on ζ as a geometric point in a tree. In particular, the relation $\xi \preceq \zeta$ corresponds to the point ζ literally lying *above* ξ in our the tree image of $\mathbb{P}^1_{\text{an}}$. For example, in Figure 6.8, we have $\alpha, \beta \preceq \gamma$ and $\alpha, \beta, \gamma, \delta \preceq \zeta$.

Proposition 6.27. *Let $\preceq$ denote the relation on $\mathbb{A}^1_{\text{an}}$ from Definition* 6.26.

(a) *The relation $\preceq$ defines a partial order on $\mathbb{A}^1_{\text{an}}$.*

(b) *All Type I and IV points are minimal with respect to $\preceq$.*

(c) *For any $\xi \in \mathbb{A}^1_{\text{an}}$ and any closed Berkovich disk $\overline{D}_{\text{an}}(a, r)$, we have $\xi \in \overline{D}_{\text{an}}(a, r)$ if and only if $\xi \preceq \zeta(a, r)$.*

(d) *For any $\xi, \zeta \in \mathbb{A}^1_{\text{an}}$ with $\xi \preceq \zeta$, we have* $\operatorname{diam}(\xi) \leq \operatorname{diam}(\zeta)$, *with equality if and only if $\xi = \zeta$.*

Proof. See Section 15.5. □

Among other things, Proposition 6.27 confirms our intuition about the locations of the various types of points in the tree. Specifically, the Type I and IV points in $\mathbb{P}^1_{\text{an}}$ are at the endpoints of their branches, while points of Type II and III have some points lying above them *and* some points lying below them.

The following definition is useful for finding the unique path between two Berkovich points.

Definition 6.28. Let $\zeta_0, \zeta_1 \in \mathbb{A}^1_{\mathrm{an}}$. The *least upper bound* of ζ_0 and ζ_1, denoted $\zeta_0 \vee \zeta_1$, is the unique element of $\mathbb{A}^1_{\mathrm{an}}$ such that

(a) $\zeta_0 \preceq \zeta_0 \vee \zeta_1$ and $\zeta_1 \preceq \zeta_0 \vee \zeta_1$;

(b) if $\xi \in \mathbb{A}^1_{\mathrm{an}}$ satisfies $\zeta_0 \preceq \xi$ and $\zeta_1 \preceq \xi$, then $\zeta_0 \vee \zeta_1 \preceq \xi$.

For example, in Figure 6.8, we have $\alpha \vee \beta = \gamma$. The point ζ also satisfies $\alpha \preceq \zeta$ and $\beta \preceq \zeta$ and hence is an upper bound for α and β, but $\gamma \preceq \zeta$ is the least upper bound. Similarly, $\alpha \vee \delta = \gamma \vee \delta = \beta \vee \delta = \zeta$.

Proposition 6.29. *Let $\zeta_0, \zeta_1 \in \mathbb{A}^1_{\mathrm{an}}$. Then the least upper bound $\zeta_0 \vee \zeta_1$ exists and is unique.*

Proof. See Section 15.5. $\square$

Proposition 6.29 provides us with the key idea for showing that $\mathbb{P}^1_{\mathrm{an}}$ is path-connected. Given two points $\zeta, \zeta' \in \mathbb{A}^1_{\mathrm{an}}$ with $\zeta \preceq \zeta'$, the set

$$I := \left\{ \xi \in \mathbb{A}^1_{\mathrm{an}} : \zeta \preceq \xi \preceq \zeta' \right\}$$

is homeomorphic to a compact real interval, as the tree image suggests; see Exercise 6.24. (That is, I is either a single point or else homeomorphic to $[0,1]$.) Thus, given any two points $\zeta_0, \zeta_1 \in \mathbb{A}^1_{\mathrm{an}}$, there is an interval I_0 from ζ_0 up to $\zeta_0 \vee \zeta_1$, and an interval I_1 from ζ_0 up to $\zeta_0 \vee \zeta_1$. Their intersection $I_0 \cap I_1$ consists solely of the point $\zeta_0 \vee \zeta_1$; the union $I_0 \cup I_1$ is then a path from ζ_0 to ζ_1.

Example 6.30. Let $x \in \mathbb{C}_v \smallsetminus \{0\}$, set $R := |x|$, and let $r \in (0, R)$. Then Figure 6.1(b) on page 124 shows the path from the Type I point 0 to the Type II or III point $\zeta(x, r)$, as we described informally in Section 6.1. More precisely, the least upper bound of the two points is

$$0 \vee \zeta(x, r) = \zeta(0, R). \tag{6.4}$$

Proposition 6.27 allows us to confirm equation (6.4), as follows. First, $\zeta(0, R)$ lies above both points, as $0 \in \overline{D}(0, R)$ and $\overline{D}(x, r) \subseteq \overline{D}(0, R)$. Second, any point ζ lying above both 0 and $\zeta(x, r)$ must be of Type II or III and correspond to a disk containing both 0 and $\overline{D}(x, r)$. That is, ζ must be of the form $\zeta(0, s)$, with $s \geq R$. Clearly, $\zeta(0, R)$ is the minimal such point, verifying equation (6.4). Thus, the path shown in Figure 6.1(b) is indeed the path described more generally above: I_0 is the leg running from 0 up to $0 \vee \zeta(x, r) = \zeta(0, R)$, and I_1 is the leg running from $\zeta(0, R)$ back down to $\zeta(x, r)$.

Of course, there are details to check: that each of I_0 and I_1 is indeed homeomorphic to a real interval, that the path lies inside any connected Berkovich affinoid that contains both ζ_0 and ζ_1, and so on. In addition, because the path is in fact unique, we need one more definition before we state the result formally.

Definition 6.31. Let X be a topological space. We say X is *uniquely path-connected* if for any two distinct points $x_0, x_1 \in X$, there is a unique subset $I \subseteq X$ containing x_0 and x_1 such that I is homeomorphic to the real closed interval $[0, 1]$, with the homeomorphism mapping x_0 to 0 and x_1 to 1.

Theorem 6.32. *Let $U \subseteq \mathbb{P}^1_{\text{an}}$ be a connected Berkovich affinoid. Then U is uniquely path-connected. Moreover, for any $\zeta_0, \zeta_1 \in U$, all points on the unique arc connecting them, other than ζ_0 and ζ_1 themselves, are of Type II or III.*

Proof. See Section 15.5. □

Corollary 6.33. *$\mathbb{P}^1_{\text{an}}$ is locally path-connected.*

Proof. Given $\zeta \in \mathbb{P}^1_{\text{an}}$ and an open set U containing ζ, there is an open connected affinoid V such that $\zeta \in V \subseteq U$, by Theorem 6.17. Then V is a path-connected neighborhood of ζ contained in U, by Theorem 6.32. □

Definition 6.34. Let $\zeta_0, \zeta_1 \in \mathbb{P}^1_{\text{an}}$. The unique arc from ζ_0 to ζ_1 is called the (*closed*) *interval* from ζ_0 to ζ_1 and is denoted $[\zeta_0, \zeta_1] \subseteq \mathbb{P}^1_{\text{an}}$. We say that any point $\xi \in [\zeta_0, \zeta_1]$ *lies between* ζ_0 and ζ_1. If $\xi \in [\zeta_0, \zeta_1]$, but $\xi \neq \zeta_0, \zeta_1$, we say that ξ *lies strictly between* ζ_0 and ζ_1.

If $\zeta_0 = \zeta_1$ in Definition 6.34, then the interval $[\zeta_0, \zeta_1]$ is the singleton $\{\zeta_0\}$. Otherwise, if $\zeta_0 \neq \zeta_1$, we can of course define the half-open and open intervals $[\zeta_0, \zeta_1)$, $(\zeta_0, \zeta_1]$, and (ζ_0, ζ_1) by removing one or both endpoints in the obvious way. The reader should be cautious about the use of the word "open" here, however. On the one hand, the closed interval $[\zeta_0, \zeta_1]$ is indeed closed in Gel'fand topology, since it is a compact subset of the Hausdorff space $\mathbb{P}^1_{\text{an}}$. On the other hand, none of $[\zeta_0, \zeta_1)$, $(\zeta_0, \zeta_1]$, or (ζ_0, ζ_1) is either open or closed in the Gel'fand topology; see Exercise 6.35.

Note that the sets $[\zeta_0, \zeta_1]$ and $[\zeta_1, \zeta_0]$ both make sense and coincide, because in contrast to $\mathbb{R}$, there is no canonical positive or negative direction on $\mathbb{P}^1_{\text{an}}$. In addition, for any distinct points $\zeta_0, \zeta_1 \in \mathbb{P}^1_{\text{an}}$, it is immediate from Definitions 6.31 and 6.34 that $[\zeta_0, \zeta_1]$ is homeomorphic to the interval $[0, 1] \subseteq \mathbb{R}$. If ξ lies strictly between ζ_0 and ζ_1, then by the final statement of Theorem 6.32, ξ must be a point of Type II or III.

Proposition 6.35. *Let $\zeta_0, \zeta_1, \zeta_2 \in \mathbb{P}^1_{\text{an}}$. Then*

$$[\zeta_0, \zeta_1] \cap [\zeta_0, \zeta_2] \cap [\zeta_1, \zeta_2]$$

consists of a single point. Moreover, if none of ζ_0, ζ_1, or ζ_2 lies between the other two, then this single point is of Type II.

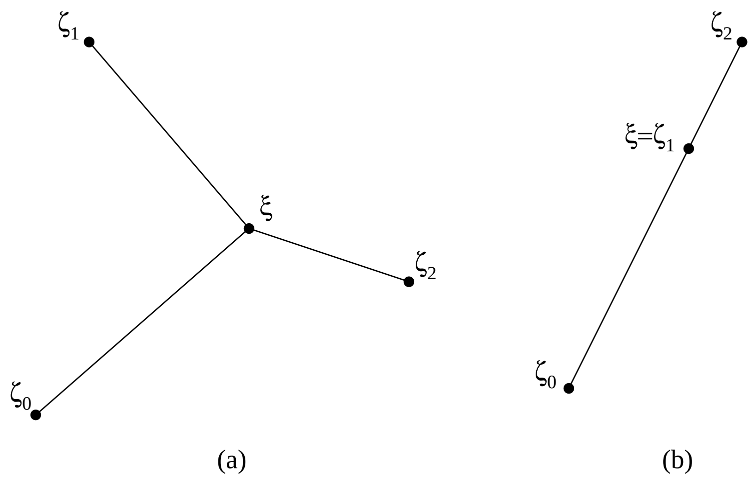

Figure 6.9. ξ lies between ζ_0, ζ_1, and ζ_2.

Proof. See Section 15.6. □

Equivalently, Proposition 6.35 says that for any $\zeta_0, \zeta_1, \zeta_2 \in \mathbb{P}^1_{\text{an}}$, there is a unique point $\xi \in \mathbb{P}^1_{\text{an}}$ that simultaneously

- lies between ζ_0 and ζ_1,
- lies between ζ_0 and ζ_2,
- lies between ζ_1 and ζ_2.

For example, in Figure 6.9(a), the point ξ is the unique point lying between any two of ζ_0, ζ_1, and ζ_2. The same is true in Figure 6.9(b), only this time the point ξ coincides with ζ_1, since ζ_1 already lies between ζ_0 and ζ_2.

Proposition 6.35 generalizes the least upper bound $\zeta_0 \vee \zeta_1$ of Definition 6.28. After all, if $\zeta_0, \zeta_1 \in \mathbb{A}^1_{\text{an}}$, then $\zeta_0 \vee \zeta_1$ is the unique point of Proposition 6.35 that lies between all three of ζ_0, ζ_1, and ∞; see Exercise 6.26.

The notion of lying between also allows us to state the following useful definition.

Definition 6.36. Let $S \subseteq \mathbb{P}^1_{\text{an}}$ be any subset of the Berkovich projective line. The *convex hull* of S is the set of all points $\xi \in \mathbb{P}^1_{\text{an}}$ for which there exist $\zeta_0, \zeta_1 \in S$ such that ξ lies between ζ_0 and ζ_1.

Equivalently, the convex hull of S is the union of all arcs that start and end at points of S. It is immediate, then, that the convex hull of S is path-connected. In fact, it is the smallest connected subset of $\mathbb{P}^1_{\text{an}}$ containing S. That is, it is the intersection of all connected subsets of $\mathbb{P}^1_{\text{an}}$ containing S.

Remark 6.37. The spaces $\mathbb{P}^1_{\text{an}}$ and $\mathbb{A}^1_{\text{an}}$, as well as every Berkovich disk and affinoid, are examples of $\mathbb{R}$-*trees*, which are certain topological inverse limits of the better-known finite trees from graph theory. Like finite trees, $\mathbb{R}$-trees are always uniquely path-connected; but like $\mathbb{P}^1_{\text{an}}$, they allow infinite branching and dense sets of branch points. For more on $\mathbb{R}$-trees, especially in the context of Berkovich disks, see [**8**, Section 1.4, Appendix B] or [**58**].

6.7. Directions at Berkovich points

The following result generalizes Corollary 6.14. It emphasizes that in the tree structure on $\mathbb{P}^1_{\text{an}}$, the points of Types I and IV are endpoints, the points of Type II are branch points with infinite branching, and the points of Type III are unbranched interior points.

Proposition 6.38. *Let $\zeta \in \mathbb{P}^1_{\text{an}}$. The number of path-connected components of $\mathbb{P}^1_{\text{an}} \smallsetminus \{\zeta\}$ is*

(a) *one, if ζ is of Type I or IV;*

(b) *two, if ζ is of Type III; or*

(c) *infinite, if ζ is of Type II.*

Proof.

(a) If ζ is of Type I or IV, then by the final statement of Theorem 6.32, ζ does not lie along the unique arc between any two points $\zeta_0, \zeta_1 \in \mathbb{P}^1_{\text{an}} \smallsetminus \{\zeta\}$. Thus, $\mathbb{P}^1_{\text{an}} \smallsetminus \{\zeta\}$ is path-connected.

(b) By Corollary 6.14, if ζ is of Type III, $\mathbb{P}^1_{\text{an}} \smallsetminus \{\zeta\}$ consists of two irrational open Berkovich disks, both of which are path-connected, by Theorem 6.32.

(c) By Corollary 6.14, if ζ is of Type II, $\mathbb{P}^1_{\text{an}} \smallsetminus \{\zeta\}$ consists of infinitely many rational open Berkovich disks, each of which is path-connected, by Theorem 6.32. □

Proposition 6.38 inspires the following definition.

Definition 6.39. Let $\zeta \in \mathbb{P}^1_{\text{an}}$. The path-connected components of $\mathbb{P}^1_{\text{an}} \smallsetminus \{\zeta\}$ are called the *residue classes* of ζ or the *directions* at ζ.

If $x \in \mathbb{P}^1_{\text{an}} \smallsetminus \{\zeta\}$, we denote the direction at ζ containing x by $\vec{v}_\zeta(x)$, or simply $\vec{v}(x)$ if the point ζ is clear from context.

The term "residue class" comes from the fact that if $\zeta = \zeta(0,1)$ is the Gauss point, then the components of $\mathbb{P}^1_{\mathrm{an}} \smallsetminus \{\zeta\}$ consist of open disks $D_{\mathrm{an}}(c,1)$ with $|c| \le 1$, as well as the disk at infinity, $\mathbb{P}^1_{\mathrm{an}} \smallsetminus \overline{D}_{\mathrm{an}}(0,1)$. The corresponding $\mathbb{P}^1(\mathbb{C}_v)$-disks are precisely the residue classes arising from the reduction map $\mathbb{P}^1(\mathbb{C}_v) \mapsto \mathbb{P}^1(\overline{k})$.

On the other hand, the term "direction" is meant to evoke the tree image: starting from ζ, one can move in any of the directions emanating from ζ. Indeed, the notation $\vec{v}_\zeta(x)$ is intended to emphasize that even though the object in question is a subset of $\mathbb{P}^1_{\mathrm{an}}$, we should think of it as a vector, pointing along an edge of the tree, from ζ toward x.

Recall from Theorem 6.17 that the open connected Berkovich affinoids form a basis for the topology on $\mathbb{P}^1_{\mathrm{an}}$. In particular, if $U \subseteq \mathbb{P}^1_{\mathrm{an}}$ is an open set and if $\zeta \in U$, then U contains an open connected affinoid containing ζ, and hence U intersects all of the residue classes at ζ. Moreover, U contains all but finitely many of the residue classes at ζ.

Remark 6.40. In [**8**], the directions (or *tangent vectors*, as Baker and Rumely call them) at a point $\zeta \in \mathbb{P}^1_{\mathrm{an}}$ are defined in a different way than in our Definition 6.39. Specifically, a tangent vector at ζ is an equivalence class of intervals of the form $[\zeta,\xi]$, where $\xi \in \mathbb{P}^1_{\mathrm{an}} \smallsetminus \{\zeta\}$, and where two intervals $[\zeta,\xi_1]$ and $[\zeta,\xi_2]$ are said to be equivalent if their intersection consists of more than just the point ζ.

It is not difficult to show that there is a natural one-to-one correspondence between our directions at ζ and the tangent vectors of Baker and Rumely at ζ: the tangent vector represented by $[\zeta,\xi]$ corresponds to the direction containing ξ; see Exercise 6.33. Our definition is perhaps a little simpler, but Baker and Rumely's definition has the advantage that it suggests the intuition of direction on a graph more explicitly.

6.8. The hyperbolic metric

In [**114**, **115**, **117**], Rivera-Letelier noted a parallel between the set of Berkovich points of Types II–IV and the three-dimensional real hyperbolic space that has the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ as its boundary. He therefore dubbed this subset of $\mathbb{P}^1_{\mathrm{an}}$ to be a non-archimedean *hyperbolic* space, and he noted that it has a natural metric, as follows.

Definition 6.41. The set

$$\mathbb{H} := \mathbb{P}^1_{\mathrm{an}} \smallsetminus \mathbb{P}^1(\mathbb{C}_v)$$

is the *hyperbolic space* over $\mathbb{C}_v$. The function $d_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to [0,\infty)$ given by

$$d_{\mathbb{H}}(\zeta,\xi) := 2\log\big(\operatorname{diam}(\zeta \vee \xi)\big) - \log\big(\operatorname{diam}(\zeta)\big) - \log\big(\operatorname{diam}(\xi)\big)$$

is the *hyperbolic metric*.

The hyperbolic metric $d_{\mathbb{H}}$ is also called the "big model metric" in [**8**], and simply "la distance" in [**115**, **117**].

Theorem 6.42. *The function* $d_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to [0, \infty)$ *is a metric on* $\mathbb{H}$.

Proof. See Section 15.6. □

Warning 6.43. Of course, the metric $d_{\mathbb{H}}$ induces a topology on $\mathbb{H}$. However, this metric topology is much stronger than the Gel'fand topology on $\mathbb{H}$; see Exercise 6.34. In the literature, therefore, the metric topology is sometimes called the "strong topology," in which case the Gel'fand topology is called the "weak topology."

To help explain the formula of Definition 6.41, first consider the special case that $\zeta = \zeta(a, R)$ and $\xi = \zeta(a, r)$ for some point $a \in \mathbb{C}_v$ and real numbers $0 < r < R$. Then ζ lies above ξ, and hence $\zeta \vee \xi = \zeta$. Since $\mathrm{diam}(\zeta) = R$ and $\mathrm{diam}(\xi) = r$, Definition 6.41 for $d_{\mathbb{H}}$ gives

$$d_{\mathbb{H}}(\zeta, \xi) = 2 \log R - \log R - \log r = \log R - \log r.$$

Thus, $d_{\mathbb{H}}(\zeta, \xi)$ is the logarithm of the ratio R/r which, as we saw in Remark 3.35, is invariant under the action of bijective Laurent series. That is, the distance from ζ to ξ is the (logarithmic) conformal invariant of the unique open Berkovich annulus bounded by ζ and ξ.

More generally, *any* two distinct points $\zeta, \xi \in \mathbb{H}$ not of Type IV define an open annulus, as follows. Let U_1 be the component of $\mathbb{P}^1_{\mathrm{an}} \smallsetminus \{\zeta\}$ containing ξ, and let U_2 be the component of $\mathbb{P}^1_{\mathrm{an}} \smallsetminus \{\xi\}$ containing ζ. Then U_1 and U_2 are both open Berkovich disks in $\mathbb{P}^1_{\mathrm{an}}$, and their intersection is an open Berkovich annulus. As we will see in Proposition 7.17, $d_{\mathbb{H}}$ is invariant under the application of linear fractional maps; hence, $d_{\mathbb{H}}(\zeta, \xi)$ is again the logarithm of the conformal invariant of this annulus.

For the moment, however, note that the formula of Definition 6.41 makes intuitive sense, even when $\zeta = \zeta(a, r)$ and $\xi = \zeta(b, s)$, but neither point lies above the other. In that case, their least upper bound is

$$\zeta \vee \xi = \zeta(a, R) = \zeta(b, R),$$

where $R := |a - b|$. We have that $R > r, s$, because of our specification that neither of ζ nor ξ lies above the other. Since the unique path from ζ to ξ passes through $\zeta \vee \xi$, it is only appropriate to define

$$\begin{aligned} d_{\mathbb{H}}(\zeta, \xi) &= d_{\mathbb{H}}(\zeta \vee \xi, \zeta) + d_{\mathbb{H}}(\zeta \vee \xi, \xi) = \log R - \log r + \log R - \log s \\ &= 2 \log\big(\mathrm{diam}(\zeta \vee \xi)\big) - \log\big(\mathrm{diam}(\zeta)\big) - \log\big(\mathrm{diam}(\xi)\big), \end{aligned}$$

thus giving precisely the formula in Definition 6.41.

Exercises for Chapter 6

6.1. For any $x \in \mathbb{C}_v$, prove that the function $\|\cdot\|_x : \mathbb{C}_v[z] \to [0,\infty)$ given by $\|f\|_x = |f(x)|$ is a multiplicative seminorm extending the absolute value $|\cdot|$ on $\mathbb{C}_v$.

6.2. Let $D_1 \supsetneq D_2 \supsetneq \cdots$ be a decreasing sequence of closed disks with empty intersection, and define $\|\cdot\| : \mathbb{C}_v[z] \to [0,\infty)$ by

$$\|f\| := \lim_{n\to\infty} \|f\|_{\zeta_n}, \tag{6.5}$$

where ζ_n is the sup-norm on D_n.

(a) Prove that for each $f \in \mathbb{C}_v[z]$, the limit in equation (6.5) converges.

(b) Prove that $\|\cdot\|$ is a multiplicative norm (not just a seminorm) extending the absolute value $|\cdot|$ on $\mathbb{C}_v$.

(c) Let $E_1 \supsetneq E_2 \supsetneq \cdots$ be an equivalent sequence of disks. That is, for every $N \geq 1$, there is some $M \geq N$ such that $E_M \subseteq D_N$ and $D_M \subseteq E_N$. Prove that the norm on $\mathbb{C}_v[z]$ corresponding to $\{E_n\}_{n\geq 1}$ equals the norm $\|\cdot\|$ corresponding to $\{D_n\}_{n\geq 1}$.

6.3. In this exercise, you will confirm some claims made after Definition 6.3.

(a) Let $\zeta = x$ be a point of Type I. Prove that

$$|\zeta| = |x| \quad \text{and} \quad \operatorname{diam}(\zeta) = 0.$$

(b) Let $\zeta = \zeta(a,r)$ be a point of Type II or III. Prove that

$$|\zeta| = \max\{|a|, r\} \quad \text{and} \quad \operatorname{diam}(\zeta) = r.$$

(c) If ζ is a Type IV point determined by a decreasing sequence

$$\overline{D}(a_1,r_1) \supsetneq \overline{D}(a_2,r_2) \supsetneq \cdots$$

of disks with empty intersection, prove that

(i) $|\zeta| = |\zeta(a_n,r_n)|$ for any n such that $0 \notin \overline{D}(a_n,r_n)$, and

(ii) $\operatorname{diam}(\zeta) = \lim_{n\to\infty} r_n$.

6.4. For any $a \in \mathbb{C}_v$, prove equation (6.1).

6.5. Prove that for any Type I point $x \in \mathbb{C}_v$, the set $\{x\}$ is closed in $\mathbb{A}^1_{\text{an}}$. That is, prove that $\mathbb{A}^1_{\text{an}} \smallsetminus \{x\}$ is open.

(Do *not* use results from Section 6.4 and beyond. Some of those results rely on this exercise, including Theorem 6.22, that $\mathbb{P}^1_{\text{an}}$ is Hausdorff.)

6.6. Given $\zeta \in \mathbb{A}^1_{\text{an}} \smallsetminus \{0\}$, define a function $\|\cdot\|_{1/\zeta} : \mathbb{C}_v[z] \to [0,\infty)$ by equation (6.2), which we reproduce here:

$$\|f\|_{1/\zeta} := \left\| z^{\deg(f)} f(1/z) \right\|_{\zeta} \Big/ \|z\|_{\zeta}^{\deg(f)} \quad \text{for } f \in \mathbb{C}_v[z].$$

(a) Prove that $\|\cdot\|_{1/\zeta}$ is a multiplicative seminorm on $\mathbb{C}_v[z]$ extending $|\cdot|$ and that $\|\cdot\|_{1/\zeta} \neq \|\cdot\|_0$. In other words, prove that $1/\zeta \in \mathbb{A}^1_{\text{an}} \smallsetminus \{0\}$.

(b) Prove that $\zeta \mapsto 1/\zeta$ is a homeomorphism from $\mathbb{A}^1_{\text{an}} \smallsetminus \{0\}$ to itself.

(Do *not* use results from Section 6.4 and beyond, as some of those results rely on this exercise.)

6.7. Consider the map $\zeta \mapsto 1/\zeta$ on $\mathbb{A}^1_{\text{an}} \smallsetminus \{0\}$ given by equation (6.2).

(a) Prove that $\zeta \mapsto 1/\zeta$ maps a Type I point $x \in \mathbb{P}^1(\mathbb{C}_v)$ to $1/x$.

(b) Prove that $\zeta \mapsto 1/\zeta$ maps a Type II or III point $\zeta(a,r)$ to

$$\begin{cases} \zeta(0,1/r) & \text{if } 0 \in \overline{D}(a,r), \\ \zeta(1/a, r/|a|^2) & \text{otherwise.} \end{cases}$$

(c) Describe the action of $\zeta \mapsto 1/\zeta$ on a Type IV point corresponding to a chain of disks $D_1 \supsetneq D_2 \supsetneq \cdots$.

(d) Use parts (a)–(c) to confirm that $\zeta \mapsto 1/\zeta$ preserves the type of the Berkovich point ζ.

(e) Prove that $|1/\zeta| = 1/|\zeta|$ for all $\zeta \in \mathbb{A}^1_{\text{an}} \smallsetminus \{0\}$.

(Do *not* use results from Section 6.4 and beyond, as some of those results rely on this exercise.)

6.8. Let $D \subseteq \mathbb{P}^1_{\text{an}}$ be an open Berkovich disk. Prove that D can be written as a nested union $\bigcup_{n\geq 1} D_n$ of closed Berkovich disks

$$D_1 \subseteq D_2 \subseteq \cdots \subseteq \mathbb{P}^1_{\text{an}}.$$

6.9. In this exercise, you will confirm several claims made in Warning 6.12.

(a) Prove that the absolute value function $|\cdot| : \mathbb{P}^1_{\text{an}} \to \mathbb{R}$ is continuous.

(b) Prove that the diameter function $\operatorname{diam} : \mathbb{P}^1_{\text{an}} \to \mathbb{R}$ is *not* continuous even when restricted to $\mathbb{A}^1_{\text{an}}$ or to any Berkovich affinoid. (*Suggestion*: Use Theorem 6.17.)

(c) Prove that $\operatorname{diam} : \mathbb{A}^1_{\text{an}} \to \mathbb{R}$ is upper semicontinuous. That is, for any $t \in \mathbb{R}$, prove that the set $\{\zeta \in \mathbb{A}^1_{\text{an}} : \operatorname{diam}(\zeta) < t\}$ is open.

(d) Prove that $\operatorname{diam} : \mathbb{P}^1_{\text{an}} \to \mathbb{R}$ is *not* upper semicontinuous.

6.10. Let $U = \{\zeta \in \mathbb{A}^1_{\text{an}} : |\zeta| = 1\}$ be the annulus of Proposition 6.13. Prove that the function $\zeta \mapsto 1/\zeta$ of Section 6.4 maps U bijectively onto itself.

6.11. Prove Proposition 6.19.

6.12. Verify Warning 6.18(a): the set of open Berkovich disks does *not* form a basis for the Gel'fand topology for either $\mathbb{A}^1_{\text{an}}$ or $\mathbb{P}^1_{\text{an}}$.

(*Suggestion*: Prove that an annulus $\{\zeta \in \mathbb{P}^1_{\text{an}} : r < |\zeta| < s\}$ is open in $\mathbb{P}^1_{\text{an}}$ but cannot be written as a union of Berkovich disks.)

6.13. Let X be a dense subset of $\mathbb{C}_v$, and let Y be a dense subset of $(0,\infty)$. Let $\mathcal{D}$ be the set of open Berkovich disks of the form

$$D_{\mathrm{an}}(a,r) \quad \text{or} \quad \mathbb{P}^1_{\mathrm{an}} \smallsetminus \overline{D}_{\mathrm{an}}(a,r) \quad \text{with } a \in X \text{ and } r \in Y.$$

Let $\mathcal{B}$ be the set of nonempty intersections of finitely many elements of $\mathcal{D}$. Prove that $\mathcal{B}$ is a basis for the Gel'fand topology on $\mathbb{P}^1_{\mathrm{an}}$.

In particular, if $\mathbb{C}_v$ is separable (i.e., has a countable dense subset), then $\mathbb{P}^1_{\mathrm{an}}$ is second countable (i.e., has a countable basis).

6.14. Let $a \in \mathbb{C}_v$ and $r > 0$.

(a) If $r \notin |\mathbb{C}_v^\times|$, prove that the interior of $\overline{D}_{\mathrm{an}}(a,r)$ is $D_{\mathrm{an}}(a,r)$.

(b) If $r \in |\mathbb{C}_v^\times|$, prove that the interior of $\overline{D}_{\mathrm{an}}(a,r)$ is the union of the infinitely many disjoint open disks $D_{\mathrm{an}}(b,r)$, for $b \in \overline{D}(a,r)$.

6.15. In this exercise, you will confirm several claims made in Remark 6.23.

(a) Prove equation (6.3): for any $a \in \mathbb{C}_v$ and $r > 0$,

$$\overline{D}_{\mathrm{an}}(a,r) \cap \mathbb{P}^1(\mathbb{C}_v) = \overline{D}(a,r), \qquad \text{and}$$
$$D_{\mathrm{an}}(a,r) \cap \mathbb{P}^1(\mathbb{C}_v) = D(a,r).$$

(b) Let $D_1 \neq D_2 \subseteq \mathbb{P}^1_{\mathrm{an}}$ be Berkovich disks that are different but for which $D_1 \cap \mathbb{P}^1(\mathbb{C}_v) = D_2 \cap \mathbb{P}^1(\mathbb{C}_v)$. Prove that there is some $a \in \mathbb{C}_v$ and $r \in (0,\infty) \smallsetminus |\mathbb{C}_v^\times|$ such that D_1 and D_2 are either

$$\begin{cases} \overline{D}_{\mathrm{an}}(a,r) \quad \text{and} \quad D_{\mathrm{an}}(a,r), & \text{or} \\ \mathbb{P}^1_{\mathrm{an}} \smallsetminus \overline{D}_{\mathrm{an}}(a,r) \quad \text{and} \quad \mathbb{P}^1_{\mathrm{an}} \smallsetminus D_{\mathrm{an}}(a,r). \end{cases}$$

6.16. Let $D \subseteq \mathbb{A}^1_{\mathrm{an}}$ be a Berkovich disk, and let $D_{\mathrm{I}} := D \cap \mathbb{C}_v$ be the associated classical disk.

(a) For any Type I point $x \in \mathbb{C}_v$, prove that $x \in D$ if and only if $x \in D_{\mathrm{I}}$.

(b) For any Type II point $\zeta = \zeta(a,r)$, prove that $\zeta \in D$ if and only if $\overline{D}(a,r) \subseteq D_{\mathrm{I}}$.

(c) For any Type III point $\zeta = \zeta(a,r)$, prove that $\zeta \in D$ if and only if $\overline{D}(a,r) \subseteq D_{\mathrm{I}}$, *unless* $D = D_{\mathrm{an}}(a,r)$.

(d) For any Type IV point ζ corresponding to $D_1 \supsetneq D_2 \supsetneq \cdots$ in $\mathbb{C}_v$, prove that $\zeta \in D$ if and only if $D_n \subseteq D_{\mathrm{I}}$ for some $n \geq 1$.

6.17. Let $W \subseteq \mathbb{P}^1_{\mathrm{an}}$ be a Berkovich connected affinoid, and let W_{I} be the associated classical affinoid $W_{\mathrm{I}} := W \cap \mathbb{P}^1(\mathbb{C}_v)$.

(a) For any Type I point $x \in \mathbb{C}_v$, prove that $x \in W$ if and only if $x \in W_{\mathrm{I}}$.

(b) For any Type IV point ζ corresponding to a decreasing sequence of disks $D_1 \supsetneq D_2 \supsetneq \cdots$ in $\mathbb{C}_v$, $\zeta \in D$ if and only if $D_n \subseteq W_{\mathrm{I}}$ for some $n \geq 1$.

(c) If W is not a disk contained in $\mathbb{A}^1_{\mathrm{an}}$, prove that there are points $\zeta = \zeta(a,r)$ of Type II such that $\zeta \in W$ but $\overline{D}(a,r) \not\subseteq W_{\mathrm{I}}$.

6.18. Let $\zeta \in \mathbb{C}_v$, and let S be a set of disks in $\mathbb{P}^1_{\mathrm{an}}$ that each contain ζ. Define

$$U := \bigcup_{D \in S} D.$$

Prove that U is

(a) $\mathbb{P}^1_{\mathrm{an}} \smallsetminus \{\xi\}$ for some point $\xi \in \mathbb{P}^1_{\mathrm{an}}$ of Type I or IV,

(b) a disk D of S,

(c) an open Berkovich disk, or

(d) all of $\mathbb{P}^1_{\mathrm{an}}$.

(See also Exercise 2.14.)

6.19. This exercise concerns boundaries of Berkovich affinoids.

(a) Let $D \subseteq \mathbb{P}^1_{\mathrm{an}}$ be either the open Berkovich disk $D_{\mathrm{an}}(a,r)$ or the closed Berkovich disk $\overline{D}_{\mathrm{an}}(a,r)$. Prove that the boundary ∂D consists of the single point $\zeta(a,r)$. Note that this single boundary point is of Type II if D is rational and of Type III if D is irrational.

(b) Let $W \subseteq \mathbb{P}^1_{\mathrm{an}}$ be a connected Berkovich affinoid. Prove that its boundary ∂W is a finite set of points of Type II and III.

(c) Let $S \subseteq \mathbb{P}^1_{\mathrm{an}}$ be a finite set of points of Type II and III, and let $\zeta \in \mathbb{P}^1_{\mathrm{an}} \smallsetminus S$. Prove that the connected component of $\mathbb{P}^1_{\mathrm{an}} \smallsetminus S$ containing ζ is a connected Berkovich affinoid.

6.20. This exercise concerns separation properties of Berkovich affinoids.

(a) Let $U, V \subseteq \mathbb{P}^1_{\mathrm{an}}$ be disjoint closed Berkovich affinoids. Prove that there are disjoint open Berkovich affinoids $U', V' \subseteq \mathbb{P}^1_{\mathrm{an}}$ such that $U \subseteq U'$ and $V \subseteq V'$.

(b) Let $U \subseteq \mathbb{P}^1_{\mathrm{an}}$ be an open connected Berkovich affinoid, and let $V \subseteq U$ be a closed connected Berkovich affinoid that is neither a disk nor all of $\mathbb{P}^1_{\mathrm{an}}$. Prove that $U \smallsetminus V$ is not connected.

6.21. Let $\zeta, \xi \in \mathbb{P}^1_{\mathrm{an}}$. Prove that $\zeta \vee \xi = \zeta$ if and only if $\xi \preceq \zeta$.

6.22. Let $\xi \in \mathbb{P}^1_{\mathrm{an}}$, $a \in \mathbb{C}_v$, and $r > 0$. Prove that

$$\operatorname{diam}\big(\zeta(a,r) \vee \xi\big) = \max\big\{r, \|z-a\|_\xi\big\}.$$

6.23. Let $\zeta, \xi \in \mathbb{P}^1_{\mathrm{an}}$.

(a) Prove that for all $a \in \mathbb{C}_v$, we have

$$\|z-a\|_{\zeta\vee\xi} = \max\left\{\|z-a\|_\zeta, \|z-a\|_\xi\right\}.$$

(b) Let $\overline{D}_{\mathrm{an}}(a,r)$ be the smallest disk containing both ζ and ξ. Prove that $\operatorname{diam}(\zeta \vee \xi) = r$.

(c) If neither ζ nor ξ lies above the other, prove that there is some $f \in \mathbb{C}_v[z]$ such that

$$\big\|f\big\|_{\zeta\vee\xi} \neq \max\left\{\|f\|_\zeta, \|f\|_\xi\right\}.$$

6.24. Let $\zeta, \zeta' \in \mathbb{A}^1_{\mathrm{an}}$ with $\zeta \preceq \zeta'$, and let

$$I := \left\{\xi \in \mathbb{A}^1_{\mathrm{an}} : \zeta \preceq \xi \preceq \zeta'\right\}.$$

(a) If $\zeta \neq \zeta'$, prove that I is homeomorphic to the real interval $[0,1]$.

(b) If $\zeta = \zeta'$, prove that $I = \{\zeta\}$ is a single point.

6.25. Let S be a connected subset of $\mathbb{P}^1_{\mathrm{an}}$. Prove that the following two conditions are equivalent.

(a) S is an interval of the form

$$[\zeta_0, \zeta_1], \quad [\zeta_0, \zeta_1), \quad \text{or} \quad (\zeta_0, \zeta_1)$$

for some $\zeta_0, \zeta_1 \in \mathbb{P}^1_{\mathrm{an}}$.

(b) For any three points in S, one lies between the other two.

6.26. In this exercise, you will verify a claim made shortly after Proposition 6.35. Let $\zeta_0, \zeta_1 \in \mathbb{A}^1_{\mathrm{an}}$. Prove that the least upper bound $\zeta_0 \vee \zeta_1$ lies between any two of ζ_0, ζ_1, and ∞.

6.27. Let S be any subset of $\mathbb{P}^1_{\mathrm{an}}$. Prove that the convex hull of S is connected; see Definition 6.36.

6.28. Let W be a connected Berkovich affinoid, and let

$$V := W \cap \mathbb{P}^1(\mathbb{C}_v).$$

(a) If W is closed, prove that W is the closure of V in $\mathbb{P}^1_{\mathrm{an}}$.

(b) If W is open, prove that W is the interior of the closure of V in $\mathbb{P}^1_{\mathrm{an}}$.

(c) If W is open or rational closed, prove that W is the convex hull of V.

(d) If W is irrational closed, prove that the interior of W is the convex hull of V.

6.29. In this exercise, we define *skeleta* of Berkovich affinoids.

(a) Let $U \subset \mathbb{P}^1_{\text{an}}$ be an open connected Berkovich affinoid. Let T be the convex hull of the boundary ∂U. (By Exercise 6.19, ∂U is a finite set of points of Type II and III.) Prove that T is a finite tree with endpoints at the points of ∂U.
(*Note*: We allow $T = \varnothing$ to be empty or $T = \{\zeta\}$ to be a single point.)
The finite tree $T \subseteq \mathbb{P}^1_{\text{an}}$ is called the *skeleton* of U.

(b) Show that the mapping from

$$\{\text{open connected Berkovich affinoids } U \subsetneq \mathbb{P}^1_{\text{an}}\}$$

to

$$\{\text{finite trees } T \subseteq \mathbb{P}^1_{\text{an}} \text{ with endpoints of Type II or III}\}$$

given in part (a) is bijective.

6.30. Let S be any subset of $\mathbb{P}^1(\mathbb{C}_v)$, and let $\overline{S}$ be its closure in $\mathbb{P}^1_{\text{an}}$. For any Type II point $\zeta = \zeta(a,r) \in \mathbb{P}^1_{\text{an}}$, prove that $\zeta \in \overline{S}$ if and only if at least one of the following two conditions holds:

(a) S intersects infinitely many residue classes of ζ; or

(b) there is some $b \in \overline{D}(a,r)$ such that the set

$$\bigl\{|x-b| \,:\, x \in S,\ |x-b| \neq r\bigr\} \subseteq \mathbb{R}$$

has an accumulation point at r.

6.31. Let S be any subset of $\mathbb{P}^1(\mathbb{C}_v)$, and let $\overline{S}$ be its closure in $\mathbb{P}^1_{\text{an}}$. For any Type III point $\zeta = \zeta(a,r) \in \mathbb{P}^1_{\text{an}}$, prove that $\zeta \in \overline{S}$ if and only if the set

$$\bigl\{|x-a| \,:\, x \in S\bigr\} \subseteq \mathbb{R}$$

has an accumulation point at r.

6.32. Let S be any subset of $\mathbb{P}^1(\mathbb{C}_v)$, and let $\overline{S}$ be its closure in $\mathbb{P}^1_{\text{an}}$. Let $\zeta \in \mathbb{P}^1_{\text{an}}$ be a Type IV point represented by a chain of disks $D_1 \supsetneq D_2 \supsetneq \cdots$. Prove that $\zeta \in \overline{S}$ if and only if $D_n \cap S \neq \varnothing$ for all $n \geq 1$.

6.33. Fix a point $\zeta \in \mathbb{P}^1_{\text{an}}$. In Remark 6.40, we noted that in [**8**], Baker and Rumely define a tangent vector at ζ to be an equivalence class of intervals of the form $[\zeta,\xi]$, where $\xi \in \mathbb{P}^1_{\text{an}} \smallsetminus \{\zeta\}$, and where two intervals $[\zeta,\xi_1]$ and $[\zeta,\xi_2]$ are said to be equivalent if their intersection consists of more than just the point ζ.

Define a map from the set of Baker–Rumely tangent vectors at ζ to the set of directions at ζ by sending the equivalence class of the interval $[\zeta,\xi]$ to the unique direction at ζ containing ξ. Prove that this map is well-defined, one-to-one, and onto.

6.34. Show that every subset $U \subseteq \mathbb{H}$ that is open in the Gel'fand (weak) topology on $\mathbb{H}$ is also open in the metric (strong) topology on $\mathbb{H}$, as mentioned in Warning 6.43.

On the other hand, show that no open ball in the metric topology is open in the Gel'fand topology.

6.35. Let ζ_0, ζ_1 be distinct points in $\mathbb{P}^1_{\text{an}}$.

(a) Prove that none of $[\zeta_0, \zeta_1)$, $(\zeta_0, \zeta_1]$, or (ζ_0, ζ_1) is open in the Gel'fand topology.

(b) Prove that none of $[\zeta_0, \zeta_1)$, $(\zeta_0, \zeta_1]$, or (ζ_0, ζ_1) is closed in the Gel'fand topology.

(c) If ζ_0 and ζ_1 both lie in $\mathbb{H}$, prove that none of the three intervals above is either open or closed in the metric topology.

Chapter 7

Rational Functions and Berkovich Space

Having discussed $\mathbb{P}^1_{\text{an}}$ as a space in Chapter 6, we now study the action of a rational function $\phi \in \mathbb{C}_v(z)$ on $\mathbb{P}^1_{\text{an}}$, which we define in Theorem 7.1 and Definition 7.2 below. (More generally, any morphism between $\mathbb{C}_v$-varieties extends to a continuous map between the corresponding Berkovich spaces; see [**26**, Section 3.4].) We also discuss local degrees at Berkovich points, generalizing the multiplicity of ϕ at a point of $\mathbb{P}^1(\mathbb{C}_v)$.

7.1. The action of rational functions

Recall from Definition 6.2 that a point in $\mathbb{A}^1_{\text{an}}$ is a multiplicative seminorm $\|\cdot\|_\zeta$ on the polynomial ring $\mathbb{C}_v[z]$ which restricts to $|\cdot|$ on $\mathbb{C}_v$. We may extend $\|\cdot\|_\zeta$ to most or all rational functions $h \in \mathbb{C}_v(z)$ as follows. Write $h = f/g$ with $f, g \in \mathbb{C}_v[z]$. If $\|g\|_\zeta \neq 0$, then define $\|h\|_\zeta := \|f\|_\zeta / \|g\|_\zeta$. See Exercise 7.1 for details.

Theorem 7.1. *Let $\phi \in \mathbb{C}_v(z)$ be a rational function, and let $\zeta \in \mathbb{A}^1_{\text{an}}$. Unless ζ is a Type I point $\zeta = a \in \mathbb{C}_v$ for which $\phi(a) = \infty$, then the function*

$$\|\cdot\|_{\phi(\zeta)} : \mathbb{C}_v[z] \to [0, \infty) \quad \text{by} \quad \|f\|_{\phi(\zeta)} := \|f \circ \phi\|_\zeta \tag{7.1}$$

is a multiplicative seminorm on $\mathbb{C}_v[z]$ extending $|\cdot|$ on $\mathbb{C}_v$.

Proof. See Section 16.1. □

We can now define the action of a rational function $\phi \in \mathbb{C}_v(z)$ on $\mathbb{P}^1_{\text{an}}$. We also denote this extended map ϕ.

Definition 7.2. Let $\phi \in \mathbb{C}_v(z)$ be a rational function, and let $\zeta \in \mathbb{P}^1_{\text{an}}$. Define $\phi(\zeta) \in \mathbb{P}^1_{\text{an}}$ to be

(a) the Type I point $\phi(\infty)$ if $\zeta = \infty$,

(b) the Type I point ∞ if $\zeta = a$ is of Type I and $\phi(a) = \infty$, or

(c) the point $\phi(\zeta) \in \mathbb{A}^1_{\text{an}}$ of equation (7.1) otherwise.

Remark 7.3. The reader can check that for the map $\eta(z) = 1/z$ and for any $\zeta \in \mathbb{A}^1_{\text{an}} \smallsetminus \{0\}$, the point $\eta(\zeta)$ given by Definition 7.2 coincides with the point $1/\zeta$ defined in equation (6.2). In fact, viewing $\mathbb{P}^1_{\text{an}}$ as two copies of $\mathbb{A}^1_{\text{an}}$ glued via $1/z$, one can alternatively define the action of $\phi \in \mathbb{C}_v(z)$ on $\mathbb{P}^1_{\text{an}}$, by gluing appropriately chosen affine patches; see Exercise 7.2. In a general treatment of Berkovich space morphisms, such a definition would be more appropriate than the ad hoc Definition 7.2.

Theorem 7.4. *Let $\phi \in \mathbb{C}_v(z)$ be a rational function. Then the function $\phi : \mathbb{P}^1_{\text{an}} \to \mathbb{P}^1_{\text{an}}$ of Definition 7.2 is the unique continuous extension of the function $\phi : \mathbb{P}^1(\mathbb{C}_v) \to \mathbb{P}^1(\mathbb{C}_v)$ to $\mathbb{P}^1_{\text{an}}$.*

Proof. See Section 16.1. ☐

Corollary 7.5. *Let $\phi, \psi \in \mathbb{C}_v(z)$ be rational functions, and let $\zeta \in \mathbb{P}^1_{\text{an}}$. Then $\psi(\phi(\zeta)) = (\psi \circ \phi)(\zeta)$.*

Proof. Recall from Theorem 6.25 that the set of Type I points is dense in $\mathbb{P}^1_{\text{an}}$. Since the desired equality holds for Type I points, the result is immediate from Theorem 7.4. ☐

Theorem 7.4 describes the action of ϕ on Type I points. For Type II and some Type III points, we have the following preliminary result.

Proposition 7.6. *Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, let $\zeta = \zeta(a, r) \in \mathbb{P}^1_{\text{an}}$ be a point of Type II or III, and suppose that ϕ has no poles in the open disk $D(a, r)$. Then there exist $b \in \mathbb{C}_v$ and $s > 0$ such that*

$$\phi\big(D(a, r)\big) = D(b, s),$$

with $s \in |\mathbb{C}_v^\times|$ if and only if $r \in |\mathbb{C}_v^\times|$. Moreover, $\phi(\zeta) = \zeta(b, s)$.

Proof. See Section 16.1. ☐

If ζ is of Type II, then we can always find a disk $D(a, r)$ satisfying the hypotheses of Proposition 7.6. After all, ζ has infinitely many residue classes (see Proposition 6.38), but ϕ has only finitely many poles. Thus, ζ must have some residue class $D(a, r)$ containing no poles of ϕ.

Since Type IV points are approximated by disks, Proposition 7.6 leads to the following result, describing the images of Type IV points.

Proposition 7.7. *Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, and let $\zeta \in \mathbb{P}^1_{\mathrm{an}}$ be a point of Type IV, corresponding to a decreasing sequence of disks*

$$D_1 \supseteq D_2 \supseteq \cdots$$

with empty intersection. Then there are positive integers $d, N \geq 1$ such that $\phi(D_n)$ is a disk in $\mathbb{C}_v$ for all $n \geq N$, and $\phi : D_n \to \phi(D_n)$ is a d-to-1 map. Moreover, $\phi(\zeta)$ is the Type IV point corresponding to the decreasing sequence of disks

$$\phi(D_N) \supseteq \phi(D_{N+1}) \supseteq \cdots .$$

Proof. See Section 16.1. □

Recall from Remark 6.23 that an open or closed affinoid in $\mathbb{P}^1(\mathbb{C}_v)$ corresponds to a unique open or closed Berkovich affinoid, respectively. The following result shows that this correspondence is respected by nonconstant rational functions acting on $\mathbb{P}^1_{\mathrm{an}}$.

Theorem 7.8. *Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, let $W \subseteq \mathbb{P}^1_{\mathrm{an}}$ be a connected Berkovich affinoid, and let*

$$W_{\mathrm{I}} := W \cap \mathbb{P}^1(\mathbb{C}_v)$$

be the corresponding connected affinoid in $\mathbb{P}^1(\mathbb{C}_v)$. Then $\phi(W)$ is the Berkovich connected affinoid of the same type (if any) corresponding to $\phi(W_{\mathrm{I}})$, and $\phi^{-1}(W)$ is the Berkovich affinoid of the same type (if any) corresponding to $\phi^{-1}(W_{\mathrm{I}})$. Moreover, the following hold.

(a) $\partial(\phi(W)) \subseteq \phi(\partial W)$.

(b) *Each of the connected components $V_1, \ldots, V_m$ of $\phi^{-1}(W)$ is a connected Berkovich affinoid mapping onto W.*

(c) *For each $i = 1, \ldots, m$,*

$$\phi(\partial V_i) = \partial W \quad \text{and} \quad \phi(\operatorname{int} V_i) = \operatorname{int} W,$$

where $\operatorname{int} X$ denotes the interior of the set X.

(d) *If W is open, then $\phi(\partial V_i) \cap W = \varnothing$.*

Proof. See Section 16.1. □

Theorem 7.8 is only a partial analogue of Proposition 3.29, because it fails to provide any degree information for components of $\phi^{-1}(W)$. After we have defined local degrees appropriately, however, Proposition 7.33 will supply the missing degree formula.

Corollary 7.9. *Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function. Then $\phi : \mathbb{P}^1_{\mathrm{an}} \to \mathbb{P}^1_{\mathrm{an}}$ is an open map. That is, for every open set $U \subseteq \mathbb{P}^1_{\mathrm{an}}$, the image $\phi(U)$ is also open.*

Proof. This is immediate from Theorem 6.17, that the open connected Berkovich affinoids form a basis for the Gel'fand topology, and Theorem 7.8. □

Corollary 7.10. *Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, and let $U \subseteq \mathbb{P}^1_{\mathrm{an}}$ be an open connected Berkovich affinoid. Then the following are equivalent.*

(a) $\phi(U) \cap \phi(\partial U) = \varnothing$.

(b) *U is a connected component of $\phi^{-1}(\phi(U))$.*

Proof. See Section 16.1. □

Remark 7.11. Any Type II point can be moved to the Gauss point $\zeta(0,1)$ by a linear fractional transformation. More precisely, if $b, c \in \mathbb{C}_v$ are distinct and $r := |b-c| \in |\mathbb{C}_v^\times|$, then

$$\eta(z) := (c-b)^{-1}(z-b)$$

takes $b \mapsto 0$, $c \mapsto 1$, and $\infty \mapsto \infty$. Thus, $\eta(D(b,r)) = D(0,1)$, and hence $\eta(\zeta(b,r)) = \zeta(0,1)$ by Proposition 7.6.

The connected components of $\mathbb{P}^1_{\mathrm{an}} \smallsetminus \zeta(0,1)$ correspond naturally to the elements $\mathbb{P}^1(\overline{k})$, where $\overline{k}$ is the residue field of $\mathbb{C}_v$. Thus, because η is a homeomorphism (as η and its inverse η^{-1} are both continuous), it follows that the connected components of $\mathbb{P}^1_{\mathrm{an}} \smallsetminus \zeta(b,s)$ also correspond to the elements of $\mathbb{P}^1(\overline{k})$, although the bijection depends on η and hence on the choice of b and c. This correspondence motivates our use of the term *residue classes* for the connected components of $\mathbb{P}^1_{\mathrm{an}} \smallsetminus \{\zeta\}$.

In addition, the elements of $\mathrm{PGL}(2,\mathbb{C}_v)$ that fix $\zeta(0,1)$ are precisely the elements of $\mathrm{PGL}(2,\mathcal{O})$, where $\mathcal{O}$ denotes the ring of integers of $\mathbb{C}_v$; see Exercise 7.3. Thus, the Type II points of $\mathbb{P}^1_{\mathrm{an}}$ correspond to the left cosets of $\mathrm{PGL}(2,\mathcal{O})$ in $\mathrm{PGL}(2,\mathbb{C}_v)$.

7.2. Images of points of Types II and III

Proposition 7.6 does not apply to all points of Type III, and even for points of Type II it requires choosing a residue class containing no poles of ϕ. The following result remedies those omissions, at the expense of requiring a slightly more complicated statement.

Theorem 7.12. *Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, and let $\zeta = \zeta(a,r) \in \mathbb{P}^1_{\mathrm{an}}$ be a point of Type II or III. Then ϕ has a Laurent series expansion $\phi(z) = \sum_{n\in\mathbb{Z}} c_n(z-a)^n$ on the annulus*

$$U_\lambda := \{\lambda r < |z-a| < r\}$$

for all $\lambda < 1$ sufficiently close to 1. Moreover, for sufficiently large $\lambda < 1$, the map $\phi - c_0$ has inner and outer Weierstrass degrees on U_λ both equal to some integer $d \neq 0$ between $-\deg(\phi)$ and $\deg(\phi)$. Setting $s := |c_d| r^d$, we have

$$\phi(U_\lambda) = \begin{cases} \{\lambda^d s < |z - c_0| < s\} & \text{if } d > 0, \\ \{s < |z - c_0| < \lambda^d s\} & \text{if } d < 0. \end{cases}$$

Finally, $\phi(\zeta) = \zeta(c_0, s)$; and $s \in |\mathbb{C}_v^\times|$ if and only if $r \in |\mathbb{C}_v^\times|$.

Proof. See Section 16.1. □

Remark 7.13. Recall that we defined a point $\zeta(a, r)$ of Type II or III to be the sup norm for polynomials on $\overline{D}(a, r)$. A more refined definition, illustrated by Theorem 7.12, is as a limsup norm for *rational* functions on thin annuli approaching the boundary of the disk. Even for polynomials, Theorem 7.12 parallels the complex analytic fact that the supremum of an analytic function f on disk is attained on the boundary. Thus, to understand the limiting behavior of f at the boundary, it suffices to understand the behavior of f on a thin annulus U_λ.

Remark 7.14. Theorem 7.12 describes an annulus extending from the point $\zeta(a, r)$ in the direction of a. Alternatively, to represent the direction containing ∞, one could use an annulus of the form

$$U = \{r < |z - a| < \lambda r\} \quad \text{with } \lambda > 1.$$

For $\lambda > 1$ sufficiently close to 1, the image $\phi(U)$ would again be of the form either

$$\{s < |z - b| < \lambda^d s\} \quad \text{or} \quad \{\lambda^d s < |z - b| < s\},$$

depending on whether the Weierstrass degree d is positive or negative, respectively. To see this fact, we need only apply Theorem 7.12 to the map $\psi := \phi \circ h$ and the point $\zeta(0, 1/r)$, where $h(z) = a + 1/z$, and then note that $h(\zeta(0, 1/r)) = \zeta(a, r)$.

Example 7.15. Pick $a \in \mathbb{C}_v$ with $0 < |a| < 1$, pick an integer $m \geq 1$, and let

$$\phi(z) = az^{m+1} + az^m + \frac{a^2}{z - a} = \frac{az^{m+2} + a(1 - a)z^{m+1} - a^2 z^m + a^2}{z - a}.$$

We now find the image $\phi(\zeta(0, 1))$ of the Gauss point via Theorem 7.12. (The reader can check that $\phi(D(0, 1)) = \mathbb{P}^1(\mathbb{C}_v)$, preventing us from applying Proposition 7.6 to the open disk $D(0, 1)$.)

Choose $\lambda \in (|a|^{1/(m+1)}, 1)$, and let $U := \{\lambda < |z| < 1\}$. Note that ϕ has no poles in the open annulus U. Moreover,

$$(7.2) \qquad |ax^m| > \max\left\{|ax^{m+1}|, \left|\frac{a^2}{x - a}\right|\right\} \quad \text{for any } x \in U,$$

and hence ϕ has inner and outer Weierstrass degree both equal to m on U. (There is no need to expand the power series for $a^2/(x-a)$, as Proposition 3.17 and inequality (7.2) guarantee that ax^m dominates all of its terms.) Let $s := |a| \cdot 1^m = |a|$, and note that the constant term c_0 of the Laurent series satisfies $|c_0| < s$. Thus, by Theorem 7.12,

$$\phi(U) = \left\{\lambda^m|a| < |z - c_0| < |a|\right\},$$

and $\phi(\zeta(0,1)) = \zeta(0,|a|)$.

Of course, what has happened here is that the az^m-term in the original definition of ϕ is dominant on the annulus U, and $z \mapsto az^m$ clearly maps $D(0,1)$ m-to-1 onto $D(0,|a|)$. Using the annulus U rather than the disk $D(0,1)$ allowed us to avoid the pole at a.

Alternatively, as we noted just after Proposition 7.6, we could determine $\phi(\zeta(0,1))$ by computing the image of a different residue class, like $D(1,1)$, that contains no poles. On the one hand, this strategy obviates the need to consider annuli. On the other hand, it would require more work, because we would need to expand ϕ as a power series centered at 1 rather than as a Laurent series centered at 0. If we had the patience to compute that power series, though, a quick application of Theorem 3.15 would show that $\phi(D(1,1)) = D(2a,|a|)$, and hence Proposition 7.6 would again show that $\phi(\zeta(0,1)) = \zeta(0,|a|)$.

Combined with the results of Section 7.1, Theorem 7.12 allows us to conclude that the type of a Berkovich point is preserved by nonconstant rational functions.

Corollary 7.16. *Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, and let $\zeta \in \mathbb{P}^1_{\text{an}}$. Then $\phi(\zeta)$ is a point of the same type as ζ.*

Proof. If ζ is of Type I, we are done by the second statement of Theorem 7.4. If ζ is of Type II or III, then Theorem 7.12 suffices. And if ζ is of Type IV, we are done by Proposition 7.7. □

The following result is useful whenever we want to change coordinates while discussing the metric $d_{\mathbb{H}}$ on $\mathbb{H}$.

Proposition 7.17. *Let $\phi \in \mathbb{C}_v(z)$ be a rational function. Then ϕ is an isometry on $\mathbb{H}$ with respect to $d_{\mathbb{H}}$ if and only if* $\deg(\phi) = 1$.

Proof. See Exercise 7.4. □

7.3. Local degrees in directions

Theorem 7.12 not only identifies the image of a given Type II or Type III point ζ under a rational function ϕ, but it also shows how each direction $\boldsymbol{v}$ at ζ is mapped to a direction $\phi_{\#}(\boldsymbol{v})$ at $\phi(\zeta)$. We make this mapping more precise now.

Definition 7.18. Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, let $\zeta \in \mathbb{P}^1_{\text{an}}$, and let $\boldsymbol{v}$ be a direction at ζ. We define a positive integer $\deg_{\zeta,\boldsymbol{v}}(\phi)$ and a direction $\phi_{\#}(\boldsymbol{v})$ at $\phi(\zeta)$ as follows.

(a) If $\zeta = x$ is of Type I, then $\phi_{\#}(\boldsymbol{v})$ is the unique direction at $\phi(\zeta)$, and $\deg_{\zeta,\boldsymbol{v}}(\phi) := \deg_x(\phi)$ is the algebraic multiplicity of the map $\phi : \mathbb{P}^1(\mathbb{C}_v) \to \mathbb{P}^1(\mathbb{C}_v)$ at x.

(b) If $\zeta = \zeta(a, r)$ is of Type II or III and $\boldsymbol{v} = \vec{v}_\zeta(a)$ is the direction at ζ containing a, then $\phi_{\#}(\boldsymbol{v})$ is the direction at $\phi(\zeta)$ containing $\phi(U)$, where $U := \{\lambda r < |z - a| < r\}$ is the annulus of Theorem 7.12, and $\deg_{\zeta,\boldsymbol{v}}(\phi)$ is the multiplicity of the mapping $\phi : U \to \phi(U)$. That is, $\deg_{\zeta,\boldsymbol{v}}(\phi) := \pm d \geq 1$, where d is the common inner and outer Weierstrass degree of ϕ on U.

(c) If $\zeta = \zeta(a, r)$ is of Type II or III and $\boldsymbol{v}$ is the direction at ζ containing ∞, then $\phi_{\#}(\boldsymbol{v})$ is the direction at $\phi(\zeta)$ containing $\phi(U)$, where $U := \{r < |z - a| < \lambda r\}$ is the annulus of Remark 7.14, and $\deg_{\zeta,\boldsymbol{v}}(\phi)$ is the multiplicity of the mapping $\phi : U \to \phi(U)$. That is, $\deg_{\zeta,\boldsymbol{v}}(\phi) := \pm d \geq 1$, where d is the common inner and outer Weierstrass degree of ϕ on U.

(d) If ζ is of Type IV, then $\phi_{\#}(\boldsymbol{v})$ is the unique direction at $\phi(\zeta)$, and $\deg_{\zeta,\boldsymbol{v}}(\phi)$ is the multiplicity d of Proposition 7.7.

Warning 7.19. Recall that we defined a direction $\boldsymbol{v}$ at $\zeta \in \mathbb{P}^1_{\text{an}}$ to be a certain subset of $\mathbb{P}^1_{\text{an}}$. However, the image direction $\phi_{\#}(\boldsymbol{v})$ *may or may not* coincide with the image $\phi(\boldsymbol{v})$ of the subset $\boldsymbol{v} \subseteq \mathbb{P}^1_{\text{an}}$. In fact, the image $\phi(\boldsymbol{v})$ of the set may be the entire space $\mathbb{P}^1_{\text{an}}$; see, for example, Theorem 7.34(b). On the other hand, there are other cases in which $\phi_{\#}(\boldsymbol{v}) = \phi(\boldsymbol{v})$; see Theorem 7.34(c).

Example 7.20. Consider again the function

$$\phi(z) = az^{m+1} + az^m + \frac{a^2}{z - a}$$

of Example 7.15, where $0 < |a| < 1$ and $m \geq 1$. Let $\boldsymbol{v} := \vec{v}(0)$ be the direction at $\zeta(0, 1)$ containing 0. Since the image of the annulus

$$U = \{\lambda < |z| < 1\}$$

is $\phi(U) = \{\lambda^m < |z| < 1\}$, Definition 7.18(b) says that $\phi_{\#}(\boldsymbol{v})$ is the direction at $\zeta(0,|a|)$ containing 0, and $\deg_{\zeta(0,1),\boldsymbol{v}}(\phi) = m$.

The following result shows that directions at Berkovich points behave well under composition of functions, and that their local degrees behave in the same way that multiplicities do.

Proposition 7.21. *Let $\phi, \psi \in \mathbb{C}_v(z)$ be nonconstant rational functions, let $\zeta \in \mathbb{P}^1_{\text{an}}$, and let $\boldsymbol{v}$ be a direction at ζ. Then*

(a) $1 \le \deg_{\zeta,\boldsymbol{v}}(\phi) \le \deg(\phi)$.

(b) $\psi_{\#}\bigl(\phi_{\#}(\boldsymbol{v})\bigr) = \bigl(\psi \circ \phi\bigr)_{\#}(\boldsymbol{v})$.

(c) $\deg_{\zeta,\boldsymbol{v}}(\psi \circ \phi) = \deg_{\zeta,\boldsymbol{v}}(\phi) \cdot \deg_{\phi(\zeta),\phi_{\#}(\boldsymbol{v})}(\psi)$.

Proof. This is left to the reader; see Exercise 7.5. □

Local degrees in directions at points in $\mathbb{H}$ can also be understood as repulsion factors with respect to the hyperbolic metric $d_{\mathbb{H}}$, as the following result shows.

Theorem 7.22. *Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, let $\zeta \in \mathbb{P}^1_{\text{an}}$, let $\boldsymbol{v}$ be a direction at ζ, and let $m := \deg_{\zeta,\boldsymbol{v}}(\phi)$. Then there is a point $\xi \in \boldsymbol{v}$ such that ϕ maps $[\zeta,\xi]$ homeomorphically onto $[\phi(\zeta),\phi(\xi)]$, and*

$$d_{\mathbb{H}}\bigl(\phi(\xi_1),\phi(\xi_2)\bigr) = m \cdot d_{\mathbb{H}}(\xi_1,\xi_2) \quad \text{for all } \xi_1,\xi_2 \in [\zeta,\xi] \cap \mathbb{H}. \tag{7.3}$$

Proof. See Section 16.2. □

Example 7.23. Fix a prime $p \ge 2$, pick $m \ge 2$, and set $\phi(z) = z^m$. Then for any $r > 0$, we have $\phi(D(0,r)) = D(0,r^m)$, and therefore

$$\phi\bigl(\zeta(0,r)\bigr) = \zeta(0,r^m)$$

by Proposition 7.6. In particular, ϕ maps the interval $[\zeta(0,r),\zeta(0,s)]$ homeomorphically onto $[\zeta(0,r^m),\zeta(0,s^m)]$, while stretching the hyperbolic metric by a factor of exactly m all along the interval, since

$$d_{\mathbb{H}}\bigl(\zeta(0,t_1),\zeta(0,t_2)\bigr) = \log t_2 - \log t_1 \quad \text{for } t_1 \le t_2.$$

Applying Theorem 7.12 to annuli emanating from $\zeta(0,r)$ toward 0, we see that $\deg_{\zeta(0,r),\vec{v}(0)}(\phi) = m$. Similarly, applying Remark 7.14 to annuli emanating toward ∞, we have $\deg_{\zeta(0,r),\vec{v}(\infty)}(\phi) = m$ as well. Thus, we have confirmed Theorem 7.22 for $\phi(z) = z^m$ at $\zeta(0,r)$ in the direction of 0 or ∞.

But what happens in other directions? For example, at the Gauss point $\zeta(0,1)$, consider the direction $\vec{v}(a)$, for some $a \in \mathbb{C}_v$ with $|a| = 1$. Expanding ϕ as a power series centered at a and noting that $\phi(a) = a^m$, we see

$$\phi(z) - a^m = \bigl((z-a)+a\bigr)^m - a^m = \sum_{n=1}^{m} \binom{m}{n} a^{m-n}(z-a)^n. \tag{7.4}$$

Write $m = p^e j$, with $p \nmid j$ and $e \geq 0$. Then by Exercise 2.15,

$$\left| \binom{m}{p^e} \right| = 1, \quad \text{but} \quad \left| \binom{m}{n} \right| < 1 \quad \text{for all } 1 \leq n < p^e.$$

Thus, for $\lambda < 1$ sufficiently close to 1, the $(z-a)^{p^e}$-term of equation (7.4) is uniquely dominant on $U_\lambda := \{\lambda < |z-a| < 1\}$. As a result,

- $\deg_{\zeta(0,1),\vec{v}(a)}(\phi) = p^e$,
- $\phi(U_\lambda) = \{\lambda^{p^e} < |z - a^m| < 1\}$, and
- for any $0 < r < 1$, we have $\phi(\zeta(a,r)) = \zeta(a^m, r^{p^e})$.

In particular, by the last bullet point,

$$\phi : [\zeta(0,1), \zeta(a,\lambda)] \to [\zeta(0,1), \zeta(a^m, \lambda^{p^e})]$$

is a homeomorphism stretching all distances by a factor of exactly p^e.

If $e = 0$, i.e., if $p \nmid m$, then ϕ is an isometry on that interval. In fact, in that case, ϕ is an isometry from the infinite-length interval $[\zeta(0,1), a)$ to $[\zeta(0,1), a^m)$. On the other hand, if $e \geq 1$, then ϕ only stretches by a constant factor of p^e on the interval

$$\left[\zeta(0,1), \zeta\left(a, |p|^{1/(p^e - p^{(e-1)})}\right)\right].$$

After all,

$$\left| \binom{m}{p^{(e-1)}} \right| = |p|,$$

and hence the $z^{p^{(e-1)}}$-term becomes dominant for $\lambda < |p|^{1/(p^e - p^{(e-1)})}$.

Theorem 7.22 also allows us to draw some broader conclusions about the action of rational functions on the metric space $(\mathbb{H}, d_{\mathbb{H}})$.

Corollary 7.24. *Let $\phi \in \mathbb{C}_v(z)$ be a rational function of degree $d \geq 1$. Then ϕ is Lipschitz on $\mathbb{H}$ with respect to the hyperbolic metric $d_{\mathbb{H}}$, with Lipschitz constant at most d.*

Proof. Given $\zeta_1, \zeta_2 \in \mathbb{H}$, we may assume without loss that $\zeta_1 \neq \zeta_2$. At each point ζ in the interval $I := [\zeta_1, \zeta_2]$, consider the two directions $\boldsymbol{v}$ and $\boldsymbol{w}$ at ζ containing ζ_1 or ζ_2, with local degrees m and n, respectively. (Of course, there is only one such direction if ζ is itself one of those two endpoints.) By Theorem 7.22, ϕ stretches distances by a factor of exactly m in a small interval extending from ζ toward ζ_1, and by exactly n in a small interval extending from ζ toward ζ_2. Since $1 \leq m, n \leq d$ by Proposition 7.21(a), there is an open subset I_ζ of I containing ζ such that

$$d_{\mathbb{H}}\big(\phi(\xi_1), \phi(\xi_2)\big) \leq d \cdot d_{\mathbb{H}}(\xi_1, \xi_2) \qquad \text{for all } \xi_1, \xi_2 \in I_\zeta.$$

Covering the compact interval I by these open subsets I_ζ and taking a finite subcover, it follows that

$$d_{\mathbb{H}}\big(\phi(\zeta_1), \phi(\zeta_2)\big) \le d \cdot d_{\mathbb{H}}(\zeta_1, \zeta_2). \qquad \square$$

We close this section with the following relationship among poles, zeros, and local degrees in directions.

Proposition 7.25. *Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, let $\zeta = \zeta(a, r) \in \mathbb{P}^1_{\text{an}}$ be a point of Type II or III, and let $\boldsymbol{v}$ be a direction at ζ. Suppose that 0 and ∞ lie in different directions at $\phi(\zeta)$.*

Let M_0 and M_∞, respectively, be the number of zeros and poles, counted with multiplicity, of ϕ in $\boldsymbol{v}$; and let N_0 and N_∞, respectively, be the number of zeros and poles, counted with multiplicity, of ϕ in $\overline{D}(a, r)$. Then

$$\deg_{\zeta,\boldsymbol{v}}(\phi) = \begin{cases} M_0 - M_\infty & \text{if } \infty \notin \boldsymbol{v} \text{ and } 0 \in \phi_\#(\boldsymbol{v}), \\ M_\infty - M_0 & \text{if } \infty \notin \boldsymbol{v} \text{ and } \infty \in \phi_\#(\boldsymbol{v}), \\ N_\infty - N_0 & \text{if } \infty \in \boldsymbol{v} \text{ and } 0 \in \phi_\#(\boldsymbol{v}), \\ N_0 - N_\infty & \text{if } \infty \in \boldsymbol{v} \text{ and } \infty \in \phi_\#(\boldsymbol{v}). \end{cases}$$

Proof. This is left to the reader; see Exercise 7.8. $\square$

7.4. Local degrees at Berkovich points

We wish to define the local degree $\deg_\zeta(\phi)$ of a rational function $\phi \in \mathbb{C}_v(z)$ at a point $\zeta \in \mathbb{P}^1_{\text{an}}$ itself. At a Type I point $\zeta = x \in \mathbb{P}^1(\mathbb{C}_v)$, the local degree $\deg_x(\phi)$ should be the algebraic multiplicity of ϕ at x, i.e., the multiplicity of x as a root of the equation $\phi(z) = \phi(x)$. However, the function $x \mapsto \deg_x(\phi)$ is not continuous on $\mathbb{P}^1(\mathbb{C}_v)$. (Consider $\phi(z) = z^d$ at $x = 0$, for example.) Thus, we cannot define $\deg_\zeta(\phi)$ as a continuous extension of $\deg_x(\phi)$ to $\mathbb{P}^1_{\text{an}}$.

Instead, note that $\deg_x(\phi)$ is the smallest integer $d \ge 1$ with the following property: x has an open neighborhood $U \subseteq \mathbb{P}^1(\mathbb{C}_v)$ such that every point in $\phi(U)$ has at most d preimages in U, counting multiplicity. Therefore, it makes sense to define $\deg_\zeta(\phi)$ as follows.

Definition 7.26. Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, and let $\zeta \in \mathbb{P}^1_{\text{an}}$. The *local degree*, or *multiplicity*, of ϕ at ζ is

$$\deg_\zeta(\phi) := \inf_U \left(\sup_y \left(\sum_{x \in U \cap \phi^{-1}(y)} \deg_x(\phi) \right) \right), \tag{7.5}$$

where the infimum is over all open $U \subseteq \mathbb{P}^1_{\text{an}}$ containing ζ, the supremum is over all $y \in \phi(U) \cap \mathbb{P}^1(\mathbb{C}_v)$, and $\deg_x(\phi)$ denotes the algebraic multiplicity of x as a root of the equation $\phi(z) = y$.

The quantity $\sum_{x\in U\cap\phi^{-1}(y)} \deg_x(\phi)$ of equation (7.5) is an integer between 1 and $\deg(\phi)$, since it is the number of preimages of y in U, counting multiplicity. Thus, $\deg_\zeta(\phi)$ is also an integer lying between 1 and $\deg(\phi)$.

Lemma 7.27. *Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, let $\zeta \in \mathbb{P}^1_{\mathrm{an}}$, and let $U \subseteq \mathbb{P}^1_{\mathrm{an}}$ be an open set containing ζ. Then there is an open connected Berkovich affinoid W such that*

(a) $\zeta \in W \subseteq U$,

(b) $\phi(\partial W) \subseteq \partial(\phi(W))$, *and*

(c) $\sum_{x\in W\cap\phi^{-1}(y)} \deg_x(\phi) = \deg_\zeta(\phi)$ *for every Type I point y in $\phi(W)$.*

Proof. See Section 16.2. □

Example 7.28. To illustrate Lemma 7.27, consider the function

$$\phi(z) = z^2 - az \in \mathbb{C}_v[z], \quad \text{where} \quad |a| = 1.$$

Fix $r > 1$, and let U be the open neighborhood

$$U := \left\{\zeta \in \mathbb{P}^1_{\mathrm{an}} : \frac{1}{r} < |\zeta| < r\right\},$$

of the Gauss point $\zeta(0,1)$. Let $U_{\mathrm{I}} := U \cap \mathbb{P}^1(\mathbb{C}_v)$ be the corresponding open annulus of Type I points. By applying Theorem 3.33 to smaller annuli contained in U_I, the reader can check that $\phi(U_{\mathrm{I}}) = D(0, r^2)$, and hence that $\phi(U) = D_{\mathrm{an}}(0, r^2)$, by Theorem 7.8. However, while Type I points in $\{1/r < |z| < r^2\}$ have two preimages in U, those in $\overline{D}(0, 1/r)$ have only one.

The problem, of course, is that both of the disks $\overline{D}(a, 1/r)$ and $\overline{D}(0, 1/r)$ map bijectively onto $\overline{D}(0, 1/r)$, but the former is contained in U_{I}, while the latter does not intersect U_{I}. Thus, if we define

$$W := D_{\mathrm{an}}(0, r) \smallsetminus \left(\overline{D}_{\mathrm{an}}(0, 1/r) \cup \overline{D}_{\mathrm{an}}(a, 1/r)\right) \subsetneq U,$$

then $\phi(W) = D_{\mathrm{an}}(0, r^2) \smallsetminus \overline{D}_{\mathrm{an}}(0, 1/r)$, and

$$\begin{aligned}\phi(\partial W) &= \phi\big(\{\zeta(0, r), \zeta(0, 1/r), \zeta(a, 1/r)\}\big) \\ &= \big\{\zeta(0, r^2), \zeta(0, 1/r)\big\} = \partial\big(\phi(W)\big).\end{aligned}$$

In addition, every Type I point in $\phi(W)$ has exactly two preimages in W, counting multiplicity. It can also be shown that $\deg_{\zeta(0,1)}(\phi) = 2$; see, for example, Theorem 7.34. Hence, W fits all three conclusions of Lemma 7.27.

In fact, the affinoid W above is precisely the one constructed in the proof of Lemma 7.27 in Section 16.2. Specifically, the original annulus U has boundary $\partial U = \{\zeta(0, 1/r), \zeta(0, r)\}$, which in turn has image

$$\phi(\partial U) = \big\{\zeta(0, 1/r), \zeta(0, r^2)\big\}.$$

The proof of Lemma 7.27 defines

$$T := \phi^{-1}\bigl(\phi(\partial U)\bigr) = \bigl\{\zeta(0,1/r), \zeta(a,1/r), \zeta(0,r)\bigr\}$$

and then defines W to be the connected component of $\mathbb{P}^1_{\mathrm{an}} \smallsetminus T$ that contains the original point ζ.

The following result gives some basic properties of local degrees. It is essentially the same as [**63**, Proposition 2.2]; it also appears as a number of separate results in [**8**, Section 9.1].

Theorem 7.29. *Let $\phi, \psi \in \mathbb{C}_v(z)$ be nonconstant rational functions, and let $\zeta \in \mathbb{P}^1_{\mathrm{an}}$. Then*

(a) *The function $\mathbb{P}^1_{\mathrm{an}} \to \{1, \dots, \deg(\phi)\}$ given by $\xi \mapsto \deg_\xi(\phi)$ is upper semicontinuous. That is, for every $m \ge 1$, the set*

$$\bigl\{\xi \in \mathbb{P}^1_{\mathrm{an}} : \deg_\xi(\phi) \ge m\bigr\}$$

is closed in the Gel'fand topology.

(b) $\deg_\zeta(\psi \circ \phi) = \deg_\zeta(\phi) \cdot \deg_{\phi(\zeta)}(\psi)$.

(c) $\sum_{\phi(\xi)=\zeta} \deg_\xi(\phi) = \deg(\phi)$.

Proof. See Section 16.2. □

Recall that $1 \le \deg_\zeta(\phi) \le \deg(\phi)$. In particular, if $\deg(\phi) = 1$, then $\deg_\zeta(\phi) = 1$ for all $\zeta \in \mathbb{P}^1_{\mathrm{an}}$. Thus, Theorem 7.29(b) implies that local degrees are invariant under coordinate changes on either the domain or range.

The following theorem relates local degrees to local degrees in directions. The summation formula (7.6) was essentially the definition of local degrees in [**115**, Section 4.2]; Baker and Rumely refer to it as Rivera-Letelier's "Repulsion Formula" in [**8**, Theorem 9.19].

Theorem 7.30. *Let $\zeta \in \mathbb{P}^1_{\mathrm{an}}$, let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, and let $\boldsymbol{w}$ be a direction at $\phi(\zeta)$. Then*

$$\deg_\zeta(\phi) = \sum_{\phi_\#(\boldsymbol{v})=\boldsymbol{w}} \deg_{\zeta,\boldsymbol{v}}(\phi), \tag{7.6}$$

where the sum is over all directions $\boldsymbol{v}$ at ζ for which $\phi_\#(\boldsymbol{v}) = \boldsymbol{w}$.

Proof. See Section 16.2. □

Example 7.31. Let $c \in \mathbb{C}_v$ with $0 < |c| < 1$, and let

$$\phi(z) = \frac{z^4 - 8c^4 z - c^4}{4z^3 + z^2 + 4c^4} \in \mathbb{C}_v(z)$$

be the Lattès map of Example 5.33. Consider $\zeta = \zeta(0, r)$ with $|c| \leq r \leq 1$. Note that ϕ has no zeros or poles in the annulus

$$U_r := \{|c| < |x| < r\} = D(0, r) \smallsetminus \overline{D}(0, |c|).$$

By Proposition 3.32, the inner and outer Weierstrass degrees of ϕ coincide on U_r. In fact, we may write

$$\phi(z) = z^2 \cdot \left(\frac{1 - 8c^4 z^{-1} - c^4 z^{-4}}{4z + 1 + 4c^4 z^{-2}} \right),$$

and because the expression in parentheses is clearly in $D(1, 1)$ for all $z \in U_r$, that common Weierstrass degree must be 2. (Put another way, the Laurent series on U_r for the expression in parentheses is dominated by its constant term. See also Exercise 3.20(b), since the numerator of ϕ has Weierstrass degree 4 on U_r, and the denominator has Weierstrass degree 2.) In fact, we have $|\phi(x)| = |x|^2$ for all $x \in U_r$.

In light of Theorems 3.33 and 7.12, then, we have

$$\phi\big(\zeta(0, r)\big) = \zeta(0, r^2).$$

By Definition 7.18, we also have

$$\phi_{\#}\big(\vec{v}(0)\big) = \vec{w}(0), \quad \text{with } \deg_{\zeta(0,r),\vec{v}(0)}(\phi) = 2$$

and

$$\phi_{\#}\big(\vec{v}(\infty)\big) = \vec{w}(\infty), \quad \text{with } \deg_{\zeta(0,r),\vec{v}(\infty)}(\phi) = 2,$$

where $\vec{v}(a)$ denotes the direction at $\zeta(0, r)$ containing a, and where $\vec{w}(a)$ denotes the direction at $\zeta(0, r^2)$ containing a.

On the other hand, $|\phi(b)| = |b|^2$ for any $b \in \mathbb{C}_v$ with $|b| = r$, and so

$$\phi_{\#}\big(\vec{v}(b)\big) \neq \vec{w}(0), \vec{w}(\infty).$$

Thus, the only direction at $\zeta(0, r)$ that maps to $\vec{w}(0)$ is $\vec{v}(0)$, and the only direction at $\zeta(0, r)$ that maps to $\vec{w}(\infty)$ is $\vec{v}(\infty)$. Using either $\boldsymbol{w} := \vec{w}(0)$ or $\boldsymbol{w} := \vec{w}(\infty)$, then, Theorem 7.30 tells us that $\deg_{\zeta(0,r)}(\phi) = 2$.

We leave it to the reader to check that for $|c|^2 < r < |c|$, we have $\phi(\zeta(0, r)) = \zeta(0, |c|^4/r^2)$, with $\deg_{\zeta(0,r)}(\phi) = 2$. This time, however,

$$\phi_{\#}\big(\vec{v}(0)\big) = \vec{w}(\infty) \quad \text{and} \quad \phi_{\#}\big(\vec{v}(\infty)\big) = \vec{w}(0),$$

with degree $\deg_{\zeta(0,r),\boldsymbol{v}}(\phi) = 2$ for both $\boldsymbol{v} = \vec{v}(0)$ and $\boldsymbol{v} = \vec{v}(\infty)$. Thus, ϕ maps each of the intervals

$$\big(\zeta(0, 1), \zeta(0, |c|)\big) \quad \text{and} \quad \big(\zeta(0, |c|), \zeta(0, |c|^2)\big)$$

bijectively onto the interval $(\zeta(0, 1), \zeta(0, |c|^2))$ in $\mathbb{P}^1_{\text{an}}$, but with local degree 2 at each point. By Theorem 7.29(c) and the fact that $\deg(\phi) = 4$, then, these two intervals make up the full preimage of $(\zeta(0, 1), \zeta(0, |c|^2))$; see Exercise 7.10.

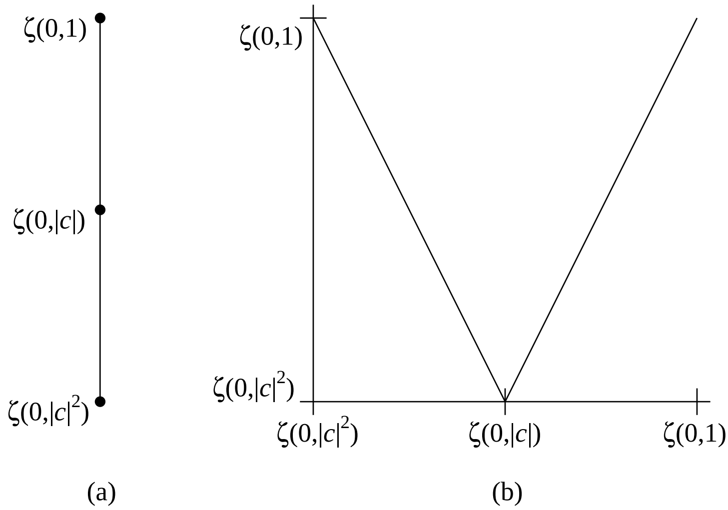

Figure 7.1. The Lattès map of Example 7.31 acting on the interval $J = [\zeta(0,1), \zeta(0,|c|^2)]$.

The reader can also check that

$$\phi\bigl(\zeta(0,1)\bigr) = \phi\bigl(\zeta(0,|c|^2)\bigr) = \zeta(0,1),$$

with

$$\deg_{\zeta(0,1)}(\phi) = \deg_{\zeta(0,|c|^2)}(\phi) = 2,$$

and that

$$\phi\bigl(\zeta(0,|c|)\bigr) = \zeta\bigl(0,|c|^2\bigr), \quad \text{with} \quad \deg_{\zeta(0,|c|)}(\phi) = 4.$$

(See Exercise 7.11.) Thus, we have accounted for the preimages of all points in the closed interval

$$J := \bigl[\zeta(0,1), \zeta(0,|c|^2)\bigr].$$

In addition, our computations above show that ϕ maps J onto itself by stretching it and then folding it in half over itself, as illustrated in Figure 7.1. The interval J, with its midpoint $\zeta(0,|c|)$, appears in Figure 7.1(a). The graph of $\phi|_J$ appears in Figure 7.1(b); the two halves

$$\bigl[\zeta(0,|c|^2), \zeta(0,|c|)\bigr] \quad \text{and} \quad \bigl[\zeta(0,|c|), \zeta(0,1)\bigr]$$

are each stretched by a factor of 2 and mapped onto J, with the lower half mapping from top to bottom, and the upper half from bottom to top.

Finally, because we accounted for all four preimages of each point $\zeta \in J$, we have $\phi^{-1}(J) = J$. We will see in Example 8.20 that J is precisely the Berkovich Julia set of ϕ.

Corollary 7.32. *Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, and let $\zeta \in \mathbb{P}^1_{\mathrm{an}}$.*

(a) *If ζ is of Type I, then $\deg_\zeta(\phi)$ is the algebraic multiplicity of ϕ at ζ.*

(b) *If ζ is of Type III, then*

$$\deg_\zeta(\phi) = \deg_{\zeta,\boldsymbol{u}}(\phi) = \deg_{\zeta,\boldsymbol{v}}(\phi),$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are the two directions at ζ.

(c) *If ζ is of Type IV, then $\deg_\zeta(\phi) = \mathrm{wdeg}_U(\phi)$ for any sufficiently small disk U containing ζ.*

Proof.

(a) and (c) This is immediate from Theorem 7.30 and Definition 7.18, since points of Types I and IV each have only one direction.

(b) We claim that $\phi_\#(\boldsymbol{u}) \neq \phi_\#(\boldsymbol{v})$. After all, $\phi(\zeta)$ is also a point of Type III, and hence if $\phi_\#(\boldsymbol{u}) = \phi_\#(\boldsymbol{v})$, then one of the two directions $\boldsymbol{w}$ at $\phi(\zeta)$ would not be the image under $\phi_\#$ of either direction at ζ. Using this direction $\boldsymbol{w}$ in Theorem 7.30, then, we would have $\deg_\zeta(\phi) = 0$, contradicting the fact that $\deg_\zeta(\phi) \geq 1$ and proving the claim.

Thus, each of the two directions at $\phi(\zeta)$ is the image under $\phi_\#$ of exactly one of the two directions at ζ. Applying Theorem 7.30 to these two directions $\phi_\#(\boldsymbol{v})$ and $\phi_\#(\boldsymbol{w})$, then, the desired equality is immediate. □

We close this section with the degree formula promised when we stated Theorem 7.8.

Proposition 7.33. *Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function of degree $d \geq 2$, and let $W \subseteq \mathbb{P}^1_{\mathrm{an}}$ be a connected Berkovich affinoid. Write*

$$\phi^{-1}(W) = V_1 \cup \cdots \cup V_m$$

as a disjoint union of connected Berkovich affinoids according to Theorem 7.8. Then for each $i = 1, \ldots, m$, there is an integer $1 \leq d_i \leq d$ such that every point in W has exactly d_i preimages in V_i, counting multiplicity. Moreover, $d_1 + \cdots + d_m = d$.

Proof. This is left to the reader; see Exercise 7.12. □

7.5. Computing local degrees

Corollary 7.32 provides an easy way to compute $\deg_\zeta(\phi)$ if the point ζ is of Type I, III, or IV. Theorem 7.34 below does the same for the remaining case where ζ is of Type II. This result, originally proven by Rivera-Letelier over $\mathbb{C}_p$, first appeared as [**114**, Corollaire 2.2, Proposition 2.4]; see also [**115**, Proposition 4.4] and [**8**, Theorem 9.26].

Recall from the discussion following Definition 6.39 that the directions at $\zeta(0,1)$ are open disks, with the direction containing $x \in \mathbb{P}^1(\mathbb{C}_v)$ denoted by $\vec{v}(x)$. Furthermore, recall that the set of directions is in one-to-one correspondence with $\mathbb{P}^1(\overline{k})$. In addition, as we did in Definition 4.6, we may write any rational function $\phi(z) \in \mathbb{C}_v(z)$ as $f(z)/g(z)$ with $f, g \in \mathcal{O}[z]$, and with at least one coefficient of f or g having absolute value 1. We can therefore define the reduction of ϕ to be the rational function

$$\overline{\phi}(z) = \frac{\overline{f}(z)}{\overline{g}(z)} \in k(z) \cup \{\infty\}.$$

We already made use of this idea in Examples 5.31 and 5.32 to understand the action of ϕ on residue classes in $\overline{D}(0,1)$. Note that if ϕ does not have explicit good reduction, then $\overline{\phi}$ has degree strictly smaller than $\deg(\phi)$; in fact, $\overline{\phi}$ may be constant.

Theorem 7.34. *Let $\phi \in \mathbb{C}_v(z)$ be a rational function of degree $d \geq 1$. Write $\phi = f/g$, where $f, g \in \mathcal{O}[z]$, and where at least one coefficient of f or g has absolute value 1. Define $\overline{\phi}(z) := \overline{f}(z)/\overline{g}(z)$. Then $\overline{\phi}$ is nonconstant if and only if $\phi(\zeta(0,1)) = \zeta(0,1)$. In that case,*

$$\deg_{\zeta(0,1)}(\phi) = \deg\left(\overline{\phi}\right). \tag{7.7}$$

Also in that case, define a direction $\boldsymbol{v}$ at $\zeta(0,1)$ to be bad *if $\boldsymbol{v}$ contains* both *a zero* and *a pole of ϕ, and define T to be the set of bad directions $\boldsymbol{v}$. Then*

(a) *T is a finite set,*

(b) *$\phi(\boldsymbol{v}) = \mathbb{P}^1_{\text{an}}$ for each direction $\boldsymbol{v} \in T$, and*

(c) *$\phi_{\#}(\boldsymbol{v}) = \phi(\boldsymbol{v}) = \vec{v}(\phi(a))$ for all directions $\boldsymbol{v}$ at $\zeta(0,1)$ not in T and all $a \in \boldsymbol{v}$. For such directions $\boldsymbol{v}$, we have*

$$\deg_{\zeta(0,1),\boldsymbol{v}}(\phi) = \mathrm{wdeg}_{\boldsymbol{v}}(\phi) \tag{7.8}$$

where $\mathrm{wdeg}_{\boldsymbol{v}}(\phi)$ is the Weierstrass degree of ϕ on the disk $\boldsymbol{v}$, and $\deg_{\zeta(0,1),\boldsymbol{v}}(\phi)$ is the local degree in the direction $\boldsymbol{v}$ at $\zeta(0,1)$.

Finally, ϕ has explicit good reduction, i.e., $\deg(\overline{\phi}) = d$, if and only if $\overline{\phi}$ is nonconstant and $T = \varnothing$.

Theorem 7.34 does even more than provide a way to compute local degrees at Type II points. In particular, if ϕ fixes the Gauss point, then it *almost* has explicit good reduction, in the following sense. Outside of the finitely many bad directions in T, ϕ maps every other residue class onto the residue class dictated by the reduced map $\overline{\phi}$.

The proof of Theorem 7.34 relies on the following lemma, which is of interest in its own right. The proofs of both results appear in Section 16.3.

Lemma 7.35. *Let $\phi \in \mathbb{C}_v(z)$ be a rational function such that $\overline{\phi}$ is nonconstant. Then*

(a) $\phi(\zeta(0,1)) = \zeta(0,1)$.

(b) *For any $a, b \in \mathbb{P}^1(\mathbb{C}_v)$, we have*

$$\overline{\phi}(\overline{a}) = \overline{b} \quad \textit{if and only if} \quad \phi_{\#}(\vec{v}(a)) = \vec{v}(b),$$

where $\overline{a}, \overline{b}$ are the reductions of a and b in $\mathbb{P}^1(k)$, and where $\vec{v}(a)$ and $\vec{v}(b)$ are the directions at $\zeta(0,1)$ containing a and b, respectively.

(c) *For any $a \in \mathbb{P}^1(\mathbb{C}_v)$, we have*

$$\deg_{\zeta(0,1),\vec{v}(a)}(\phi) = \deg_{\overline{a}}(\overline{\phi}),$$

where $\vec{v}(a)$ is the direction of a at $\zeta(0,1)$, and $\deg_{\overline{a}}(\overline{\phi})$ is the multiplicity of $\overline{\phi} \in k(z)$ at the point $\overline{a} \in \mathbb{P}^1(k)$.

Warning 7.36. Because ϕ may not have explicit good reduction, many points $a \in \mathbb{P}^1(\mathbb{C}_v)$ do *not* satisfy $\overline{\phi}(\overline{a}) = \overline{\phi(a)}$. After all, according to Theorem 7.34(b), for each of the the finitely many bad directions $\boldsymbol{v} \in T$, we have $\phi(\boldsymbol{v}) = \mathbb{P}^1_{\text{an}}$. Thus, for Type I points $a \in \boldsymbol{v}$, $\phi(a)$ could be any point in $\mathbb{P}^1(\mathbb{C}_v)$, but $\overline{\phi}(\overline{a})$ is only a single residue class in $\mathbb{P}^1(\mathbb{C}_v)$.

Of course, Theorem 7.34 computes $\deg_\zeta(\phi)$ only in the case that ζ and its image $\phi(\zeta)$ are both the Gauss point. To do the same computation for a more general type II point ζ, we need to make coordinate changes on both the domain and range to move ζ and $\phi(\zeta)$ to $\zeta(0,1)$. We illustrate this technique with the following example.

Example 7.37. Fix a prime number p, let $\mathbb{C}_v = \mathbb{C}_p$, and define

$$\phi(z) = z^2 - z + 1 + \frac{p^2}{z} = \frac{z^3 - z^2 + z + p^2}{z}.$$

Let's use Theorem 7.34 to compute $\phi(\zeta)$ and $\deg_\zeta(\phi)$, where $\zeta = \zeta(0,|p|)$.

First, we can move $\zeta(0,1)$ to ζ with the linear fractional transformation $\eta(z) = pz$. Thus, $\phi(\zeta) = \phi(\eta(\zeta(0,1)))$. We have

$$\phi \circ \eta(z) = \phi(pz) = \frac{p^2z^3 - pz^2 + z + p}{z}.$$

Note that $\overline{\phi \circ \eta}(z) = z/z = 1$ is constant; by Theorem 7.34, $\phi \circ \eta$ does not map $\zeta(0,1)$ to itself.

In fact, since the constant value of $\overline{\phi \circ \eta}$ is 1, the constant value of $\overline{\phi \circ \eta - 1}$ is 0, and so we can scale by some power of p, as follows:

$$\phi \circ \eta(z) - 1 = \frac{p^2 z^3 - pz^2 + p}{z} = p\left(\frac{pz^3 - z^2 + 1}{z}\right).$$

Thus, setting $\theta(z) = (z-1)/p$, we have

$$\theta \circ \phi \circ \eta(z) = \frac{pz^3 - z^2 + 1}{z},$$

and hence

$$\overline{\theta \circ \phi \circ \eta}(z) = \frac{1 - z^2}{z},$$

which is nonconstant and, in fact, of degree 2. By Theorem 7.34, then, $\theta \circ \phi \circ \eta$ maps the Gauss point $\zeta(0,1)$ to itself, with multiplicity 2. Thus, ϕ maps the original point $\zeta = \zeta(0,|p|) = \eta(\zeta(0,1))$ to the point $\theta^{-1}(\zeta(0,1)) = \zeta(1,|p|)$, and $\deg_\zeta(\phi) = 2$.

Remark 7.38. Our approach to the action of $\phi(z) \in \mathbb{C}_v(z)$ on $\mathbb{P}^1_{\text{an}}$ in this chapter, and especially our treatment of multiplicities, is a hybrid of various other equivalent definitions. Our initial definition in Section 7.1 of this action via homogeneous coordinates roughly follows the strategy laid out in [**8**, Section 2.3]. On the other hand, our emphasis in Section 7.3 on local degrees in directions at Berkovich points comes from Rivera-Letelier's constructions in [**114**, Section 2.1] and [**115**, Section 4.1], although this concept also appears in [**8**, Section 9.1] with different terminology and notation.

As for local degrees themselves, our Definition 7.26 is inspired by [**8**, Proposition 9.15 and Corollary 9.17]. However, there are other equivalent ways of defining $\deg_\zeta(\phi)$. Baker and Rumely's version, which appears as [**8**, Definition 9.7], is measure-theoretic, while Favre and Rivera-Letelier's version, embedded in the proof of [**63**, Proposition-Définition 2.1], is algebraic. In his earlier papers, [**114**, Section 2.2] and [**115**, Section 4.2], Rivera-Letelier defined the local degree on a type-by-type basis, in line with our Corollary 7.32 and Theorem 7.34, but his definition essentially amounts to the sum of directional local degrees found in our Theorem 7.30.

7.6. The injectivity and ramification loci

Definition 7.39. Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function. The *ramification locus* of ϕ is the set

$$\text{Ram}(\phi) := \big\{\zeta \in \mathbb{P}^1_{\text{an}} : \deg_\zeta(\phi) \geq 2\big\},$$

and the *injectivity locus* of ϕ is the complement

$$\text{Inj}(\phi) := \mathbb{P}^1_{\text{an}} \smallsetminus \text{Ram}(\phi) = \big\{\zeta \in \mathbb{P}^1_{\text{an}} : \deg_\zeta(\phi) = 1\big\}.$$

By Theorem 7.29(a), $\mathrm{Inj}(\phi)$ is an open set and $\mathrm{Ram}(\phi)$ is closed. In complex analysis, of course, the ramification locus of $\phi \in \mathbb{C}(z)$ is the finite set of critical points of ϕ. However, in our setting, the ramification locus is always infinite if $\deg(\phi) \geq 2$. Indeed, by the Riemann–Hurwitz formula (Theorem 1.9), ϕ has at least one critical point $a \in \mathbb{P}^1(\mathbb{C}_v)$, with some local degree $m \geq 2$. After a change of coordinates, we may assume that $a = 0$, and that $\phi(z) = c_0 + c_m z^m + \cdots$ on some disk $D(0,r)$. Thus, $\deg_{\zeta(0,s)}(\phi) = m$ for all sufficiently small s. In particular, $\mathrm{Ram}(\phi)$ contains a line segment extending from a.

Example 7.40. Let $\phi(z) = z^d$ with $d \geq 2$. Then $\deg_{\zeta(0,r)}(\phi) = d$ for all $r > 0$, and hence the Berkovich interval $[0,\infty]$ is contained in $\mathrm{Ram}(\phi)$.

If $|d| = 1$, i.e., if d is not divisible by the residue characteristic $\operatorname{char} k$, then $\mathrm{Ram}(\phi) = [0,\infty]$. Indeed, on any disk of the form $D(a,r)$ with $r < |a|$, we have

$$\phi(z) = a^d + da^{d-1}(z-a) + \sum_{i=2}^{d} \binom{d}{i} a^{d-i}(z-a)^i, \tag{7.9}$$

which has Weierstrass degree 1 on $D(a,r)$. Hence, every point of $D_{\mathrm{an}}(a,r)$ belongs to $\mathrm{Inj}(\phi)$.

If $|d| = 0$, i.e., if d is divisible by $p := \operatorname{char} \mathbb{C}_v$, then $\mathrm{Ram}(\phi) = \mathbb{P}^1_{\mathrm{an}}$. After all, the expansion of equation (7.9) has Weierstrass degree at least p on any disk $D(a,r)$.

If $0 < |d| < 1$, i.e., if $\mathbb{C}_v$ has mixed characteristic, with d divisible by the residue characteristic $p := \operatorname{char} k$, then

$$\mathrm{Ram}(\phi) = \left\{ \zeta \in \mathbb{A}^1_{\mathrm{an}} \,:\, \mathrm{diam}(\zeta) \geq |p|^{1/(p-1)} |\zeta| \right\} \cup \{\infty\},$$

as we leave the reader to check in Exercise 7.24. This set properly contains the interval $[0,\infty]$, with various extra points of Types II, III, and IV. In particular, the Type II and III points in $\mathrm{Ram}(\phi)$ are precisely those of the form $\zeta(a,r)$ with $r \geq |p|^{1/(p-1)}|a|$.

In Example 7.40, we see that even for the simple map z^d, the injectivity locus $\mathrm{Inj}(\phi)$ can consist of infinitely many connected components. In that example, all such components are disks; more generally, the components of $\mathrm{Inj}(\phi)$ are at worst affinoids, as the next result shows.

Theorem 7.41. *Let $\phi \in \mathbb{C}_v(z)$ be a rational function of degree $d \geq 2$. Then every connected component of the injectivity locus* $\mathrm{Inj}(\phi)$ *is a rational open connected affinoid.*

Proof. By Theorem 7.29(a), $\mathrm{Inj}(\phi)$ is an open set. Thus, a connected component $U \subseteq \mathrm{Inj}(\phi)$ is an open connected subset of $\mathbb{P}^1_{\mathrm{an}}$. We claim that every point of ∂U is of Type II and is isolated in ∂U.

Pick $\xi \in U$. Given $\zeta \in \partial U$, we have $\zeta \notin U$, since U is open, and hence $\deg_\zeta(\phi) = m \geq 2$. However, the half-open interval $(\zeta, \xi]$ must be contained in U, and hence in $\operatorname{Inj}(\phi)$.

Suppose first that ζ is of Type I. Change coordinates on the domain and the range separately, so that $\zeta = \phi(\zeta) = 0$. The path $(0, \xi] \subseteq U$ must contain all points of the form $\zeta(0, r)$ for $r > 0$ small enough. However, by definition of $\deg_0(\phi)$, the power series expansion of ϕ is of the form $\phi(z) = a_m z^m + \cdots$ with $a_m \neq 0$. For all $r > 0$ small enough, then, $\deg_{\zeta(0,r)}(\phi) = m \geq 2$, contradicting the fact that $\zeta(0, r) \in U \subseteq \operatorname{Inj}(\phi)$.

Suppose next that ζ is of Type III. After a change of coordinates, we may assume that $\zeta = \zeta(0, s)$ for some $s \in (0, \infty) \smallsetminus |\mathbb{C}_v^\times|$, and that $\xi \in D_{\mathrm{an}}(0, s)$. The interval $(\zeta, \xi]$ must therefore contain all points of the form $\zeta(0, r)$ for $r < s$ large enough. By definition of $\deg_\zeta(\phi)$, however, the Laurent series expansion of ϕ on a thin enough open annulus $\{\lambda s < |\zeta| < s\}$ has inner and outer Weierstrass degree equal to $\pm m$. Thus,

$$\deg_{\zeta(0,r)}(\phi) = m \geq 2$$

for all $r < s$ large enough, contradicting the fact that $\zeta(0, r) \in U \subseteq \operatorname{Inj}(\phi)$.

Similarly, if ζ is of Type IV, represented by a decreasing sequence of disks $D(a_1, r_1) \supseteq D(a_2, r_r) \supseteq \cdots$, then for all sufficiently large n, the point $\zeta(a_n, r_n)$ must lie in the path $(\zeta, \xi] \subseteq U$. By definition, however, we have $\deg_{\zeta(a_n, r_n)}(\phi) = m \geq 2$ for all sufficiently large n, contradicting the fact that $\zeta(a_n, r_n) \in U \subseteq \operatorname{Inj}(\phi)$.

Thus, ζ is indeed of Type II. Let $\boldsymbol{v}$ be the direction at ζ containing ξ. Then $U \subseteq \boldsymbol{v}$, since $\zeta \notin \operatorname{Inj}(\phi)$ and U is a a connected component of $\operatorname{Inj}(\phi)$. After a change of coordinates, we may assume that $\zeta = \zeta(0, 1)$, and that $\boldsymbol{v} = \vec{v}_\zeta(0)$. Hence, there is some $\lambda < 1$ sufficiently large that ϕ has inner and outer Weiestrass degree equal to some $m \in \mathbb{Z}$ on the annulus

$$V := \big\{\lambda < |\zeta| < 1\big\}.$$

The interval $(\zeta, \xi] \subseteq U$ contains every point of the form $\zeta(0, r)$ for $r < 1$ sufficiently large. Because $\deg_{\zeta(0,r)}(\phi) = 1$ for all such r, we must have $m = \pm 1$. Thus, $\deg_\eta(\phi) = 1$ for all $\eta \in V$, and therefore $V \subseteq \operatorname{Inj}(\phi)$. Because V is connected, in fact we have $V \subseteq U$. As a result,

$$(\partial U) \cap \big(\mathbb{P}^1_{\mathrm{an}} \smallsetminus \overline{D}_{\mathrm{an}}(0, \lambda)\big) = \{\zeta\},$$

and hence ζ is an isolated point of ∂U, proving our claim.

On the other hand, ∂U is a closed subset of $\mathbb{P}^1_{\mathrm{an}}$, and therefore it is compact. Since every point is isolated, it follows that ∂U is finite. Thus, U is a connected component of $\mathbb{P}^1_{\mathrm{an}} \smallsetminus \partial U$, where ∂U is a finite set of Type II points. Hence, U is a rational open connected affinoid. $\square$

Any rational open connected affinoid can be realized as a component of $\mathrm{Inj}(\phi)$ for some $\phi \in \mathbb{C}_v(z)$; see Exercises 7.28 and 7.29. For more on the ramification locus $\mathrm{Ram}(\phi)$, see [**52**, **53**].

Exercises for Chapter 7

7.1. Let $\zeta \in \mathbb{A}^1_{\mathrm{an}}$, define $I := \{g \in \mathbb{C}_v[z] : \|g\|_\zeta = 0\}$, and let

$$A := \mathbb{C}_v[z]_I = \big\{ f/g \;:\; f, g \in \mathbb{C}_v[z] \text{ and } g \notin I \big\} \subseteq \mathbb{C}_v(z)$$

be the associated localization. Extend $\|\cdot\|_\zeta$ to A by

$$\left\| \frac{f}{g} \right\|_\zeta := \frac{\|f\|_\zeta}{\|g\|_\zeta} \quad \text{for} \quad f \in \mathbb{C}_v[z] \text{ and } g \in \mathbb{C}_v[z] \smallsetminus I. \tag{7.10}$$

(a) Prove that I is a prime ideal of $\mathbb{C}_v[z]$.

(b) If $\zeta = a \in \mathbb{C}_v$ is of Type I, prove that $I = \langle z - a \rangle$.

(c) If ζ is not of Type I, prove that $I = \{0\}$, and hence $A = \mathbb{C}_v(z)$.

(d) Prove that equation (7.10) is a well-defined multiplicative seminorm on A extending the original seminorm $\|\cdot\|_\zeta$ on $\mathbb{C}_v[z]$.

7.2. In this exercise, we flesh out the construction hinted at in Remark 7.3. Write $\mathbb{P}^1_{\mathrm{an}}$ as the gluing of two copies $\mathbb{A}^1_{\mathrm{an},1}$ and $\mathbb{A}^1_{\mathrm{an},2}$ of the affine line, as in Definition 6.11. Let $\phi \in \mathbb{C}_v[z]$, write $\phi = g/h$ for relatively prime polynomials $g, h \in \mathbb{C}_v[z]$, and let

$$d := \deg(\phi) = \max\big\{ \deg(g), \deg(h) \big\}.$$

(a) Define the following four subsets of $\mathbb{A}^1_{\mathrm{an},1} \amalg \mathbb{A}^1_{\mathrm{an},2}$:
- $U_{11} := \{\zeta \in \mathbb{A}^1_{\mathrm{an},1} : \|h\|_\zeta > 0\}$,
- $U_{12} := \{\zeta \in \mathbb{A}^1_{\mathrm{an},1} : \|g\|_\zeta > 0\}$,
- $U_{21} := \{\zeta \in \mathbb{A}^1_{\mathrm{an},2} : \|z^d h(1/z)\|_\zeta > 0\}$,
- $U_{22} := \{\zeta \in \mathbb{A}^1_{\mathrm{an},2} : \|z^d g(1/z)\|_\zeta > 0\}$.

Prove that each U_{ij} is an open subset of $\mathbb{A}^1_{\mathrm{an},i}$ and that together the four sets cover $\mathbb{A}^1_{\mathrm{an},1} \amalg \mathbb{A}^1_{\mathrm{an},2}$.

(b) Define maps $\phi_{ij} : U_{ij} \to \mathbb{A}^1_{\mathrm{an},j}$ by
- $\|f(z)\|_{\phi_{11}(\zeta)} := \|f(\phi(z))\|_\zeta$,
- $\|f(z)\|_{\phi_{12}(\zeta)} := \|f(1/\phi(z))\|_\zeta$,
- $\|f(z)\|_{\phi_{21}(\zeta)} := \|f(\phi(1/z))\|_\zeta$,
- $\|f(z)\|_{\phi_{22}(\zeta)} := \|f(1/\phi(1/z))\|_\zeta$,

for $f \in \mathbb{C}_v[z]$. Prove that for each $i, j \in \{1, 2\}$ and any $\zeta \in U_{ij}$, the expression $\phi_{ij}(\zeta)$ above is indeed defined and is a seminorm on $\mathbb{C}_v[z]$ extending $|\cdot|$, and hence an element of $\mathbb{A}^1_{\mathrm{an}}$.
(*Suggestion*: For ϕ_{21} and ϕ_{22}, write $\phi(1/z) = [z^d g(1/z)]/[z^d h(1/z)]$.)

(c) Prove that the maps ϕ_{ij} induce a well-defined map from $\mathbb{P}^1_{\text{an}}$ to $\mathbb{P}^1_{\text{an}}$. That is, prove that the four maps agree on their overlaps, after gluing $\mathbb{A}_{\text{an},1}$ to $\mathbb{A}_{\text{an},2}$ via $\zeta \mapsto 1/\zeta$ on both the domain and range.

(d) Prove that the map from part (c) above coincides with the description of $\phi : \mathbb{P}^1_{\text{an}} \to \mathbb{P}^1_{\text{an}}$ of Definition 7.2.

7.3. Prove the following claim made in Remark 7.11: The set of linear fractional transformations $(az+b)/(cz+d) \in \text{PGL}(2, \mathbb{C}_v)$ that fix $\zeta(0,1)$ is precisely $\text{PGL}(2, \mathcal{O})$.

7.4. Prove Proposition 7.17. (Use nothing beyond Section 7.2, as Proposition 7.17 was used in the proof of Theorem 7.22 and subsequent results.)

7.5. Prove Proposition 7.21.

7.6. Prove that local degrees in directions are invariant under coordinate changes. That is, if $\phi \in \mathbb{C}_v(z)$ and $\eta, \theta \in \text{PGL}(2, \mathbb{C}_v)$, then for any $\zeta \in \mathbb{P}^1_{\text{an}}$ and any direction $\boldsymbol{v}$ at ζ, if we set $\boldsymbol{w}$ to be the direction $\eta_{\#}(\boldsymbol{v})$ at $\eta(\zeta)$, then

$$\deg_{\eta(\zeta),\boldsymbol{w}}(\phi) = \deg_{\zeta,\boldsymbol{v}}(\theta \circ \phi \circ \eta).$$

7.7. Prove the following strengthening of Theorem 7.22 for points of Type II. Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, and let $\zeta \in \mathbb{P}^1_{\text{an}}$ be a point of Type II. Prove that there exists $\delta > 0$ such that for *any* direction $\boldsymbol{v}$ at ζ, there is an integer $m \geq 1$ so that for any point $\xi \in \boldsymbol{v}$ with $d_{\mathbb{H}}(\zeta, \xi) \leq \delta$,

- ϕ maps $[\zeta, \xi]$ homeomorphically onto $[\phi(\zeta), \phi(\xi)]$, and
- $d_{\mathbb{H}}(\phi(\xi_1), \phi(\xi_2)) = m \cdot d_{\mathbb{H}}(\xi_1, \xi_2)$ for all $\xi_1, \xi_2 \in [\zeta, \xi]$.

That is, show that the quantity λ of Theorem 7.12 can be chosen independently of the direction $\boldsymbol{v}$.

7.8. Prove Proposition 7.25.

(*Suggestion*: Use Exercise 3.21.)

7.9. Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, and let $\zeta \in \mathbb{P}^1_{\text{an}}$ be a point of Type III. Prove that $\deg_\zeta(\phi)$ is the Weierstrass degree of ϕ on any small enough annulus containing ζ.

More precisely, prove that for any $\xi_1, \xi_2 \in \mathbb{P}^1_{\text{an}}$ with $\zeta \in (\xi_1, \xi_2)$ and $d_{\mathbb{H}}(\zeta, \xi_i)$ small enough, if we define U to be the annulus that is the connected component of $\mathbb{P}^1_{\text{an}} \smallsetminus \{\xi_1, \xi_2\}$ containing ζ, then the inner and outer Weierstrass degrees of ϕ on U both equal $\deg_\zeta(\phi)$.

7.10. Fix $c \in \mathbb{C}_v$ with $0 < |c| < 1$, and let

$$\phi(z) = \frac{z^4 - 8c^4 z - c^4}{4z^3 + z^2 + 4c^4} \in \mathbb{C}_v(z)$$

as in Example 7.31. Prove the following claims made in that example.

For $|c|^2 < r < |c|$,

(a) $\deg_{\zeta(0,r)}(\phi) = 2$;

(b) $\phi(\zeta(0,r)) = \zeta(0, |c|^4/r^2)$;

(c) $\phi_{\#}$ maps the direction at $\zeta(0,r)$ containing 0 to the direction at $\zeta(0, |c|^4/r^2)$ containing ∞; and

(d) $\phi_{\#}$ maps the direction at $\zeta(0,r)$ containing ∞ to the direction at $\zeta(0, |c|^4/r^2)$ containing 0.

7.11. For the same function ϕ as in Exercise 7.10, prove the following claims made in Example 7.31:

(a) $\phi(\zeta(0,1)) = \phi(\zeta(0,|c|^2)) = \zeta(0,1)$;

(b) $\deg_{\zeta(0,1)}(\phi) = \deg_{\zeta(0,|c|^2)}(\phi) = 2$;

(c) $\phi(\zeta(0,|c|)) = \zeta(0,|c|^2)$; and

(d) $\deg_{\zeta(0,|c|)}(\phi) = 4$.

7.12. Prove Proposition 7.33.

(*Suggestion*: Use Lemma 7.27, Theorem 7.29(c), and Theorem 7.30.)

7.13. Let $\phi \in \mathbb{C}_v(z)$ be a nonconstant rational function, and let $\zeta_1, \zeta_2 \in \mathbb{P}^1_{\text{an}}$. Prove that the image $\phi([\zeta_1, \zeta_2])$ of the interval $[\zeta_1, \zeta_2] \subseteq \mathbb{P}^1_{\text{an}}$ is a finite union of closed intervals.

7.14. Let $\phi \in \mathbb{C}_v(z)$ be a rational function of degree $d \geq 1$, and let $\zeta_1, \zeta_2 \in \mathbb{H}$. Suppose that ϕ is injective on the interval $[\zeta_1, \zeta_2]$. Let $b := d_{\mathbb{H}}(\zeta_1, \zeta_2)$, and for each real number $t \in [0, b]$, define $\zeta(t)$ to be the unique point in $[\zeta_1, \zeta_2]$ for which $d_{\mathbb{H}}(\zeta_1, \zeta(t)) = t$. Prove that

$$d_{\mathbb{H}}\big(\phi(\zeta_1), \phi(\zeta_2)\big) = \int_0^b \deg_{\zeta(t)}(\phi)\, dt.$$

In particular, $d_{\mathbb{H}}(\phi(\zeta_1), \phi(\zeta_2)) \geq d_{\mathbb{H}}(\zeta_1, \zeta_2)$.

7.15. Fix a prime $p \geq 2$, and let $\mathbb{C}_v = \mathbb{C}_p$. Let $\phi(z) = z^p$. Fix $r > 0$.

(a) Let $\zeta = \zeta(0,r)$. Prove that $\phi(\zeta) = \zeta(0, r^p)$ and that $\deg_\zeta(\phi) = p$.

(b) Suppose that $r \geq |p|^{1/(p-1)}$ and let $\zeta = \zeta(1,r)$. Prove that $\phi(\zeta) = \zeta(1, r^p)$ and that $\deg_\zeta(\phi) = p$.

(c) Suppose that $r < |p|^{1/(p-1)}$, and let $\zeta = \zeta(1,r)$. Prove that $\phi(\zeta) = \zeta(1, |p|r)$ and that $\deg_\zeta(\phi) = 1$.

(d) Let $D_1 \supsetneq D_2 \supsetneq \cdots$ be a decreasing sequence of closed disks in $\mathbb{C}_p$ with empty intersection. Let $\zeta \in \mathbb{P}^1_{\text{an}}$ be the corresponding Type IV point. Writing $D_i = \overline{D}(a_i, r_i)$, assume that $|a_i| = 1$ and $|p|^{1/(p-1)} < r_i < 1$ for all i. Prove that $\phi(\zeta)$ is the Type IV point corresponding to the sequences of disks

$$\overline{D}(a_1^p, r_1^p) \supsetneq \overline{D}(a_2^p, r_2^p) \supsetneq \cdots$$

and that $\deg_\zeta(\phi) = p$.

7.16. Let $\phi \in \mathbb{C}_v(z)$ be nonconstant, let $d = \deg(\phi)$, and let p be the residue characteristic of $\mathbb{C}_v$. Suppose either that $p = 0$ or that $d < p$. Prove that for any Type IV point $\zeta \in \mathbb{P}^1_{\mathrm{an}}$, we have $\deg_\zeta(\phi) = 1$.

(*Suggestion*: Use Proposition 3.9.)

7.17. Suppose $|3| = 1$. Fix $a \in \mathbb{C}_v$ with $0 < |a| < 1$, and let

$$\phi(z) = \frac{z+2}{z^3 + 3z^2 + a^2 z - 5}.$$

Let $\xi_0 := \zeta(-1, |a|)$ and $\xi_1 := \zeta(0,1)$. Prove that

$$\phi\big([\xi_0, \xi_1]\big) = \left[\zeta\left(-\frac{1}{3}, |a|^3\right), \zeta(0,1)\right],$$

with $\deg_\zeta(\phi) = 3$ for all $\zeta \in [\xi_0, \xi_1]$.

7.18. Suppose that $\operatorname{char} \mathbb{C}_v = p > 0$, let $\phi(z) \in \mathbb{C}_v(z)$ be nonconstant, and define $\psi(z) := (\phi(z))^p$. Prove that

$$\deg_\zeta(\psi) = p \deg_\zeta(\phi) \quad \text{and} \quad \deg_{\zeta,\boldsymbol{v}}(\psi) = p \deg_{\zeta,\boldsymbol{v}}(\phi)$$

for all $\zeta \in \mathbb{P}^1_{\mathrm{an}}$ and all directions $\boldsymbol{v}$ at ζ.

7.19. Fix $a \in \mathbb{C}_v$ with $0 < |a| < 1$, and let

$$\phi(z) = \frac{z-1}{az^3 + z^2 + 1}.$$

Find all $\xi \in \mathbb{P}^1_{\mathrm{an}}$ for which $\phi(\xi) = \zeta(0,1)$, and confirm Theorem 7.29(c) for ϕ and $\zeta(0,1)$ by computing $\deg_\xi(\phi)$ for each such ξ. Do this in each of the following cases.

(a) Assume the residue characteristic of $\mathbb{C}_v$ is not 2.

(b) Assume the residue characteristic of $\mathbb{C}_v$ *is* 2.
(*Note*: There are a number of different possibilities in this case, depending on the exact value of a.)

7.20. Let $\psi(z) := (pz^3 - z^2 + 1)/z \in \mathbb{C}_p(z)$. We saw in Example 7.37 that $\psi(\zeta(0,1)) = \zeta(0,1)$.

(a) Find the absolute values of all zeros of ψ, and show that the associated set T from Theorem 7.34 of bad directions at $\zeta(0,1)$ is $T = \{\vec{v}(\infty)\}$.

(b) Show directly that $\psi\big(\mathbb{P}^1(\mathbb{C}_v) \smallsetminus \overline{D}(0,1)\big) = \mathbb{P}^1(\mathbb{C}_v)$, but that

$$\psi_\#\big(\vec{v}(a)\big) = \vec{v}\big(\psi(a)\big) \quad \text{for all} \quad a \in \overline{D}(0,1).$$

(c) Find all ramification points of the reduction $\overline{\psi}(z) = (1 - z^2)/z$ in $\mathbb{P}^1(k)$.
(*Note*: The answer is different if the residue characteristic is $p = 2$.)

(d) Let $a \in \overline{D}(0,1)$. Confirm that the mapping

$$\psi : D(a,1) \to D(\psi(a),1)$$

is two-to-one if $\overline{a}$ is one of the ramification points of $\overline{\psi}$ from part (c), and is one-to-one otherwise.

(e) Show that ψ maps the annulus $\{1 < |z| < r\}$ bijectively onto itself for $r > 1$ sufficiently small. Find the largest such value of r.

7.21. Fix $a \in \mathbb{C}_v$ with $0 < |a| < 1$, and let

$$\phi(z) = \frac{az^3 + z^2 - a^2}{z^2 + a}.$$

For each of the following Berkovich points ζ, use Theorem 7.34 and the method of Example 7.37 to compute $\phi(\zeta)$ and $\deg_\zeta(\phi)$. In addition, compute $\deg_{\zeta,\boldsymbol{v}}(\phi)$ for each direction $\boldsymbol{v}$ at ζ.

(a) $\zeta = \zeta(0,1)$

(b) $\zeta = \zeta(0,|a|)$

(c) $\zeta = \zeta\left(\frac{-1}{a}, 1\right)$

(d) $\zeta = \zeta(1,|a|)$, assuming the residue characteristic of $\mathbb{C}_v$ is not 3.

(e) $\zeta = \zeta(1,|a|)$, assuming the residue characteristic of $\mathbb{C}_v$ is 3, and $|a| > |3|$.

(f) $\zeta = \zeta(1,|a|)$, assuming the residue characteristic of $\mathbb{C}_v$ is 3, and $|a| < |3|$.

(g) $\zeta = \zeta(1,|a|)$, assuming the residue characteristic of $\mathbb{C}_v$ is 3, and $|a| = |3|$.
(*Caution*: There are *many* further subcases in part (g).)

7.22. Fix $a \in \mathbb{C}_v$ with $0 < |a| < 1$, and let

$$\phi(z) = \frac{z - 1}{az^3 + z^2 + a}.$$

Let $\zeta := \zeta(0,1)$, and for any $x \in \mathbb{P}^1(\mathbb{C}_v)$, let $\vec{v}(x)$ denote the direction at ζ containing x.

(a) Confirm that $\phi(\zeta) = \zeta$ in two ways: once by Theorem 7.34, and once by Theorem 7.12.

(b) Compute $\deg_{\zeta,\vec{v}(x)}(\phi)$ for $x = 0, 1, \infty$. In each case, do the computation in in two different ways: once by finding an annulus as in Theorem 7.12 or Remark 7.14, and once by Lemma 7.35(c).

(c) Let $\boldsymbol{w} := \vec{v}(0)$. Find all directions $\boldsymbol{v}$ at ζ for which $\phi_{\#}(\boldsymbol{v}) = \boldsymbol{w}$. For each such $\boldsymbol{v}$, compute $\deg_{\zeta,\boldsymbol{v}}(\phi)$ in two different ways: once by finding an annulus as in Theorem 7.12 or Remark 7.14, and once by Lemma 7.35(c). Then confirm that Theorem 7.30 holds for ϕ, ζ, and $\boldsymbol{w}$.

(d) Repeat part (c) for $\boldsymbol{w} := \vec{v}(\infty)$.

7.23. Let $a \in \mathbb{C}_v$, $\phi(z) \in \mathbb{C}_v(z)$, and ζ be as in Exercise 7.22.

(a) Find all $\xi \in \mathbb{P}^1_{\text{an}}$ for which $\phi(\xi) = \zeta$, and compute $\deg_\xi(\phi)$ at each by Theorem 7.34. Then confirm that Theorem 7.29(c) holds for ϕ and ζ.

(b) Fix $r \in \mathbb{R}$ with $1 < r < 1/|a|$, and let $U = D_{\text{an}}(0, r)$. Compute $\phi^{-1}(U)$.

(c) With r as in part (b), let $V = \{1/r < |z| < r\}$. Compute $\phi^{-1}(V)$.

(d) With V as in part (c), let W be the connected component of $\phi^{-1}(V)$ containing ζ. Determine how many connected components $\phi^{-1}(W)$ has.

(*Note*: You will get slightly different answers in parts (b) and (c) depending on whether or not the residue characteristic is 2.)

7.24. Suppose that $\operatorname{char} \mathbb{C}_v = 0$ but $\operatorname{char} k = p > 0$, and let $\phi(z) = z^d$ for some integer $d \geq 2$ divisible by p. Prove that the ramification locus of ϕ is

$$\operatorname{Ram}(\phi) = \left\{\zeta \in \mathbb{A}^1_{\text{an}} : \operatorname{diam}(\zeta) \geq |p|^{1/(p-1)}|\zeta|\right\} \cup \{\infty\},$$

as claimed in Example 7.40.

(*Suggestion*: Use Exercise 7.15.)

7.25. Let $\phi(z) \in \mathbb{C}_v(z)$ be a rational function of degree $d \geq 2$ and of explicit good reduction. Suppose either that $\operatorname{char} k = 0$ or that $d < \operatorname{char} k$.

(a) Prove that $\operatorname{Ram}(\phi)$ is the union of all line segments of the form $[a, \zeta(0,1)]$, where $a \in \mathbb{P}^1(\mathbb{C}_v)$ is a (Type I) critical point of ϕ.

(b) Prove, equivalently, that $\operatorname{Ram}(\phi)$ is the convex hull of the set of Type I critical points of ϕ.

7.26. Let $\phi(z) \in \mathbb{C}_v(z)$ be a rational function of degree 2, and suppose that $\operatorname{char} k \neq 2$. Then ϕ has two distinct Type I critical points $a, b \in \mathbb{P}^1(\mathbb{C}_v)$. Prove that $\operatorname{Ram}(\phi) = [a, b]$.

7.27. Let $\phi(z) \in \mathbb{C}_v(z)$ be a rational function of degree $d \geq 2$, and suppose either that $\operatorname{char} k = 0$ or that $d < \operatorname{char} k$. Let S be the (finite) set of Type I critical points of ϕ. Prove that $\operatorname{Ram}(\phi)$ is contained in the convex hull of S.

(*Note*: This is a weak version of [**52**] Corollary 7.13.)

7.28. Fix $a \in \mathbb{C}_v$ with $0 < |a| < 1$, and let

$$\phi(z) = \frac{a^2}{z} + z + z^2.$$

Prove that the open affinoid $\{|a| < |\zeta| < 1\}$ is a connected component of the injectivity locus $\mathrm{Inj}(\phi)$.

7.29. Let $D_1, \ldots, D_m$ be rational closed Berkovich disks, all contained in $D_{\mathrm{an}}(0,1)$. Let $U := D_{\mathrm{an}}(0,1) \smallsetminus (D_1 \cup \cdots \cup D_m)$. Find a rational function $\phi(z) \in \mathbb{C}_v(z)$ such that U is a connected component of $\mathrm{Inj}(\phi)$.