PREFACE

Mathematicians are Frenchmen of sorts: whatever one says to them they translate into their own language and then it becomes something entirely different.

Johann Wolfgang von Goethe, *Maximen und Reflexionen*, 1840

The purpose of this book is to provide a broad introduction to the theory of scattering resonances.

Scattering resonances appear in many branches of mathematics, physics and engineering. They generalize eigenvalues or bound states for systems in which energy can scatter to infinity. A typical state has then a rate of oscillation (just as a bound state does) and a rate of decay. Although the notion is intrinsically dynamical, an elegant mathematical formulation comes from considering meromorphic continuations of Green's functions or scattering matrices. The poles of these meromorphic continuations capture the physical information by identifying the rate of oscillations with the real part of a pole and the rate of decay with its imaginary part. The resonant state, which is the corresponding wave function, then appears in the residue of the meromorphically continued operator. An example from pure mathematics is given by the zeros of the Riemann zeta function: they are, essentially, the resonances of the Laplacian on the modular surface. The Riemann hypothesis then states that the decay rates for the modular surface are all either 0 or $\frac{1}{4}$. A standard example from physics is given by shape resonances created when the interaction region is separated from free space by a potential barrier. The decay rate is then exponentially small in a way depending on the width of the barrier.

In this book we provide an introduction to mathematical techniques used in the study of scattering resonances, concentrating on the simplest models but providing references to modern literature and indications of what happens in more general situations. Some chapters (such as Chapters 2 and 3) are meant to be easily accessible and others (such as Chapter 5) somewhat more demanding. The rather substantial set of appendices provides detailed accounts of most methods needed in the text. A diagram representing the dependencies of various sections is presented at the end of Chapter 1. The choice of topics is necessarily determined by the research interests of the authors, and many important aspects of the subject are not covered. We also stayed away from exciting but technical developments such as precise asymptotics for shape resonances, fractal Weyl laws, resonance gaps for chaotic systems, or the applications of scattering theory to hyperbolic dynamical systems – see the survey [**Zw17**] for an overview and references.

SD was introduced to scattering resonances by MZ, who in turn had the good fortune to be introduced to this field by Richard Melrose. We would like to thank him for his generous guidance and insights and for his foundational results on resonance counting and trace formulas.

The viewpoint and many discoveries of Johannes Sjöstrand changed the subject in a profound way. MZ was privileged to maintain a long collaboration with Sjöstrand and would like to thank him for sharing his ideas and expertise over the years.

Many other colleagues and collaborators have contributed to our understanding of the subject, and special thanks are due to Ivana Alexandrova, Jean-François Bony, David Bindel, Paul Brumer, Nicolas Burq, Tanya Christiansen, Kiril Datchev, Frédéric Faure, Jeff Galkowski, Colin Guillarmou, Laurent Guillopé, Bernard Helffer, Peter Hintz, Michael Hitrik, Long Jin, Ulrich Kuhl, André Martinez, William H. Miller, Shu Nakamura, Frédéric Naud, Stéphane Nonnenmacher, Galina Perelman, Vesselin Petkov, Jim Ralston, Antônio Sá Barreto, Hart Smith, Plamen Stefanov, Siu-Hung Tang, Jared Wunsch, András Vasy, and Georgi Vodev.

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Chapter 2 developed from notes on one-dimensional scattering written by Siu-Hung Tang and MZ in 2001 [**TZ01**] – we are grateful for Tang's help on the project and for allowing us to use that material.

Simon Becker's careful reading of the final version eliminated countless mistakes: we and the readers of this book owe him a great debt. We are also particularly grateful to Alexis Drouot, Benjamin Küster, Hari Manoharan, Alberto Parmeggiani, Euan Spence, Jian Wang, Tobias Weich, Mengxuan Yang and the three anonymous reviewers of the book for their helpful comments and corrections.

Peter Hintz helped us with the translation of Goethe's maxim for the epigraph.

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