Preface

This text is designed for a first course in complex analysis for beginning graduate students or for well-prepared undergraduates whose background includes multivariable calculus, linear algebra, and advanced calculus. In this course the student will learn that all the basic functions that arise in calculus, first derived as functions of a real variable—such as powers and fractional powers, exponentials and logs, trigonometric functions and their inverses, as well as many new functions that the student will meet—are naturally defined for complex arguments. Furthermore, this expanded setting reveals a much richer understanding of such functions.

Care is taken to first introduce these basic functions in real settings. In the opening section on complex power series and exponentials, in Chapter 1, the exponential function is first introduced for real values of its argument as the solution to a differential equation. This is used to derive its power series, and from there extend it to complex argument. Similarly $\sin t$ and $\cos t$ are first given geometrical definitions for real angles and the Euler identity is established based on the geometrical fact that e^{it} is a unit-speed curve on the unit circle for real t. Then one sees how to define $\sin z$ and $\cos z$ for complex z.

The central objects in complex analysis are functions that are complex-differentiable (i.e., holomorphic). One goal in the early part of the text is to establish an equivalence between being holomorphic and having a convergent power series expansion. Half of this equivalence, namely the holomorphy of convergent power series, is established in Chapter 1.

Chapter 2 starts with two major theoretical results: the Cauchy integral theorem and its corollary, the Cauchy integral formula. These theorems have a major impact on the rest of the text, including the demonstration that if a function $f(z)$ is holomorphic on a disk, then it is given by a convergent power series on that disk. A useful variant of such power series is the Laurent series for a function holomorphic on an annulus.

The text segues from Laurent series to Fourier series in Chapter 3 and from there to the Fourier transform and the Laplace transform. These three topics have many applications in analysis, such as constructing harmonic functions and providing other tools for differential equations. The Laplace transform of a function has the important property of being holomorphic on a half-space. It is convenient to have a treatment of the Laplace transform after the Fourier transform, since the Fourier inversion formula serves to motivate and provide a proof of the Laplace inversion formula.

Results on these transforms illuminate the material in Chapter 4. For example, these transforms are a major source of important definite integrals that one cannot evaluate by elementary means, but that are amenable to analysis by residue calculus, a key application of the Cauchy integral theorem. Chapter 4 starts with this and proceeds to the study of two important special functions: the Gamma function and the Riemann zeta function.

The Gamma function, which is the first "higher" transcendental function, is essentially a Laplace transform. The Riemann zeta function is a basic object of analytic number theory arising in the study of prime numbers. One sees in Chapter 4 the roles of Fourier analysis, residue calculus, and the Gamma function in the study of the zeta function. For example, a relation between Fourier series and the Fourier transform, known as the Poisson summation formula, plays an important role in its study.

In Chapter 5, the text takes a geometrical turn, viewing holomorphic functions as conformal maps. This notion is pursued not only for maps between planar domains but also for maps to surfaces in $\mathbb{R}^3$. The standard case is the unit sphere S^2 and the associated stereographic projection. The text also considers other surfaces. It constructs conformal maps from planar domains to general surfaces of revolution, deriving for the map a first-order differential equation, nonlinear but separable. These surfaces are discussed as examples of Riemann surfaces. The Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is also discussed as a Riemann surface, conformally equivalent to S^2. One sees the group of linear fractional transformations as a group of conformal automorphisms of $\widehat{\mathbb{C}}$ and certain subgroups as groups of conformal automorphisms of the unit disk and of the upper half-plane.

We also bring in the theory of normal families of holomorphic maps. We use this to prove the Riemann mapping theorem, which states that if $\Omega \subset \mathbb{C}$ is simply connected and $\Omega \neq \mathbb{C}$, then there is a holomorphic diffeomorphism $\Phi : \Omega \to D$, the unit disk. Application of this theorem to a special domain, together with a reflection argument, shows that there is a holomorphic covering of $\mathbb{C} \setminus \{0, 1\}$ by the unit disk. This leads to key results of Picard and Montel, applications to the behavior of iterations of holomorphic maps $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, and the study of the associated Fatou and Julia sets, on which these iterates behave tamely and wildly, respectively.

The treatment of Riemann surfaces includes some differential geometric material. In an appendix to Chapter 5, we introduce the concept of a metric tensor and show how it is associated to a surface in Euclidean space and how the metric tensor behaves under smooth mappings, in particular how this behavior characterizes conformal mappings. We discuss the notion of metric tensors beyond the setting of metrics induced on surfaces in Euclidean space. In particular, we introduce a special metric on the unit disk, called the Poincaré metric, which has the property of being invariant under all conformal automorphisms of the disk. We show how the geometry of the Poincaré metric leads to another proof of Picard's theorem and also provides a different perspective on the proof of the Riemann mapping theorem.

The text next examines elliptic functions in Chapter 6. These are doubly periodic functions on $\mathbb{C}$, holomorphic except at poles (that is, meromorphic). Such a function can be regarded as a meromorphic function on the torus $\mathbb{T}_\Lambda = \mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice. A prime example is the Weierstrass function $\wp_\Lambda(z)$, defined by a double series. Analysis shows that $\wp'_\Lambda(z)^2$ is a cubic polynomial in $\wp_\Lambda(z)$, so the Weierstrass function inverts an elliptic integral. Elliptic integrals arise in many situations in geometry and mechanics, including arclengths of ellipses and pendulum problems, to mention two basic cases. The analysis of general elliptic integrals leads to the problem of finding the lattice whose associated elliptic functions are related to these integrals. This is the Abel inversion problem. Section 6.5 of the text tackles this problem by constructing the Riemann surface associated to $\sqrt{p(z)}$, where $p(z)$ is a cubic or quartic polynomial.

Early in this text, the exponential function was defined by a differential equation and given a power series solution, and these two characterizations were used to develop its properties. Coming full circle, we devote Chapter 7 to other classes of differential equations and their solutions. We first study a special class of functions known as Bessel functions, characterized as solutions to Bessel equations. Part of the central importance of these functions arises from their role in producing solutions to partial differential equations in several variables, as explained in an appendix. The Bessel functions for real values of their arguments arise as solutions to wave equations, and for imaginary values of their arguments they arise as solutions to diffusion equations. Thus it is very useful that they can be understood as holomorphic functions of a complex variable. Next, Chapter 7 deals with more general differential equations on a complex domain. Results include constructing solutions as convergent power series and the analytic continuation of such solutions to larger domains. General results here are used to put the Bessel equations in a larger context. This includes a study of equations with "regular singular points." Other classes of equations with regular singular points are presented, particularly hypergeometric equations.

The text ends with a short collection of appendices. Some of these survey background material that the reader might have seen in an advanced calculus course, including material on convergence and compactness, and differential calculus of several variables. Others develop tools that prove useful in the text, such as the Laplace asymptotic method, the Stieltjes integral, and results on Abelian and Tauberian theorems. The last appendix shows how to solve cubic and quartic equations via radicals and introduces a special function, called the Bring radical, to treat quintic equations. (In Chapter 7 the Bring radical is shown to be given in terms of a generalized hypergeometric function.)

As indicated in the discussion above, while the first goal of this text is to present the beautiful theory of functions of a complex variable, we have the further objective of placing this study within a broader mathematical framework. Examples of how this text differs from many others in the area include the following.

1) A greater emphasis on Fourier analysis, both as an application of basic results in complex analysis and as a tool of more general applicability in analysis. We see the use of Fourier series in the study of harmonic functions. We see the influence of the Fourier transform on the study of the Laplace transform, and then the Laplace transform as a tool in the study of differential equations.

2) The use of geometrical techniques in complex analysis. This clarifies the study of conformal maps, extends the usual study to more general surfaces, and shows how geometrical concepts are effective in classical problems from the Riemann mapping theorem to Picard's theorem. An appendix discusses applications of the Poincaré metric on the disk.

3) Connections with differential equations. The use of techniques of complex analysis to study differential equations is a strong point of this text. This important area is frequently neglected in complex analysis texts, and the treatments one sees in many differential equations texts are often confined to solutions for real variables and may furthermore lack a complete analysis of crucial convergence issues. Material here also provides a more detailed study than one usually sees of significant examples, such as Bessel functions.

4) Special functions. In addition to material on the Gamma function and the Riemann zeta function, the text has a detailed study of elliptic functions

and Bessel functions and also material on Airy functions, Legendre functions, and hypergeometric functions.

We follow this introduction with a record of some standard notation that will be used throughout this text.

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