Preface

The mathematical study of general relativity is a large and active field. This book is an attempt to introduce students to just one part of this field. Specifically, as the title suggests, this book deals primarily with problems in general relativity that are essentially *geometric* in character, meaning that they can be attacked using the methods of Riemannian geometry and partial differential equations. However, since there are still so many topics that match this description, we have chosen to further narrow the focus of this book to the following concept. This book is primarily about the positive mass theorem and the various ideas that surround it and have grown from it. It is about understanding the interplay between mass, scalar curvature, minimal surfaces, and related concepts.

Many geometric problems in general relativity specialize to problems in pure Riemannian geometry. The most famous of these is the positive mass theorem, first proved by Richard Schoen and Shing-Tung Yau in 1979 [**SY79c**, **SY81a**], and later by Edward Witten using an unrelated method [**Wit81**]. Around two decades later, Gerhard Huisken and Tom Ilmanen proved a generalization of the positive mass theorem called the Penrose inequality [**HI01**], which was later proved using a different approach by Hubert Bray [**Bra01**]. The goal of this book is to explain the background context and proofs of all of these theorems, while introducing various related concepts along the way. Unfortunately, there are many topics and results that would fit together nicely with the material in this book, and an argument could certainly be made that they belong in this book, but for one reason or another, we had to leave them out. At the top of the wish list for topics we would have liked to include are: a thorough discussion of the Jang equation as in [**SY81b**, **Eic13**, **Eic09**, **AM09**], a complete proof of the rigidity of the spacetime positive mass theorem as in [**BC96**, **HL17**] (see Section 8.2), compactly supported scalar curvature deformations as in [**Cor00**, **CS06**, **Cor17**] (see Theorems 3.51 and 6.14), and a tour of constant mean curvature foliations and their relationship to center of mass [**HY96**, **QT07**, **Hua09**, **EM13**].

The main prerequisite for this book is a working understanding of Riemannian geometry (from books such as [**Cha06**, **dC92**, **Jos11**, **Lee97**, **Pet16**, **Spi79**]) and basic knowledge of elliptic linear partial differential equations, especially Sobolev spaces (various parts of [**Eva10**, **GT01**, **Jos13**]). Certain facts from partial differential equations are recalled in the Appendix, with special attention given to the topics which are the least "standard"—most notably the theory of weighted spaces on asymptotically flat manifolds. A modest amount of knowledge of algebraic topology is assumed (at the level of a typical one-year graduate course such as [**Hat02**, **Bre97**]) and will typically only be used on a superficial level. No knowledge of physics at all is required. In fact, the book has been structured in such a way that Part 1 contains almost no physics. Although the Riemannian positive mass theorem was originally motivated by physical considerations, it is the author's conviction that it eventually would have been discovered for purely mathematical reasons. Part 2 includes a short crash course in general relativity, but again, only the most shallow understanding of physics is involved.

Despite the level of prerequisites, this book is still, unfortunately, not self-contained. We will typically skip arguments that rely on a large body of specialized knowledge (e.g., geometric measure theory). More generally, there are many places in the book where we only give sketches of proofs. This is sometimes because the results draw upon a wide variety of facts in geometric analysis, and it is not realistic to include all relevant background material. In other cases, it is because our goal is less to give a complete proof than to give the reader a guide for how to understand those proofs. For example, we avoid the most technical details in the two proofs of the Penrose inequality in Chapter 4, partly because the author has little to offer in terms of improved exposition of those details. The interested reader can and should consult the original papers [**HI01**, **Bra01**, **BL09**]. Since this book is intended to be an introduction to a field of active research, we are not shy about presenting statements of some theorems without any proof at all. We hope that this will help the reader to understand the current state of what is known and offer directions for further study and research.

In order to simplify the discussion, most definitions and theorems will be stated for manifolds, metrics, functions, vector fields, etc., which are *smooth*. Except where explicitly stated otherwise, the reader should assume that everything is smooth. (Despite this, because of the use of elliptic theory, we will of course still need to use Sobolev spaces for our proofs.) The reason for this is to prevent having to discuss what the optimal regularity is for the hypotheses of each theorem. The reader will have to refer to the research literature if interested in more precise statements.

When we refer to concepts or ideas that are especially common or well known, instead of citing a textbook, we will sometimes cite Wikipedia. The reasoning is that in today's world, although Wikipedia is rarely the *best* source, it is often the *fastest* source. Here, the reader can get a quick introduction (or refresher) on the concept and then seek a more traditional mathematical text as desired. These citations will be marked with the name of the relevant article. For example, the citation [**Wik**, Riemannian_geometry] means that the reader should visit

`http://en.wikipedia.org/wiki/Riemannian_geometry`.

There are many exercises sprinkled throughout the text. Some of them are routine computations of facts and formulas that are used heavily throughout the text. Others serve as simple "reality checks" to make sure the reader understands statements of definitions or theorems on a basic level. Finally, there are some exercises (and "check this" statements) that ask the reader to fill in the details of some proof—these are meant to mimic the sort of routine computations that tend to come up in research.

The motivation for writing this book came from the fact that, to the author's knowledge, there is no graduate-level text that gives a full account of the positive mass theorem and related theorems. This presents an unnecessarily high barrier to entry into the field, despite the fact that the core material in this book is now quite well understood by the research community. A fair amount of the material in Part 1 was presented as a series of lectures during the Fall of 2015 as part of the General Relativity and Geometric Analysis seminar at Columbia University.

I would like to thank Hubert Bray, who is the person most responsible for shepherding me into this field of research. He taught me much of what I know about the subject matter of this book and strongly shaped my intuition and perspective. He also encouraged me to write this book and came up with the title. I thank Richard Schoen, my doctoral advisor, for teaching me about geometric analysis and supporting my research in geometric relativity. I have also learned a great deal about this subject from him through many private conversations, unpublished lecture notes, and talks I have attended over the years. Similarly, I thank my other collaborators in the field, who have taught me so much throughout my career: André Neves, Jeffrey Jauregui, Christina Sormani, Michael Eichmair, Philippe LeFloch, and especially Lan-Hsuan Huang, who kindly discussed certain technical issues related to this book.

I also thank Mu-Tao Wang for inviting me to give lectures at Columbia on the positive mass theorem at the very beginning of this project, and Greg Galloway for explaining to me various things that made their way into the introduction to general relativity in Part 2. Indeed, the exposition there owes a great deal to his excellent lecture notes [**Gal14**]. I thank Pengzi Miao for some helpful conversations while writing this book, as well as the anonymous reviewers who offered constructive feedback on an earlier draft. As an undergraduate, I wrote my senior thesis on Witten's proof of the positive mass theorem under the direction of Peter Kronheimer, and in some sense this book might be thought of as the culmination of that project, which began nearly two decades ago.