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Preface

The past two decades represent a period of explosive growth in 4-manifold theory. From a desert of nearly complete ignorance, the theory has flourished into a virtual rain forest of ideas and techniques, a lush ecosystem supporting complex interactions between diverse fields such as gauge theory, algebraic geometry and symplectic topology, in addition to more topological ideas. Numerous books are appearing that discuss smooth 4-manifolds from the viewpoint of other disciplines. The present volume is intended to introduce the subject from a topologist's viewpoint, bridging the gaps to other disciplines and presenting classical but important topological techniques that have not previously appeared in expository literature.

For a better perspective on the rise of 4-manifold theory, it is useful to consider the history of topology. Manifolds have been a central theme of mathematics for over a century. The topology of manifolds of dimensions ≤ 2 (curves and surfaces) has been well understood since the nineteenth century. Although 3-manifold topology is much harder, there has been steady progress in the field for most of the twentieth century. High-dimensional manifold topology was revolutionized by the s-cobordism and surgery theorems, which were developed in the 1960's into powerful tools for analyzing existence and uniqueness questions about manifolds of dimension ≥ 5. The resulting theory has long since matured into a subject with a very algebraic flavor. In dimension 4, however, there was not enough room to apply the fundamental "Whitney trick" to prove these theorems, and as a result, very little was known about 4-manifold topology through the 1970's. The first revolution came in 1981 with Michael Freedman's discovery that the Whitney trick could be performed in dimension 4, provided that we ignore smooth structures and work with the underlying topological manifolds up to homeomorphism (and provided that the fundamental group is suitably "small"). The resulting theory [**FQ**] led quickly to a complete classification of closed, simply connected topological 4-manifolds, and topological 4-manifold theory now seems closely related to the theory of high-dimensional manifolds. Freedman's revolution was immediately followed by the 1982 counterrevolution of Simon Donaldson. Using gauge theory (differential geometry and nonlinear analysis), Donaldson showed that smooth 4-manifolds are much different from their high-dimensional counterparts. In fact, the predictions made by the s-cobordism and surgery conjectures for smooth 4-manifolds failed miserably, resulting in a dramatic clash between the theories of smooth and topological manifolds in this dimension. For example, this is the only dimension in which a fixed homeomorphism type of closed manifold is represented by infinitely many diffeomorphism types, or where there are manifolds homeomorphic but not diffeomorphic to $\mathbb{R}^n$. (In fact, there are uncountably many such "exotic $\mathbb{R}^4$'s".) One might think of dimension 4 as representing a phase transition between low- and high-dimensional topology, where we find uniquely complicated phenomena and diverse connections with other fields. Donaldson's program of analyzing the self-dual Yang-Mills equations [**DK**] was central to smooth 4-manifold theory for 12 years, until it was superseded in 1994 (several revolutions later) by analysis of the Seiberg-Witten equations [**KKM**], [**Mr1**], [**Sa**], which simplifies and expands Donaldson's original approach and results.

The results of gauge theory, from Donaldson through the Seiberg-Witten equations, are primarily in a negative direction, and require balance by positive results. That is, gauge theory proves the nonexistence of smooth manifolds satisfying various constraints, the nonexistence of connected-sum splittings, and the nonexistence of diffeomorphisms between pairs of manifolds. One needs a different approach for the corresponding existence results. While many useful examples come from algebraic geometry [**BPV**] and symplectic topology [**McS1**], perhaps the most powerful general technique for existence results (particularly for manifolds with small Betti numbers) is Kirby calculus. This technique, which allows one to see the internal structure of a 4-manifold (or its boundary 3-manifold) without loss of information, was created and developed into a fine art in the late 1970's by topologists such as Akbulut, Fenn, Harer, Kaplan, Kirby, Melvin, Rourke, Rolfsen and Stern. However, the theory was handicapped by the pre-Donaldson absence of any way to prove negative results. Much time was spent on ambitious goals that gauge theory now shows are impossible. Eventually, the theory was abandoned by all but the most stalwart practitioners. Since the advent of gauge theory, however, Kirby calculus has entered a Renaissance. Armed with the knowledge of what *not* to attempt, topologists are using the calculus to construct new manifolds with novel gauge-theoretic properties, some of which are nonalgebraic or even nonsymplectic, and to show that other examples are diffeomorphic or to decompose them into simple pieces. The insight provided by the calculus into the internal structure of manifolds meshes with gauge theory to create an even more powerful tool for analyzing 4-manifolds. In addition, surprising connections have emerged with affine complex analysis and contact topology [**G13**], [**G14**] since a discovery of Eliashberg led to a theory of Kirby diagrams for representing Stein surfaces.

One of the main goals of the present book is to provide an exposition of Kirby calculus that is both elementary and comprehensive, since there appears to be no previous reference in the literature that satisfies either of these conditions. We have attempted a complete exposition, providing careful proofs of the main theorems and constructions, either directly or through references to the literature (notably to [**M4**] and [**RS**] for careful treatments of handlebody theory in general dimensions). This is at least partly to avoid conveying a false impression of Kirby calculus as being "just pictures and not proofs". For easy reference, we have included an index of important diagrams, following the glossary of notation in Chapter 13. The reader should note that we have included Kirby diagrams representing all of the main types of closed, simply connected 4-manifolds (as viewed from the current perspective of the theory), namely complex surfaces of rational, elliptic and general type, a symplectic but noncomplex manifold and an irreducible nonsymplectic one. (We have also included an example with even $b_2^{\pm}$ that might be irreducible.) Chapter 13 also provides an index for Kirby moves and related operations such as Rolfsen moves, Gluck twists and logarithmic transformations. The text has been liberally sprinkled with exercises intended to increase the reader's comprehension; many of these are labelled with an asterisk and solved in Chapter 12.

The remaining goal of the book is to introduce 4-manifold theory in its current state. There are many books avaliable on the subject, but ours is almost unique in describing the theory from the point of view of differential topology. The other reference from this viewpoint is Kirby [**K2**]; our text is intended to be complementary to it. Parts of the text were inspired by Harer, Kas and Kirby [**HKK**]; where overlap occurs we have tried to choose a more elementary and leisurely approach. There are many references for gauge theory as applied to 4-manifolds, notably [**DK**] (one of the most recent references from the viewpoint of the self-dual equations), and [**KKM**], [**Mr1**], [**Sa**] on Seiberg-Witten theory. These provide detailed treatments, so our approach to gauge theory is to sketch the main ideas and applications with references for details. Similarly, the theory of complex surfaces is covered in detail in [**BPV**], and symplectic topology is carefully treated in

[**McS1**], so we again focus on the main applications to 4-manifold topology while avoiding unnecessary coverage of other aspects of these theories. For topological 4-manifolds, the reader is referred to [**FQ**] after our brief discussions. Although we treat Rolfsen calculus in some detail, the reader is also referred to [**Ro**] for this 3-dimensional technique related to Kirby calculus. One other noteworthy reference is Kirby's latest list [**K4**] of problems in low-dimensional topology; many of these problems are directly related to 4-manifolds and Kirby calculus.

This book is divided into four parts. The first part covers introductory material and basic techniques for later use, as well as an outline of the current state of the theory of 4-manifolds and surfaces contained in them. Part 2 is our main exposition of Kirby calculus. It is essentially independent of Part 1, except for such elementary notions as intersection forms. The logical dependence of the sections of Part 2 is approximately given by Figure 0.1. (Dashed arrows indicate only occasional or minor dependence.) Part 3 ties together the two previous parts by presenting more advanced applications of Kirby calculus, and consists of five mostly independent chapters intended to cover current research areas within 4-manifold theory and their connections to other disciplines. While we have attempted to include the most recent developments, such a goal is inevitably doomed by the rapid change of the field. Solutions to exercises and the tables described above comprise Part 4. The book can be used as a graduate text, with each of the first two parts providing enough material for nearly a semester. The topics in the third part provide supplementary material intended to introduce a student to research in 4-manifold topology.

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Robert E. Gompf and András I. Stipsicz

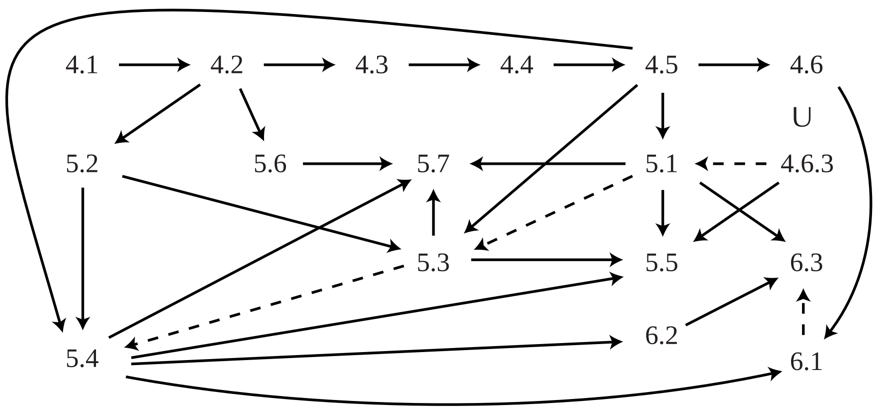

Figure 0.1. Logical dependence of the sections of Part 2

Chapter 1

Introduction

1.1. Manifolds

We assume that the reader is familiar with the basics of algebraic topology, such as homotopy theory and singular homology and cohomology. We will use the terms *principal G-bundle*, *tangent bundle* of a manifold, *associated vector bundle* and *section* of a bundle without defining them. Similarly, various forms of *Poincaré duality* will be used without explicit description of the theorems. (Detailed treatments of these topics can be found in, e.g., [**GP**], [**MS**] and [**Sp**].) Definitions of *topological* and *smooth manifolds*, *orientability*, *complex structures* and *ambient isotopy* are given to serve as a reference in forthcoming discussions. For similar reasons, we have included two sections (Sections 1.4 and 5.6) introducing *characteristic classes* and *spin structures*. An n-dimensional manifold (with possibly nonempty boundary) is usually denoted by X; we use M in contexts where the manifold is conveniently thought of as a boundary component of some other manifold. If we wish to emphasize that the 4-manifold is equipped with a complex structure, we denote it by S (being a complex *surface*). We define $\mathbb{R}^n_+$ as the upper half space of $\mathbb{R}^n$, in coordinates, $\mathbb{R}^n_+ = \{(x_1, \dots , x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$; otherwise we use standard notation.

Definition 1.1.1. A separable Hausdorff topological space X is an n-dimensional *topological (or C^0-) manifold* if for every point $p \in X$ there is an open neighborhood U of p in X which is homeomorphic to an open subset of $\mathbb{R}^n_+$. A pair (U_α, ϕ_α) of such a neighborhood and homeomorphism is called a *chart*. A collection of charts $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ is an *atlas* if it is a cover of X, that is, $\bigcup\{U_\alpha \mid \alpha \in A\} = X$. The map $\phi_\beta \circ \phi_\alpha^{-1}$ (on $\phi_\alpha(U_\alpha \cap U_\beta)$) is the *transition function* between the charts (U_α, ϕ_α) and (U_β, ϕ_β). The points of

X corresponding to points in $\{(x_1, \dots, x_n) \mid x_n = 0\} = \mathbb{R}^{n-1} \subset \mathbb{R}^n_+$ form a submanifold ∂X of dimension $n-1$, which is called the *boundary* of X. The topological manifold X with an atlas $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ is a *C^r-manifold* ($r = 1, 2, 3, \dots, \infty$) if the transition functions $\phi_\beta \circ \phi_\alpha^{-1}$ ($\alpha, \beta \in A$) are C^r-maps. In the case $r = \infty$, X is called a *smooth* manifold. In the notation, we will not specify the atlas giving the C^r-structure of X. Note that by definition a C^r-structure specifies a C^s-structure on X for all $0 \le s \le r$. A map $f \colon X \to X'$ is a *C^r-map* between the two C^r-manifolds X and X' if f is C^r on every chart of the given C^r-atlases. A homeomorphism $f \colon X \to X'$ is a *C^r-diffeomorphism* if both f and f^{-1} are C^r-maps. Two C^r-structures on X are *isotopic* if the identity map id_X is isotopic (homotopic through homeomorphisms) to a C^r-diffeomorphism between the structures. We will not distinguish between isotopic C^r-structures, and frequently use C^r-diffeomorphisms to identify C^r-manifolds with each other. We say that X is *closed* if it is compact and $\partial X = \emptyset$. We call a space X a *singular manifold* if there is a finite subset $Sing \subset X$ such that $X - Sing$ is a smooth manifold. The *n-dimensional disk* $\{x \in \mathbb{R}^n \mid ||x|| \le 1\}$ will be denoted by D^n; in particular, D^n is a compact manifold and the boundary ∂D^n is the *$(n-1)$-dimensional sphere* S^{n-1}.

An *orientation* of the Euclidean vector space $\mathbb{R}^n$ is simply a choice of one orbit of the set of ordered bases under the action of the connected group $GL^+(n;\mathbb{R}) = \{A \in GL(n;\mathbb{R}) \mid \det(A) > 0\}$. The chosen equivalence class is referred to as the set of *positive* bases; the rest of the bases (which form the other equivalence class) are the *negative* bases. An orientation can be given by fixing an ordering (up to even permutations) of a given basis.

Definition 1.1.2. Let X be a smooth n-dimensional manifold, hence it admits a *tangent bundle* $TX \to X$ with fibers isomorphic to $\mathbb{R}^n$ (see, e.g., [**GP**], [**MS**]). A consistent choice of orientation of the tangent space at every point of X is called an *orientation* of X. (By consistent we mean that at each point in X there is a chart (V, φ) mapping into $\mathbb{R}^n$ with its standard orientation, such that $d\varphi_p$ preserves orientation at every point p of V.) This choice, however, cannot be made for every manifold. X is called *orientable* if it admits an orientation, and X is *oriented* if an orientation of X is fixed. The standard convention of "outward normal first" provides an orientation of ∂X induced by an orientation of X [**GP**]: At $p \in \partial X$ the basis $(v_1, \dots, v_{n-1})$ of $T_p\partial X$ is positive if $(\nu, v_1, \dots, v_{n-1})$ is a positive basis of T_pX, where the vector ν stands for the outward normal vector, which is tangent to X but not to ∂X and points out of X. Since orientation plays a key role throughout 4-manifold theory, we will always be careful about specifying a fixed orientation of the manifold X. If we change the orientation on every component of X, the new object will be denoted by $\overline{X}$.

Remark 1.1.3. It can be shown that an orientation specifies a class $[X]$ in $H_n(X, \partial X; \mathbb{Z})$, called the *fundamental class* of the oriented manifold (see, e.g., the appendix of [**MS**]). For X noncompact, we should use "locally finite" homology on infinite chains. In this way, orientability can easily be extended to topological manifolds as well: An n-dimensional connected manifold X is orientable if $H_n(X, \partial X; \mathbb{Z}) \cong \mathbb{Z}$, and an orientation of X is simply a choice of a generator of the group $H_n(X, \partial X; \mathbb{Z})$. The induced orientation of ∂X can be seen as the image of $[X] \in H_n(X, \partial X; \mathbb{Z})$ under the map $H_n(X, \partial X; \mathbb{Z}) \to H_{n-1}(\partial X; \mathbb{Z})$ of the long exact sequence of the pair $(X, \partial X)$; moreover $[\overline{X}] = -[X] \in H_n(X, \partial X; \mathbb{Z})$. In terms of bundles, the orientability of X can be rephrased as follows. The smooth n-dimensional manifold X is orientable if the structure group of the tangent bundle $TX \to X$ (which is $GL(n; \mathbb{R})$) can be reduced to its connected component $GL^+(n; \mathbb{R})$; by fixing a reduction we specify an orientation for X. Note that this approach to orientability can easily be extended to arbitrary (real) vector bundles (cf. Lemma 1.4.23).

Definition 1.1.4. An atlas $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A_{\mathbb{C}}\}$ on a (real) $2n$-dimensional manifold X is a *complex structure* if each ϕ_α is a homeomorphism between U_α and an open subset of $\mathbb{C}^n$ (identified with $\mathbb{R}^{2n}$), and the transition functions $\phi_\beta \circ \phi_\alpha^{-1}$ are holomorphic. Complex manifolds are canonically oriented. This is because the connected group $GL(n; \mathbb{C})$ lies in $GL^+(2n; \mathbb{R})$, so any complex n-dimensional vector space is canonically oriented — by choosing a complex isomorphism with $\mathbb{C}^n = \mathbb{C} \times \ldots \times \mathbb{C}$, with $\mathbb{C}$ oriented as a real vector space by the ordered basis $(1, i)$.

Definition 1.1.5. A smooth (resp. topological) *isotopy* between embeddings $\varphi_0, \varphi_1 \colon Y \to X$ is a smooth (resp. topological) homotopy $\varphi_t \colon Y \to X$ $(0 \leq t \leq 1)$ through embeddings. If an isotopy exists, then φ_0 and φ_1 are *isotopic*. By the Isotopy Extension Theorem (Theorem 5.8 of [**M4**]), if Y is compact then any smooth isotopy $Y \to \operatorname{int} X$ can be extended to an *ambient isotopy*, an isotopy $\Phi_t \colon X \to X$ through diffeomorphisms such that $\Phi_0 = \mathrm{id}_X$ and $\varphi_t = \Phi_t \circ \varphi_0$ for each t. Two (possibly singular) submanifolds Y_1, Y_2 in a manifold X are *ambiently isotopic* if there is a diffeomorphism of X homotopic to id_X through diffeomorphisms which maps Y_1 to Y_2.

Having dispensed with the preliminaries, we state a few theorems and conjectures related to the classification problem of n-dimensional manifolds. Theorem 1.1.6 demonstrates the fact that although the notion of C^r-manifold is defined for every integer r $(0 \leq r \leq \infty)$, the $r = 0$ and $r = \infty$ cases are the only interesting ones in terms of classification.

Theorem 1.1.6. ([**Mu**]) *Suppose that X is a C^r-manifold and $1 \leq r \leq k$ (including $k = \infty$). Then there is a C^k-atlas of X for which the induced*

C^r-structure is isotopic to the original C^r-structure of X. (In fact, id_X is a C^r-diffeomorphism between them.) Moreover, this C^k-structure is unique up to isotopy (through C^r-diffeomorphisms); consequently the C^r-manifold X admits a unique induced C^k-structure for every $k \geq r$. (As we will see, this statement does not hold for $r = 0$.) □

Our primary aim in manifold theory is to classify topological manifolds, i.e., to give a complete list of n-dimensional (closed) topological manifolds, and to find a way to tell which topological manifolds carry smooth structures (have C^∞-atlases). Furthermore, if there is one such atlas, we would like to determine the total number of these up to diffeomorphism. In most dimensions this aim cannot be achieved for algebraic reasons (cf. Theorem 1.2.33 and Exercise 5.1.10(c)); in those cases we will impose further conditions (like simple connectivity) for the manifolds at hand. For a better understanding of results concerning 4-manifolds, we will conclude this section with theorems concerning manifolds of dimension different from 4. Assume that the manifolds we are working with are closed, connected and oriented. The classification problem is easy in dimension 1 and classical in dimension 2. Up to homeomorphism there is only one topological 1-manifold with the above properties, and this is the circle $S^1 = \{x \in \mathbb{R}^2 \mid ||x|| = 1\}$; it admits a unique isotopy class of smooth structures. For $n = 2$, the (oriented) topological 2-manifolds are precisely the surfaces Σ_g with genus g ($g = 0, 1, 2, \dots$); in particular, Σ_0 is the sphere S^2 and Σ_1 is the 2-dimensional torus $T^2 = S^1 \times S^1$. All these topological manifolds carry unique smooth structures (up to isotopy); actually these manifolds carry complex structures as well. The classification problem in dimension 3 is among the most popular in contemporary mathematics. Although we will discuss constructions of 3-manifolds in the present volume, we will not address the classification problem in dimension 3. It is known that every topological 3-manifold admits a unique smooth structure [**Mo**]; the classification problem of topological 3-manifolds is, however, still unsolved. Understanding 3-manifolds homotopy equivalent to the 3-dimensional sphere S^3 would be a major step in this direction; see the conjecture below. (For further discussion of 3-manifolds, see for example [**He**], [**N**], [**Ro**], [**Th2**].)

Conjecture 1.1.7. (Poincaré Conjecture) *A simply connected closed 3-manifold is homeomorphic to S^3.*

For topological manifolds of dimension $n \geq 5$ there is sophisticated machinery for dealing with both the existence problem and the number of nonisotopic smooth structures on a given topological manifold. Parts of this theory will be mentioned later on in this volume (e.g., Theorem 9.2.2), so here we only mention one theorem (which is false in dimension 4):

Theorem 1.1.8. *If X^n is a compact n-dimensional topological manifold and $n \geq 6$ (or $n \geq 5$ and $\partial X = \emptyset$), then there are only finitely many smooth structures on X (up to isotopy). In particular, there are smooth manifolds $Y_1, \ldots, Y_k$ homeomorphic to X such that any smooth manifold homeomorphic to X is diffeomorphic to some Y_i. (Here k might be 0, meaning that there is no smooth structure on X.)* □

Finally, we quote a theorem which emphasizes the special behavior of 4-dimensional manifolds. Let X be a smooth noncompact n-dimensional manifold.

Theorem 1.1.9. *If X is homeomorphic to $\mathbb{R}^n$ and $n \neq 4$ then X is diffeomorphic to $\mathbb{R}^n$. If $n = 4$, this statement is false, there are "exotic" $\mathbb{R}^4$'s; such examples will be given in Sections 9.3 and 9.4.* □

In general, the term *exotic smooth structure* is used to refer to smooth structures not diffeomorphic to the given one on a smooth manifold X. These correspond to manifolds homeomorphic to X but not diffeomorphic to it.

Remark 1.1.10. Another sort of structure frequently used by topologists is a *piecewise linear (PL-) structure*, which is defined by an atlas whose transition functions respect a suitable triangulation of $\mathbb{R}^n$ (e.g., [**RS**]). Any smooth structure determines a PL-structure, and the converse holds for $n \leq 6$ [**HM**], so for our purposes PL-structures are equivalent to smooth structures.

1.2. 4-manifolds

We begin our discussion about 4-dimensional manifolds by defining the *intersection form* of a compact, oriented, topological 4-manifold X. Recall that when X is oriented, it admits a fundamental class $[X] \in H_4(X, \partial X; \mathbb{Z})$.

Definition 1.2.1. The symmetric bilinear form

$$Q_X \colon H^2(X, \partial X; \mathbb{Z}) \times H^2(X, \partial X; \mathbb{Z}) \to \mathbb{Z}$$

defined by $Q_X(a, b) = \langle a \cup b, [X] \rangle = a \cdot b \in \mathbb{Z}$ is called the *intersection form* of X. Since by Poincaré duality $H_2(X; \mathbb{Z}) \cong H^2(X, \partial X; \mathbb{Z})$, Q_X is defined on $H_2(X; \mathbb{Z}) \times H_2(X; \mathbb{Z})$ as well.

Note that for this definition of Q_X we only need the topological structure of X. Clearly, $Q_X(a, b) = 0$ if a or b is a torsion element, hence Q_X descends to a pairing on homology mod torsion. By choosing a basis of $H_2(X; \mathbb{Z})/\text{Torsion}$, we can represent Q_X by a matrix. The matrix M of Q_X transforms under a basis transformation C as $C^T M C$. Consequently the determinant $\det M$ is independent of the choice of the basis over $\mathbb{Z}$; we sometimes denote this by $\det Q_X$.

Remarks 1.2.2. **(a)** The above definition of Q_X can be extended to cohomology with arbitrary coefficient ring R. When $R = \mathbb{Z}_2$, the theory generalizes in the obvious way to nonorientable manifolds. For X orientable, the inclusion $r\colon H_2(X;\mathbb{Z}) \otimes \mathbb{Z}_2 \to H_2(X;\mathbb{Z}_2)$ given by the Universal Coefficient Theorem preserves the intersection form; note that r is an isomorphism if $H_1(X;\mathbb{Z})$ has no 2-torsion. The definition also generalizes to noncompact manifolds if we use compactly supported cohomology (which is Poincaré dual to ordinary homology). The intersection form of a compact manifold will not change if we remove its boundary.

(b) Using the same idea, another pairing Q'_X can be defined for the oriented manifold X: The map $Q'_X\colon H^2(X;\mathbb{Z}) \times H^2(X,\partial X;\mathbb{Z}) \to \mathbb{Z}$ is given as $Q'_X(a,b) = \langle a \cup b, [X]\rangle$ (and the second group is compactly supported in the noncompact case). In our discussion we will mainly use Q_X.

(c) By the definition of the intersection form we have $Q_{\overline{X}} = -Q_X$.

If X is a smooth manifold, then $Q_X(a,b)$ can be interpreted as the intersection number of certain submanifolds in X. For a better understanding of this relation we need a little preparation. Let X^n be a smooth n-dimensional manifold. A class $\alpha \in H_2(X^n;\mathbb{Z})$ is *represented by* a closed, oriented surface Σ if there is an embedding $i\colon \Sigma \hookrightarrow X$ such that $i_*([\Sigma]) = \alpha$. (Again, $[\Sigma] \in H_2(\Sigma;\mathbb{Z})$ is the fundamental class of Σ).

Proposition 1.2.3. *Let X be a closed, oriented, smooth 4-manifold. Then every element of $H_2(X;\mathbb{Z})$ can be represented by an embedded surface.*

Proof. Elements of $H^2(X;\mathbb{Z})$ are in 1-1 correspondence with $U(1)$-bundles over X (cf. Proposition 1.4.1). For $\alpha \in H_2(X;\mathbb{Z})$ take its Poincaré dual $a = PD(\alpha) \in H^2(X;\mathbb{Z})$ and denote the corresponding $U(1)$-bundle by $L_\alpha \to X$. The zero set of a generic section of the bundle $L_\alpha \to X$ will be a smooth surface representing α. □

Remark 1.2.4. The proposition is also true if X has boundary or is noncompact or nonorientable, and the analogous statement holds with $\mathbb{Z}_2$-coefficients if we allow Σ to be nonorientable. (See Exercise 4.5.12(b).) Note that if X is simply connected, then by the Hurewicz Theorem $\pi_2(X) \cong H_2(X;\mathbb{Z})$. This implies that for a simply connected 4-manifold every second homology element can be represented by an *immersed* sphere. Such an immersion will not be an embedding in general, but one can assume that an immersion $S^2 \to X^4$ intersects itself only in transverse double points.

Suppose that X^4 is closed and oriented. For $a, b \in H^2(X;\mathbb{Z})$ take surface representatives Σ_α and Σ_β of the Poincaré duals $\alpha = PD(a)$ and $\beta = PD(b)$. Suppose furthermore that Σ_α and Σ_β have been chosen generically, so that their intersections are all transverse. The orientations of Σ_α and Σ_β —

together with the fixed orientation of the ambient 4-manifold X — assign a sign ± 1 to every intersection point of Σ_α and Σ_β in the following way [**GP**]. By concatenating positive bases of the tangent spaces $T_p\Sigma_\alpha$ and $T_p\Sigma_\beta$ at a point $p \in \Sigma_\alpha \cap \Sigma_\beta$ we get a basis of T_pX. The sign of the intersection at p is positive if this basis is positive, and negative otherwise. Note that the sign does not depend on the order of $\{\alpha, \beta\}$, but does depend on the orientations of Σ_α and Σ_β. Now we are in the position to give the geometric interpretation of Q_X — this description explains the name of it.

Proposition 1.2.5. *For $a, b \in H^2(X;\mathbb{Z})$ and $\alpha, \beta \in H_2(X;\mathbb{Z})$ as above, $Q_X(a,b)$ is the number of points in $\Sigma_\alpha \cap \Sigma_\beta$, counted with sign.*

Proof. Assume that η_1 is a smooth 2-form on X representing the image of $a \in H^2(X;\mathbb{Z})$ under the map $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R})$ induced by $\mathbb{Z} \hookrightarrow \mathbb{R}$. Assume furthermore that η_1 is supported in a small tubular neighborhood of Σ_α. Choose a 2-form η_2 similarly for b and Σ_β. Find a coordinate chart (x, y, u, v) around each intersection point $p \in \Sigma_\alpha \cap \Sigma_\beta$ in which Σ_α is given by $\{x = 0, y = 0\}$ and Σ_β is given by $\{u = 0, v = 0\}$, and assume that $\eta_1 = f(x,y)dx \wedge dy$, $\eta_2 = f(u,v)du \wedge dv$ (where f is a bump function localized around $0 \in \mathbb{R}^2$ with integral 1). Following [**BT**] it is easy to see that $Q_X(a,b) = \int_X \eta_1 \wedge \eta_2$, hence the assertion easily follows. Note that for the last two propositions we assumed that X is a smooth 4-manifold. ☐

Remarks 1.2.6. (a) Again, the above proposition applies if X^4 has boundary or is noncompact (using suitable versions of Poincaré duality), and a similar statement holds over $\mathbb{Z}_2$ without the orientability hypotheses. In arbitrary dimensions, the same method of counting intersections gives the intersection pairing (including relative versions) for any two homology classes of complementary dimension, and this is again dual to the cup product pairing [**GH**]. The only difference is that high-dimensional homology classes cannot always be represented by submanifolds, so one must allow smooth cycles with singularities.

(b) Recall that a complex structure on X gives an orientation on it, and if Σ_α and Σ_β are complex submanifolds, then they are also canonically oriented. An easy argument shows that the transverse intersection of complex submanifolds is always positive. In particular, $Q_S([C_1],[C_2]) \geq 0$ if $C_1, C_2 \subset S$ are transversely intersecting complex curves in a complex surface S. Applying more delicate arguments, one can prove the same positivity result for any pair of embedded complex curves, provided C_1 and C_2 have no common component. (As we will see, the self-intersection of a complex curve can be negative, cf. Section 2.2.)

We make a short digression and briefly recall the classification of integral forms. (For a more detailed treatment, see [**MH**]). For a given symmetric, bilinear form Q on the finitely generated free abelian group A the rank, signature and parity of Q are defined in the following way: The *rank* $\mathrm{rk}(Q)$ of Q is the dimension of A. Extend and diagonalize Q over $A \otimes_{\mathbb{Z}} \mathbb{R}$. The number of $+1$'s (-1's respectively) on the diagonal is denoted by b_2^+ (resp. b_2^-); the difference $b_2^+ - b_2^-$ is the *signature* $\sigma(Q)$ of Q. Finally, Q is *even* if $Q(\alpha, \alpha) \equiv 0 \pmod 2$ for every $\alpha \in A$; Q is *odd* otherwise.

Exercises 1.2.7. Suppose that Q is an integral form.

(a) Show that Q is even iff $Q(\alpha_i, \alpha_i) \equiv 0 \pmod 2$ $(i = 1, \ldots, n)$ for *every* basis $\{\alpha_1, \ldots, \alpha_n\}$.

(b) Prove that Q is even iff *every* matrix representing Q has even diagonal.

(c) Prove the above statements after replacing *every* with *at least one*.

(d) Let $\langle n \rangle$ be the bilinear form over $\mathbb{Z}$ represented by the 1×1 matrix $[n]$ $(n \in \mathbb{Z})$. Prove that $\langle n \rangle$ and $\langle m \rangle$ are equivalent iff $m = n$.

Definitions 1.2.8. (a) Q is *positive (negative) definite* if $\mathrm{rk}(Q) = \sigma(Q)$ ($\mathrm{rk}(Q) = -\sigma(Q)$ resp.). Q is *indefinite* otherwise.

(b) The *direct sum* $Q = Q_1 \oplus Q_2$ of the forms Q_1 and Q_2 (given on A_1, A_2 respectively) is defined on $A_1 \oplus A_2$ in the following way. If $a, b \in A = A_1 \oplus A_2$ split as $a = a_1 + a_2$ and $b = b_1 + b_2$ with $a_i, b_i \in A_i$, then $Q(a, b) = Q_1(a_1, b_1) + Q_2(a_2, b_2)$. If $k > 0$ then kQ denotes the k-fold sum $\oplus_k Q$; for negative k we take kQ to be $|k|(-Q)$; finally if $k = 0$, then the form kQ equals the zero form on the trivial group (represented by the empty matrix $\emptyset$) by definition. The form on $A = \mathbb{Z} \oplus \mathbb{Z}$ represented by the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ will be denoted by H.

(c) An element $x \in A$ is called a *characteristic element* if $Q(\alpha, x) \equiv Q(\alpha, \alpha) \pmod 2$ for all $\alpha \in A$. Note that Q is even iff $0 \in A$ is characteristic. An element $\alpha \in A$ is *primitive* if we cannot write α as $d\beta$ $(\beta \in A,\ d \in \mathbb{Z})$ unless $d = \pm 1$. For any $x \in A$ there is a primitive element α such that $x = d\alpha$; the integer $|d|$ is called the *divisibility* of x.

(d) Q is called *unimodular* if $\det Q = \pm 1$.

For an element $x \in A$ define $L_x \in A^*$ by $L_x(y) = Q(x, y)$. In this way we get a homomorphism $L \colon A \to A^*$.

Lemma 1.2.9. *The form Q is unimodular iff L is an isomorphism.*

Proof. Fix a basis $a_1, \ldots, a_n \in A$ and take the dual basis $a_i^* \in A^*$ (given by $a_i^*(a_j) = \delta_{ij} \in \mathbb{Z}$). Since $L(a_i) = \sum_j Q(a_j, a_i) a_j^*$, the matrix of L in these bases is $[Q(a_i, a_j)]_{ij}$. A matrix B over $\mathbb{Z}$ is invertible (over $\mathbb{Z}$) iff $\det B = \pm 1$, hence the lemma follows. □

Exercise 1.2.10. * Prove that if X is a closed 4-manifold (so $\partial X = \emptyset$), then Q_X is unimodular. (*Hint*: Apply Poincaré duality.)

Remark 1.2.11. Exercise 1.2.10 can be extended to show that Q_X is unimodular when ∂X is a homology sphere. (A 3-manifold M is a *homology sphere* iff $H_*(M;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$, or equivalently, iff it is closed, orientable and connected with $H_1(M;\mathbb{Z}) = 0$.) One simply observes that by the long exact homology sequence, inclusion induces an isomorphism $H_2(X;\mathbb{Z}) \to H_2(X,\partial X;\mathbb{Z})$. In fact, for a compact, oriented 4-manifold X with $H_1(X;\mathbb{Z}) = 0$, Q_X is unimodular if and only if ∂X is a disjoint union of homology spheres. We consider the relation between Q_X and $H_1(\partial X;\mathbb{Z})$ in more detail in Corollary 5.3.12 and Exercise 5.3.13(f). Note that the pairing Q'_X described in Remark 1.2.2(b) is unimodular for every pair $(X,\partial X)$.

Lemma 1.2.12. *Suppose that the restriction of the symmetric bilinear form Q to the subgroup $A_1 \subset A$ is unimodular. Then (A,Q) can be split as the sum of forms $(A,Q) = (A_1, Q|A_1) \oplus (A_1^\perp, Q|A_1^\perp)$, where $A_1^\perp = \{y \in A \mid Q(x,y) = 0 \; for \; all \; x \in A_1\}$. Moreover, $Q|A_1^\perp$ is unimodular iff Q is.*

Proof. If $0 \neq a \in A_1 \cap A_1^\perp$, then $Q(a,b) = 0$ for all $b \in A_1$, contradicting the fact that $Q|A_1$ is unimodular. For any $x \in A$ we can take the linear function $a \mapsto Q(x,a)$ on A_1; by the unimodularity of $Q|A_1$ there is a unique element $b \in A_1$ with $Q(x,a) = Q(b,a)$ for all $a \in A_1$ (cf. Lemma 1.2.9). Hence $x - b \in A_1^\perp$, so $x = b + (x-b) \in A_1 + A_1^\perp$. We have now proved that $A = A_1 \oplus A_1^\perp$; the unimodularity of $(A_1^\perp, Q|A_1^\perp)$ follows from the fact that $\det Q = \det Q|A_1 \cdot \det Q|A_1^\perp = \pm \det Q|A_1^\perp$. □

As a corollary, the following useful observation can be made:

Corollary 1.2.13. *Suppose that* $\dim A$ *is equal to n and the determinant of the matrix of Q on the set $\{a_1, \dots, a_n\}$ is ± 1. Then $\{a_1, \dots, a_n\}$ is a basis of the free group A.* □

For the rest of the section, we only consider unimodular forms. As we will see (cf. Theorem 1.2.30), indefinite forms will be more interesting for our purposes. At the same time, indefinite forms admit a very nice classification scheme.

Theorem 1.2.14. *If indefinite unimodular forms Q_1, Q_2 (defined on A_1, A_2 respectively) have the same rank, signature and parity, then they are equivalent.*

In the following — through a series of exercises — we will give an outline of the proof of Theorem 1.2.14. The proof rests on the following theorem, whose proof requires some difficult algebraic geometry (cf. [**MH**]).

Theorem 1.2.15. *If $A \neq 0$ and $\sigma(Q) = 0$, then there exists a nonzero $\alpha \in A$ with $Q(\alpha,\alpha) = 0$.* □

Using the above result, we can easily classify intersection forms with signature 0.

Lemma 1.2.16. *If $\sigma(Q) = 0$, then Q is equivalent to kH if Q is even, and to $l\langle 1\rangle \oplus l\langle -1\rangle$ if Q is odd ($k, l \in \mathbb{N}$).*

Proof. Take $x \in A$ with $Q(x,x) = 0$; we can assume that x is primitive. Since Q is unimodular, there is $y \in A$ with $Q(x,y) = 1$. Now split A as $\operatorname{span}(x,y) \oplus \operatorname{span}(x,y)^{\perp}$. Since $Q|\operatorname{span}(x,y)$ is unimodular, by Lemma 1.2.12 Q on $\operatorname{span}(x,y)^{\perp}$ is unimodular and obviously has 0 signature. Hence if $\operatorname{span}(x,y)^{\perp}$ is nonzero, the above splitting process can be repeated.

Exercises 1.2.17. Prove that

(a)* If $Q(y,y)$ is even, then $Q|\operatorname{span}(x,y) \cong H$.

(b)* If $Q(y,y)$ is odd, then $Q|\operatorname{span}(x,y) \cong \langle 1\rangle \oplus \langle -1\rangle$.

(c)* $H \oplus \langle -1\rangle \cong 2\langle -1\rangle \oplus \langle 1\rangle$.

The solutions of the above exercises complete the proof of Lemma 1.2.16. □

Proof of Theorem 1.2.14. Note that Lemma 1.2.16 covers the case when $\sigma(Q_1) = 0$ in Theorem 1.2.14. Thus, without loss of generality, we can assume that $\sigma(Q_1) > 0$, and prove the theorem by induction on $\sigma(Q_1)$. By induction we know that $Q_1 \oplus \langle -1\rangle$ and $Q_2 \oplus \langle -1\rangle$ are both isomorphic to $Q = b_2^+\langle 1\rangle \oplus (b_2^- + 1)\langle -1\rangle$. Assume that $x \in A = A_1 \oplus \mathbb{Z} \cong A_2 \oplus \mathbb{Z}$ is the vector spanning the orthogonal complement of Q_1 and $y \in A$ spans the complement of Q_2 in (A, Q). All we need is an automorphism of (A, Q) mapping x to y — hence Q_1 to Q_2. For the proof of the following proposition see [**W1**].

Proposition 1.2.18. *Suppose that $Q \cong n\langle 1\rangle \oplus m\langle -1\rangle$ $(n, m > 1)$ on A, and x, y are primitive elements in A such that $Q(x,x) = Q(y,y)$. If both x and y are characteristic elements, then there is an automorphism of (A, Q) mapping x to y. A similar automorphism exists if neither x nor y is characteristic.* □

The above proposition concludes the proof of Theorem 1.2.14, since x, y in A are characteristic iff Q_1 and Q_2 are even, and the equalities $Q(x,x) = Q(y,y) = -1$ show that x, y are primitive elements. □

Remark 1.2.19. The proof of Proposition 1.2.18 given in [**W1**] goes as follows. The case $n = m = 2$ is proved first, by explicit construction of automorphisms. In this case it is also shown that if x is characteristic, it can be mapped to a canonical element depending only on $Q(x,x)$. This idea extends to general n and m, and proves Proposition 1.2.18 in the characteristic case. For x, y not characteristic, the proof of the $n = m = 2$ case provides an automorphism mapping the noncharacteristic vectors either into the subspace $2\langle 1\rangle \oplus \langle -1\rangle$ or into $\langle 1\rangle \oplus 2\langle -1\rangle$. Now for general n and m, the splitting $n\langle 1\rangle \oplus m\langle -1\rangle = (2\langle 1\rangle \oplus 2\langle -1\rangle) \oplus ((n-2)\langle 1\rangle \oplus (m-2)\langle -1\rangle)$ combined with induction gives the result in the noncharacteristic case. For details see [**W1**].

To finish the description of indefinite forms we must determine the triples (rank, signature, parity) for which a form Q with these invariants exists. Since $\operatorname{rk}(Q) = b_2^+ + b_2^-$ and $\sigma(Q) = b_2^+ - b_2^-$, obviously $|\sigma(Q)| < \operatorname{rk}(Q)$ and $\sigma(Q) \equiv \operatorname{rk}(Q) \pmod 2$. The next lemma gives one further restriction for invariants of even intersection forms.

Lemma 1.2.20. *If $x \in A$ is characteristic, then $Q(x,x) \equiv \sigma(Q) \pmod 8$; in particular, if Q is even, then the signature $\sigma(Q)$ is divisible by 8.*

Proof. Note that if x is characteristic in (A,Q), then $x + e_+ + e_-$ is characteristic in $(A \oplus \mathbb{Z} \oplus \mathbb{Z}, Q \oplus \langle 1\rangle \oplus \langle -1\rangle)$, where $e_\pm$ generate the $\mathbb{Z}$ summands. By Theorem 1.2.14, $Q' = Q \oplus \langle 1\rangle \oplus \langle -1\rangle \cong (b_2^+ + 1)\langle 1\rangle \oplus (b_2^- + 1)\langle -1\rangle$, and a characteristic vector has odd components in this new basis. Since the square of an odd number is congruent to 1 modulo 8, we have that $Q(x,x) = Q'(x+e_++e_-, x+e_++e_-) \equiv (b_2^+ + 1) - (b_2^- + 1) = \sigma(Q) \pmod 8$. If Q is even, then 0 is a characteristic element, which implies that $\sigma(Q) \equiv 0 \pmod 8$. □

To show that all constraints on the triple (rank, signature, parity) have been found, we define a particular 8-dimensional intersection form. Consider the matrix corresponding to the Dynkin diagram of the exceptional Lie algebra E_8:

$$E_8 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}.$$

As the matrix of a bilinear form Q on $\mathbb{Z}^8$, E_8 gives a positive definite, even, unimodular form with $\sigma(Q) = 8$. (Check these statements by diagonalizing E_8 over $\mathbb{Q}$; beware that the determinant is not an invariant over $\mathbb{Q}$.) By a slight abuse of notation, from now on E_8 will denote that bilinear form. Recall that H is used for the form corresponding to the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Using E_8 and H as building blocks, for every pair $(\sigma, r) \in \mathbb{Z} \times \mathbb{N}$ with $\sigma \equiv 0 \pmod 8$, $r > |\sigma|$ and $r \equiv \sigma \pmod 2$ one can build up an indefinite unimodular form $Q = aE_8 \oplus bH$ with $\sigma = \sigma(Q)$ and $r = \operatorname{rk}(Q)$. (Take $a = \frac{\sigma}{8}$ and $b = \frac{r - |\sigma|}{2}$.) Consequently Theorem 1.2.14 implies the following.

Theorem 1.2.21. *Suppose that Q is an indefinite, unimodular form. If Q is odd, then it is isomorphic to $b_2^+\langle 1\rangle \oplus b_2^-\langle -1\rangle$; if Q is even then it is isomorphic to $\frac{\sigma(Q)}{8}E_8 \oplus \frac{\mathrm{rk}(Q)-|\sigma(Q)|}{2}H$.* □

Remark 1.2.22. Since the negative definite form $-E_8$ appears more commonly in 4-manifold theory than E_8, some authors use the notation E_8 for the negative definite form. A matrix for the latter form is obtained from the above matrix by reversing all signs on the diagonal. (Check this by a basis change reversing the signs of 4 vectors.) We will call this matrix the $-E_8$-matrix.

Exercises 1.2.23. (a) Let Q be an indefinite, unimodular form. Find a characteristic element $x \in A$ with $Q(x,x) = \sigma(Q)$. (*Hint*: Solve the problem for $\pm E_8, H, \langle \pm 1\rangle$ and apply the Classification Theorem 1.2.21.)

(b) Prove that $H \cong -H$ and $E_8 \oplus (-E_8) \cong 8H$.

In the definite case there is no such nice description of all unimodular forms. For a given rank there are only finitely many definite symmetric unimodular forms (see [**MH**]); this number, however, can be very large. (For example, there are more than 10^{50} definite forms of rank 40.)

Exercise 1.2.24. Prove that the positive definite forms $Q_1 = E_8 \oplus n\langle 1\rangle$ and $Q_2 = (8+n)\langle 1\rangle$ are not equivalent, although they have equal rank, signature and parity for $n > 0$. (*Hint*: Count the number of vectors of length 1 with respect to Q_1 and Q_2.)

We now consider the intersection form of a closed 4-manifold X; recall that this is always unimodular. For the sake of simplicity, let us restrict ourselves to the simply connected case. Since $\pi_1(X) = 0$, the first and the third homologies and cohomologies vanish (by Poincaré duality), and $H_2(X;\mathbb{Z}) \cong H^2(X;\mathbb{Z}) \cong \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z})$ has no torsion, so Q_X contains all the (co)homological information about X. As the next theorem shows, Q_X classifies topological 4-manifolds up to homotopy.

Theorem 1.2.25. (Whitehead) *The simply connected, closed, topological 4-manifolds X_1 and X_2 are homotopy equivalent iff $Q_{X_1} \cong Q_{X_2}$.*

Proof (sketch). Using homotopy theoretic arguments one can show that a simply connected topological 4-manifold X is homotopy equivalent to a CW-complex of the form $\bigvee_{i=1}^k S_i^2 \cup_g D^4$ — here $\bigvee_{i=1}^k S_i^2$ denotes the wedge (or bouquet) of k 2-spheres and g is the gluing map $S^3 \to \bigvee_{i=1}^k S_i^2$ of the 4-cell, hence defines a class $[g] \in \pi_3(\bigvee_{i=1}^k S_i^2)$. Denoting the additive group of symmetric $k \times k$ matrices with integer entries by $M(k \times k)$, we can obtain an isomorphism $L\colon \pi_3(\bigvee_{i=1}^k S_i^2) \to M(k \times k)$ as follows. Take $x_i \in S_i^2$ and assume that g is transverse to x_i and smooth in a neighborhood of $g^{-1}(x_i)$.

Define the matrix $L([g]) = [\lambda_{ij}(g)]$ as $\lambda_{ij}(g) = \ell k(g^{-1}(x_i), g^{-1}(x_j))$ (and $\lambda_{ii}(g) = \ell k(g^{-1}(x_i), g^{-1}(x_i'))$ for x_i' close to x_i), where $\ell k(L_1, L_2)$ denotes the linking number of the two oriented links L_1 and L_2. (For more about linking numbers see Section 4.5.) It is not hard to see that $L([g])$ represents Q_X in an appropriate basis, hence if $Q_{X_1} \cong Q_{X_2}$, the gluing maps g_1 and g_2 corresponding to X_1 and X_2 are homotopic. The proof of the theorem now easily follows. □

Assume for a moment that X is a finite CW complex with cells of dimension ≤ 4. For a fixed homology element $[X] \in H_4(X;\mathbb{Z})$ the pair $(X, [X])$ is called a (4-dimensional oriented) *Poincaré duality space* if the map $\alpha \mapsto \alpha \cap [X]$ (with $\alpha \in H^i(X;\mathbb{Z})$) defines an isomorphism $H^i(X;\mathbb{Z}) \to H_{4-i}(X;\mathbb{Z})$ for $i = 0, \dots, 4$. (It can be shown that any oriented topological 4-manifold is homotopy equivalent to a finite CW complex, hence, to a Poincaré duality space.) Note that the formula $Q_X(a, b) = \langle a \cup b, [X]\rangle$ extends the definition of intersection forms to Poincaré duality spaces. Since the solution of Exercise 1.2.10 applies without change, the intersection form Q_X of a Poincaré duality space is unimodular. The proof of Theorem 1.2.25 can be easily modified to show the following result.

Theorem 1.2.26. *Two simply connected (4-dimensional) Poincaré duality spaces X_1 and X_2 are homotopy equivalent iff $Q_{X_1} \cong Q_{X_2}$. Moreover, for each unimodular form Q there exists a (4-dimensional) Poincaré duality space X with $Q_X \cong Q$.*

Proof. The proof of the first statement proceeds verbatim as the proof of Theorem 1.2.25. If Q is represented by the symmetric matrix $B \in M(k \times k)$ in a basis, then take $g \in L^{-1}(B) \in \pi_3(\bigvee_{i=1}^k S_i^2)$ and form $X = \bigvee_{i=1}^k S_i^2 \cup_g D^4$. Because of the unimodularity of Q (i.e., $\det B = \pm 1$), X is a (4-dimensional) Poincaré duality space satisfying $Q_X \cong Q$. □

The following theorem — due to M. Freedman — can be regarded as the topological strengthening of the above homotopy theoretic classification results. Some ideas of the proof of Theorem 1.2.27 will be discussed in later chapters.

Theorem 1.2.27. (Freedman, [**F**], [**FQ**]) *For every unimodular symmetric bilinear form Q there exists a simply connected, closed, topological 4-manifold X such that $Q_X \cong Q$. If Q is even, this manifold is unique (up to homeomorphism). If Q is odd, there are exactly two different homeomorphism types of manifolds with the given intersection form. At most one of these homeomorphism types carries a smooth structure. Consequently, simply connected, smooth 4-manifolds are determined up to homeomorphism by their intersection forms.* □

One special case (the topological 4-dimensional Poincaré Conjecture) deserves a corollary:

Corollary 1.2.28. *If a topological 4-manifold X is homotopy equivalent to S^4, then X is homeomorphic to the 4-sphere.* □

Regarding smooth structures, the two main questions (existence and uniqueness) can now be formulated as follows.

- **Q1**. Existence: Which simply connected topological manifolds (or equivalently, intersection forms) carry smooth structures?
- **Q2**. Uniqueness: If the intersection form Q does carry a smooth structure, how many nondiffeomorphic smooth manifolds can be found with the same intersection form Q?

The following theorems illustrate what we know about the answers for **Q1** and **Q2**. Assume that X is a simply connected, closed, oriented, smooth 4-manifold.

Theorem 1.2.29. (Rohlin, [**R2**]) *If Q_X is even, then the signature $\sigma(X)$ is divisible by 16.* □

This theorem tells us, for example, that the topological manifold corresponding to E_8 does not carry any smooth structure. Another constraint on the intersection form of a simply connected smooth 4-manifold was found by Donaldson, cf. also Corollary 2.4.29.

Theorem 1.2.30. (Donaldson, [**D1**]) *If the intersection form Q_X of a smooth, simply connected, closed 4-manifold X is negative definite, then Q_X is equivalent to $n\langle -1\rangle$.* □

Note that by Remark 1.2.2(c) this theorem takes care of manifolds with positive definite intersection forms as well. As we will soon see, Theorem 1.2.30 answers **Q1** for definite and odd intersection forms. For indefinite even intersection forms — besides Theorem 1.2.29 implying that the coefficient of E_8 is even — the following estimate has been proved:

Theorem 1.2.31. (Furuta, [**Fur**]) *If X is a simply connected, closed, oriented, smooth 4-manifold and Q_X is equivalent to $2kE_8 \oplus lH$, then we have $l \geq 2|k| + 1$.* □

The $\frac{11}{8}$*-Conjecture* states that in the above theorem $l \geq 3|k|$ should be the right answer — this conjecture, however, is still open. On the other hand (as we will see in the next section), all intersection forms allowed by Theorems 1.2.29, 1.2.30 and the $\frac{11}{8}$-Conjecture can be represented as intersection forms of simply connected, smooth 4-manifolds. Thus the only remaining question for answering **Q1** in the simply connected case lies in the difference between Furuta's result and the $\frac{11}{8}$-Conjecture.

The next result indicates how much we know about the answer of **Q2**. As a consequence of Theorem 1.2.27, the homeomorphism type of a smooth, simply connected, oriented, closed 4-manifold X is determined by the parity of Q_X and the two numerical invariants $\sigma(X)$ and $b_2(X) = \operatorname{rk} H_2(X;\mathbb{Z})$. In contrast to Theorem 1.1.8, there is no finiteness result on the number of nondiffeomorphic smooth structures on a topological 4-manifold. The known results are of the following type. (An indication of the proof of this result (and similar ones) will be given later on, cf. Corollary 3.3.7 and subsequent text. See also Theorem 10.3.9.)

Theorem 1.2.32. ([**FM1**]) *The (simply connected) topological manifolds corresponding to the intersection forms* $2n(-E_8) \oplus (4n-1)H$ $(n \geq 1)$ *and* $(2k-1)\langle 1\rangle \oplus N\langle -1\rangle$ $(k \geq 2,\ N \geq 10k-1)$ *each carry infinitely many distinct (nondiffeomorphic) smooth structures.* □

Throughout the last part of this section we always assumed that the 4-manifolds we considered were simply connected. This assumption can be relaxed in some cases, but the general case (arbitrary fundamental group) is too difficult to study, since:

Theorem 1.2.33. *For every finitely presented group G there is a smooth, closed, oriented 4-manifold X with $\pi_1(X) \cong G$.* □

In Part 2 of this volume we will prove Theorem 1.2.33 from two different points of view (Exercises 4.6.4(b) and 5.2.2(c)) and deduce a theorem of Markov (Exercise 5.1.10(c)) that there can be no algorithm for classifying closed 4-manifolds (or n-manifolds for any fixed $n \geq 4$). Thus, the difficulty of understanding finitely presented groups leads us to focus mainly on simply connected 4-manifolds.

The invariants we have discussed until now — the intersection form Q_X, or more generally the cohomology ring $H^*(X;\mathbb{Z})$, and the fundamental group $\pi_1(X)$ — depend only on the homeomorphism type (in fact, homotopy type) of the manifold. Our ultimate goal, however, is to study *smooth* 4-manifolds. On one hand, we need finer invariants and ways to compute them in order to distinguish nondiffeomorphic 4-manifolds. Seiberg-Witten invariants and Seiberg-Witten basic classes will be introduced in Section 2.4, and we will see some applications of the knowledge of the Seiberg-Witten function to the geometry of the underlying 4-manifold. In this way we will distinguish homeomorphic but nondiffeomorphic 4-manifolds. On the other hand, we also need a method to decide when 4-manifolds given by different constructions result in diffeomorphic manifolds. Part 2 — about

Kirby calculus — will give a way to deal with 4-manifolds defined by various standard constructions. Using Kirby calculus one can (under favorable circumstances) prove that 4-manifolds defined by different constructions are actually diffeomorphic.

In the next section we present some familiar examples of (simply connected) 4-manifolds and determine the corresponding intersection forms. More complicated and more interesting constructions will be shown in Part 3. For our occasional use of characteristic classes, the reader is referred to Section 1.4 for an overview of background material or to [**MS**] for more details.

1.3. Examples

We now present some basic examples of closed, simply connected manifolds. The simplest example of such a 4-manifold is the 4-dimensional sphere $S^4 = \{x \in \mathbb{R}^5 \mid ||x|| = 1\}$; since $H_2(S^4;\mathbb{Z}) = 0$, the intersection form Q_{S^4} is trivial. Other examples are provided by the *complex projective spaces*. Given the obvious free action of $\mathbb{C}^* = \mathbb{C} - \{0\}$ on $\mathbb{C}^{n+1} - \{0\}$ (that is, $\lambda(z_0, \dots, z_n) = (\lambda \cdot z_0, \dots, \lambda \cdot z_n)$ for $\lambda \in \mathbb{C}^*$), one can take the quotient $\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$. The resulting space is the n-dimensional complex projective space $\mathbb{CP}^n$; $\mathbb{CP}^1 = S^2$ is the *complex projective line* and $\mathbb{CP}^2$ is the *complex projective plane*. Using $\mathbb{R}$ instead of $\mathbb{C}$, one defines the real projective spaces $\mathbb{RP}^n$. If $P \in \mathbb{CP}^n$ and $(z_0, \dots, z_n) \in P$, then we can denote P by its *homogeneous coordinates* $[z_0 : z_1 : \dots : z_n]$, which are defined up to multiplication by $\lambda \in \mathbb{C}^*$. Note that $\mathbb{CP}^n$ can be covered by the *affine coordinate* charts $\psi_i\colon \mathbb{C}^n \to \mathbb{CP}^n$ $(i = 0, \dots, n)$, where $\psi_i(z_1, \dots, z_n) = [z_1 : \dots : z_i : 1 : z_{i+1} : \dots : z_n]$. The 2-dimensional sphere S^2 has a unique complex structure as $\mathbb{CP}^1$, so we can use the symbols S^2 and $\mathbb{CP}^1$ interchangeably.

Exercises 1.3.1. **(a)*** Prove that $\mathbb{CP}^n$ is compact and $\pi_1(\mathbb{CP}^n) = 1$. Consequently, $\mathbb{CP}^2$ is a closed, simply connected 4-manifold.

(b) Prove that $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$ if $n > 1$. What is $\mathbb{RP}^1$? For which values of n is $\mathbb{RP}^n$ orientable?

(c) Prove that $H_i(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$ if $i = 2d$ $(d = 0, \dots, n)$ and $H_i(\mathbb{CP}^n;\mathbb{Z}) = 0$ otherwise. (For a solution, see Example 4.2.4.)

(d) Determine the functions $\psi_i^{-1} \circ \psi_j$ for the affine coordinate charts of $\mathbb{CP}^n$. (*Hint*: See Example 4.2.4.)

(e)* Show that the homology class $h \in H_2(\mathbb{CP}^2;\mathbb{Z})$ given as the fundamental class of the submanifold $H = \{[x : y : z] \in \mathbb{CP}^2 \mid x = 0\}$ generates $H_2(\mathbb{CP}^2;\mathbb{Z})$. Show furthermore that $Q_{\mathbb{CP}^2}(h,h) = 1$ and conclude that $Q_{\mathbb{CP}^2} = \langle 1 \rangle$.

(f) We let $\overline{\mathbb{CP}^2}$ denote the manifold $\mathbb{CP}^2$ with the opposite orientation, hence (by Remark 1.2.2(c)) $Q_{\overline{\mathbb{CP}^2}} = -Q_{\mathbb{CP}^2} = \langle -1 \rangle$. (*Caveat*: Do not confuse this notation with complex conjugation, which *preserves* orientation on $\mathbb{CP}^{2k}$.) Prove that there is no orientation-preserving diffeomorphism between $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$.

As in the real case, it is easy to see that any two distinct points of $\mathbb{CP}^2$ lie on a unique projective line ($\approx \mathbb{CP}^1$), and any two (distinct) projective lines in $\mathbb{CP}^2$ intersect each other in exactly one point.

The cartesian product $\mathbb{CP}^1 \times \mathbb{CP}^1$ provides the next example of a simply connected 4-manifold. By the Künneth formula $H_2(\mathbb{CP}^1 \times \mathbb{CP}^1; \mathbb{Z}) = H_2(S^2 \times S^2; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, and it is not hard to see that $Q_{S^2 \times S^2} = H$: Choose the homology elements $\alpha = [S^2 \times \{\text{pt.}\}]$ and $\beta = [\{\text{pt.}\} \times S^2]$ as a basis for $H_2(S^2 \times S^2; \mathbb{Z})$; the matrix of $Q_{S^2 \times S^2}$ in this basis is equal to $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. To construct other 4-manifolds from the above ones we introduce a general operation for two smooth n-dimensional manifolds with boundary.

Definition 1.3.2. Let X_1, X_2 be oriented n-dimensional manifolds, and assume that $Z_i \subset \partial X_i$ $(i = 1, 2)$ are compact, codimension-zero submanifolds of the boundaries. Assume furthermore that $\varphi\colon Z_1 \to Z_2$ is an orientation-reversing diffeomorphism. By identifying Z_1 with $\overline{Z}_2$ via φ (and smoothing the corners) we get a new oriented manifold, denoted by $X_1 \cup_\varphi X_2$ (or by $X_1 \cup_Z X_2$ if $Z = Z_1 = \overline{Z}_2$ and $\varphi = \mathrm{id}_Z$).

Remark 1.3.3. The operation of smoothing corners is easy in dimension 2: Replace an angular boundary such as $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq 0 \text{ or } x_2 \leq 0\}$ by a smooth one, using compactly supported smooth functions; see Figure 1.1. In higher dimensions, the same can be done (canonically) by multiplying the previous model by the extra dimensions of ∂Z_i.

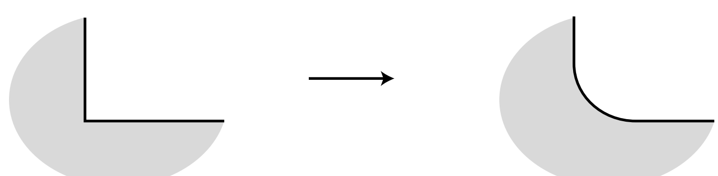

Figure 1.1. Model for smoothing corners in dimension 2.

A special case of the construction of Definition 1.3.2 is the *boundary sum* — when we glue along the $(n-1)$-dimensional ball $Z_1 \approx Z_2 \approx D^{n-1}$. The result is denoted by $X_1 \natural X_2$ and is well-defined (independent of the embeddings of D^{n-1}) whenever each ∂X_i is connected. The boundary sum of m copies of the manifold is denoted by $\natural m X$; if $m = 0$, then $\natural m X = D^n$ by definition. Another special case of the construction given in Definition 1.3.2 is the connected sum of two connected, oriented n-dimensional manifolds X_1 and X_2:

Definition 1.3.4. For $i = 1, 2$, let $D_i^n \subset X_i$ be embedded disks, and let $\varphi\colon D_1^n \to D_2^n$ be an orientation-reversing diffeomorphism. The smooth manifold $(X_1 - \text{int } D_1) \cup_{\varphi|\partial D_1} (X_2 - \text{int } D_2)$ is called the *connected sum* $X_1 \# X_2$ of X_1 and X_2; it does not depend on the choices of D_i^n or φ (since any two orientation-preserving embeddings of a disk are smoothly isotopic). In particular, $\# mX$ denotes the manifold we get by the connected sum of m ($m \geq 0$) copies of the same manifold X. (Again, if $m = 0$, then $\# mX = S^n$ by definition.)

Note that, by definition, the boundary sum of X_1 and X_2 has the connected sum $\partial X_1 \# \partial X_2$ as boundary, so $\partial(X_1 \natural X_2) = \partial X_1 \# \partial X_2$ — this relation might explain the names of the operations. The iterated application of the connected sum operation for $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$ and $S^2 \times S^2$ gives other examples of simply connected 4-manifolds.

Exercises 1.3.5. (a)* Given 4-manifolds X_1 and X_2 with intersection forms Q_{X_i}, show that $Q_{X_1 \# X_2} = Q_{X_1} \oplus Q_{X_2}$.

(b)* More generally, prove that if $X = X_1 \cup_N X_2$ and N is a homology 3-sphere, then $Q_X = Q_{X_1} \oplus Q_{X_2}$.

Remark 1.3.6. The converse of the above exercise is also true ([**FT1**], Theorem 1), namely that if X is a closed, smooth, simply connected 4-manifold and Q_X splits as $Q_1 \oplus Q_2$, then there are $X_1, X_2 \subset X$ such that $X = X_1 \cup_N X_2$ giving the splitting $Q_1 \oplus Q_2$ for Q_X (as $Q_i = Q_{X_i}$); moreover N is a (smoothly embedded) homology sphere. Note that by applying the result of Exercises 1.3.1(e) and 1.3.5(a) one can easily prove that the intersection form of $\# n\mathbb{CP}^2 \# m\overline{\mathbb{CP}^2}$ is equivalent to $n\langle 1\rangle \oplus m\langle -1\rangle$ $(n, m \geq 0)$, so these intersection forms — which cover all possible definite (cf. Theorem 1.2.30) and odd indefinite candidates — can be realized by smooth manifolds. Note also that the intersection form of $S^2 \times S^2 \# \overline{\mathbb{CP}^2}$ is isomorphic to the intersection form of $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ (cf. Exercise 1.2.17(c)). By Theorem 1.2.27, this implies that these manifolds are homeomorphic. As we will see later, $S^2 \times S^2 \# \overline{\mathbb{CP}^2}$ is, in fact, diffeomorphic to $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$.

Polynomials in the variables $\{z_0, \ldots, z_n\}$ are not well-defined functions on $\mathbb{CP}^n$, but for a *homogeneous* polynomial p of degree d, i.e., a polynomial satisfying $p(\lambda z) = \lambda^d p(z)$ for all $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^{n+1}$, the *zero set* of p is well-defined. (If p vanishes on a point $z \in \mathbb{C}^{n+1} - \{0\}$ then it vanishes on its entire equivalence class $[z] \in \mathbb{CP}^n$.)

Definition 1.3.7. If p is a homogeneous polynomial of degree d then the set $V_p = \{[z] \in \mathbb{CP}^n \mid p(z) = 0\}$ is called the *hypersurface* corresponding to the polynomial p. The complex submanifolds of $\mathbb{CP}^n$ are called *complex projective* manifolds.

It has been proved [**GH**] that any complex projective manifold can be written as the zero set of a collection of homogeneous polynomials. Not every complex manifold, however, can be embedded in $\mathbb{CP}^n$, so not all complex manifolds are projective.

By considering hypersurfaces in $\mathbb{CP}^3$, we will provide further examples of 4-manifolds with even Q_X; in particular, we will show that if the indefinite even form Q satisfies the constraints posed by Theorem 1.2.29 and the $\frac{11}{8}$-Conjecture, then there exists a smooth 4-manifold X with $Q \cong Q_X$ (cf. Exercise 1.3.12(a)). Consider the hypersurface

$$S_d = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid \sum z_i^d = 0\} \subset \mathbb{CP}^3,$$

where d is a positive integer.

Theorem 1.3.8. (See also [**McS1**].) *The hypersurface S_d is a smooth, simply connected, complex surface. If d is odd, then Q_{S_d} is equivalent to $\lambda_d\langle 1\rangle \oplus \mu_d\langle -1\rangle$, where $\lambda_d = \frac{1}{3}(d^3-6d^2+11d-3)$ and $\mu_d = \frac{1}{3}(d-1)(2d^2-4d+3)$; if d is even, then Q_{S_d} is equivalent to $l_d(-E_8)\oplus m_dH$, where $l_d = \frac{1}{24}d(d^2-4)$ and $m_d = \frac{1}{3}(d^3-6d^2+11d-3)$.*

Proof. The Implicit Function Theorem shows that $S_d \subset \mathbb{CP}^3$ is a smooth 4-manifold; the fact that $\pi_1(S_d) = 1$ follows from the Lefschetz Hyperplane Theorem 1.4.22 (see Exercise 8.1.1(b)). For determining Q_{S_d} we must compute its parity, rank and signature (cf. Theorem 1.2.21). Note that S_d is a complex surface, hence it admits Chern classes $c_1(S_d) \in H^2(S_d;\mathbb{Z})$ and $c_2(S_d) \in H^4(S_d;\mathbb{Z})$. Since $c_2[S_d] = \chi(S_d) = 2 + \mathrm{rk}(Q_{S_d})$ and $c_1^2[S_d] = 3\sigma(S_d) + 2\chi(S_d)$, the classes $c_2(S_d)$ and $c_1(S_d)$ determine the rank and the signature of S_d. (Here $\chi(S_d)$ denotes the topological Euler characteristic of S_d.) Moreover $c_1(S_d) \equiv w_2(S_d) \pmod 2$, and (since $\pi_1(S_d) = 1$) Q_{S_d} is even iff $w_2(S_d) = 0$; hence the parity of Q_{S_d} is determined by $c_1(S_d)$. Consequently we only need to determine $c_1(S_d)$ and $c_2(S_d)$ in order to compute Q_{S_d}. Recall that the total Chern class of $\mathbb{CP}^3$ is $c(\mathbb{CP}^3) = (1+g)^4 \in H^*(\mathbb{CP}^3;\mathbb{Z})$, where g denotes the generator of $H^2(\mathbb{CP}^3;\mathbb{Z})$ satisfying $\langle g, [\mathbb{CP}^1]\rangle = 1$ [**MS**]. In the next lemma, x will denote the pullback $i^*(g) \in H^2(S_d;\mathbb{Z})$ of $g \in H^2(\mathbb{CP}^3;\mathbb{Z})$ via the embedding $i\colon S_d \hookrightarrow \mathbb{CP}^3$.

Lemma 1.3.9. *The Chern classes of S_d are given by $c_1(S_d) = (4-d)x$ and $c_2(S_d) = (d^2-4d+6)x^2$. Moreover, $\langle x^2, [S_d]\rangle = d$, hence $\chi(S_d) = (d^2-4d+6)d$ and $c_1^2[S_d] = (4-d)^2 d$. In addition, Q_{S_d} is even iff d is even.*

Proof. Restrict the tangent bundle $T\mathbb{CP}^3$ of $\mathbb{CP}^3$ to S_d. It splits as the tangent bundle TS_d of S_d and the normal bundle νS_d: $T\mathbb{CP}^3|S_d = TS_d \oplus \nu S_d$.

By the Whitney product formula we have that $c(T\mathbb{CP}^3|S_d) = (1+x)^4 = (1+c_1(S_d)+c_2(S_d)) \cdot (1+c_1(\nu S_d))$, so

$$\begin{aligned} 1 + c_1(S_d) + c_2(S_d) &= (1+x)^4(1+c_1(\nu S_d))^{-1} \\ &= (1+4x+6x^2)(1-c_1(\nu S_d)+c_1^2(\nu S_d)). \end{aligned}$$

Hence, to prove Lemma 1.3.9 we only need to determine $c_1(\nu S_d)$.

Lemma 1.3.10. *The first Chern class of the normal bundle νS_d equals dx.*

Proof. Suppose that S'_d is a hypersurface of degree d (defined by another homogeneous polynomial of degree d) intersecting S_d transversally in $V = S_d \cap S'_d$. Since $c_1(\nu S_d) = e(\nu S_d)$ and S'_d can be chosen to be a section of $\nu S_d \to S_d$, we get that $c_1(\nu S_d) = PD([V])$. Since $[S_d] = [S'_d] = d \cdot [S_1] \in H_4(\mathbb{CP}^3;\mathbb{Z})$, and in $H^2(S_d;\mathbb{Z})$ one has $PD[S_1 \cap S_d] = i^*(PD[S_1]) = i^*(g) = x$, the lemma follows. □

Consequently,

$$\begin{aligned} 1 + c_1(S_d) + c_2(S_d) &= (1+4x+6x^2)(1-dx+d^2x^2) \\ &= 1 + (4-d)x + (d^2-4d+6)x^2, \end{aligned}$$

and this implies the first statement of Lemma 1.3.9. The term $\langle x^2, [S_d]\rangle$ can be computed in the following way: $\langle x^2, [S_d]\rangle = \langle (i^*g)^2, [S_d]\rangle = \langle g^2, i_*[S_d]\rangle = \langle g^2 \cup PD(i_*[S_d]), [\mathbb{CP}^3]\rangle = \langle g^2 \cup dg, [\mathbb{CP}^3]\rangle = d \cdot \langle g^3, [\mathbb{CP}^3]\rangle = d$. Note that for odd d the term $\langle x^2, [S_d]\rangle = Q_{S_d}(x,x)$ is odd, hence Q_{S_d} is odd as well. If d is even, then $c_1(S_d) = (4-d)x \equiv 0 \pmod 2$, so $w_2(S_d) = 0$. Consequently, Proposition 1.4.18 implies that Q_{S_d} is even iff d is even, which concludes the proof of Lemma 1.3.9. □

With the results of Lemma 1.3.9, the proof of Theorem 1.3.8 is just a simple computation. □

Our particular choice of the homogeneous polynomial in the definition of S_d has no importance, since

Claim 1.3.11. *If p_1 and p_2 are two homogeneous polynomials with equal degree (and not powers of other polynomials) and the hypersurfaces $F_i = \{P \in \mathbb{CP}^n \mid p_i(P) = 0\}$ are smooth submanifolds of $\mathbb{CP}^n$ $(i = 1,2)$, then F_1 is diffeomorphic to F_2.*

Proof. By taking the coefficients of the monomials in a (homogeneous) polynomial of degree d in $n+1$ variables, one defines a point in $\mathbb{C}^N$ (where N depends on n and d). Conversely, a point $z \in \mathbb{C}^N$ defines a polynomial p_z by specifying the coefficients. The set $V_z = \{P \in \mathbb{CP}^n \mid p_z(P) = 0\}$ is a hypersurface unless $z = 0$, and clearly $V_{\lambda z} = V_z$ for all $\lambda \in \mathbb{C}^*$, hence the hypersurfaces of degree d in $\mathbb{CP}^n$ are parametrized by the points of

$(\mathbb{C}^N - \{0\})/\mathbb{C}^* = \mathbb{CP}^{N-1}$. Singular hypersurfaces correspond to points of a complex codimension-1 subspace of $\mathbb{CP}^{N-1}$, since the singular objects can be described by equations (specifying that the Implicit Function Theorem fails). Since this subspace has real codimension 2, the points of $\mathbb{CP}^{N-1}$ corresponding to smooth hypersurfaces form a connected subset, which means that one smooth hypersurface can be smoothly deformed to any other smooth one, and this proves that F_1 is diffeomorphic to F_2. The above proof, in fact, shows that F_1 is ambiently isotopic to F_2 in $\mathbb{CP}^n$. □

In the light of Claim 1.3.11, it is easy to see that $S_1 = \mathbb{CP}^2$. (Define $S_1' = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_3 = 0\}$.) By taking the quadric surface $S_2' = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_0 z_3 = z_1 z_2\}$ (which is diffeomorphic to S_2 by Claim 1.3.11), we see that $S_2 \approx \mathbb{CP}^1 \times \mathbb{CP}^1$. (Notice that the map $([s_0 : s_1], [t_0 : t_1]) \mapsto [s_0 t_0 : s_0 t_1 : s_1 t_0 : s_1 t_1]$ gives an isomorphism between $\mathbb{CP}^1 \times \mathbb{CP}^1$ and S_2'. See also Exercise 3.2.1.) We need additional tools to show that $S_3 = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$ (Lemma 3.1.17 and subsequent text). The case $d = 4$ gives an example of a simply connected complex surface with $c_1 = 0$; such a surface is called a *K3-surface*. By algebraic geometric methods it can be shown that all $K3$-surfaces are diffeomorphic (cf. Theorem 3.4.9), so from the differential topological point of view we can call S_4 *the* $K3$-surface. By the previous formula $Q_{S_4} = 2(-E_8) \oplus 3H$.

Exercises 1.3.12. **(a)*** Realize all the (indefinite) even unimodular forms allowed by Theorem 1.2.29 and the $\frac{11}{8}$-Conjecture as intersection forms of simply connected, *smooth* 4-manifolds.

(b)* Show that if X is a simply connected, smooth 4-manifold with even intersection form and $b_2^+(X) = 0$, then X is homeomorphic to S^4. (*Hint*: Apply Theorems 1.2.30 and 1.2.27.)

More examples of simply connected 4-manifolds can be given by generalizing the above construction of S_d. Take homogeneous polynomials p_i of degree d_i in $n+1$ variables ($i = 1, \ldots, n-2$). Note that each p_i defines a hypersurface in $\mathbb{CP}^n$. Suppose that their intersection, $S = S(d_1, \ldots, d_{n-2}) = \{P \in \mathbb{CP}^n \mid p_i(P) = 0 \ (i = 1, \ldots, n-2)\}$, is a smooth submanifold of complex dimension 2 in $\mathbb{CP}^n$. In this case S is called a *complete intersection* surface of multidegree $(d_1, \ldots, d_{n-2})$. By generalizing Claim 1.3.11 it can be proved that the diffeomorphism type of $S(d_1, \ldots, d_{n-2})$ depends only on the multidegree $(d_1, \ldots, d_{n-2})$. Note that without loss of generality we can assume that each $d_i \geq 2$. By the Lefschetz Hyperplane Theorem 1.4.22 (cf. Exercise 8.1.1(b)), we have that $\pi_1(S(d_1, \ldots, d_{n-2})) = 1$.

Exercises 1.3.13. For $S = S(d_1, \ldots, d_{n-2})$ prove that

(a) $c_2[S] = (\frac{n(n+1)}{2} - (n+1)\sum d_i + \sum d_i^2 + \sum_{i<j} d_i d_j) \prod d_i$;

(b) $c_1^2[S] = (\sum d_i - (n+1))^2 \prod d_i$;

(c) $\sigma(S) = \frac{1}{3}((n+1) - \sum d_i^2) \prod d_i$;

(d) the second Stiefel-Whitney class $w_2(S)$ vanishes (hence Q_S is even) iff $\sum d_i - (n+1)$ is even.

(e) Show that $c_1(S(d_1, \ldots, d_{n-2})) = n_S h_S$, where $n_S = \sum d_i - (n+1)$ and $h_S \in H^2(S(d_1, \ldots, d_{n-2}); \mathbb{Z})$ is a primitive class. (*Hint*: Let $x = i^* g$ in $H^2(S; \mathbb{Z})$, where g generates $H^2(\mathbb{CP}^n; \mathbb{Z})$ and $i\colon S \hookrightarrow \mathbb{CP}^n$ is the embedding. Show that $\nu S = L_1 \oplus \ldots \oplus L_{n-2}$, where $L_i \to S$ is a complex line bundle with $c_1(L_i) = d_i x$; moreover $\langle x^2, [S] \rangle = \prod d_i$ and x is primitive. Characteristic class computations and easy arithmetic yield the solution.)

The surfaces $S(2,3)$ and $S(2,2,2)$ are $K3$-surfaces (hence diffeomorphic to $S(4) = S_4$); $S(2,2)$ is diffeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$. (Recall also that $S(1) = S_1 = \mathbb{CP}^2$, $S(2) = S_2 = \mathbb{CP}^1 \times \mathbb{CP}^1$ and $S(3) = S_3 = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$.) All other complete intersection surfaces are surfaces of general type. (For the definition of surfaces of general type see Section 3.4.)

Other complex surfaces can be given by taking $n-2$ hypersurfaces of $\mathbb{CP}^{i_1} \times \ldots \times \mathbb{CP}^{i_k}$ ($\sum i_j = n$) intersecting each other in a smooth complex surface. These surfaces will be complex projective manifolds (since the product $\mathbb{CP}^{i_1} \times \cdots \times \mathbb{CP}^{i_k}$ embeds holomorphically in some $\mathbb{CP}^N$) but usually not complete intersections, and they need not be simply connected. A hypersurface of $\mathbb{CP}^{i_1} \times \ldots \times \mathbb{CP}^{i_k}$ can be defined by a multi-homogeneous polynomial of degree $(d_1, \ldots, d_k)$: such a polynomial is homogeneous of degree d_1 in the first (i_1+1) variables, homogeneous of degree d_2 in the next (i_2+1) variables, and so on. The computation of the characteristic numbers of a surface given by the above construction is left to the reader.

1.4. Appendix

In this appendix we will present the very basics of characteristic classes — for a more detailed treatment see [**MS**]. For alternative definitions see [**St**], [**We**] or Section 5.6 of this volume. At the end of the section we will give a quick review of spin structures and Dirac operators (see also Sections 2.4 and 5.6). In the following, X^m will denote an m-dimensional manifold. We will spell out the special relations among the characteristic classes of tangent bundles of 4-manifolds.

1.4.1. Characteristic classes. The set of isomorphism classes of $U(1)$-bundles ($O(1)$-bundles resp.) over X will be denoted by $\mathcal{L}_X$ ($\mathcal{R}_X$ resp.). Obviously $\mathcal{L}_X$ and $\mathcal{R}_X$ admit group structures with the tensor product of line bundles as multiplication.

Proposition 1.4.1. *The groups $\mathcal{L}_X$ and $H^2(X;\mathbb{Z})$ are canonically isomorphic; similarly, $\mathcal{R}_X$ and $H^1(X;\mathbb{Z}_2)$ are canonically isomorphic groups.*

Proof. We only give a hint for proving that $\mathcal{L}_X \cong H^2(X;\mathbb{Z})$; the proof of the other statement follows the same pattern. From algebraic topology we know that $H^2(X;\mathbb{Z}) = [X, K(\mathbb{Z},2)]$, where $[X,Y]$ is the set of homotopy classes of maps from X to Y, and $K(\mathbb{Z},2)$ is the Eilenberg-MacLane space with $\pi_i(K(\mathbb{Z},2)) = 0$ for $i \neq 2$ and $\pi_2(K(\mathbb{Z},2)) = \mathbb{Z}$. Bundle theory tells us that $\mathcal{L}_X$ and $[X, BU(1)]$ are isomorphic (where $BU(1)$ is the classifying space for $U(1)$-bundles). An easy argument shows that both $K(\mathbb{Z},2)$ and $BU(1)$ are homotopy equivalent to $\mathbb{CP}^\infty$, and this gives the desired isomorphism. The proof that $\mathcal{R}_X \cong H^1(X;\mathbb{Z}_2)$ rests on the fact that both $K(\mathbb{Z}_2,1)$ and $BO(1)$ are homotopy equivalent to $\mathbb{RP}^\infty$. □

The isomorphism $\mathcal{L}_X \to H^2(X;\mathbb{Z})$ (with suitably chosen sign) is usually called c_1, and $c_1(L)$ is the *first Chern class* of the complex line bundle L. Similarly, $w_1 \colon \mathcal{R}_X \to H^1(X;\mathbb{Z}_2)$ is the isomorphism given by Proposition 1.4.1, and $w_1(R)$ is the *first Stiefel-Whitney class* of the real line bundle $R \to X$. An alternative — obstruction theoretic — description of c_1 (and of w_1) can be given in the following way. (Compare with Section 5.6.) Suppose that X has a CW-decomposition and $L \to X$ is a $U(1)$-bundle; note that for each cell $f \colon D^k \to X$, the bundle f^*L over D^k is canonically trivial. Obviously L is trivial on the 0-skeleton of X, and since $U(1)$ is connected, such a trivialization can be extended over the 1-skeleton. Comparing this trivialization with the canonical trivialization over each 2-cell defines a map $\varphi \colon \partial D^2 \to U(1) = S^1$ for every 2-cell D^2, hence associates a number (the degree of φ) to every 2-cell. In this way we define a cochain c.

Claim 1.4.2. *The cochain c is a cocycle, and the class $[c] \in H^2(X;\mathbb{Z})$ depends only on the bundle $L \to X$ (and is independent of the CW-decomposition and the trivialization).* □

Note that if $[c] = 0$, so the trivialization can be changed over the 1-skeleton in such a way that it extends over the 2-skeleton, then L is trivial. This follows from the fact that when we want to extend the trivialization to higher dimensional cells, we do not find any more obstructions, since all maps $\partial D^k \to U(1) = S^1$ are nullhomotopic once $k > 2$.

Theorem 1.4.3. *The above-defined class $[c] \in H^2(X;\mathbb{Z})$ of $L \to X$ coincides with the Chern class $c_1(L)$ defined by the isomorphism of Proposition 1.4.1.*

Proof. Both c_1 and the above obstruction class $[c]$ are natural with respect to continuous maps. Since each line bundle can be regarded as the pull-back of the tautological line bundle $\tau \to \mathbb{CP}^\infty$, Theorem 1.4.3 has to be proved only for τ. That proof, however, is essentially contained in the proof of Claim 2.2.1. □

A similar argument gives an obstruction theoretic description for $w_1(R)$ of a real line bundle $R \to X$. (For more about obstruction theory see also [**St**] or [**FFG**].)

Next we outline the definition of the other Chern classes $c_i(E)$ in $H^{2i}(X;\mathbb{Z})$ $(i = 2, \dots, n)$ for a complex n-plane bundle E. (For a real n-plane bundle $F \to X$ one can proceed similarly and get the Stiefel-Whitney classes $w_i(F) \in H^i(X;\mathbb{Z}_2)$, $i = 1, \dots, n$.) If $E = L_1 \oplus \dots \oplus L_n$ is a sum of complex line bundles L_i, take $c(E) \in H^*(X;\mathbb{Z})$ (the *total Chern class*) to be the cup product $c(E) = (1 + c_1(L_1)) \cup \dots \cup (1 + c_1(L_n)) \in H^*(X;\mathbb{Z})$. The component of $c(E)$ in $H^{2i}(X;\mathbb{Z})$ is called the $i^{\underline{th}}$ *Chern class* $c_i(E)$ of E, so $c(E) = 1 + c_1(E) + \dots + c_n(E)$. Hence $c_i(E)$ is the value of the $i^{\underline{th}}$ elementary symmetric polynomial of n variables evaluated on $c_1(L_1), \dots, c_1(L_n)$. Not all n-plane bundles are sums of complex line bundles, however. For the definition of Chern classes in those cases we need a theorem.

Theorem 1.4.4. (Splitting Principle) *For a given complex n-plane bundle $E \to X$ there is a space Y and a map $g\colon Y \to X$ such that g^*E splits as $L_1 \oplus \dots \oplus L_n$ (where $L_i \to Y$ are complex line bundles), $g^*\colon H^*(X;\mathbb{Z}) \to H^*(Y;\mathbb{Z})$ is a monomorphism and the elementary symmetric polynomials of the classes $c_1(L_i)$ are in* $\operatorname{Im} g^*$. □

For the proof and an analogous statement for real bundles see [**Sha**], [**Hu**]. Here we only give the inductive step of the construction of Y and g; the properties should be checked by the reader. Projectivize $\pi\colon E \to X$ (replace the fiber $\mathbb{C}^n$ with $\mathbb{CP}^{n-1}$), and get the new bundle $p\colon \mathbb{CP}(E) \to X$. Pull $E \to X$ back via the map $p\colon \mathbb{CP}(E) \to X$ and get the $\mathbb{C}^n$-bundle $p^*(E) \to \mathbb{CP}(E)$. It is easy to see that $p^*(E) = L \oplus E_1$, where L is a line bundle and $E_1 \to \mathbb{CP}(E)$ is a $\mathbb{C}^{n-1}$-bundle. Applying the Leray-Hirsch Theorem [**Hu**] we get that $p^*\colon H^*(X;\mathbb{Z}) \to H^*(\mathbb{CP}(E);\mathbb{Z})$ is a monomorphism. Repeat this inductive step and split E into line bundles; the composition of the corresponding projections will give the desired g. The classes $c(g^*(E)) = 1 + c_1(g^*(E)) + \dots + c_n(g^*(E)) \in H^*(Y;\mathbb{Z})$ are defined, and since these classes are elementary symmetric polynomials of the $c_1(L_i)$'s, they are in $\operatorname{Im} g^*$. The homomorphism g^* is injective, so $g^{*-1}(c_i(g^*(E)))$ is a well-defined cohomology class, which is, by definition, the $i^{\underline{th}}$ Chern class of $E \to X$. The real analogue of the above process defines Stiefel-Whitney classes $w_i(F) \in H^i(X;\mathbb{Z}_2)$ of a given $\mathbb{R}^n$-bundle $F \to X$. For a real n-plane bundle $F \to X$ the *Pontrjagin classes* $p_i(F) \in H^{4i}(X;\mathbb{Z})$ can be defined by the formula $p_i(F) = (-1)^i c_{2i}(F \otimes_{\mathbb{R}} \mathbb{C})$. The next proposition summarizes the most important properties of the Chern, Stiefel-Whitney and Pontrjagin classes. Let $E \to X$ denote a $\mathbb{C}^n$-bundle and $F \to X$ an arbitrary $\mathbb{R}^n$-bundle.

Proposition 1.4.5. (a) *The above-defined characteristic classes* $c_i(E) \in H^{2i}(X;\mathbb{Z})$, $w_i(F) \in H^i(X;\mathbb{Z}_2)$ *and* $p_i(F) \in H^{4i}(X;\mathbb{Z})$ *are well-defined cohomology elements. (They depend only on the bundles, not on the particular splitting used in the definition.)*

(b) *These classes are natural with respect to continuous maps:* $c_i(f^*E) = f^*(c_i(E))$, $w_i(f^*F) = f^*(w_i(F))$ *and* $p_i(f^*F) = f^*(p_i(F))$ *for a continuous map* $f\colon X' \to X$.

(c) (Whitney product formula) *For direct sums of bundles we have the identities* $c(E \oplus E') = c(E) \cup c(E')$, $w(F \oplus F') = w(F) \cup w(F')$, *and* $2(p(F \oplus F') - p(F) \cup p(F')) = 0$. □

Exercises 1.4.6. (a) Show that if $E \to X$ is a complex n-plane bundle, then $E \otimes \mathbb{C} \cong E \oplus \overline{E}$. Here $\overline{E}$ stands for the conjugate bundle of $E \to X$. (See [**MS**] for a solution.)

(b) Prove that $c_i(\overline{E}) = (-1)^i c_i(E)$.

There is one more characteristic class that will be used in our arguments, namely, the Euler class of an oriented real n-plane bundle. We define it only in the case where X^m is an m-dimensional closed, smooth manifold and $F_1 \to X$ is a smooth, oriented $\mathbb{R}^n$-bundle. Let $s\colon X \to F_1$ be a generic smooth section of $F_1 \to X$ and $Z = s^{-1}(0)$ be its zero set. (By generic we mean that the image $s(X)$ intersects the image of the zero section of $F_1 \to X$ transversally.) The fundamental class of the zero set Z defines a homology class $[Z] \in H_{m-n}(X;\mathbb{Z})$, so its Poincaré dual $PD([Z])$ gives rise to an element in $H^n(X;\mathbb{Z})$.

Claim 1.4.7. *The class* $e(F_1) = PD([Z]) \in H^n(X;\mathbb{Z})$ *depends only on the bundle* $F_1 \to X$*, and by definition this cohomology class is the Euler class of* $F_1 \to X$. □

For a manifold X, the $i^{\underline{th}}$ Stiefel-Whitney class of its tangent bundle TX is denoted by $w_i(X)$. Similarly, for an oriented manifold one defines $p_i(X)$ and $e(X)$. If X is a complex manifold, so TX admits a canonical complex structure, $c_i(X)$ is defined as well. For the following computations see [**MS**].

Examples 1.4.8. (a) The total Chern class of the complex projective space $\mathbb{CP}^n$ is given as $c(\mathbb{CP}^n) = (1+g)^{n+1}$, where g is the standard generator of $H^2(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$. For example, $c_1(\mathbb{CP}^n) = (n+1)g$ and $c_1(\mathbb{CP}^2) = 3g$, $c_2(\mathbb{CP}^2) = 3g^2$.

(b) Using Exercises 1.4.6(a) and (b) we can see that $p(\mathbb{CP}^n) = (1+g^2)^{n+1}$; in particular, $p_1(\mathbb{CP}^2) = 3g^2$.

(c) The total Stiefel-Whitney class of $\mathbb{RP}^n$ is $w(\mathbb{RP}^n) = (1+a)^{n+1}$, where $a \in H^1(\mathbb{RP}^n;\mathbb{Z}_2)$ is the generator of $H^1(\mathbb{RP}^n;\mathbb{Z}_2) = \mathbb{Z}_2$.

The following proposition describes the most important relations among characteristic classes.

Proposition 1.4.9. *If $E \to X$ is an n-dimensional complex bundle, then the relations $c_n(E) = e(E)$ and $c_i(E) \equiv w_{2i}(E) \pmod 2$ hold for all $i \leq n$; moreover $w_{2i+1}(E) = 0$. For a smooth, closed, oriented n-dimensional manifold X we have $e(X) \equiv w_n(X) \pmod 2$, while $\langle e(X), [X] \rangle$ is equal to the Euler characteristic $\chi(X)$ of X.* □

Exercise 1.4.10. Find the relation among the Pontrjagin and Chern classes of a complex bundle $E \to X$. In particular, show that $p_1(E)$ is given by $c_1^2(E) - 2c_2(E)$. (*Hint*: Use Exercises 1.4.6 or see [**MS**].)

Note that we denote the Euler characteristic of a manifold X by $\chi(X)$ — while the Euler class of a bundle E is denoted by $e(E)$. (The *holomorphic* Euler characteristic of a complex manifold will be denoted by χ_h.) If X is a closed, complex n-dimensional manifold, and $\{i_1, \dots , i_k\}$ is a partition of n (that is, i_j are positive integers and $i_1 + \ldots + i_k = n$), then the product $c_{i_1}(X) \cup \ldots \cup c_{i_k}(X)$ can be evaluated on the fundamental class $[X]$, defining the *Chern number* corresponding to the partition $\{i_1, \dots , i_k\}$. A similar definition gives Stiefel-Whitney (and Pontrjagin) numbers of closed, smooth (oriented) manifolds. If S is a complex surface, the two Chern numbers are denoted by $c_2[S]$ and $c_1^2[S]$. (By slight abuse of notation we will often confuse the Chern numbers $c_2[S]$ and $c_1^2[S] \in \mathbb{Z}$ with the corresponding Chern classes $c_2(S), c_1^2(S) \in H^4(S; \mathbb{Z})$. In this introductory chapter, however, we would like to make the distinction clear.)

Exercises 1.4.11. (a) Prove that if X is a closed, orientable manifold of odd dimension, then $\chi(X) = 0$.

(b)* Show that if Σ is a closed, oriented surface embedded in an arbitrary oriented 4-manifold X, then $e(\nu\Sigma)[\Sigma] = Q_X([\Sigma], [\Sigma]) = [\Sigma]^2$.

Now we turn our attention to the 4-dimensional case, so X denotes a 4-dimensional (closed, oriented, smooth) manifold. The next theorem gives a relation between the signature of X and the first Pontrjagin class of its tangent bundle TX.

Theorem 1.4.12. (Hirzebruch signature theorem for 4-manifolds) *If X is a smooth, closed, oriented 4-dimensional manifold, then its signature $\sigma(X)$ is equal to $\frac{1}{3}\langle p_1(X), [X] \rangle$.* □

To emphasize that a certain smooth 4-manifold X admits a complex structure (so it is a *complex surface*), we will denote it by S. For a complex surface S we have that $p_1(S) = c_1^2(S) - 2c_2(S)$ (cf. Exercise 1.4.10), and so from Theorem 1.4.12 (and from the identity $c_2[S] = e[S] = \chi(S)$) it follows that the Chern number $c_1^2[S]$ equals $3\sigma(S) + 2\chi(S)$. Below, we will show that $c_1(S) \in H^2(S;\mathbb{Z})$ is characteristic for the intersection form (because it reduces mod 2 to $w_2(S)$), so Lemma 1.2.20 implies that $c_1^2[S] \equiv \sigma(S) \pmod 8$, and thus $\sigma(S) + \chi(S) \equiv 0 \pmod 4$. We obtain the following theorem by expressing this in terms of Chern numbers, and also observing that for any closed, oriented 4-manifold we have $\sigma(X) + \chi(X) = b_2^+(X) - b_2^-(X) + (b_2^+(X) + b_2^-(X) - 2b_1(X) + 2) = 2(1 - b_1(X) + b_2^+(X))$.

Theorem 1.4.13. (Noether formula) *For a complex surface S the integer $c_1^2[S] + c_2[S] = 3(\sigma(S) + \chi(X))$ is divisible by 12, or equivalently, $1 - b_1(S) + b_2^+(S)$ is even. In particular, if S is a simply connected complex surface then $b_2^+(S)$ is odd.* □

Note that for defining $c_i(S)$ we do not really need S to be a complex manifold. If $TX \to X$ is a $\mathbb{C}^n$-bundle, c_i already makes sense. Of course, if X is a complex manifold then $TX \to X$ has a natural complex structure. However, a $\mathbb{C}^n$-structure on the fibers (defining multiplication by i fiberwise) can be defined for a much wider class of manifolds. Such a structure on a manifold is called an *almost-complex structure*. Formally, we have

Definition 1.4.14. An *almost-complex structure* on the bundle $TX \to X$ is a smooth, fiberwise linear map $J\colon TX \to TX$ covering id_X such that $J^2 = -\mathrm{id}_{TX}$.

An almost-complex structure defines a natural orientation on the smooth manifold X, since the choice of J reduces the structure group of the tangent bundle to $GL(n;\mathbb{C}) \subset GL^+(2n;\mathbb{R})$. Once an almost-complex structure is specified, Chern classes c_i make sense. In this latter case the Chern classes will also depend on the almost-complex structure chosen; for an oriented manifold X we only consider almost-complex structures generating the given orientation. For a given J the corresponding Chern classes of (TX, J) are denoted by $c_i(X, J) \in H^{2i}(X;\mathbb{Z})$. The following theorem provides a necessary and sufficient condition for the existence of an almost-complex structure on the 4-manifold X.

Theorem 1.4.15. (Wu [**Wu**], see also [**HH**]) *For a given 4-manifold X and almost-complex structure J on X we have $c_2(X, J) = e(X) \in H^4(X;\mathbb{Z})$, $c_1(X, J) \equiv w_2(X) \pmod 2$ and $c_1^2[X, J] = 3\sigma(X) + 2\chi(X)$. Conversely, if for $h \in H^2(X;\mathbb{Z})$ the equation $h^2 = 3\sigma(X) + 2\chi(X)$ and the congruence $h \equiv w_2(X) \pmod 2$ hold, then there is an almost-complex structure J on TX with $h = c_1(X, J)$.* □

(For the proof see Exercise 1.4.21(c); cf. also Exercise 10.1.3(a).) Since the proof of the Noether formula only used properties of $c_1(S)$ satisfied by the first Chern class $c_1(X, J)$ of an almost-complex structure J — namely that $c_1^2[X,J] = 3\sigma(X) + 2\chi(X)$ and $c_1(X,J) \equiv w_2(X) \pmod 2$ — Theorem 1.4.13 holds for an almost-complex manifold as well. By Theorem 1.4.15, the existence of an almost-complex structure is a cohomological question.

Exercises 1.4.16. (a) Prove that S^4, $(S^2 \times S^2)\#(S^2 \times S^2)$ and $\mathbb{CP}^2\#\mathbb{CP}^2$ do not admit any almost-complex structure.

(b)* More generally, prove (using Theorem 1.4.15) that a simply connected, smooth, closed 4-manifold X admits an almost-complex structure iff $b_2^+(X)$ is odd.

The definitions $c_2[X] = \chi(X)$ and $c_1^2[X] = 3\sigma(X) + 2\chi(X)$ extend the notions of c_2 and c_1^2 to all closed, oriented 4-manifolds (even to manifolds without an almost-complex structure). The next formula has fundamental importance in the study of the smooth structures of 4-manifolds. (For generalizations see Theorems 2.4.8 and 11.4.7; for applications see, e.g., Theorem 2.1.6.) Let S be a complex surface (so a real 4-dimensional manifold) with $i\colon C \hookrightarrow S$ a smooth (nonsingular), connected complex curve in it.

Theorem 1.4.17. (Adjunction Formula) *Denoting the genus of C by $g(C)$ and the self-intersection by $[C]^2$, we have $2g(C)-2 = [C]^2 - c_1(S)[C]$ (where $c_1(S)[C]$ means $\langle c_1(S), [C]\rangle \in \mathbb{Z}$).*

Proof. Restrict the tangent bundle of S to C and apply characteristic class computations: $TS|C = TC \oplus \nu C$, where $\nu C \to C$ is the normal bundle of C in S; hence $c_1(S)[C] = c_1(TS|C)[C] = c_1(TC)[C] + c_1(\nu C)[C] = e(TC)[C] + e(\nu C)[C] = \chi(C) + e(\nu C)[C] = 2 - 2g(C) + e(\nu C)[C]$. The solution of Exercise 1.4.11(b) now gives $c_1(S)[C] = 2 - 2g(C) + [C]^2$, which proves the adjunction formula. □

A similar argument shows:

Proposition 1.4.18. *For a given oriented 4-manifold X and $\alpha \in H_2(X;\mathbb{Z})$ we have that $\langle w_2(X), \alpha\rangle \equiv Q_X(\alpha,\alpha) \pmod 2$.*

Proof. Represent $\alpha \in H_2(X;\mathbb{Z})$ by an embedded orientable surface $\Sigma \subset X$. Then $\langle w_2(X), \alpha\rangle = \langle w_2(TX|\Sigma), [\Sigma]\rangle = w_2(T\Sigma)[\Sigma] + w_2(\nu\Sigma)[\Sigma] + (w_1(T\Sigma) \cup w_1(\nu\Sigma))[\Sigma] \equiv e(T\Sigma)[\Sigma] + e(\nu\Sigma)[\Sigma] \equiv e(\nu\Sigma)[\Sigma] = Q_X([\Sigma],[\Sigma]) = \alpha^2 \pmod 2$. Note that $w_1(T\Sigma) = 0$ and $e(T\Sigma)[\Sigma] = \chi(\Sigma) \equiv 0 \pmod 2$, since Σ is orientable. (In the expression $\langle w_2(X), \alpha\rangle$ we took the mod 2 reduction of the integral homology class α.) □

Assuming that X is orientable, the same relation holds for homology elements with $\mathbb{Z}_2$-coefficients: For $a \in H_2(X;\mathbb{Z}_2)$ one has $\langle w_2(X), a\rangle = Q_X(a,a)$. This equation is called the *Wu formula* (cf. Exercise 5.7.3). Note also that Proposition 1.4.18 holds even if X has boundary or is noncompact.

Exercise 1.4.19. Prove the Wu formula using Remark 1.2.4. (For a nonorientable $\mathbb{R}^n$-bundle F, $w_n(F)$ can be defined like the Euler class in Claim 1.4.7, using $\mathbb{Z}_2$-coefficients, and $\langle w_n(X), [X]\rangle \equiv \chi(X) \pmod 2$ for X closed.)

Proposition 1.4.18 shows that if there is no 2-torsion in $H^2(X;\mathbb{Z})$ (for example, if X is simply connected), then $w_2(X)$ vanishing is equivalent to Q_X being even. This is not true for all 4-manifolds, since in the presence of 2-torsion the mod 2 reduction $H_2(X;\mathbb{Z}) \to H_2(X;\mathbb{Z}_2)$ is not onto. There exists a manifold X (e.g., the Enriques surface, cf. Section 3.4) with nontrivial $w_2(X)$, $Q_X \cong (-E_8) \oplus H$ and $\pi_1(X) \cong \mathbb{Z}_2$. (In fact, $w_2(X)$ lifts to a class of order 2 in $H^2(X;\mathbb{Z})$ and pairs nontrivially with a class in $H_2(X;\mathbb{Z}_2)$ with no integer lift.) Note also that since $c_1(X,J) \equiv w_2(X) \pmod 2$, Proposition 1.4.18 implies that the first Chern class of an almost-complex structure is a characteristic element, as we needed when proving Theorem 1.4.13.

To demonstrate how useful characteristic classes are, we describe the classification of $U(2)$, $SU(2)$, $SO(3)$ and $SO(4)$-bundles over a 4-manifold X. (Note that the case $SO(2) \cong U(1)$ was already discussed at the beginning of this section.)

Theorem 1.4.20. (a) *Two $U(2)$-bundles E_1 and E_2 on X are isomorphic iff $c_1(E_1) = c_1(E_2)$ and $c_2(E_1) = c_2(E_2)$. Moreover, for every pair $(c_1, c_2) \in H^2(X;\mathbb{Z}) \times H^4(X;\mathbb{Z})$ there is a $U(2)$-bundle E with $c_1 = c_1(E)$ and $c_2 = c_2(E)$. Furthermore, a $U(2)$-bundle E can be reduced to an $SU(2)$-bundle iff $c_1(E) = 0$. Consequently, two $SU(2)$-bundles E_1 and E_2 are isomorphic iff $c_2(E_1) = c_2(E_2)$.*

(b) *Two $SO(4)$-bundles F_1 and F_2 are isomorphic iff $w_2(F_1) = w_2(F_2)$, $p_1(F_1) = p_1(F_2)$ and $e(F_1) = e(F_2)$.*

(c) *Two $SO(3)$-bundles $F_1, F_2 \to X$ are isomorphic iff $w_2(F_1) = w_2(F_2)$ and $p_1(F_1) = p_1(F_2)$. Moreover $p_1(F_1) \equiv \mathcal{P}(w_2(F_1)) \pmod 4$, and for every pair $(p_1, w_2) \in H^4(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}_2)$ with $p_1 \equiv \mathcal{P}(w_2) \pmod 4$ there is an $SO(3)$-bundle $F \to X$ such that $p_1 = p_1(F)$ and $w_2 = w_2(F)$.* □

In the above theorem, the map $\mathcal{P}\colon H^2(X;\mathbb{Z}_2) \to H^4(X;\mathbb{Z}_4)$ denotes the *Pontrjagin square*. (If the cohomology class $c \in H^2(X;\mathbb{Z})$ is an integral lift of w_2 (so $c \equiv w_2 \pmod 2$), then $c^2 \equiv \mathcal{P}(w_2) \pmod 4$. We do not give the definition of $\mathcal{P}$ in full generality.)

Exercises 1.4.21. (a) Prove that if $c, c' \in H^2(X;\mathbb{Z})$ are lifts of a fixed element $w_2 \in H^2(X;\mathbb{Z}_2)$, then $c^2 \equiv (c')^2 \pmod 4$.

(b) Prove Theorem 1.4.20(a). (*Hint*: Take a CW decomposition of X and apply obstruction theory. For Theorem 1.4.20(b) and (c) see [**DW**].)

(c)* Using Theorem 1.4.20, prove Theorem 1.4.15.

(d) Show that for an $SO(n)$-bundle F we have $w_{2i}^2(F) \equiv p_i(F) \pmod 2$. (*Hint*: Recall that $p_i(F) = (-1)^i c_{2i}(F \otimes_{\mathbb{R}} \mathbb{C}) \equiv w_{4i}(F \otimes_{\mathbb{R}} \mathbb{C}) \pmod 2$; by the Whitney product formula determine $w_{4i}(F \otimes_{\mathbb{R}} \mathbb{C})$ in terms of the $w_j(F)$'s.)

We close this subsection by quoting the *Lefschetz Hyperplane Theorem* used in various places in the text. (The proof appears in [**M2**], see also Exercise 11.2.3(b).)

Theorem 1.4.22. (Lefschetz Hyperplane Theorem) *Let X be a compact, complex n-dimensional submanifold of $\mathbb{CP}^N$. If H is a hyperplane in $\mathbb{CP}^N$, then the homomorphisms $\pi_i(X \cap H) \to \pi_i(X)$ and $H_i(X \cap H) \to H_i(X)$ are isomorphisms for $i < n-1$ and surjections for $i = n-1$.* □

1.4.2. Spin structures. We will now sketch the theory of spin structures and their relation to $w_2(X)$. For an obstruction-theoretic approach of the same notions see Section 5.6.

For a given real n-plane bundle $F \to X$ one can always reduce the structure group $GL(n;\mathbb{R})$ to $O(n)$ by introducing a Riemannian metric on F. The Lie group $O(n)$ is not connected, however, and the possibility of a further reduction of the structure group to $SO(n)$ (a connected component of $O(n)$) depends on a characteristic class of F. If such a reduction exists, $F \to X$ is an *orientable* bundle, and the choice of a reduction is an *orientation* for F (cf. Remark 1.1.3; note that by definition $GL^+(n;\mathbb{R}) \cap O(n) = SO(n)$). Since $w_1(\det F) = w_1(F)$ and the line bundle $\det F$ is trivial iff the structure group of F can be reduced to $SO(n)$, we have

Lemma 1.4.23. *The bundle $F \to X$ is orientable iff $w_1(F) \in H^1(X;\mathbb{Z}_2)$ vanishes; the orientations are parametrized by $H^0(X;\mathbb{Z}_2)$ (which is isomorphic to $\mathbb{Z}_2$ if X is connected).* □

If F is the tangent bundle TX, the orientability of F means that X is an orientable manifold (cf. Section 1.1). Although $SO(n)$ is connected, it is not a simply connected group; for $n \geq 3$ we have $\pi_1(SO(n)) \cong \mathbb{Z}_2$. (We always reduce to the case $n \geq 3$ by summing F with a trivial bundle if necessary.) The universal (double) cover of $SO(n)$ is the *spin group* $Spin(n)$. Let $F \to X$ be an oriented Riemannian (i.e., $SO(n)$-) bundle; the corresponding principal frame bundle will be denoted by $P_{SO(n)} \to X$.

Definition 1.4.24. The bundle $F \to X$ is *spinnable* if $P_{SO(n)} \to X$ can be covered by a principal $Spin(n)$-bundle $P_{Spin(n)} \to X$ such that the double covering $P_{Spin(n)} \to P_{SO(n)}$ is the universal cover $\rho\colon Spin(n) \to SO(n)$ fiberwise, i.e., $P_{Spin(n)} \times_\rho SO(n) \cong P_{SO(n)}$. (Hence a spin structure comprises a principal $Spin(n)$-bundle $P_{Spin(n)} \to X$ together with an identification $c\colon P_{Spin(n)} \times_\rho SO(n) \cong P_{SO(n)}$.) Fixing such a cover of $P_{SO(n)}$ — a *spin* *structure* — realizes F as a *spin bundle*. A spin structure on $F = TX$ turns X into a *spin manifold*.

From spectral sequences we get an exact sequence

$$0 \to H^1(X;\mathbb{Z}_2) \to H^1(P_{SO(n)};\mathbb{Z}_2) \xrightarrow{i^*} H^1(SO(n);\mathbb{Z}_2) \xrightarrow{\delta} H^2(X;\mathbb{Z}_2),$$

where $H^1(SO(n);\mathbb{Z}_2) \cong \mathbb{Z}_2$ and $\delta(1)$ equals $w_2(F)$ in $H^2(X;\mathbb{Z}_2)$ (cf. Remark 5.6.9(b)). The exercise below shows that the double covers of $P_{SO(n)}$ are in 1-1 correspondence with elements of $H^1(P_{SO(n)};\mathbb{Z}_2)$. By the assumption that fiberwise the cover needs to be $Spin(n) \to SO(n)$, the spin structures are in 1-1 correspondence with elements of $(i^*)^{-1}(1) \subset H^1(P_{SO(n)};\mathbb{Z}_2)$. This means that a required cover exists iff $1 \in \text{Im } i^*$, hence (by exactness) iff $\delta(1) = w_2(F) = 0$.

Proposition 1.4.25. *The $SO(n)$-bundle $\pi\colon F \to X$ is spinnable iff its second Stiefel-Whitney class $w_2(F) \in H^2(X;\mathbb{Z}_2)$ vanishes. If so, then the different spin structures are parametrized by $(i^*)^{-1}(1) \cong \ker\ i^*$, which is isomorphic to $H^1(X;\mathbb{Z}_2)$. (The identification of $H^1(X;\mathbb{Z}_2)$ with the set of spin structures is not canonical; it becomes canonical only after choosing a "base spin structure" corresponding to $0 \in H^1(X;\mathbb{Z}_2)$.)* □

Exercises 1.4.26. (a)* Show that the double covers of a manifold X are in 1-1 correspondence with elements of $H^1(X;\mathbb{Z}_2)$.

(b) Using obstruction theory, show that if G is a connected Lie group and X is a CW complex, then any principal G-bundle P_G is trivial over the 1-skeleton X_1. If G is simply connected, show that $P_G|X_2$ is trivial. (See Section 5.6 for related discussions.) What can we say about $P_G|X_3$ if $\pi_1(G) = 1$? (*Hint*: Use the fact that for any Lie group, $\pi_2(G) = 0$.)

We say that the oriented manifold X is spinnable if $w_2(X) = 0$. Note that when X is simply connected (so $H^1(X;\mathbb{Z}_2) = 0$), the spin structure is unique.

Remarks 1.4.27. (a) By slight (and standard) abuse of terminology we will refer to a manifold with $w_2(X) = 0$ as a *spin* manifold, although technically this implies the choice of a particular spin structure.

(b) For a 3-manifold X^3 we have $w_1^2(X) = w_2(X)$ [**MS**], so an orientable 3-manifold always has a spin structure. (In fact, if X^3 is orientable, then its tangent bundle is trivial, since $\pi_2(SO(3)) = 0$.)

(c) A simply connected (not necessarily closed) 4-manifold X is spin iff Q_X is even (since both statements are equivalent to $w_2(X)$ vanishing). If $H_1(X;\mathbb{Z})$ has 2-torsion, this equivalence no longer holds. (See also Corollary 5.7.6.)

Rohlin's Theorem (Theorem 1.2.29) was stated only for simply connected 4-manifolds. It can be generalized to arbitrary (closed, smooth) 4-manifolds with the following theorem — replacing the assumption about Q_X with the assumption that X is spin. (See [**K2**] or [**LaM**] for a proof.)

Theorem 1.4.28. (Rohlin, [**R2**]) *If X is a smooth, closed, spin 4-manifold, then $\sigma(X) \equiv 0 \pmod{16}$.* □

In the rest of this section we will give various bundle constructions and define operators one can associate to a spin structure. (For further details see [**LaM**], [**Mr2**].) The importance of these constructions in dimension four becomes clear once we generalize the notion of spin structures to *spin^c structures* and list the spectacular results based on that theory (cf. Section 2.4). The group $Spin(n)$ can be constructed as a subgroup of a 2^n-dimensional real algebra, the *Clifford algebra Cl_n*. By definition $Cl_n = T(\mathbb{R}^n)/I(\mathbb{R}^n)$, where $T(\mathbb{R}^n)$ is the tensor algebra $\bigoplus_k (\mathbb{R}^n)^{\otimes k}$ and $I(\mathbb{R}^n)$ is the ideal generated by elements of the form $v \otimes v + \langle v, v\rangle 1 \in T(\mathbb{R}^n),\ v \in \mathbb{R}^n$. We denote the complexification of Cl_n by $\mathbb{C}l_n$. The following algebraic statement can be found, e.g., in [**LaM**].

Proposition 1.4.29. *If $n = 2k+1$ is odd, then the complex Clifford algebra $\mathbb{C}l_n$ is isomorphic to the direct sum of two isomorphic matrix algebras: $\mathbb{C}l_n \cong M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C})$ (where $M_m(\mathbb{C}) = \{m \times m$ complex matrices$\}$). In this way we get two complex 2^k-dimensional representations of $Spin(n) \subset \mathbb{C}l_n$; these are irreducible and isomorphic. (We will denote them by S_n.) If $n = 2k$ is even, then $\mathbb{C}l_n \cong M_{2^k}(\mathbb{C})$; the corresponding complex 2^k-dimensional representation S_n of $Spin(n) \subset \mathbb{C}l_n$ splits into two (nonisomorphic) irreducible representations, hence $S_n = S_n^+ \oplus S_n^-$ as a $Spin(n)$-module.* □

Exercise 1.4.30. Check that for the real Clifford algebras we have $Cl_1 \cong \mathbb{C}, Cl_2 \cong \mathbb{H}, Cl_3 \cong M_2(\mathbb{C})$ and $Cl_4 \cong M_2(\mathbb{H}) = \{2 \times 2$ quaternionic matrices$\}$. Determine $\mathbb{C}l_n$ for $n \leq 4$.

Assume now that X is a spin manifold with a fixed spin structure $P_{Spin(n)} \to X$ and $c\colon P_{Spin(n)} \times_\rho SO(n) \cong P_{SO(n)}$. Using the above complex representation S_n of $Spin(n)$, one can associate the vector bundle $S \to X$ to $P_{Spin(n)} \to X$. Sections of $S \to X$ are the *spinors* over X. If $\dim X$ is even, then the bundle S splits as $S = S^+ \oplus S^-$ (corresponding to the decomposition $S_n = S_n^+ \oplus S_n^-$); sections of S^+ (S^-) are the *positive (negative) spinors*, respectively. The bundle associated to $P_{Spin(n)} \to X$ by the $Spin(n)$-representation $\mathbb{C}l_n$ is the *Clifford bundle* $\mathbb{C}l(X)$ of the spin structure. The action of $\mathbb{C}l_n$ on S_n induces an action of the Clifford bundle $\mathbb{C}l(X)$ on the spinor bundle $S \to X$; this action is called the *Clifford multiplication*. Recall that for defining the principal bundle $P_{SO(n)} \to X$ of $TX \to X$ we fixed a metric g and an orientation on X (or equivalently, reduced the structure group of TX from $GL(n;\mathbb{R})$ to $SO(n)$). There is a canonical object associated to the metric g on X, which is the *Levi-Civita connection* $\nabla_g\colon \Gamma(X;TX) \to \Gamma(X;TX \otimes T^*X)$. (As usual, $\Gamma(X;F)$ denotes the vector space of C^∞-sections of the vector bundle $F \to X$.) This connection can be pulled back to the $Spin(n)$-bundle $P_{Spin(n)} \to X$, defining a covariant differentiation $\nabla\colon \Gamma(X;S) \to \Gamma(X;S \otimes T^*X)$ on the associated bundle $S \to X$.

Remark 1.4.31. It is a standard fact of differential geometry (cf. [**DK**]) that a covariant differentiation on the associated bundle F determines a $Lie(G)$-valued 1-form on the principal G-bundle P_G corresponding to F, and vice versa. (Here $Lie(G)$ denotes the Lie algebra of the Lie group G.) Hence the Levi-Civita connection ∇_g determines a $Lie(SO(n))$-valued 1-form on $P_{SO(n)}$, which can be pulled back to $P_{Spin(n)}$. (Note that the corresponding Lie algebras satisfy $Lie(SO(n)) = Lie(Spin(n))$.) In the following ∇ denotes the associated covariant differentiation on the associated bundle $S \to X$.

Since T^*X is a subbundle of the Clifford bundle $\mathbb{C}l(X)$, T^*X acts on the spinor bundle S. Hence one can define a map (the Clifford multiplication) $C\colon \Gamma(X;S \otimes T^*X) \to \Gamma(X;S)$.

Definition 1.4.32. For a given Riemannian manifold X with a fixed spin structure $P_{Spin(n)} \to X$, the composition

$$\partial\!\!\!/ = C \circ \nabla\colon \Gamma(X;S) \to \Gamma(X;S)$$

is called the *Dirac operator* of the spin manifold X. If $\dim X$ is even (so $S = S^+ \oplus S^-$), the Dirac operator $\partial\!\!\!/\colon \Gamma(X;S^\pm) \to \Gamma(X;S^\mp)$ interchanges spinors of opposite sign.

Finally, we examine the above constructions on 4-manifolds. Note that $Spin(3) \cong SU(2) \cong \{\text{unit quaternions}\} = Sp(1) = S^3$. (For $q \in Sp(1)$ associate the map $q\colon \mathbb{H} \to \mathbb{H}$ given by $x \mapsto qxq^{-1}$ (with quaternionic multiplication); this determines an action of $Sp(1)$ on the imaginary quaternions $\operatorname{Im}\mathbb{H}$, giving the double cover $Sp(1) = SU(2) \to SO(3) \approx \mathbb{RP}^3$.) Similarly, we have that $Spin(4) = SU(2)\times SU(2)$: For a pair $(q_+, q_-) \in SU(2)\times SU(2)$, take the linear transformation $\mathbb{H} \to \mathbb{H}$ defined by $x \mapsto q_+ x q_-^{-1}$, and get the desired universal (double) cover $SU(2) \times SU(2) \to SO(4)$.

In dimension 4 there is an alternative (and obviously equivalent) way of defining spin structures. Let V be a 4-dimensional oriented Euclidean vector space — so the symmetry group of V is isomorphic to $SO(4)$. We define a *spin structure* for V as a pair of 1-dimensional quaternionic vector spaces V^+, V^- with hermitian metrics and a fixed isomorphism $\gamma\colon V \to \operatorname{Hom}_{\mathbb{H}}(V^+, V^-)$ compatible with the metrics.

Exercise 1.4.33. Prove that the symmetry group of a spin structure (V^+, V^-, γ) is isomorphic to $SU(2) \times SU(2) \cong Spin(4)$.

Applying the above definition fiberwise, we get an alternative definition of spin structures over a 4-manifold: A *spin structure* for the 4-dimensional (oriented) Riemannian manifold X^4 is a pair of $SU(2)$-bundles $S^\pm \to X$ and an isomorphism $\gamma\colon TX \to \mathrm{Hom}_{\mathbb{H}}(S^+, S^-)$ compatible with the metrics. Once the $SU(2)$-bundles $S^\pm$ are given, the principal bundle $P_{Spin(4)} \to X$ can be easily constructed. (Put the cocycle structures of S^+ and S^- together to map into $SU(2) \times SU(2) \cong Spin(4)$ and get $P_{Spin(4)} \to X$; the isomorphism $c\colon P_{Spin(4)} \times_\rho SO(4) \cong SO(4)$ can be derived from γ.) This shows that the triple $(S^\pm, \gamma)$ determines a spin structure on X in the previous sense. On the other hand, we have already seen how to derive $S^\pm$ and γ (which corresponds to the Clifford multiplication) from $P_{Spin(4)}$, so the two definitions of spin structures over a 4-dimensional manifold X are obviously equivalent.

Remark 1.4.34. It is known that $n = 4$ is the unique dimension in which $Lie(SO(n))$ splits as a Lie algebra. The splitting of $Lie(SO(4))$ as $Lie(SO(3)) \oplus Lie(SO(3))$ will be exploited in the definition of the 4-manifold invariants discussed in Section 2.4.