for an $n \times n$ matrix A. This is of use in the analysis in §3.3 of the Laplace transform approach to solutions of systems of differential equations, and the equivalence of this result to an identity known as Duhamel's formula.

This chapter is capped by Appendix 3.6, which derives two important approximation results. One, due to Weierstrass, says any continuous function on an interval $[a, b] \subset \mathbb{R}$ is a uniform limit of polynomials. The other, due to Runge, says that if $K \subset \mathbb{C}$ is compact and if f is holomorphic on a neighborhood of K, then, on K, f is a limit of rational functions. These results could have been pushed to Chapter 2, but the arguments involved seem natural in light of material developed in this chapter. This appendix also contains an important extension of the Weierstrass approximation result, known as the Stone-Weierstrass theorem. This will play a useful role in the proof of Karamata's Tauberian theorem, in Appendix A.5.

3.1. Fourier series and the Poisson integral

Given an integrable function $f : S^1 \to \mathbb{C}$, we desire to write

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\, e^{ik\theta}, \tag{3.1.1}$$

for some coefficients $\hat{f}(k) \in \mathbb{C}$. We identify the unit circle S^1 with $\mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. If (3.1.1) is absolutely convergent, we can multiply both sides by $e^{-in\theta}$ and integrate. A change in order of summation and integration is then justified, and using

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\ell\theta}\, d\theta &= 0 \quad \text{if } \ \ell \neq 0, \\ &\ 1 \quad \text{if } \ \ell = 0, \end{aligned} \tag{3.1.2}$$

we see that

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-ik\theta}\, d\theta. \tag{3.1.3}$$

The series on the right side of (3.1.1) is called the Fourier series of f.

If $\hat{f}$ is given by (3.1.3) and if (3.1.1) holds, it is called the Fourier inversion formula. To examine whether (3.1.1) holds, we first sneak up on the sum on the right side. For $0 < r < 1$, set

$$J_r f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k) r^{|k|} e^{ik\theta}. \tag{3.1.4}$$

Note that

$$|\hat{f}(k)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|\, d\theta, \tag{3.1.5}$$

so whenever the right side of (3.1.5) is finite we see that the series (3.1.4) is absolutely convergent for each $r \in [0,1)$. Furthermore, we can substitute (3.1.3) for $\hat{f}$ in (3.1.4) and change the order of summation and integration to obtain

$$J_r f(\theta) = \int_{S^1} f(\theta') p_r(\theta - \theta')\, d\theta', \tag{3.1.6}$$

where

$$\begin{aligned} p_r(\theta) = p(r,\theta) &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta} \\ &= \frac{1}{2\pi} \Big[1 + \sum_{k=1}^{\infty} (r^k e^{ik\theta} + r^k e^{-ik\theta}) \Big], \end{aligned} \tag{3.1.7}$$

and summing these geometrical series yields

$$p(r,\theta) = \frac{1}{2\pi} \frac{1-r^2}{1 - 2r\cos\theta + r^2}. \tag{3.1.8}$$

Let us examine $p(r,\theta)$. It is clear that the numerator and denominator on the right side of (3.1.8) are positive, so $p(r,\theta) > 0$ for each $r \in [0,1)$, $\theta \in S^1$. As $r \nearrow 1$, the numerator tends to 0; as $r \nearrow 1$, the denominator tends to a nonzero limit, except at $\theta = 0$. Since we have

$$\int_{S^1} p(r,\theta)\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum r^{|k|} e^{ik\theta}\, d\theta = 1, \tag{3.1.9}$$

we see that, for r close to 1, $p(r,\theta)$ as a function of θ is highly peaked near $\theta = 0$ and small elsewhere, as in Figure 3.1.1. Given these facts, the following is an exercise in real analysis.

Proposition 3.1.1. *If $f \in C(S^1)$, then*

$$J_r f \to f \quad \text{uniformly on} \quad S^1 \quad \text{as} \quad r \nearrow 1. \tag{3.1.10}$$

Proof. We can rewrite (3.1.6) as

$$J_r f(\theta) = \int_{-\pi}^{\pi} f(\theta - \theta') p_r(\theta')\, d\theta'. \tag{3.1.11}$$

The results (3.1.8)–(3.1.9) imply that for each $\delta \in (0,\pi)$,

$$\int_{|\theta'| \le \delta} p_r(\theta')\, d\theta' = 1 - \varepsilon(r,\delta), \tag{3.1.12}$$

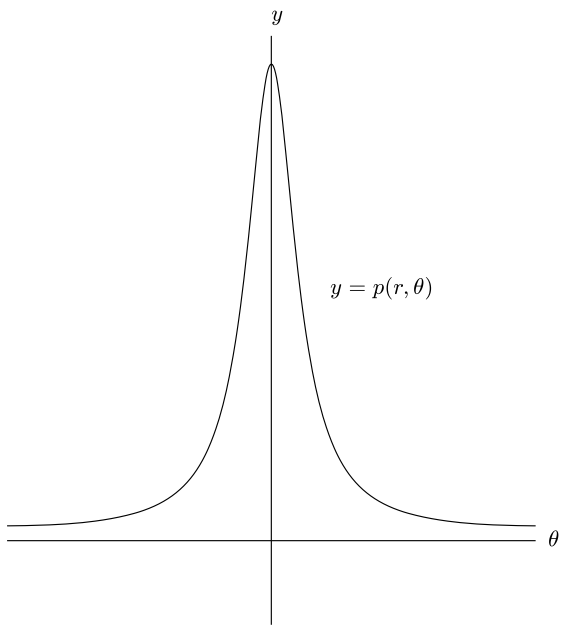

Figure 3.1.1. Poisson kernel

with $\varepsilon(r,\delta) \to 0$ as $r \nearrow 1$. Now, we break (3.1.11) into three pieces:

$$
\begin{aligned}
J_r f(\theta) &= f(\theta) \int_{-\delta}^{\delta} p_r(\theta')\, d\theta' \\
&\quad + \int_{-\delta}^{\delta} [f(\theta-\theta') - f(\theta)] p_r(\theta')\, d\theta' \\
&\quad + \int\limits_{\delta \le |\theta'| \le \pi} f(\theta-\theta') p_r(\theta')\, d\theta' \\
&= I + II + III.
\end{aligned} \tag{3.1.13}
$$

We have

$$
\begin{aligned}
I &= f(\theta)(1-\varepsilon(r,\delta)), \\
|II| &\le \sup_{|\theta'|\le\delta} |f(\theta-\theta') - f(\theta)|, \\
|III| &\le \varepsilon(r,\delta) \sup |f|.
\end{aligned} \tag{3.1.14}
$$

These estimates yield (3.1.10). □

From (3.1.10) the following is an elementary consequence.

Proposition 3.1.2. *Assume $f \in C(S^1)$. If the Fourier coefficients $\hat{f}(k)$ form a summable series, i.e., if*

$$\sum_{k=-\infty}^{\infty} |\hat{f}(k)| < \infty, \tag{3.1.15}$$

then the identity (3.1.1) *holds for each $\theta \in S^1$.*

Proof. What is to be shown is that if $\sum_k |a_k| < \infty$, then

$$\sum_k a_k = S \Longrightarrow \lim_{r \nearrow 1} \sum_k r^{|k|} a_k = S. \tag{3.1.16}$$

To get this, let $\varepsilon > 0$ and pick N such that

$$\sum_{|k|>N} |a_k| < \varepsilon. \tag{3.1.17}$$

Then

$$S_N = \sum_{k=-N}^{N} a_k \Longrightarrow |S - S_N| < \varepsilon \tag{3.1.18}$$

and

$$\Big| \sum_{|k|>N} r^{|k|} a_k \Big| < \varepsilon, \quad \forall\, r \in (0,1). \tag{3.1.19}$$

Since clearly

$$\lim_{r \nearrow 1} \sum_{k=-N}^{N} r^{|k|} a_k = \sum_{k=-N}^{N} a_k, \tag{3.1.20}$$

the conclusion in (3.1.16) follows. □

Remark. A stronger result, due to Abel, is that the implication (3.1.16) holds without the requirement of absolute convergence. This is treated in Appendix A.5.

Note that if (3.1.15) holds, then the right side of (3.1.1) is absolutely and uniformly convergent, and its sum is continuous on S^1.

We seek conditions on f that imply (3.1.15). Integration by parts for $f \in C^1(S^1)$ gives, for $k \neq 0$,

$$\begin{aligned} \hat{f}(k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, \frac{i}{k}\, \frac{\partial}{\partial \theta}(e^{-ik\theta})\, d\theta \\ &= \frac{1}{2\pi i k} \int_{-\pi}^{\pi} f'(\theta) e^{-ik\theta}\, d\theta, \end{aligned} \tag{3.1.21}$$

and hence

$$|\hat{f}(k)| \le \frac{1}{2\pi|k|}\int_{-\pi}^{\pi} |f'(\theta)|\,d\theta. \tag{3.1.22}$$

If $f \in C^2(S^1)$, we can integrate by parts a second time and get

$$\hat{f}(k) = -\frac{1}{2\pi k^2}\int_{-\pi}^{\pi} f''(\theta)e^{-ik\theta}\,d\theta, \tag{3.1.23}$$

and hence

$$|\hat{f}(k)| \le \frac{1}{2\pi k^2}\int_{-\pi}^{\pi} |f''(\theta)|\,d\theta. \tag{3.1.24}$$

In concert with (3.1.5), we have

$$|\hat{f}(k)| \le \frac{1}{2\pi(k^2+1)}\int_{S^1} [|f''(\theta)| + |f(\theta)|]\,d\theta. \tag{3.1.25}$$

Hence

$$f \in C^2(S^1) \Longrightarrow \sum |\hat{f}(k)| < \infty. \tag{3.1.26}$$

We will produce successive sharpenings of (3.1.26) below. We start with an interesting example. Consider

$$f(\theta) = |\theta|, \quad -\pi \le \theta \le \pi, \tag{3.1.27}$$

continued to be periodic of period 2π. This defines a Lipschitz function on S^1 whose Fourier coefficients are

$$\begin{aligned} \hat{f}(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta|\,e^{-ik\theta}\,d\theta \\ &= -\big[1-(-1)^k\big]\frac{1}{\pi k^2}, \end{aligned} \tag{3.1.28}$$

for $k \ne 0$, while $\hat{f}(0) = \pi/2$. It is clear that this forms a summable series, so Proposition 3.1.2 implies that, for $-\pi \le \theta \le \pi$,

$$\begin{aligned} |\theta| &= \frac{\pi}{2} - \sum_{k\,\mathrm{odd}} \frac{2}{\pi k^2}e^{ik\theta} \\ &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^2}\,\cos(2\ell+1)\theta. \end{aligned} \tag{3.1.29}$$

We note that evaluating this at $\theta = 0$ yields the identity

$$\sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^2} = \frac{\pi^2}{8}. \tag{3.1.30}$$

Writing

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \tag{3.1.31}$$

as a sum of two series, one for $k \ge 1$ odd and one for $k \ge 2$ even, yields an evaluation of (3.1.31). (See Exercise 1 below.)

We see from (3.1.29) that if f is given by (3.1.27), then $\hat{f}(k)$ satisfies

$$|\hat{f}(k)| \le \frac{C}{k^2+1}. \tag{3.1.32}$$

This is a special case of the following generalization of (3.1.26).

Proposition 3.1.3. *Let f be continuous and piecewise C^2 on S^1. Then* (3.1.32) *holds.*

Proof. Here we are assuming f is C^2 on $S^1 \setminus \{p_1, \dots, p_\ell\}$, and f' and f'' have limits at each of the endpoints of the associated intervals in S^1, but f is not assumed to be differentiable at the endpoints p_ℓ. We can write f as a sum of functions f_ν, each of which is Lipschitz on S^1, C^2 on $S^1 \setminus p_\nu$, and f'_ν and f''_ν have limits as one approaches p_ν from either side. It suffices to show that each $\hat{f}_\nu(k)$ satisfies (3.1.32). Now $g(\theta) = f_\nu(\theta + p_\nu - \pi)$ is singular only at $\theta = \pi$, and $\hat{g}(k) = \hat{f}_\nu(k)e^{ik(p_\nu - \pi)}$, so it suffices to prove Proposition 3.1.3 when f has a singularity only at $\theta = \pi$. In other words, $f \in C^2([-\pi, \pi])$ and $f(-\pi) = f(\pi)$.

In this case, we still have (3.1.21), since the endpoint contributions from integration by parts still cancel. A second integration by parts gives, in place of (3.1.23),

$$\begin{aligned} \hat{f}(k) &= \frac{1}{2\pi i k} \int_{-\pi}^{\pi} f'(\theta)\, \frac{i}{k}\, \frac{\partial}{\partial\theta}(e^{-ik\theta})\, d\theta \\ &= -\frac{1}{2\pi k^2} \Big[\int_{-\pi}^{\pi} f''(\theta) e^{-ik\theta}\, d\theta + f'(\pi) - f'(-\pi) \Big], \end{aligned} \tag{3.1.33}$$

which yields (13.26). □

Given $f \in C(S^1)$, let us say

$$f \in \mathcal{A}(S^1) \Longleftrightarrow \sum |\hat{f}(k)| < \infty. \tag{3.1.34}$$

Proposition 3.1.2 applies to elements of $\mathcal{A}(S^1)$, and Proposition 3.1.3 gives a sufficient condition for a function to belong to $\mathcal{A}(S^1)$. A more general sufficient condition will be given in Proposition 3.1.6.

We next make use of (3.1.2) to produce results on $\int_{S^1} |f(\theta)|^2\, d\theta$, starting with the following.

Proposition 3.1.4. *Given $f \in \mathcal{A}(S^1)$,*

$$\sum |\hat{f}(k)|^2 = \frac{1}{2\pi} \int_{S^1} |f(\theta)|^2 \, d\theta. \tag{3.1.35}$$

More generally, if also $g \in \mathcal{A}(S^1)$, then

$$\sum \hat{f}(k)\overline{\hat{g}(k)} = \frac{1}{2\pi} \int_{S^1} f(\theta)\overline{g(\theta)} \, d\theta. \tag{3.1.36}$$

Proof. Switching the order of summation and integration and using (3.1.2), we have

$$\begin{aligned} \frac{1}{2\pi} \int_{S^1} f(\theta)\overline{g(\theta)} \, d\theta &= \frac{1}{2\pi} \int_{S^1} \sum_{j,k} \hat{f}(j)\overline{\hat{g}(k)} e^{-i(j-k)\theta} \, d\theta \\ &= \sum_k \hat{f}(k)\overline{\hat{g}(k)}, \end{aligned} \tag{3.1.37}$$

giving (3.1.36). Taking $g = f$ gives (3.1.35). ☐

We will extend the scope of Proposition 3.1.4 below. Closely tied to this is the issue of convergence of $S_N f$ to f as $N \to \infty$, where

$$S_N f(\theta) = \sum_{|k| \le N} \hat{f}(k) e^{ik\theta}. \tag{3.1.38}$$

Clearly $f \in \mathcal{A}(S^1) \Rightarrow S_N f \to f$ uniformly on S^1 as $N \to \infty$. Here, we are interested in convergence in the L^2-norm, where

$$\|f\|_{L^2}^2 = \frac{1}{2\pi} \int_{S^1} |f(\theta)|^2 \, d\theta. \tag{3.1.39}$$

Given f and $|f|^2$ integrable on S^1 (we say f is square integrable), this defines a "norm," satisfying the following result, called the triangle inequality:

$$\|f + g\|_{L^2} \le \|f\|_{L^2} + \|g\|_{L^2}. \tag{3.1.40}$$

See Appendix 3.4 for details on this. Behind these results is the fact that

$$\|f\|_{L^2}^2 = (f, f)_{L^2}, \tag{3.1.41}$$

where, when f and g are square integrable, we define the inner product

$$(f, g)_{L^2} = \frac{1}{2\pi} \int_{S^1} f(\theta)\overline{g(\theta)} \, d\theta. \tag{3.1.42}$$

Thus the content of (3.1.35) is that

$$\sum |\hat{f}(k)|^2 = \|f\|_{L^2}^2, \tag{3.1.43}$$

and that of (3.1.36) is that

$$\sum \hat{f}(k)\overline{\hat{g}(k)} = (f, g)_{L^2}. \tag{3.1.44}$$

The left side of (3.1.43) is the square norm of the sequence $(\hat{f}(k))$ in ℓ^2. Generally, a sequence (a_k) $(k \in \mathbb{Z})$ belongs to ℓ^2 if and only if

$$\|(a_k)\|_{\ell^2}^2 = \sum |a_k|^2 < \infty. \tag{3.1.45}$$

There is an associated inner product

$$((a_k), (b_k)) = \sum a_k \overline{b_k}. \tag{3.1.46}$$

As in (3.1.40), one has (see Appendix 3.4)

$$\|(a_k) + (b_k)\|_{\ell^2} \le \|(a_k)\|_{\ell^2} + \|(b_k)\|_{\ell^2}. \tag{3.1.47}$$

As for the notion of L^2-norm convergence, we say

$$f_\nu \to f \ \text{ in } \ L^2 \Longleftrightarrow \|f - f_\nu\|_{L^2} \to 0. \tag{3.1.48}$$

There is a similar notion of convergence in ℓ^2. Clearly

$$\|f - f_\nu\|_{L^2} \le \sup_\theta |f(\theta) - f_\nu(\theta)|. \tag{3.1.49}$$

In view of the uniform convergence $S_N f \to f$ for $f \in \mathcal{A}(S^1)$ noted above, we have

$$f \in \mathcal{A}(S^1) \Longrightarrow S_N f \to f \text{ in } L^2, \text{ as } N \to \infty. \tag{3.1.50}$$

The triangle inequality implies

$$\Big| \|f\|_{L^2} - \|S_N f\|_{L^2} \Big| \le \|f - S_N f\|_{L^2}, \tag{3.1.51}$$

and clearly (by Proposition 3.1.4)

$$\|S_N f\|_{L^2}^2 = \sum_{k=-N}^{N} |\hat{f}(k)|^2, \tag{3.1.52}$$

so

$$\|f - S_N f\|_{L^2} \to 0 \ \text{ as } \ N \to \infty \Longrightarrow \|f\|_{L^2}^2 = \sum |\hat{f}(k)|^2. \tag{3.1.53}$$

We now consider more general square integrable functions f on S^1. With $\hat{f}(k)$ and $S_N f$ defined by (3.1.3) and (3.1.38), we define $R_N f$ by

$$f = S_N f + R_N f. \tag{3.1.54}$$

Note that $\int_{S^1} f(\theta) e^{-ik\theta}\, d\theta = \int_{S^1} S_N f(\theta) e^{-ik\theta}\, d\theta$ for $|k| \le N$. Hence

$$(f, S_N f)_{L^2} = (S_N f, S_N f)_{L^2}, \tag{3.1.55}$$

and hence

$$(S_N f, R_N f)_{L^2} = 0. \tag{3.1.56}$$

Consequently,

$$\|f\|_{L^2}^2 = (S_N f + R_N f, S_N f + R_N f)_{L^2} \\ = \|S_N f\|_{L^2}^2 + \|R_N f\|_{L^2}^2. \tag{3.1.57}$$

In particular,

$$\|S_N f\|_{L^2} \leq \|f\|_{L^2}. \tag{3.1.58}$$

We are now in a position to prove the following.

Lemma 3.1.5. *Let f, f_ν be square integrable on S^1. Assume*

$$\lim_{\nu\to\infty} \|f - f_\nu\|_{L^2} = 0, \tag{3.1.59}$$

and, for each ν,

$$\lim_{N\to\infty} \|f_\nu - S_N f_\nu\|_{L^2} = 0. \tag{3.1.60}$$

Then

$$\lim_{N\to\infty} \|f - S_N f\|_{L^2} = 0. \tag{3.1.61}$$

In such a case, (3.1.53) *holds.*

Proof. Writing $f - S_N f = (f - f_\nu) + (f_\nu - S_N f_\nu) + S_N(f_\nu - f)$, and using the triangle inequality, we have, for each ν,

$$\|f - S_N f\|_{L^2} \leq \|f - f_\nu\|_{L^2} + \|f_\nu - S_N f_\nu\|_{L^2} + \|S_N(f_\nu - f)\|_{L^2}. \tag{3.1.62}$$

Taking $N \to \infty$ and using (3.1.58), we have

$$\limsup_{N\to\infty} \|f - S_N f\|_{L^2} \leq 2\|f - f_\nu\|_{L^2}, \tag{3.1.63}$$

for each ν. Then (3.1.59) yields the desired conclusion (3.1.61). ☐

Given $f \in C(S^1)$, we have seen that $J_r f \to f$ uniformly (hence in the L^2-norm) as $r \nearrow 1$. Also, $J_r f \in \mathcal{A}(S^1)$ for each $r \in (0, 1)$. Thus (3.1.50) and Lemma 3.1.5 yield the following:

$$f \in C(S^1) \Longrightarrow S_N f \to f \quad \text{in} \quad L^2 \text{ and} \\ \sum |\hat{f}(k)|^2 = \|f\|_{L^2}^2. \tag{3.1.64}$$

Lemma 3.1.5 also applies to many discontinuous functions. Consider, for example

$$f(\theta) = 0 \quad \text{for} \quad -\pi < \theta < 0, \\ 1 \quad \text{for} \quad 0 < \theta < \pi. \tag{3.1.65}$$

We can set, for $\nu \in \mathbb{N}$,

$$
\begin{aligned}
f_\nu(\theta) &= 0 \quad \text{for} \quad -\pi \le \theta \le 0, \\
&\nu\theta \quad \text{for} \quad 0 \le \theta \le \frac{1}{\nu}, \\
&1 \quad \text{for} \quad \frac{1}{\nu} \le \theta \le \pi - \frac{1}{\nu} \\
&\nu(\pi - \theta) \quad \text{for} \quad \pi - \frac{1}{\nu} \le \theta \le \pi.
\end{aligned}
\tag{3.1.66}
$$

Then each $f_\nu \in C(S^1)$. (In fact, $f_\nu \in \mathcal{A}(S^1)$, by Proposition 3.1.3.). Also, one can check that $\|f - f_\nu\|_{L^2}^2 \le 2/\nu$. Thus the conclusion in (3.1.64) holds for f given by (3.1.65).

More generally, any piecewise continuous function on S^1 is an L^2 limit of continuous functions, so the conclusion of (3.1.64) holds for them. To go further, let us recall the class of Riemann integrable functions. (Details can be found in Chapter 4, §2 of [**44**] or in §0 of [**45**].) A function $f : S^1 \to \mathbb{R}$ is Riemann integrable provided f is bounded (say $|f| \le M$) and, for each $\delta > 0$, there exist piecewise constant functions g_δ and h_δ on S^1 such that

$$
g_\delta \le f \le h_\delta \quad \text{and} \quad \int_{S^1} \big(h_\delta(\theta) - g_\delta(\theta)\big)\, d\theta < \delta.
\tag{3.1.67}
$$

Then

$$
\int_{S^1} f(\theta)\, d\theta = \lim_{\delta \to 0} \int_{S^1} g_\delta(\theta)\, d\theta = \lim_{\delta \to 0} \int_{S^1} h_\delta(\theta)\, d\theta.
\tag{3.1.68}
$$

Note that we can assume $|h_\delta|, |g_\delta| < M + 1$, and so

$$
\begin{aligned}
\frac{1}{2\pi} \int_{S^1} |f(\theta) - g_\delta(\theta)|^2\, d\theta &\le \frac{M+1}{\pi} \int_{S^1} |h_\delta(\theta) - g_\delta(\theta)|\, d\theta \\
&< \frac{M+1}{\pi} \delta,
\end{aligned}
\tag{3.1.69}
$$

so $g_\delta \to f$ in the L^2-norm. A function $f : S^1 \to \mathbb{C}$ is Riemann integrable provided its real and imaginary parts are as well. In such a case, there are also piecewise constant functions $f_\nu \to f$ in the L^2-norm, so

$$
\begin{aligned}
f \text{ Riemann integrable on } S^1 \Longrightarrow S_N f \to f \ \text{ in } \ L^2,& \text{ and} \\
\sum |\hat{f}(k)|^2 = \|f\|_{L^2}^2&.
\end{aligned}
\tag{3.1.70}
$$

This is not the end of the story. There are unbounded functions on S^1 that are square integrable, such as

$$
f(\theta) = |\theta|^{-\alpha} \quad \text{on} \quad [-\pi, \pi], \quad 0 < \alpha < \frac{1}{2}.
\tag{3.1.71}
$$

In such a case, one can take $f_\nu(\theta) = \min(f(\theta), \nu),\ \nu \in \mathbb{N}$. Then each f_ν is continuous and $\|f - f_\nu\|_{L^2} \to 0$ as $\nu \to \infty$. Hence the conclusion of (13.64) holds for such f.

The ultimate theory of functions for which the result

$$S_N f \longrightarrow f \quad \text{in the} \quad L^2\text{-norm} \tag{3.1.72}$$

holds was produced by H. Lebesgue in what is now known as the theory of Lebesgue measure and integration. There is the notion of measurability of a function $f : S^1 \to \mathbb{C}$. One says $f \in L^2(S^1)$ provided f is measurable and $\int_{S^1} |f(\theta)|^2 \, d\theta < \infty$, the integral here being the Lebesgue integral. Actually, $L^2(S^1)$ consists of equivalence classes of such functions, where $f_1 \sim f_2$ if and only if $\int |f_1(\theta) - f_2(\theta)|^2 \, d\theta = 0$. With ℓ^2 as in (3.1.45), it is then the case that

$$\mathcal{F} : L^2(S^1) \longrightarrow \ell^2, \tag{3.1.73}$$

given by

$$(\mathcal{F}f)(k) = \hat{f}(k), \tag{3.1.74}$$

is one-to-one and onto, with

$$\sum |\hat{f}(k)|^2 = \|f\|_{L^2}^2, \quad \forall\, f \in L^2(S^1), \tag{3.1.75}$$

and

$$S_N f \longrightarrow f \quad \text{in} \quad L^2, \quad \forall\, f \in L^2(S^1). \tag{3.1.76}$$

For the reader who has not seen Lebesgue integration, we refer to books on the subject (e.g., [**47**]) for more information.

We mention two key propositions which, together with the arguments given above, establish these results. The fact that $\mathcal{F}f \in \ell^2$ for all $f \in L^2(S^1)$ and (3.1.75)–(3.1.76) hold follows via Lemma 3.1.5 from the following.

Proposition A. Given $f \in L^2(S^1)$, there exist $f_\nu \in C(S^1)$ such that $f_\nu \to f$ in L^2.

As for the surjectivity of $\mathcal{F}$ in (3.1.73), note that, given $(a_k) \in \ell^2$, the sequence

$$f_\nu(\theta) = \sum_{|k| \le \nu} a_k e^{ik\theta} \tag{3.1.77}$$

satisfies, for $\mu > \nu$,

$$\|f_\mu - f_\nu\|_{L^2}^2 = \sum_{\nu < |k| \le \mu} |a_k|^2 \to 0 \quad \text{as} \quad \nu \to \infty. \tag{3.1.78}$$

That is to say, (f_ν) is a Cauchy sequence in $L^2(S^1)$. Surjectivity follows from the fact that Cauchy sequences in $L^2(S^1)$ always converge to a limit:

Proposition B. If (f_ν) is a Cauchy sequence in $L^2(S^1)$, there exists $f \in L^2(S^1)$ such that $f_\nu \to f$ in the L^2-norm.

Proofs of these results can be found in the standard texts on measure theory and integration, such as [**47**].

We now establish a sufficient condition for a function f to belong to $\mathcal{A}(S^1)$, more general than that in Proposition 3.1.3.

Proposition 3.1.6. *If f is a continuous, piecewise C^1 function on S^1, then $\sum |\hat{f}(k)| < \infty$.*

Proof. As in the proof of Proposition 3.1.3, we can reduce the problem to the case $f \in C^1([-\pi,\pi])$, $f(-\pi) = f(\pi)$. In such a case, with $g = f' \in C([-\pi,\pi])$, the integration by parts argument (3.1.21) gives

$$\hat{f}(k) = \frac{1}{ik}\hat{g}(k), \quad k \neq 0. \tag{3.1.79}$$

By (3.1.70),

$$\sum |\hat{g}(k)|^2 = \|g\|_{L^2}^2. \tag{3.1.80}$$

Also, by Cauchy's inequality (cf. Appendix 3.4),

$$\begin{aligned}\sum_{k\neq 0} |\hat{f}(k)| &\le \Bigl(\sum_{k\neq 0} \frac{1}{k^2}\Bigr)^{1/2} \Bigl(\sum_{k\neq 0} |\hat{g}(k)|^2\Bigr)^{1/2} \\ &\le C\|g\|_{L^2}.\end{aligned} \tag{3.1.81}$$

This completes the proof. □

There is a great deal more that can be said about convergence of the Fourier series. For example, the material presented in the appendix to the next section has an analogue for the Fourier series. We also mention Chapter 5, §4 in [**44**]. For further results, one can consult treatments of Fourier analysis such as Chapter 3 of [**46**].

The Fourier series connects with the theory of harmonic functions, as follows. Taking $z = re^{i\theta}$ in the unit disk, we can write (3.1.4) as

$$J_r f(\theta) = \sum_{k=0}^{\infty} \hat{f}(k) z^k + \sum_{k=1}^{\infty} \hat{f}(-k)\overline{z}^k. \tag{3.1.82}$$

We write this as

$$(\mathrm{PI}\, f)(z) = (\mathrm{PI}_+\, f)(z) + (\mathrm{PI}_-\, f)(z), \tag{3.1.83}$$

a sum of a holomorphic function and a conjugate-holomorphic function on the unit disk D. Thus the left side is a *harmonic* function, called the Poisson integral (PI) of f.

Given $f \in C(S^1)$, PI f is the unique function in $C^2(D) \cap C(\overline{D})$ equal to f on $\partial D = S^1$ (uniqueness being a consequence of Proposition 2.3.4). Using (3.1.6)–(3.1.8), we can write the following Poisson integral formula:

$$\text{PI } f(z) = \frac{1 - |z|^2}{2\pi} \int_{S^1} \frac{f(w)}{|w - z|^2}\, ds(w), \tag{3.1.84}$$

the integral being with respect to the arclength on S^1. To see this, note that if $w = e^{i\theta'}$ and $z = re^{i\theta}$, then $ds(w) = d\theta'$ and

$$\begin{aligned} |w - z|^2 &= (e^{i\theta'} - re^{i\theta})(e^{-i\theta'} - re^{-i\theta}) \\ &= 1 - r(e^{i(\theta-\theta')} + e^{-i(\theta-\theta')}) + r^2. \end{aligned} \tag{3.1.85}$$

Since solutions to $\Delta u = 0$ remain solutions upon translation and dilation of coordinates, we have the following result.

Proposition 3.1.7. *If $D \subset \mathbb{C}$ is an open disk and $f \in C(\partial D)$ is given, there exists a unique $u \in C(\overline{D}) \cap C^2(D)$ satisfying*

$$\Delta u = 0 \quad \text{on} \quad D, \quad u\big|_{\partial D} = f. \tag{3.1.86}$$

We call (3.1.86) the Dirichlet boundary problem.

Now we make use of Proposition 3.1.7 to improve the version of the Schwarz reflection principle given in Proposition 2.4.2. As in the discussion of the Schwarz reflection principle in §2.4, we assume $\Omega \subset \mathbb{C}$ is a connected, open set that is symmetric with respect to the real axis, so $z \in \Omega \Rightarrow \overline{z} \in \Omega$. We set $\Omega^{\pm} = \{z \in \Omega : \pm \text{Im } z > 0\}$ and $L = \Omega \cap \mathbb{R}$.

Proposition 3.1.8. *Assume $u : \Omega^+ \cup L \to \mathbb{C}$ is continuous, harmonic on Ω^+, and $u = 0$ on L. Define $v : \Omega \to \mathbb{C}$ by*

$$\begin{aligned} v(z) = \ & u(z), \quad z \in \Omega^+ \cup L, \\ & -u(\overline{z}), \quad z \in \Omega^-. \end{aligned} \tag{3.1.87}$$

Then v is harmonic on Ω.

Proof. It is readily verified that v is harmonic in $\Omega^+ \cup \Omega^-$ and continuous on Ω. We need to show that v is harmonic on a neighborhood of each point $p \in L$. Let $D = D_r(p)$ be a disk centered at p such that $\overline{D} \subset \Omega$. Let $f \in C(\partial D)$ be given by $f = v|_{\partial D}$. Let $w \in C^2(D) \cap C(\overline{D})$ be the unique harmonic function on D equal to f on ∂D.

Since f is odd with respect to reflection about the real axis, so is w, so $w = 0$ on $\overline{D} \cap \mathbb{R}$. Thus both v and w are harmonic on $D^+ = D \cap \{\text{Im } z > 0\}$, and continuous on $\overline{D}^+$, and agree on ∂D^+, so the maximum principle implies $w = v$ on $\overline{D}^+$. Similarly $w = v$ on $\overline{D}^-$, and this gives the desired harmonicity of v. ☐

Using Proposition 3.1.8, we establish the following stronger version of Proposition 2.4.2, the Schwarz reflection principle, weakening the hypothesis that f is continuous on $\Omega^+ \cup L$ to the hypothesis that $\operatorname{Im} f$ is continuous on $\Omega^+ \cup L$ (and vanishes on L). While this improvement may seem to be a small thing, it can be quite useful, as we will see in §5.5.

Proposition 3.1.9. *Let Ω be as in Proposition 3.1.8, and assume $f : \Omega^+ \to \mathbb{C}$ is holomorphic. Assume $\operatorname{Im} f$ extends continuously to $\Omega^+ \cup L$ and vanishes on L. Define $g : \Omega^+ \cup \Omega^-$ by*

$$g(z) = f(z), \quad z \in \Omega^+, \qquad \overline{f(\overline{z})}, \quad z \in \Omega^-. \tag{3.1.88}$$

Then g extends to a holomorphic function on Ω.

Proof. It suffices to prove this under the additional assumption that Ω is a disk. We apply Proposition 3.1.8 to $u(z) = \operatorname{Im} f(z)$ on Ω^+, 0 on L, obtaining a harmonic extension $v : \Omega \to \mathbb{R}$. By Proposition 2.3.1, v has a harmonic conjugate $w : \Omega \to \mathbb{R}$, so $v + iw$ is holomorphic, and hence $h : \Omega \to \mathbb{C}$, given by

$$h(z) = -w(z) + iv(z), \tag{3.1.89}$$

is holomorphic. Now $\operatorname{Im} h = \operatorname{Im} f$ on Ω^+, so $g - h$ is real valued on Ω^+, so, being holomorphic, it must be constant. Thus, altering w by a real constant, we have

$$h(z) = g(z), \quad z \in \Omega^+. \tag{3.1.90}$$

Also, $\operatorname{Im} h(z) = v(z) = 0$ on L, so (cf. Exercise 1 in §2.6)

$$h(z) = \overline{h(\overline{z})}, \quad \forall\, z \in \Omega. \tag{3.1.91}$$

It follows from this and (3.1.88) that

$$h(z) = g(z), \quad \forall\, z \in \Omega^+ \cup \Omega^-, \tag{3.1.92}$$

so h is the desired holomorphic extension. □

Exercises

1. Verify the evaluation of the integral in (3.1.28). Use the evaluation of (3.1.29) at $\theta = 0$ (as done in (3.1.30)) to show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{3.1.93}$$

2. Compute $\hat{f}(k)$ when

$$f(\theta) = 1 \quad \text{for } 0 < \theta < \pi, \qquad 0 \quad \text{for } -\pi < \theta < 0.$$

Then use (3.1.70) to obtain another proof of (3.1.93).

3. Apply (3.1.35) when $f(\theta)$ is given by (3.1.27). Use this to show that

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

4. Give the details for (3.1.84), as a consequence of (3.1.6) and (3.1.8).

5. Suppose f is holomorphic on an annulus Ω containing the unit circle S^1, with Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

Show that

$$a_n = \hat{g}(n), \quad g = f\big|_{S^1}.$$

Compare this with (2.8.10), with $z_0 = 0$ and γ the unit circle S^1.

Exercises 6–8 deal with the convolution of functions on S^1, defined by

$$f * g(\theta) = \frac{1}{2\pi} \int_{S^1} f(\varphi) g(\theta - \varphi)\, d\varphi.$$

6. Show that

$$h = f * g \Longrightarrow \hat{h}(k) = \hat{f}(k)\hat{g}(k).$$

7. Show that

$$f, g \in L^2(S^1), \quad h = f * g \Longrightarrow \sum_{k=-\infty}^{\infty} |\hat{h}(k)| < \infty.$$

8. Let χ be the characteristic function of $[-\pi/2, \pi/2]$, regarded as an element of $L^2(S^1)$. Compute $\hat{\chi}(k)$ and $\chi * \chi(\theta)$. Relate these computations to (3.1.27)–(3.1.29).

9. Show that a formula equivalent to (3.1.84) is

$$\text{PI}\, f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(\text{Re}\, \frac{e^{i\theta} + z}{e^{i\theta} - z}\Big) f(\theta)\, d\theta. \tag{3.1.94}$$

(We abuse notation by confounding $f(\theta)$ and $f(e^{i\theta})$, identifying S^1 and $\mathbb{R}/(2\pi\mathbb{Z})$.)

10. Give the details for the improvement of Proposition 2.4.2 mentioned right after the proof of Proposition 3.1.8. Proposition 2.3.6 may come in handy.

11. Given $f(\theta) = \sum_k a_k e^{ik\theta}$, show that f is real valued on S^1 if and only if

$$\overline{a}_k = a_{-k}, \quad \forall\, k \in \mathbb{Z}.$$

12. Let $f \in C(S^1)$ be real valued. Show that $\mathrm{PI}\, f$ and $(1/i)\,\mathrm{PI}\, g$ are harmonic conjugates, provided

$$\begin{aligned} \hat{g}(k) = \hat{f}(k) \qquad & \text{for } \ k > 0, \\ -\hat{f}(-k) \quad & \text{for } \ k < 0. \end{aligned}$$

13. Using Exercise 1 and (3.1.16), in concert with Exercise 19 of §1.5, show that

$$\begin{aligned} \frac{\pi^2}{6} &= \int_0^1 \frac{1}{x} \log \frac{1}{1-x}\, dx \\ &= \int_0^\infty \frac{t}{e^t - 1}\, dt. \end{aligned}$$

14. Let Ω be symmetric about $\mathbb{R}$, as in Proposition 3.1.8. Suppose f is holomorphic and nowhere vanishing on Ω^+ and $|f(z)| \to 1$ as $z \to L$. Show that f extends to be holomorphic on Ω, with $|f(z)| = 1$ for $z \in L$.

Hint. Consider the harmonic function $u(z) = \log |f(z)| = \mathrm{Re}\, \log f(z)$.

15. Establish the variant of Exercise 14 (and the strengthening of Exercise 3 from §2.4). Take $a > 1$. Suppose f is holomorphic and nowhere vanishing on the annulus $1 < |z| < a$ and that $|f(z)| \to 1$ as $|z| \to 1$. Show that f extends to a holomorphic function g on $1/a < |z| < a$, satisfying $g(z) = f(z)$ for $1 < |z| < a$ and

$$g(z) = \frac{1}{\overline{f(1/\overline{z})}}, \qquad \frac{1}{a} < |z| < 1.$$

3.2. Fourier transforms

Take a function f that is integrable on $\mathbb{R}$, so

$$\|f\|_{L^1} = \int_{-\infty}^{\infty} |f(x)|\, dx < \infty. \tag{3.2.1}$$

We define the Fourier transform of f to be

$$\hat{f}(\xi) = \mathcal{F} f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi}\, dx, \quad \xi \in \mathbb{R}. \tag{3.2.2}$$

Similarly, we set

$$\mathcal{F}^* f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ix\xi}\, dx, \quad \xi \in \mathbb{R}, \tag{3.2.3}$$

and ultimately plan to identify $\mathcal{F}^*$ as the inverse Fourier transform.

Clearly the Fourier transform of an integrable function is bounded:

$$|\hat{f}(\xi)| \le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\, dx. \tag{3.2.4}$$

We also have continuity.

Proposition 3.2.1. *If f is integrable on $\mathbb{R}$, then $\hat{f}$ is continuous on $\mathbb{R}$.*

Proof. Given $\varepsilon > 0$, pick $N < \infty$ such that $\int_{|x|>N} |f(x)|\,dx < \varepsilon$. Write $f = f_N + g_N$, where $f_N(x) = f(x)$ for $|x| \le N$, 0 for $|x| > N$. Then

$$\hat{f}(\xi) = \hat{f}_N(\xi) + \hat{g}_N(\xi) \tag{3.2.5}$$

and

$$|\hat{g}_N(\xi)| < \frac{\varepsilon}{\sqrt{2\pi}}, \quad \forall\,\xi. \tag{3.2.6}$$

Meanwhile, for $\xi, \zeta \in \mathbb{R}$,

$$\hat{f}_N(\xi) - \hat{f}_N(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} f(x)\Big(e^{-ix\xi} - e^{-ix\zeta}\Big)\,dx, \tag{3.2.7}$$

and

$$\begin{aligned} |e^{-ix\xi} - e^{-ix\zeta}| &\le |\xi - \zeta|\, \max_{\eta}\Big|\frac{\partial}{\partial\eta} e^{-ix\eta}\Big| \\ &\le |x| \cdot |\xi - \zeta| \\ &\le N|\xi - \zeta| \end{aligned} \tag{3.2.8}$$

for $|x| \le N$, so

$$|\hat{f}_N(\xi) - \hat{f}_N(\zeta)| \le \frac{N}{\sqrt{2\pi}} \|f\|_{L^1}\,|\xi - \zeta|, \tag{3.2.9}$$

where $\|f\|_{L^1}$ is defined by (3.2.1). Hence each $\hat{f}_N$ is continuous, and, by (3.2.6), $\hat{f}$ is a uniform limit of continuous functions, so it is continuous. ☐

The Fourier inversion formula asserts that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi}\,d\xi, \tag{3.2.10}$$

in appropriate senses, depending on the nature of f. We approach this in a spirit similar to the Fourier inversion formula (3.1.1) of the previous section. First we sneak up on (3.2.10) by inserting a factor of $e^{-\varepsilon\xi^2}$. Set

$$J_\varepsilon f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-\varepsilon\xi^2} e^{ix\xi}\,d\xi. \tag{3.2.11}$$

Note that, by (3.2.4), whenever $f \in L^1(\mathbb{R})$, $\hat{f}(\xi)e^{-\varepsilon\xi^2}$ is integrable for each $\varepsilon > 0$. Furthermore, we can plug in (3.2.2) for $\hat{f}(\xi)$ and switch the order of integration, getting

$$\begin{aligned} J_\varepsilon f(x) &= \frac{1}{2\pi} \iint f(y) e^{i(x-y)\xi} e^{-\varepsilon\xi^2}\,dy\,d\xi \\ &= \int_{-\infty}^{\infty} f(y) H_\varepsilon(x-y)\,dy, \end{aligned} \tag{3.2.12}$$

where

$$H_\varepsilon(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\varepsilon\xi^2 + ix\xi}\, d\xi. \tag{3.2.13}$$

A change of variable shows that $H_\varepsilon(x) = (1/\sqrt{\varepsilon})H_1(x/\sqrt{\varepsilon})$, and the computation of $H_1(x)$ is accomplished in §2.6; we see that $H_1(x) = (1/2\pi)G(ix)$, with $G(z)$ defined by (2.6.3) and computed in (2.6.8). We obtain

$$H_\varepsilon(x) = \frac{1}{\sqrt{4\pi\varepsilon}} e^{-x^2/4\varepsilon}. \tag{3.2.14}$$

The computation $\int e^{-x^2}\, dx = \sqrt{\pi}$ done in (2.6.6) implies

$$\int_{-\infty}^{\infty} H_\varepsilon(x)\, dx = 1, \quad \forall\, \varepsilon > 0. \tag{3.2.15}$$

We see that $H_\varepsilon(x)$ is highly peaked near $x = 0$ as $\varepsilon \searrow 0$. An argument parallel to that used to prove Proposition 3.1.1 then establishes the following.

Proposition 3.2.2. *Assume f is integrable on $\mathbb{R}$. Then*

$$J_\varepsilon f(x) \to f(x) \quad \text{whenever } f \text{ is continuous at } x. \tag{3.2.16}$$

If, in addition, f is continuous on $\mathbb{R}$ and $f(x) \to 0$ as $|x| \to \infty$, then $J_\varepsilon f(x) \to f(x)$ uniformly on $\mathbb{R}$.

From here, parallel to Proposition 3.1.2, we have:

Corollary 3.2.3. *Assume f is bounded and continuous on $\mathbb{R}$, and f and $\hat{f}$ are integrable on $\mathbb{R}$. Then* (3.2.10) *holds for all $x \in \mathbb{R}$.*

Proof. If $f \in L^1(\mathbb{R})$, then $\hat{f}$ is bounded and continuous. If also $\hat{f} \in L^1(\mathbb{R})$, then $\mathcal{F}^*\hat{f}$ is continuous. Furthermore, arguments similar to those used to prove Proposition 3.1.2 show that the right side of (3.2.11) converges to the right side of (3.2.10) as $\varepsilon \searrow 0$. That is to say,

$$J_\varepsilon f(x) \longrightarrow \mathcal{F}^*\hat{f}(x), \quad \text{as } \varepsilon \to 0. \tag{3.2.17}$$

It follows from (3.2.16) that $f(x) = \mathcal{F}^*\hat{f}(x)$. ☐

Remark. With some more work, one can omit the hypothesis in Corollary 3.2.3 that f be bounded and continuous, and use (3.2.17) to deduce these properties as a conclusion. This sort of reasoning is best carried out in a course on measure theory and integration.

At this point, we take the time to discuss integrable functions and square integrable functions on $\mathbb{R}$. Examples of integrable functions on $\mathbb{R}$ are bounded, piecewise continuous functions satisfying (3.2.1). More generally, f could be Riemann integrable on each interval $[-N, N]$, and satisfy

$$\lim_{N\to\infty} \int_{-N}^{N} |f(x)|\, dx = \|f\|_{L^1} < \infty, \tag{3.2.18}$$

where Riemann integrability on $[-N, N]$ has a definition similar to that given in (3.1.67)–(3.1.68) for functions on S^1. Still more general is Lebesgue's class, consisting of measurable functions $f : \mathbb{R} \to \mathbb{C}$ satisfying (3.2.1), where the Lebesgue integral is used. An element of $L^1(\mathbb{R})$ consists of an equivalence class of such functions, where we say $f_1 \sim f_2$ provided $\int_{-\infty}^{\infty} |f_1 - f_2|\, dx = 0$. The quantity $\|f\|_{L^1}$ is called the L^1-norm of f. It satisfies the triangle inequality

$$\|f + g\|_{L^1} \le \|f\|_{L^1} + \|g\|_{L^1}, \tag{3.2.19}$$

as an easy consequence of the pointwise inequality $|f(x) + g(x)| \le |f(x)| + |g(x)|$ (cf. (1.1.15)). Thus $L^1(\mathbb{R})$ has the structure of a metric space, with $d(f, g) = \|f - g\|_{L^1}$. We say $f_\nu \to f$ in L^1 if $\|f_\nu - f\|_{L^1} \to 0$. Parallel to Propositions A and B of §3.1, we have the following.

Proposition A1. Given $f \in L^1(\mathbb{R})$ and $k \in \mathbb{N}$, there exist $f_\nu \in C_0^k(\mathbb{R})$ such that $f_\nu \to f$ in L^1.

Here, $C_0^k(\mathbb{R})$ denotes the space of functions with compact support whose derivatives of order $\le k$ exist and are continuous. There is also the following completeness result.

Proposition B1. If (f_ν) is a Cauchy sequence in $L^1(\mathbb{R})$, there exists $f \in L^1(\mathbb{R})$ such that $f_\nu \to f$ in L^1.

As in §3.1, we will also be interested in square integrable functions. A function $f : \mathbb{R} \to \mathbb{C}$ is said to be square integrable if f and $|f|^2$ are integrable on each finite interval $[-N, N]$ and

$$\|f\|_{L^2}^2 = \int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty. \tag{3.2.20}$$

Taking the square root gives $\|f\|_{L^2}$, called the L^2-norm of f. Parallel to (3.1.40) and (3.2.19), there is the triangle inequality

$$\|f + g\|_{L^2} \le \|f\|_{L^2} + \|g\|_{L^2}. \tag{3.2.21}$$

The proof of (3.2.21) is not as easy as that of (3.2.19), but, like (3.1.40), it follows, via results of Appendix 3.4, from the fact that

$$\|f\|_{L^2}^2 = (f, f)_{L^2}, \tag{3.2.22}$$

where, for square integrable functions f and g, we define the inner product

$$(f, g)_{L^2} = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx. \tag{3.2.23}$$

The triangle inequality (3.2.21) makes $L^2(\mathbb{R})$ a metric space, with distance function $d(f, g) = \|f - g\|_{L^2}$, and we say $f_\nu \to f$ in L^2 if $\|f_\nu - f\|_{L^2} \to 0$. Parallel to Propositions A1 and B1, we have the following.

Proposition A2. Given $f \in L^2(\mathbb{R})$ and $k \in \mathbb{N}$, there exist $f_\nu \in C_0^k(\mathbb{R})$ such that $f_\nu \to f$ in L^2.

Proposition B2. If (f_ν) is a Cauchy sequence in $L^2(\mathbb{R})$, there exists $f \in L^2(\mathbb{R})$ such that $f_\nu \to f$ in L^2.

As in §3.1, we refer to books on measure theory, such as [**47**], for further material on $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, including proofs of results stated above.

Somewhat parallel to (3.1.34), we set

$$\mathcal{A}(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : f \text{ bounded and continuous, } \hat{f} \in L^1(\mathbb{R})\}. \tag{3.2.24}$$

By Corollary 3.2.3, the Fourier inversion formula (3.2.10) holds for all $f \in \mathcal{A}(\mathbb{R})$. It also follows that $f \in \mathcal{A}(\mathbb{R}) \Rightarrow \hat{f} \in \mathcal{A}(\mathbb{R})$. Note also that

$$\mathcal{A}(\mathbb{R}) \subset L^2(\mathbb{R}). \tag{3.2.25}$$

In fact, if $f \in \mathcal{A}(\mathbb{R})$,

$$\begin{aligned} \|f\|_{L^2}^2 &= \int |f(x)|^2 \, dx \\ &\le \sup |f(x)| \cdot \int |f(x)| \, dx \\ &\le \frac{1}{\sqrt{2\pi}} \|\hat{f}\|_{L^1} \|f\|_{L^1}. \end{aligned} \tag{3.2.26}$$

It is of interest to know when $f \in \mathcal{A}(\mathbb{R})$. Here we mention one simple result. Namely, if $f \in C^k(\mathbb{R})$ has compact support (we say $f \in C_0^k(\mathbb{R})$), then integration by parts yields

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^{(j)}(x) e^{-ix\xi} \, dx = (i\xi)^j \hat{f}(\xi), \quad 0 \le j \le k. \tag{3.2.27}$$

Hence

$$C_0^2(\mathbb{R}) \subset \mathcal{A}(\mathbb{R}). \tag{3.2.28}$$

While (3.2.28) is crude, it will give us a start on the L^2-theory of the Fourier transform. Let us denote by $\mathcal{F}$ the map $f \mapsto \hat{f}$, and by $\mathcal{F}^*$ the map you get upon replacing $e^{-ix\xi}$ by $e^{ix\xi}$. Then, with respect to the inner product (3.2.23), we have, for $f, g \in \mathcal{A}(\mathbb{R})$,

$$(\mathcal{F}f, g) = (f, \mathcal{F}^* g). \tag{3.2.29}$$

Now combining (3.2.29) with Corollary 3.2.3 we have

$$f, g \in \mathcal{A}(\mathbb{R}) \Longrightarrow (\mathcal{F}f, \mathcal{F}g) = (\mathcal{F}^*\mathcal{F}f, g) = (f, g). \tag{3.2.30}$$

One readily obtains a similar result with $\mathcal{F}$ replaced by $\mathcal{F}^*$. Hence

$$\|\mathcal{F}f\|_{L^2} = \|\mathcal{F}^* f\|_{L^2} = \|f\|_{L^2}, \tag{3.2.31}$$

for $f, g \in \mathcal{A}(\mathbb{R})$.

The result (3.2.31) is called the Plancherel identity. Using it, we can extend $\mathcal{F}$ and $\mathcal{F}^*$ to act on $L^2(\mathbb{R})$, obtaining (3.2.31) and the Fourier inversion formula on $L^2(\mathbb{R})$.

Proposition 3.2.4. *The maps $\mathcal{F}$ and $\mathcal{F}^*$ have unique continuous linear extensions from*

$$\mathcal{F}, \mathcal{F}^* : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R}) \tag{3.2.32}$$

to

$$\mathcal{F}, \mathcal{F}^* : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \tag{3.2.33}$$

and the identities

$$\mathcal{F}^*\mathcal{F}f = f, \quad \mathcal{F}\mathcal{F}^*f = f \tag{3.2.34}$$

hold for all $f \in L^2(\mathbb{R})$, as does (3.2.31).

This result can be proven using Propositions A2 and B2, and the inclusion (3.2.28), which together with Proposition A2 implies that

$$\text{For each } f \in L^2(\mathbb{R}),\ \exists f_\nu \in \mathcal{A}(\mathbb{R}) \text{ such that } f_\nu \to f \text{ in } L^2. \tag{3.2.35}$$

The argument proceeds as follows. Given $f \in L^2(\mathbb{R})$, take $f_\nu \in \mathcal{A}(\mathbb{R})$ such that $f_\nu \to f$ in L^2. Then $\|f_\mu - f_\nu\|_{L^2} \to 0$ as $\mu, \nu \to \infty$. Now (3.2.31), applied to $f_\mu - f_\nu \in \mathcal{A}(\mathbb{R})$, gives

$$\|\mathcal{F}f_\mu - \mathcal{F}f_\nu\|_{L^2} = \|f_\mu - f_\nu\|_{L^2} \to 0, \tag{3.2.36}$$

as $\mu, \nu \to \infty$. Hence $(\mathcal{F}f_\mu)$ is a Cauchy sequence in $L^2(\mathbb{R})$. By Proposition B2, there exists a limit $h \in L^2(\mathbb{R})$; $\mathcal{F}f_\nu \to h$ in L^2. One gets the same element h regardless of the choice of (f_ν) such that (3.2.35) holds, and so we set $\mathcal{F}f = h$. The same argument applies to $\mathcal{F}^*f_\nu$, which hence converges to $\mathcal{F}^*f$. We have

$$\|\mathcal{F}f_\nu - \mathcal{F}f\|_{L^2},\ \|\mathcal{F}^*f_\nu - \mathcal{F}^*f\|_{L^2} \to 0. \tag{3.2.37}$$

From here, the result (3.2.34) and the extension of (3.2.31) to $L^2(\mathbb{R})$ follow.

Given $f \in L^2(\mathbb{R})$, we have

$$\chi_{[-R,R]}\hat{f} \longrightarrow \hat{f} \text{ in } L^2, \quad \text{as } R \to \infty, \tag{3.2.38}$$

so Proposition 3.2.4 yields the following.

Proposition 3.2.5. *Define S_R by*

$$S_R f(x) = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \hat{f}(\xi) e^{ix\xi}\, d\xi. \tag{3.2.39}$$

Then

$$f \in L^2(\mathbb{R}) \Longrightarrow S_R f \to f \quad \text{in} \quad L^2(\mathbb{R}), \tag{3.2.40}$$

as $R \to \infty$.

Having Proposition 3.2.4, we can sharpen (3.2.28) as follows.

Proposition 3.2.6. *There is the inclusion*

$$C_0^1(\mathbb{R}) \subset \mathcal{A}(\mathbb{R}). \tag{3.2.41}$$

Proof. Given $f \in C_0^1(\mathbb{R})$, we can use (14.24) with $j = k = 1$ to get

$$g = f' \Longrightarrow \hat{g}(\xi) = i\xi \hat{f}(\xi). \tag{3.2.42}$$

Proposition 3.2.4 implies

$$\|(1+i\xi)\hat{f}\|_{L^2} = \|f + f'\|_{L^2}. \tag{3.2.43}$$

Now, parallel to the proof of Proposition 3.1.6, we have

$$\begin{aligned} \|\hat{f}\|_{L^1} &= \int \bigl|(1+i\xi)^{-1}\bigr| \cdot \bigl|(1+i\xi)\hat{f}(\xi)\bigr|\, d\xi \\ &\le \Bigl\{\int \frac{d\xi}{1+\xi^2}\Bigr\}^{1/2} \Bigl\{\int \bigl|(1+i\xi)\hat{f}(\xi)\bigr|^2\, d\xi\Bigr\}^{1/2} \\ &= \sqrt{\pi}\, \|f + f'\|_{L^2}, \end{aligned} \tag{3.2.44}$$

the inequality in (3.2.44) by Cauchy's inequality (cf. (3.4.18)) and the last identity by (3.2.42)–(3.2.43). This proves (3.2.41). ☐

Remark. Parallel to Proposition 3.1.6, one can extend Proposition 3.2.6 to show that if f has compact support, is continuuous, and is piecewise C^1 on $\mathbb{R}$, then $f \in \mathcal{A}(\mathbb{R})$. In conjunction with (3.2.44), the following is useful for identifying other elements of $\mathcal{A}(\mathbb{R})$.

Proposition 3.2.7. *Let $f_\nu \in \mathcal{A}(\mathbb{R})$ and $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$. Assume*

$$f_\nu \to f \ \text{ in } L^1\text{-norm, and } \ \|\hat{f}_\nu\|_{L^1} \le A, \tag{3.2.45}$$

for some $A < \infty$. Then $f \in \mathcal{A}(\mathbb{R})$.

Proof. Clearly $\hat{f}_\nu \to \hat{f}$ uniformly on $\mathbb{R}$. Hence, for each $R < \infty$,

$$\int_{-R}^{R} |\hat{f}_\nu(\xi) - \hat{f}(\xi)|\, d\xi \longrightarrow 0, \quad \text{as } \ \nu \to \infty. \tag{3.2.46}$$

Thus

$$\int_{-R}^{R} |\hat{f}(\xi)|\, d\xi \le A, \quad \forall\, R < \infty, \tag{3.2.47}$$

and it follows that $\hat{f} \in L^1(\mathbb{R})$, completing the proof. ☐

After the exercises, we provide a still sharper condition guaranteeing that $f \in \mathcal{A}(\mathbb{R})$.

Exercises

In Exercises 1–2, assume $f : \mathbb{R} \to \mathbb{C}$ is a C^2 function satisfying

$$|f^{(j)}(x)| \le C(1+|x|)^{-2}, \quad j \le 2. \tag{3.2.48}$$

1. Show that

$$|\hat{f}(\xi)| \le \frac{C'}{\xi^2+1}, \quad \xi \in \mathbb{R}. \tag{3.2.49}$$

Deduce that $f \in \mathcal{A}(\mathbb{R})$.

2. With $\hat{f}$ given as in (3.2.1), show that

$$\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx} = \sum_{\ell=-\infty}^{\infty} f(x+2\pi\ell). \tag{3.2.50}$$

This is known as the Poisson summation formula.
Hint. Let g denote the right side of (3.2.50), pictured as an element of $C^2(S^1)$. Relate the Fourier series of g (à la §3.1) to the left side of (3.2.50).

3. Use $f(x) = e^{-x^2/4t}$ in (3.2.50) to show that, for $\tau > 0$,

$$\sum_{\ell=-\infty}^{\infty} e^{-\pi\ell^2\tau} = \sqrt{\frac{1}{\tau}} \sum_{k=-\infty}^{\infty} e^{-\pi k^2/\tau}. \tag{3.2.51}$$

This is a Jacobi identity.
Hint. Use (3.2.13)–(3.2.14) to get $\hat{f}(\xi) = \sqrt{2t}\, e^{-t\xi^2}$. Take $t = \pi\tau$, and set $x = 0$ in (3.2.50).

4. For each of the following functions $f(x)$, compute $\hat{f}(\xi)$.
   a) $f(x) = e^{-|x|}$,
   b) $f(x) = \frac{1}{1+x^2}$,
   c) $f(x) = \chi_{[-1/2,1/2]}(x)$,
   d) $f(x) = (1-|x|)\chi_{[-1,1]}(x)$.

   Here $\chi_I(x)$ is the characteristic function of a set $I \subset \mathbb{R}$. Reconsider the computation of (b) when you get to §4.1.

5. In each case, (a)–(d) of Exercise 4, record the identity that follows from the Plancherel identity (3.2.31). In particular, show that

$$\int_{-\infty}^{\infty} \frac{\sin^2 \xi}{\xi^2}\, d\xi = \pi.$$

Exercises 6–8 deal with the convolution of functions on $\mathbb{R}$, defined by

$$f * g(x) = \int_{-\infty}^{\infty} f(y) g(x-y)\, dy. \tag{3.2.52}$$

6. Show that

$$\|f * g\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1}, \quad \sup |f * g| \le \|f\|_{L^2} \|g\|_{L^2}.$$

7. Show that

$$\widehat{(f * g)}(\xi) = \sqrt{2\pi}\hat{f}(\xi)\hat{g}(\xi).$$

8. Compute $f * f$ when $f(x) = \chi_{[-1/2,1/2]}(x)$, the characteristic function of the interval $[-1/2, 1/2]$. Compare the result of Exercise 7 with the computation of (d) in Exercise 4.

9. Prove the following result, known as the Riemann-Lebesgue lemma:

$$f \in L^1(\mathbb{R}) \Longrightarrow \lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0. \tag{3.2.53}$$

Hint. (3.2.49) gives the desired conclusion for $\hat{f}_\nu$ when $f_\nu \in C_0^2(\mathbb{R})$. Then use Proposition A1 and apply (3.2.4) to $f - f_\nu$, to uniformly get $\hat{f}_\nu \to \hat{f}$.

10. Sharpen the result of Exercise 1 as follows, using the reasoning in the proof of Proposition 3.2.6. Assume $f : \mathbb{R} \to \mathbb{C}$ is a C^1 function satisfying

$$|f^{(j)}(x)| \le C(1 + |x|)^{-2}, \quad j \le 1. \tag{3.2.54}$$

Then show that $f \in \mathcal{A}(\mathbb{R})$. More generally, show that $f \in \mathcal{A}(\mathbb{R})$ provided that f is Lipschitz continuous on $\mathbb{R}$, C^1 on $(-\infty, 0]$ and on $[0, \infty)$, and (3.2.54) holds for all $x \neq 0$.

More general sufficient condition for $f \in \mathcal{A}(\mathbb{R})$

Here we establish a result substantially sharper than Proposition 3.2.6. We mention that an analogous result holds for the Fourier series. The interested reader can investigate this.

To set things up, given $f \in L^2(\mathbb{R})$, let

$$f_h(x) = f(x + h). \tag{3.2.55}$$

Our goal here is to prove the following.

Proposition 3.2.8. *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and there exists $C < \infty$ such that*

$$\|f - f_h\|_{L^2} \le Ch^\alpha, \quad \forall\, h \in [0, 1], \tag{3.2.56}$$

with

$$\alpha > \frac{1}{2}, \tag{3.2.57}$$

then $f \in \mathcal{A}(\mathbb{R})$.

Proof. A calculation gives

$$\hat{f}_h(\xi) = e^{ih\xi}\hat{f}(\xi), \tag{3.2.58}$$

so, by the Plancherel identity,

$$\|f - f_h\|_{L^2}^2 = \int_{-\infty}^{\infty} |1 - e^{ih\xi}|^2\, |\hat{f}(\xi)|^2\, d\xi. \tag{3.2.59}$$

Now,

$$\frac{\pi}{2} \le |h\xi| \le \frac{3\pi}{2} \Longrightarrow |1 - e^{ih\xi}|^2 \ge 2, \tag{3.2.60}$$

so

$$\|f - f_h\|_{L^2}^2 \ge 2 \int\limits_{\frac{\pi}{2} \le |h\xi| \le \frac{3\pi}{2}} |\hat{f}(\xi)|^2 \, d\xi. \tag{3.2.61}$$

If (3.2.56) holds, we deduce that, for $h \in (0, 1]$,

$$\int\limits_{\frac{2}{h} \le |\xi| \le \frac{4}{h}} |\hat{f}(\xi)|^2 \, d\xi \le C h^{2\alpha}, \tag{3.2.62}$$

and hence (setting $h = 2^{-\ell+1}$), for $\ell \ge 1$,

$$\int\limits_{2^\ell \le |\xi| \le 2^{\ell+1}} |\hat{f}(\xi)|^2 \, d\xi \le C 2^{-2\alpha\ell}. \tag{3.2.63}$$

Cauchy's inequality gives

$$\begin{aligned} &\int\limits_{2^\ell \le |\xi| \le 2^{\ell+1}} |\hat{f}(\xi)| \, d\xi \\ &\le \Big\{ \int\limits_{2^\ell \le |\xi| \le 2^{\ell+1}} |\hat{f}(\xi)|^2 \, d\xi \Big\}^{1/2} \times \Big\{ \int\limits_{2^\ell \le |\xi| \le 2^{\ell+1}} 1 \, d\xi \Big\}^{1/2} \\ &\le C 2^{-\alpha\ell} \cdot 2^{\ell/2} \\ &= C 2^{-(\alpha - 1/2)\ell}. \end{aligned} \tag{3.2.64}$$

Summing over $\ell \in \mathbb{N}$ and using (again by Cauchy's inequality)

$$\int\limits_{|\xi| \le 2} |\hat{f}| \, d\xi \le C \|\hat{f}\|_{L^2} = C \|f\|_{L^2}, \tag{3.2.65}$$

then gives the proof. □

To see how close to sharp Proposition 3.2.8 is, consider

$$\begin{aligned} f(x) = \chi_I(x) = 1 \quad &\text{if } \ 0 \le x \le 1, \\ 0 \quad &\text{otherwise.} \end{aligned} \tag{3.2.66}$$

We have, for $0 \le h \le 1$,

$$\|f - f_h\|_{L^2}^2 = 2h, \tag{3.2.67}$$

so (3.2.56) holds, with $\alpha = 1/2$. Since $\mathcal{A}(\mathbb{R}) \subset C(\mathbb{R})$, this function does not belong to $\mathcal{A}(\mathbb{R})$, so the condition (3.2.57) is about as sharp as it could be.

Remark. Using

$$\int |uv|\,dx \le \sup|u| \int |v|\,dx, \tag{3.2.68}$$

we have the estimate

$$\|f - f_h\|_{L^2}^2 \le \sup_x |f(x) - f_h(x)| \cdot \|f - f_h\|_{L^1}, \tag{3.2.69}$$

so, with

$$\begin{aligned} \|f\|_{BV} &= \sup_{0<h\le 1} \|h^{-1}(f - f_h)\|_{L^1}, \\ \|f\|_{C^r} &= \sup_{x\in\mathbb{R},0<h\le 1} h^{-r}|f(x) - f_h(x)|, \end{aligned} \tag{3.2.70}$$

for $0 < r < 1$, we have

$$\|f - f_h\|_{L^2}^2 \le h^{1+r}\|f\|_{BV}\|f\|_{C^r}, \tag{3.2.71}$$

which can be applied to the hypothesis (3.2.56) in Proposition 3.2.8.

Fourier uniqueness

The Fourier inversion formula established in Corollary 3.2.3 yields

$$f \in \mathcal{A}(\mathbb{R}),\ \hat{f} = 0 \Longrightarrow f = 0. \tag{3.2.72}$$

Similarly, Proposition 3.2.4 yields

$$f \in L^2(\mathbb{R}),\ \hat{f} = 0 \Longrightarrow f = 0. \tag{3.2.73}$$

We call these Fourier uniqueness results. An extension of (3.2.72) is the following consequence of Proposition 3.2.2:

$$f \in L^1(\mathbb{R}) \cap C(\mathbb{R}),\ \hat{f} = 0 \Longrightarrow f = 0. \tag{3.2.74}$$

Here, we advertize the following strengthening of (3.2.72).

Proposition 3.2.9. *We have the implication*

$$f \in L^1(\mathbb{R}),\ \hat{f} = 0 \Longrightarrow f = 0. \tag{3.2.75}$$

We indicate a proof of this result, starting with the following variant of (3.2.29). If $f \in L^1(\mathbb{R})$ and also $g \in L^1(\mathbb{R})$, then

$$\begin{aligned} \int_{-\infty}^{\infty} \hat{f}(\xi) g(\xi)\,d\xi &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-ix\xi} g(\xi)\,dx\,d\xi \\ &= \int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx, \end{aligned} \tag{3.2.76}$$

where the second identity uses a change in the order of integration. Thus

$$f \in L^1(\mathbb{R}),\ \hat{f} = 0 \Longrightarrow \int_{-\infty}^{\infty} f(x)h(x)\,dx = 0, \tag{3.2.77}$$

for all $h = \hat{g},\ g \in L^1(\mathbb{R})$. In particular, (3.2.73) holds for all $h \in \mathcal{A}(\mathbb{R})$, and so, by (3.2.28), it holds for all $h \in C_0^2(\mathbb{R})$. The implication

$$f \in L^1(\mathbb{R}),\ \int f(x)h(x)\,dx = 0\ \forall\, h \in C_0^2(\mathbb{R}) \Longrightarrow f = 0 \tag{3.2.78}$$

is a basic result in a course in measure theory and integration.

3.3. Laplace transforms and Mellin transforms

Suppose we have a function $f : \mathbb{R}^+ \to \mathbb{C}$ that is integrable on $[0, R]$ for all $R < \infty$ and satisfies

$$\int_0^\infty |f(t)|e^{-at}\,dt < \infty, \quad \forall\ a > A, \tag{3.3.1}$$

for some $A \in (-\infty, \infty)$. We define the Laplace transform of f by

$$\mathcal{L}f(s) = \int_0^\infty f(t)e^{-st}\,dt, \quad \text{Re } s > A. \tag{3.3.2}$$

It is clear that this integral is absolutely convergent for each s in the half-plane $H_A = \{z \in \mathbb{C} : \text{Re } z > A\}$ and defines a continuous function $\mathcal{L}f : H_A \to \mathbb{C}$. Also, if γ is a closed curve (e.g., the boundary of a rectangle) in H_A, we can change the order of integration to see that

$$\int_\gamma \mathcal{L}f(s)\,ds = \int_0^\infty \int_\gamma f(t)e^{-st}\,ds\,dt = 0. \tag{3.3.3}$$

Hence Morera's theorem implies that $\mathcal{L}f$ is holomorphic on H_A. We have

$$\frac{d}{ds}\mathcal{L}f(s) = \mathcal{L}g(s), \quad g(t) = -tf(t). \tag{3.3.4}$$

On the other hand, if $f \in C^1([0,\infty))$ and $\int_0^\infty |f'(t)|e^{-at}\,dt < \infty$ for all $a > A$, then we can integrate by parts and get

$$\mathcal{L}f'(s) = s\mathcal{L}f(s) - f(0), \tag{3.3.5}$$

and a similar hypothesis on higher derivatives of f gives

$$\mathcal{L}f^{(k)}(s) = s^k\mathcal{L}f(s) - s^{k-1}f(0) - \cdots - f^{(k-1)}(0). \tag{3.3.6}$$

Thus, if f satisfies an ODE of the form

$$c_n f^{(n)}(t) + c_{n-1}f^{(n-1)}(t) + \cdots + c_0 f(t) = g(t) \tag{3.3.7}$$

for $t \geq 0$, with initial data

$$f(0) = a_0, \ldots, f^{(n-1)}(0) = a_{n-1}, \tag{3.3.8}$$

and hypotheses yielding (3.3.6) hold for all $k \leq n$, we have

$$p(s)\mathcal{L}f(s) = \mathcal{L}g(s) + q(s), \tag{3.3.9}$$

with

$$p(s) = c_n s^n + c_{n-1} s^{n-1} + \cdots + c_0, \tag{3.3.10}$$
$$q(s) = c_n(a_0 s^{n-1} + \cdots + a_{n-1}) + \cdots + c_1 a_0.$$

If all the roots of $p(s)$ satisfy $\operatorname{Re} s \le B$, we have

$$\mathcal{L}f(s) = \frac{\mathcal{L}g(s) + q(s)}{p(s)}, \quad s \in H_C,\ C = \max\{A, B\}, \tag{3.3.11}$$

and we are motivated to seek an inverse Laplace transform.

We can get this by relating the Laplace transform to the Fourier transform. In fact, if (3.3.1) holds, and if $B > A$, then

$$\mathcal{L}f(B + i\xi) = \sqrt{2\pi}\,\hat{\varphi}(\xi), \quad \xi \in \mathbb{R}, \tag{3.3.12}$$

with

$$\begin{aligned} \varphi(x) &= f(x)e^{-Bx}, \quad x \ge 0, \\ &\qquad 0 \quad\ , \quad x < 0. \end{aligned} \tag{3.3.13}$$

In §3.2 we have seen several senses in which

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}(\xi) e^{ix\xi}\, d\xi, \tag{3.3.14}$$

hence giving, for $t > 0$,

$$\begin{aligned} f(t) &= \frac{e^{Bt}}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}f(B + i\xi) e^{i\xi t}\, d\xi \\ &= \frac{1}{2\pi i} \int\limits_{\gamma} \mathcal{L}f(s) e^{st}\, ds, \end{aligned} \tag{3.3.15}$$

where γ is the vertical line $\gamma(\xi) = B + i\xi,\ -\infty < \xi < \infty$.

For example, if φ in (3.3.13) belongs to $L^2(\mathbb{R})$, then (3.3.15) holds in the sense of Proposition 3.2.5. If φ belongs to $\mathcal{A}(\mathbb{R})$, then (3.3.15) holds in the sense of Corollary 3.2.3. Frequently, f is continuous on $[0, \infty)$ but $f(0) \ne 0$. Then φ in (3.3.13) has a discontinuity at $x = 0$, so $\varphi \notin \mathcal{A}(\mathbb{R})$. However, sometimes one has $\psi(x) = x\varphi(x)$ in $\mathcal{A}(\mathbb{R})$, which is obtained as in (3.3.13) by replacing $f(t)$ with $tf(t)$. (See Exercise 7 below.) In light of (3.3.4), we obtain

$$-tf(t) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{d}{ds} \mathcal{L}f(s)\, e^{st}\, ds, \tag{3.3.16}$$

with an absolutely convergent integral, provided $\psi \in \mathcal{A}(\mathbb{R})$.

Related to such inversion formulas is the following uniqueness result, which, via (3.3.12)–(3.3.13), is an immediate consequence of Proposition 3.2.9.

Proposition 3.3.1. *If f_1 and f_2 are integrable on $[0,R]$ for all $R<\infty$ and satisfy* (3.3.1), *then*

$$\mathcal{L}f_1(s)=\mathcal{L}f_2(s),\quad \forall\, s\in H_A \Longrightarrow f_1=f_2 \quad on \quad \mathbb{R}^+. \tag{3.3.17}$$

We can also use material in §2.6 to deduce that $f_1=f_2$, given $\mathcal{L}f_1(s)=\mathcal{L}f_2(s)$ on a set with an accumulation point in H_A.

We next introduce a transform called the Mellin transform:

$$\mathcal{M}f(z)=\int_0^\infty f(t)t^{z-1}\,dt, \tag{3.3.18}$$

defined and holomoprhic on $A<\operatorname{Re} z<B$, provided $f(t)t^{x-1}$ is integrable for real $x\in(A,B)$. This is related to the Laplace transform via a change of variable, $t=e^x$:

$$\mathcal{M}f(z)=\int_{-\infty}^\infty f(e^x)e^{zx}\,dx. \tag{3.3.19}$$

Assuming $0\in(A,B)$, evaluation of these integrals for z on the imaginary axis yields

$$\begin{aligned}\mathcal{M}^\# f(\xi)&=\int_0^\infty f(t)t^{i\xi-1}\,dt\\ &=\int_{-\infty}^\infty f(e^x)e^{ix\xi}\,dx.\end{aligned} \tag{3.3.20}$$

The Fourier inversion formula and Plancherel formula imply

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^\infty (\mathcal{M}^\# f)(\xi)t^{-i\xi}\,d\xi \tag{3.3.21}$$

and

$$\int_{-\infty}^\infty |\mathcal{M}^\# f(\xi)|^2\,d\xi=2\pi\int_0^\infty |f(t)|^2\,\frac{dt}{t}. \tag{3.3.22}$$

If $0\notin(A,B)$ but $\tau\in(A,B)$, one can evaluate $\mathcal{M}f(z)$ on the vertical axis $z=\tau+i\xi$, and modify (3.3.20)–(3.3.22) accordingly.

Exercises

1. Show that the Laplace transform of $f(t)=t^{z-1}$,

$$\mathcal{L}f(s)=\int_0^\infty e^{-st}t^{z-1}\,dt,\quad \operatorname{Re} z>0, \tag{3.3.23}$$

is given by

$$\mathcal{L}f(s)=\Gamma(z)s^{-z}, \tag{3.3.24}$$

where $\Gamma(z)$ is the Gamma function:

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\,dt. \tag{3.3.25}$$

For more on this function, see §4.3.

Remark. The integral (3.3.23) has the remarkable property of being simultaneously a Laplace transform and a Mellin transform. It is the simplest integral with this property.

2. Compute the Laplace transforms of the following functions (defined for $t \geq 0$):

$$\begin{aligned} &e^{at}, \\ &\cosh at, \\ &\sinh at, \\ &\sin at, \\ &t^{z-1}\, e^{at}. \end{aligned}$$

3. Compute the inverse Laplace transforms of the following functions (defined in appropriate right half-spaces):

$$\begin{aligned} &\frac{1}{s-a}, \\ &\frac{s}{s^2-a^2}, \\ &\frac{a}{s^2-a^2}, \\ &\frac{a}{s^2+a^2}, \\ &\frac{1}{\sqrt{s+1}}. \end{aligned}$$

Reconsider these problems when you read §4.1.

Exercises 4–6 deal with the convolution of functions on $\mathbb{R}^+$, defined by

$$f * g(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau. \tag{3.3.26}$$

4. Show that (3.3.26) coincides with the definition (3.2.52) of convolution, provided $f(t)$ and $g(t)$ vanish for $t < 0$.

5. Show that if $f_z(t) = t^{z-1}$ for $t > 0$, with $\operatorname{Re} z > 0$, and if also $\operatorname{Re} \zeta > 0$, then

$$f_z * f_\zeta(t) = B(z,\zeta)\, f_{z+\zeta}(t), \tag{3.3.27}$$

with

$$B(z,\zeta) = \int_0^1 s^{z-1}(1-s)^{\zeta-1}\, ds. \tag{3.3.28}$$

6. Show that

$$\mathcal{L}(f * g)(s) = \mathcal{L}f(s) \cdot \mathcal{L}g(s). \tag{3.3.29}$$

See §4.3 for an identity resulting from applying (3.3.29) to (3.3.27).

7. Assume $f \in C^1([0,\infty))$ satisfies

$$|f(x)|,\ |f'(x)| \le Ce^{Ax}, \quad x \ge 0.$$

Take $B > A$ and define $\varphi(x)$ as in (3.3.13). Show that

$$\psi(x) = x\varphi(x) \Longrightarrow \psi \in \mathcal{A}(\mathbb{R}).$$

Hint. Use the result of Exercise 10 in §3.2.

The matrix Laplace transform and Duhamel's formula

The matrix Laplace transform allows one to treat $n \times n$ first-order systems of differential equations, of the form

$$f'(t) = Kf(t) + g(t), \quad f(0) = a, \tag{3.3.30}$$

in a fashion parallel to (3.3.7)–(3.3.11). Here K is an $n \times n$ matrix,

$$K \in M(n,\mathbb{C}), \quad a \in \mathbb{C}^n, \quad g(t) \in \mathbb{C}^n, \quad \forall\, t, \tag{3.3.31}$$

and we seek a solution $f(t)$, taking values in $\mathbb{C}^n$.

A key ingredient in this study is the matrix exponential

$$e^{tK} = \sum_{j=0}^{\infty} \frac{t^j}{j!} K^j, \quad K \in M(n,\mathbb{C}), \tag{3.3.32}$$

which has a development parallel to that of the exponential (for $K \in \mathbb{C}$) given in §1.1. In particular, we have

$$\frac{d}{dt} e^{tK} = Ke^{tK} = e^{tK}K, \quad e^{(t+\tau)K} = e^{tK}e^{\tau K}. \tag{3.3.33}$$

See Appendix 3.5 for more details and for definitions of some of the concepts used below, such as the norm $\|K\|$ of the matrix K. Using the notation

$$E_K(t) = e^{tK}, \tag{3.3.34}$$

we have

$$\mathcal{L}E_K(s) = \int_0^\infty e^{tK} e^{-st}\, dt, \tag{3.3.35}$$

valid whenever

$$\|e^{tK}\| \le ce^{\alpha t}, \quad \Re s > \alpha. \tag{3.3.36}$$

In particular, the first estimate always holds for $\alpha = \|K\|$.

Turning to (3.3.30), if we assume g and f have Laplace transforms and apply $\mathcal{L}$ to both sides, then (3.3.5) continues to apply, and we get

$$s\mathcal{L}f(s) - a = K\mathcal{L}f(s) + \mathcal{L}g(s). \tag{3.3.37}$$

Hence

$$\mathcal{L}f(s) = (sI - K)^{-1}(a + \mathcal{L}g(s)), \tag{3.3.38}$$

if $\operatorname{Re} s$ is sufficiently large. To solve (3.3.30), it suffices to identify the right side of (3.3.38) as the Laplace transform of a function.

To start, we assert that, with $E_K(t) = e^{tK}$,

$$\mathcal{L}E_K(s) = (sI - K)^{-1}, \tag{3.3.39}$$

whenever s satisfies (3.3.36). To see this, let $L \in M(n, \mathbb{C})$ and note that the identity $(d/dt)e^{-tL} = -Le^{-tL}$ implies

$$L \int_0^T e^{-tL}\, dt = I - e^{-TL}, \tag{3.3.40}$$

for each $T \in (0, \infty)$. If L satisfies

$$\|e^{-tL}\| \le ce^{-\delta t}, \quad \forall\, t > 0, \tag{3.3.41}$$

for some $\delta > 0$, then we can take $T \to \infty$ in (3.3.40) and deduce that

$$L \int_0^\infty e^{-tL}\, dt = I, \quad \text{i.e.,} \quad \int_0^\infty e^{-tL}\, dt = L^{-1}. \tag{3.3.42}$$

Clearly (3.3.41) applies to $L = sI - K$ as long as (3.3.36) holds, since

$$\|e^{t(K-sI)}\| = e^{-t\operatorname{Re} s}\|e^{tK}\|, \tag{3.3.43}$$

so we have (3.3.39). This gives

$$(sI - K)^{-1}a = \mathcal{L}(E_K a)(s). \tag{3.3.44}$$

Also, by (3.3.29),

$$(sI - K)^{-1}\mathcal{L}g(s) = \mathcal{L}(E_K * g)(s), \tag{3.3.45}$$

where

$$\begin{aligned} E_K * g(t) &= \int_0^t E_K(t-\tau)g(\tau)\, d\tau \\ &= \int_0^t e^{(t-\tau)K} g(\tau)\, d\tau. \end{aligned} \tag{3.3.46}$$

Now (3.3.38) yields $f(t) = E_K(t)a + E_K * g(t)$. In conclusion, the solution to (3.3.30) is given by

$$f(t) = e^{tK}a + \int_0^t e^{(t-\tau)K} g(\tau)\, d\tau. \tag{3.3.47}$$

This is known as Duhamel's formula.

Here is another route to the derivation of Duhamel's formula. We seek the solution to (3.3.30) in the form

$$f(t) = e^{tK}F(t), \tag{3.3.48}$$

and find that $F(t)$ satisfies a differential equation that is simpler than (3.3.30). In fact, differentiating (3.3.48) and using (3.3.33) gives

$$\begin{aligned} \frac{df}{dt} &= e^{tK}\Big(\frac{dF}{dt} + KF(t)\Big), \\ Kf(t) + g(t) &= Ke^{tK}F(t) + g(t), \end{aligned} \tag{3.3.49}$$

and hence

$$\frac{dF}{dt} = e^{-tK}g(t), \quad F(0) = a. \tag{3.3.50}$$

Simply integrating this gives

$$F(t) = a + \int_0^t e^{-\tau K}g(\tau)\,d\tau, \tag{3.3.51}$$

and applying e^{tK} to both sides (and using the last identity in (3.3.33)) again gives (3.3.47).

3.4. Inner product spaces

On occasion, particularly in §§3.1–3.2, we have looked at norms and inner products on spaces of functions, such as $C(S^1)$ and $\mathcal{S}(\mathbb{R})$, which are vector spaces. Generally, a complex vector space V is a set on which there are operations of vector addition:

$$f, g \in V \Longrightarrow f + g \in V, \tag{3.4.1}$$

and multiplication by an element of $\mathbb{C}$ (called scalar multiplication):

$$a \in \mathbb{C},\ f \in V \Longrightarrow af \in V, \tag{3.4.2}$$

satisfying the following properties. For vector addition, we have

$$\begin{gathered} f + g = g + f,\ (f + g) + h = f + (g + h), \\ f + 0 = f,\ f + (-f) = 0. \end{gathered} \tag{3.4.3}$$

For multiplication by scalars, we have

$$a(bf) = (ab)f, \quad 1 \cdot f = f. \tag{3.4.4}$$

Furthermore, we have two distributive laws:

$$a(f + g) = af + ag, \quad (a + b)f = af + bf. \tag{3.4.5}$$

These properties are readily verified for the function spaces arising in §§3.1–3.2.

6. Show that $Y_{1/2}(t) = -J_{-1/2}(t)$, and deduce that

$$H^{(1)}_{1/2}(t) = -i\sqrt{\frac{2}{\pi t}}e^{it}, \quad H^{(2)}_{1/2}(t) = i\sqrt{\frac{2}{\pi t}}e^{-it}.$$

7. Show that if $\nu \in \mathbb{R}$, then $H^{(1)}_\nu(t)$ does not vanish at any $t \in \mathbb{R} \setminus 0$.

7.2. Differential equations on a complex domain

Here we study differential equations, such as

$$a_n(z)\frac{d^n u}{dz^n} + a_{n-1}(z)\frac{d^{n-1}u}{dz^{n-1}} + \cdots + a_1(z)\frac{du}{dz} + a_0(z)u = g(z), \tag{7.2.1}$$

given a_j and g holomorphic on a connected, open domain $\Omega \subset \mathbb{C}$, and $a_n(z) \neq 0$ for $z \in \Omega$. An example, treated in §7.1, is Bessel's equation, i.e.,

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \Bigl(1 - \frac{\nu^2}{z^2}\Bigr)u = 0, \tag{7.2.2}$$

for which $\Omega = \mathbb{C} \setminus 0$. We confine our attention to linear equations. Some basic material on nonlinear holomorphic differential equations can be found in Chapter 1 (particularly Sections 6 and 9) of [**46**].

We will work in the setting of a first-order $n \times n$ system,

$$\frac{dv}{dz} = A(z)v + f(z), \quad v(z_0) = v_0, \tag{7.2.3}$$

given $z_0 \in \Omega$, $v_0 \in \mathbb{C}^n$, and

$$A : \Omega \longrightarrow M(n, \mathbb{C}), \quad f : \Omega \longrightarrow \mathbb{C}^n, \quad \text{holomorphic}. \tag{7.2.4}$$

A standard conversion of (7.2.1) to (7.2.3) takes $v = (u_0, \dots, u_{n-1})^t$, with

$$u_j = \frac{d^j u}{dz^j}, \quad 0 \le j \le n-1. \tag{7.2.5}$$

Then v satisfies the $n \times n$ system

$$\frac{du_j}{dz} = u_{j+1}, \quad 0 \le j \le n-2, \tag{7.2.6}$$

$$\frac{du_{n-1}}{dz} = -a_n(z)^{-1}\Bigl[g(z) - \sum_{j=0}^{n-1} a_j(z)u_j\Bigr]. \tag{7.2.7}$$

We pick $z_0 \in \Omega$ and impose on this system the initial condition

$$v(z_0) = (u(z_0), u'(z_0), \dots, u^{(n-1)}(z_0))^t = v_0 \in \mathbb{C}^n. \tag{7.2.8}$$

In the general framework of (7.2.3)–(7.2.4), we seek a solution $v(z)$ that is holomorphic on a neighborhood of z_0 in Ω. Such $v(z)$ would have a power series expansion

$$v(z) = \sum_{k=0}^{\infty} v_k (z - z_0)^k, \quad v_k \in \mathbb{C}^n. \tag{7.2.9}$$

Assume $A(z)$ and $f(z)$ are given by convergent power series,

$$A(z) = \sum_{k=0}^{\infty} A_k (z - z_0)^k, \quad f(z) = \sum_{k=0}^{\infty} f_k (z - z_0)^k, \tag{7.2.10}$$

with

$$A_k \in M(n, \mathbb{C}), \quad f_k \in \mathbb{C}^n. \tag{7.2.11}$$

If (7.2.9) is a convergent power series, then the coefficients v_k are obtained, recursively, as follows. We have

$$\frac{dv}{dz} = \sum_{k=1}^{\infty} k v_k (z - z_0)^{k-1} = \sum_{k=0}^{\infty} (k+1) v_{k+1} z^k \tag{7.2.12}$$

and

$$\begin{aligned} A(z)v(z) &= \sum_{j=0}^{\infty} A_j (z - z_0)^j \sum_{\ell=0}^{\infty} v_\ell (z - z_0)^\ell \\ &= \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} A_{k-j} v_j \Big) (z - z_0)^k, \end{aligned} \tag{7.2.13}$$

so the power series for the left side and the right side of (7.2.3) agree if and only if, for each $k \geq 0$,

$$(k+1) v_{k+1} = \sum_{j=0}^{k} A_{k-j} v_j + f_k. \tag{7.2.14}$$

The first three iterations of this recursive formula are

$$\begin{aligned} v_1 &= A_0 v_0 + f_0, \\ 2v_2 &= A_1 v_0 + A_0 v_1 + f_1, \\ 3v_3 &= A_2 v_0 + A_1 v_1 + A_0 v_2 + f_2. \end{aligned} \tag{7.2.15}$$

To start the recursion, the initial condition in (7.2.3) specifies v_0.

The following key result addresses the issue of convergence of the power series thus produced for $v(z)$.

Proposition 7.2.1. *Assume the power series for $A(z)$ and $f(z)$ in* (7.2.10) *converge for*

$$|z - z_0| < R_0, \tag{7.2.16}$$

and let v_k *be defined recursively by* (7.2.14). *Then the power series* (7.2.9) *for* $v(z)$ *also converges in the region* (7.2.16).

Proof. To estimate the terms in (7.2.9), we use matrix and vector norms, as treated in Appendix 3.5. The hypotheses on (7.2.10) imply that for each $R < R_0$, there exist $a, b \in (0, \infty)$ such that

$$\|A_k\| \le aR^{-k}, \quad \|f_k\| \le bR^{-k}, \quad \forall\, k \in \mathbb{Z}^+. \tag{7.2.17}$$

We will show that, given $r \in (0, R)$, there exists $C \in (0, \infty)$ such that

$$\|v_j\| \le Cr^{-j}, \quad \forall\, j \in \mathbb{Z}^+. \tag{7.2.18}$$

Such estimates imply that the power series (7.2.9) converges for $|z - z_0| < r$, for each $r < R_0$, and hence for $|z - z_0| < R_0$.

We will prove (7.2.18) by induction. The inductive step is to assume it holds for all $j \le k$ and to deduce it holds for $j = k + 1$. This deduction proceeds as follows. We have, by (7.2.14), (7.2.17), and (7.2.18) for $j \le k$,

$$\begin{aligned}(k+1)\|v_{k+1}\| &\le \sum_{j=0}^{k} \|A_{k-j}\| \cdot \|v_j\| + \|f_k\| \\ &\le aC \sum_{j=0}^{k} R^{j-k} r^{-j} + bR^{-k} \\ &= aCr^{-k} \sum_{j=0}^{k} \Bigl(\frac{r}{R}\Bigr)^{k-j} + bR^{-k}.\end{aligned} \tag{7.2.19}$$

Now, given $0 < r < R$,

$$\sum_{j=0}^{k} \Bigl(\frac{r}{R}\Bigr)^{k-j} < \sum_{j=0}^{\infty} \Bigl(\frac{r}{R}\Bigr)^{j} = \frac{1}{1 - r/R} = M(R, r) < \infty. \tag{7.2.20}$$

Hence

$$(k+1)\|v_{k+1}\| \le aCM(R, r)r^{-k} + br^{-k}. \tag{7.2.21}$$

We place on C the constraint that

$$C \ge b, \tag{7.2.22}$$

and obtain

$$\|v_{k+1}\| \le \frac{aM(R, r) + 1}{k+1} r \cdot Cr^{-k-1}. \tag{7.2.23}$$

This gives the desired result

$$\|v_{k+1}\| \le Cr^{-k-1}, \tag{7.2.24}$$

as long as

$$\frac{aM(R,r)+1}{k+1} r \leq 1. \tag{7.2.25}$$

Thus, to finish the argument, we pick $K \in \mathbb{N}$ such that

$$K+1 \geq \big[aM(R,r)+1\big]r. \tag{7.2.26}$$

Recall that we have a, R, r and $M(R,r)$. Then we pick $C \in (0,\infty)$ large enough that (7.2.18) holds for all $j \in \{0,1,\dots,K\}$, i.e., we take (in addition to (7.2.22))

$$C \geq \max_{0\leq j\leq K} r^j \|v_j\|. \tag{7.2.27}$$

Then for all $k \geq K$, the inductive step yielding (7.2.24) from the validity of (7.2.18) for all $j \leq k$ holds, and the inductive proof of (7.2.18) is complete. □

Under the hypothesis (7.2.4), Proposition 7.2.1 applies whenever

$$D_{R_0}(z_0) = \{z \in \mathbb{C} : |z-z_0| < R_0\} \subset \Omega. \tag{7.2.28}$$

Having the solution to (7.2.3) on $D_{R_0}(z_0)$, we next want to analytically continue this solution to a larger domain in Ω. To start, we discuss analytic continuation of v along a continuous curve

$$\gamma : [a,b] \longrightarrow \Omega, \quad \gamma(a) = z_0, \quad \gamma(b) = z_1. \tag{7.2.29}$$

To get this, using compactness we pick $R_1 > 0$ and a partition

$$a = a_0 \leq a_1 \leq \cdots \leq a_{n+1} = b \tag{7.2.30}$$

of $[a,b]$ such that, for $0 \leq j \leq n$,

$$D_j = D_{R_1}(\gamma(a_j)) \text{ contains } \gamma([a_j,a_{j+1}]) \text{ and is contained in } \Omega. \tag{7.2.31}$$

Proposition 7.2.1 implies that the power series (7.2.9), given via (7.2.14), is convergent and holomorphic on D_0. Call this solution v_{D_0}. Now $\gamma(a_1) \in D_0$, and we can apply Proposition 7.2.1, with z_0 replaced by $\gamma(a_1)$ and R_0 replaced by R_1, to produce a holomorphic function v_{D_1} on D_1, satisfying (7.2.9), and agreeing with v_{D_0} on $D_0 \cap D_1$. We can continue this construction for each j, obtaining at the end v_{D_n}, holomorphic on D_n, which contains $\gamma(a_{n+1}) = z_1$. Also, v_{D_n} solves (7.2.9) on D_n.

It is useful to put this notion of analytic continuation along γ in a more general context, as follows. Given a continuous path γ as in (7.2.29), we say a chain along γ is a partition of $[a,b]$ as in (7.2.30) together with a collection of open, convex sets D_j, satisfying

$$\gamma([a_j,a_{j+1}]) \subset D_j \subset \Omega, \quad \text{for } 0 \leq j \leq n. \tag{7.2.32}$$

Given $v = v_{D_0}$, holomorphic on D_0, we say an analytic continuation of v along this chain is a collection

$$\begin{aligned} &v_{D_j} : D_j \to \mathbb{C}, \text{ holomorphic, such that} \\ &v_{D_j} = v_{D_{j+1}} \text{ on } D_j \cap D_{j+1}, \text{ for } 0 \le j \le n-1. \end{aligned} \tag{7.2.33}$$

Note that (7.2.32) implies $D_j \cap D_{j+1} \neq \emptyset$.

The following is a key uniqueness result.

Proposition 7.2.2. *Let $\{\widetilde{D}_0, \dots, \widetilde{D}_m\}$ be another chain along γ, associated to a partition*

$$a = \tilde{a}_0 \le \tilde{a}_1 \le \cdots \le \tilde{a}_{m+1} = b. \tag{7.2.34}$$

Assume we also have an analytic continuation of v along this chain, given by holomorphic functions $\tilde{v}_{\widetilde{D}_j}$ on $\widetilde{D}_j$. Then

$$\tilde{v}_{\widetilde{D}_m} = v_{D_n} \quad \text{on the neighborhood } \widetilde{D}_m \cap D_n \text{ of } \gamma(b). \tag{7.2.35}$$

Proof. We first show that the conclusion holds when the two partitions are equal, i.e., $m = n$ and $\tilde{a}_j = a_j$, but the sets D_j and $\widetilde{D}_j$ may differ. In such a case, $D_j^\# = D_j \cap \widetilde{D}_j$ is also an open, convex set satisfying $D_j^\# \supset \gamma([a_j, a_{j+1}])$. The application of Proposition 2.6.1 and induction on j shows that if $v_{D_0} = \tilde{v}_{\widetilde{D}_0}$ on $D_0^\#$, then $v_{D_j} = \tilde{v}_{\widetilde{D}_j}$ on $D_j^\#$, for each j, which yields (7.2.35).

It remains to show that two analytic continuations of v on D_0, along chains associated to two different partitions of $[a, b]$, agree on a neighborhood of $\gamma(b)$. To see this, note that two partitions of $[a, b]$ have a common refinement, so it suffices to show such agreement when the partition (7.2.30) is augmented by adding one element, say $\tilde{a}_\ell \in (a_\ell, a_{\ell+1})$. We can obtain a chain associated to this partition by taking the chain $\{D_0, \dots, D_n\}$ associated to the partition (7.2.30), as in (7.2.32), and "adding" $\widetilde{D}_\ell = D_\ell \supset \gamma([a_\ell, a_{\ell+1}]) = \gamma([a_\ell, \tilde{a}_\ell]) \cup \gamma([\tilde{a}_\ell, a_{\ell+1}])$. That the conclusion of Proposition 7.2.2 holds in this case is clear. ☐

Proposition 7.2.2 motivates us to introduce the following notation. Let $\mathcal{O}_0$ and $\mathcal{O}_1$ be open convex neighborhoods of z_0 and z_1 in Ω, and let γ be a continuous path from z_0 to z_1, as in (7.2.29). Let v be holomorphic on $\mathcal{O}_0$. Assume there exists a chain $\{D_0, \dots, D_n\}$ along γ such that $D_0 = \mathcal{O}_0$ and $D_n = \mathcal{O}_1$, and an analytic continuation of v along this chain, as in (7.2.33). We set

$$v_\gamma : \mathcal{O}_1 \longrightarrow \mathbb{C} \tag{7.2.36}$$

equal to v_{D_n} on $D_n = \mathcal{O}_1$. By Proposition 7.2.2, this is independent of the choice of chain along γ for which such an analytic continuation exists.

We next supplement Proposition 7.2.2 with another key uniqueness result, called the *monodromy theorem.* To formulate it, suppose we have a homotopic family of curves

$$\gamma_s : [a,b] \longrightarrow \Omega, \quad \gamma_s(a) \equiv z_0,\ \gamma_s(b) \equiv z_1, \quad 0 \le s \le 1, \tag{7.2.37}$$

so $\gamma(s,t) = \gamma_s(t)$ defines a continuous map $\gamma : [0,1] \times [a,b] \to \Omega$. As above, assume $\mathcal{O}_1$ and $\mathcal{O}_1$ are convex open neighbrhoods of z_0 and z_1 in Ω, and that v is holomorphic on $\mathcal{O}_0$.

Proposition 7.2.3. *Assume that for each $s \in [0,1]$ there is a chain along γ_s from $\mathcal{O}_0$ to $\mathcal{O}_1$ and an analytic continuation of v along this chain, producing*

$$v_{\gamma_s} : \mathcal{O}_1 \longrightarrow \mathbb{C}. \tag{7.2.38}$$

Then v_{γ_s} is independent of $s \in [0,1]$.

Proof. Take $s_0 \in [0,1]$, and let $\{D_0, \dots, D_n\}$ be a chain along γ_{s_0}, associated to a partition of $[a,b]$ of the form (7.2.30), satisfying $D_0 = \mathcal{O}_0$, $D_n = \mathcal{O}_1$, along which v has an analytic continuation. Then there exists $\varepsilon > 0$ such that, for each $s \in [0,1]$ such that $|s - s_0| \le \varepsilon$, $\{D_0, \dots, D_n\}$ is also a chain along γ_s. The fact that $v_{\gamma_s} = v_{\gamma_{s_0}}$ for all such s follows from Proposition 7.2.2. This observation readily leads to the conclusion of Proposition 7.2.3. □

Let us return to the setting of Proposition 7.2.1 and the discussion of analytic continuation along a curve in Ω, involving (7.2.28)–(7.2.31). Combining these results with Proposition 7.2.3, we have the following.

Proposition 7.2.4. *Consider the differential equation* (7.2.3)*, where A and f satisfy* (7.2.4) *and $z_0 \in \Omega$. If Ω is simply connected, then the solution v on $D_{R_0}(z_0)$ produced in Proposition* 7.2.1 *has a unique extension to a holomorphic function $v : \Omega \to \mathbb{C}$, satisfying* (7.2.3) *on Ω.*

Returning to Bessel's equation (7.2.2) and the first-order 2×2 system arising from it, we see that Proposition 7.2.4 applies, not to $\Omega = \mathbb{C} \setminus 0$, but to a simply connected subdomain of $\mathbb{C} \setminus 0$, such as $\Omega = \mathbb{C} \setminus (-\infty, 0]$. Indeed, as we saw in (7.1.17)–(7.1.18), the solution $J_\nu(z)$ to (7.2.2) has the form

$$J_\nu(z) = z^\nu \mathcal{J}_\nu(z), \quad \text{for } z \in \mathbb{C} \setminus (-\infty, 0], \tag{7.2.39}$$

with $\mathcal{J}_\nu(z)$ holomorphic in $z \in \mathbb{C}$, for each $\nu \in \mathbb{C}$. Subsequent calculations, involving (7.1.35)–(7.1.44), show that the solution $Y_\nu(z)$ to (7.2.2) has the form

$$Y_\nu(z) = z^\nu \mathcal{A}_\nu(z) + z^{-\nu} \mathcal{B}_\nu(z), \quad \text{for } z \in \mathbb{C} \setminus (-\infty, 0], \tag{7.2.40}$$

with $\mathcal{A}_\nu(z)$ and $\mathcal{B}_\nu(z)$ holomorphic in $z \in \mathbb{C}$, for each $\nu \in \mathbb{C} \setminus \mathbb{Z}$, and, for $n \in \mathbb{Z}$,

$$Y_n(z) = \frac{2}{\pi}\Bigl(\log \frac{z}{2}\Bigr) J_n(z) + \mathcal{A}_n(z), \quad \text{for } \ z \in \mathbb{C} \setminus (-\infty, 0], \tag{7.2.41}$$

where $\mathcal{A}_n(z)$ is holomorphic on $\mathbb{C}$ if $n = 0$, and is meromorphic with a pole of order n at $z = 0$ if $n \in \mathbb{N}$.

Bessel's equation (7.2.2) belongs to the more general class of second order linear differential equations of the form

$$z^2 u''(z) + z b(z) u'(z) + c(z) u(z) = 0, \tag{7.2.42}$$

where $c(z)$ and $b(z)$ are holomorphic on some disk $D_a(0)$. For (7.2.2),

$$b(z) = 1, \quad c(z) = z^2 - \nu^2. \tag{7.2.43}$$

To convert (7.2.42) to a first-order system, instead of (7.2.5), it is convenient to set

$$v(z) = \begin{pmatrix} u(z) \\ z u'(z) \end{pmatrix}. \tag{7.2.44}$$

Then the differential equation for v becomes

$$z \frac{dv}{dz} = A(z) v(z), \tag{7.2.45}$$

with

$$A(z) = \begin{pmatrix} 0 & 1 \\ -c(z) & 1 - b(z) \end{pmatrix}. \tag{7.2.46}$$

Generally, if

$$A : D_a(0) \longrightarrow M(n, \mathbb{C}) \ \text{ is holomorphic}, \tag{7.2.47}$$

the $n \times n$ system (7.2.45) is said to have a regular singular point at $z = 0$. By Proposition 7.2.4, if (7.2.47) holds and we set

$$\Omega = D_a(0) \setminus (-a, 0], \quad z_0 \in \Omega, \tag{7.2.48}$$

and take $v_0 \in \mathbb{C}^n$, then the system (7.2.45) has a unique solution, holomorphic on Ω, satisfying $v(z_0) = v_0$. We now desire to understand more precisely how such a solution $v(z)$ behaves as $z \to 0$, in light of the results of (7.2.39)–(7.2.41).

As a simple example, take $A(z) \equiv A_0 \in M(n, \mathbb{C})$, so (7.2.45) becomes

$$z \frac{dv}{dz} = A_0 v. \tag{7.2.49}$$

If we set $z_0 = 1$ and $v(1) = v_0$, the solution is

$$v(z) = e^{(\log z) A_0} v_0, \quad z \in \mathbb{C} \setminus (-\infty, 0], \tag{7.2.50}$$

where $e^{\zeta A_0}$ is the matrix exponential, as discussed in Appendix 3.5. If v_0 is an eigenvector of A_0,

$$A_0 v_0 = \lambda v_0 \Longrightarrow v(z) = z^\lambda v_0. \tag{7.2.51}$$

From (7.2.42)–(7.2.46), we see that, for the system arising from Bessel's equation,

$$A(z) = A_0 + A_2 z^2, \quad A_0 = \begin{pmatrix} 0 & 1 \\ \nu^2 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{7.2.52}$$

and the eigenvalues of A_0 are $\pm\nu$. This is a preliminary indication of a connection between (7.2.51) and (7.2.39)–(7.2.41). The following result will lead to a strengthening of this connection.

Proposition 7.2.5. *Assume $B : D_a(0) \to M(n, \mathbb{C})$ is holomorphic, and set $B_0 = B(0)$. Assume that 0 is an eigenvalue of B_0 but that B_0 has no eigenvalues that are positive integers. If $B_0 v_0 = 0$, then the system*

$$z\frac{dv}{dz} = B(z)v \tag{7.2.53}$$

has a solution v, holomorphic on $D_a(0)$, satisfying $v(0) = v_0$.

Proof. Parallel to the proof of Proposition 7.2.1, we set

$$v(z) = \sum_{k=0}^{\infty} z^k v_k, \tag{7.2.54}$$

and produce a recursive formula for the coefficients v_k, given $v_0 \in \mathbb{C}^n$ such that $B_0 v_0 = 0$. Granted convergence of (7.2.54), we have

$$z\frac{dv}{dz} = \sum_{k\ge 1} k v_k z^k \tag{7.2.55}$$

and

$$\begin{aligned} B(z)v &= \sum_{j\ge 0} B_j z^j \sum_{\ell \ge 0} v_\ell z^\ell \\ &= B_0 v_0 + \sum_{k\ge 1}\sum_{\ell=0}^{k} B_{k-\ell} v_\ell z^k. \end{aligned} \tag{7.2.56}$$

Having $B_0 v_0 = 0$, we obtain the following recursive formula, for $k \ge 1$:

$$k v_k = B_0 v_k + \sum_{\ell=0}^{k-1} B_{k-\ell} v_\ell, \tag{7.2.57}$$

or equivalently

$$(kI - B_0) v_k = \sum_{\ell=0}^{k-1} B_{k-\ell} v_\ell. \tag{7.2.58}$$

We can solve uniquely for v_k provided k is not an eigenvalue of B_0. If no positive integer is an eigenvalue of B_0, we can solve for the coefficients v_k for all $k \in \mathbb{N}$, obtaining the series (7.2.54). Estimates on these coefficients, verifying that (7.2.54) converges, are similar to those made in the proof of Proposition 7.2.1. □

To apply Proposition 7.2.5 to the system (7.2.45), under the hypothesis (7.2.47), assume λ is an eigenvalue of $A_0 = A(0)$, so

$$(A_0 - \lambda)v_0 = 0, \tag{7.2.59}$$

for some nonzero $v_0 \in \mathbb{C}^n$. If we set

$$v(z) = z^\lambda w(z), \quad z \in \Omega = D_a(0) \setminus (-a, 0], \tag{7.2.60}$$

the equation (7.2.45) is converted to

$$z\frac{dw}{dz} = (A(z) - \lambda)w. \tag{7.2.61}$$

Hence Proposition 7.2.5 applies, with $B(z) = A(z) - \lambda$, and we have:

Proposition 7.2.6. *Assume $A : D_a(0) \to M(n, \mathbb{C})$ is holomorphic and λ is an eigenvalue of $A_0 = A(0)$, with eigenvector v_0. Assume that $\lambda + k$ is not an eigenvalue of A_0 for any $k \in \mathbb{N}$. Then the system* (7.2.45) *has a solution of the form* (7.2.60)*, where*

$$w : D_a(0) \to \mathbb{C}^n \;\; \text{is holomorphic and} \;\; w(0) = v_0. \tag{7.2.62}$$

For Bessel's equation, whose associated first-order system (7.2.45) takes $A(z)$ as in (7.2.52), we have seen that

$$A_0 = \begin{pmatrix} 0 & 1 \\ \nu^2 & 0 \end{pmatrix} \;\; \text{has eigenvalues } \pm\nu \tag{7.2.63}$$

and

$$(A_0 - \nu I)\begin{pmatrix} 1 \\ \nu \end{pmatrix} = 0. \tag{7.2.64}$$

Thus Proposition 7.2.6 provides a solution to (7.2.45) of the form

$$v_\nu(z) = z^\nu w_\nu(z), \quad w_\nu \text{ holomorphic on } \mathbb{C}, \quad w_\nu(0) = \begin{pmatrix} 1 \\ \nu \end{pmatrix}, \tag{7.2.65}$$

as long as

$$-2\nu \notin \mathbb{N}. \tag{7.2.66}$$

Comparison with (7.2.39)–(7.2.41), and the supporting calculations from §7.1, shows that the condition (7.2.66) is stronger than necessary for the conclusion given in (7.2.65). The condition ought to be

$$-\nu \notin \mathbb{N}. \tag{7.2.67}$$

To get this stronger conclusion, we can take into account the special structure of (7.2.52) and bring in the following variant of Proposition 7.2.5.

Proposition 7.2.7. *In the setting of Proposition* 7.2.5*, assume that* $B : D_a(0) \to M(n, \mathbb{C})$ *is holomorphic and even in* z*, and set* $B_0 = B(0)$*. Assume that* 0 *is an eigenvalue of* B_0 *(with associated eigenvector* v_0*) and that* B_0 *has no eigenvalues that are positive even integers. Then the equation* (7.2.53) *has a solution* v*, holomorphic and even on* $D_a(0)$*, satisfying* $v(0) = v_0$*.*

Proof. This is a simple variant of the proof of Proposition 7.2.5. Replace (7.2.54) by

$$v(z) = \sum_{k=0}^{\infty} z^{2k} v_{2k}, \tag{7.2.68}$$

and replace (7.2.56) by

$$\begin{aligned} B(z)v &= \sum_{j\ge 0} B_{2j} z^{2j} \sum_{\ell \ge 0} v_{2\ell} z^{2\ell} \\ &= B_0 v_0 + \sum_{k\ge 1} \sum_{\ell=0}^{k} B_{2k-2\ell} v_{2\ell} z^{2k}. \end{aligned} \tag{7.2.69}$$

Then the recursion (7.2.58) is replaced by

$$(2kI - B_0) v_{2k} = \sum_{\ell=0}^{k-1} B_{2k-2\ell} v_{2\ell}, \tag{7.2.70}$$

and the result follows. □

Proposition 7.2.8. *In the setting of Proposition* 7.2.6*, assume* $A : D_a(0) \to M(n, \mathbb{C})$ *is holomorphic and even in* z*, and* λ *is an eigenvalue of* $A_0 = A(0)$*, with eigenvector* v_0*. Assume* $\lambda + 2k$ *is not an eigenvalue of* A_0 *for any* $k \in \mathbb{N}$*. Then the system* (7.2.45) *has a solution of the form* (7.2.60)*, with* (7.2.62) *holding.*

We see that Proposition 7.2.8 applies to the Bessel system

$$z\frac{dv}{dz} = \begin{pmatrix} 0 & 1 \\ \nu^2 - z^2 & 0 \end{pmatrix} v, \tag{7.2.71}$$

to produce a 2D space of solutions, holomorphic on $\Omega = \mathbb{C} \setminus (-\infty, 0]$, if $\nu \in \mathbb{C} \setminus \mathbb{Z}$, and a 1D space of solutions to this system if $\nu = n \in \mathbb{Z}$ (which one verifies to be the same for $\nu = n$ and $\nu = -n$). Results of §7.1 show that the 2D space of solutions to (7.2.71) is spanned by

$$\begin{pmatrix} J_\nu(z) \\ zJ'_\nu(z) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} J_{-\nu}(z) \\ zJ'_{-\nu}(z) \end{pmatrix}, \quad \text{for } \nu \in \mathbb{C} \setminus \mathbb{Z}, \tag{7.2.72}$$

and the 1D space of solutions given by Proposition 7.2.8 is spanned by

$$\begin{pmatrix} J_n(z) \\ zJ_n'(z) \end{pmatrix}, \quad \text{for} \quad \nu = n \in \mathbb{Z}. \tag{7.2.73}$$

Also results of §7.1 imply that the 2D space of solutions to (7.2.71) is spanned by (7.2.73) and

$$\begin{pmatrix} Y_n(z) \\ zY_n'(z) \end{pmatrix}, \quad \text{for} \quad \nu = n \in \mathbb{Z}. \tag{7.2.74}$$

As seen in (7.2.41), these last solutions do not behave as in (7.2.65), since factors of $\log z$ appear.

To obtain results more general than those of Propositions 7.2.6 and 7.2.8, we take the following approach. We seek a holomorphic map

$$U : D_a(0) \longrightarrow M(n, \mathbb{C}), \quad U(0) = I, \tag{7.2.75}$$

such that, under the change of variable

$$v(z) = U(z)w(z), \tag{7.2.76}$$

(7.2.45) becomes

$$z\frac{dw}{dz} = A_0 w, \tag{7.2.77}$$

a case already treated in (7.2.49)–(7.2.50). To reiterate, the general solution to (7.2.77) on $\mathbb{C} \setminus (-\infty, 0]$ is

$$w(z) = e^{(\log z)A_0} v_0 = z^{A_0} v_0, \quad v_0 \in \mathbb{C}^n, \tag{7.2.78}$$

the latter identity defining z^{A_0} for $z \in \mathbb{C} \setminus (-\infty, 0]$. To construct such $U(z) = \sum_{k \geq 0} U_k z^k$, we start with

$$A(z)U(z)w = z\frac{dv}{dz} = zU(z)\frac{dw}{dz} + zU'(z)w, \tag{7.2.79}$$

which leads to (7.2.77) provided $U(z)$ satisfies

$$z\frac{dU}{dz} = A(z)U(z) - U(z)A_0. \tag{7.2.80}$$

This equation has the same form as (7.2.45), i.e.,

$$z\frac{dU}{dz} = \mathcal{A}(z)U(z), \tag{7.2.81}$$

where $U(z)$ takes values in $M(n, \mathbb{C})$ and $\mathcal{A}(z)$ is a linear transformation on the vector space $M(n, \mathbb{C})$:

$$\begin{aligned} \mathcal{A}(z)U &= A(z)U - UA_0 \\ &= \sum_{k \geq 0} \mathcal{A}_k z^k U. \end{aligned} \tag{7.2.82}$$

In particular,

$$\mathcal{A}_0 U = A_0 U - U A_0 = [A_0, U] = C_{A_0} U, \tag{7.2.83}$$

the latter identity defining

$$C_{A_0} : M(n, \mathbb{C}) \longrightarrow M(n, \mathbb{C}). \tag{7.2.84}$$

Note that $C_{A_0} I = [A_0, I] = 0$, so 0 is an eigenvalue of C_{A_0}, with eigenvector $I = U(0)$. Hence Proposition 7.2.5 applies, with $\mathbb{C}^n$ replaced by $M(n, \mathbb{C})$ and $B_0 = C_{A_0}$. The hypothesis that B_0 has no eigenvalue that is a positive integer is hence that C_{A_0} has no eigenvalue that is a positive integer. One readily verifies that the set $\operatorname{Spec} C_{A_0}$ of eigenvalues of C_{A_0} is described as follows:

$$\operatorname{Spec} A_0 = \{\lambda_j\} \Longrightarrow \operatorname{Spec} C_{A_0} = \{\lambda_j - \lambda_k\}. \tag{7.2.85}$$

Thus the condition that $\operatorname{Spec} C_{A_0}$ contains no positive integer is equivalent to the condition that A_0 have no pair of eigenvalues that differ by a nonzero integer. This establishes the following.

Proposition 7.2.9. *Assume $A : D_a(0) \to M(n, \mathbb{C})$ is holomorphic, and $A(0) = A_0$. Assume A_0 has no two eigenvalues that differ by a nonzero integer. Then there is a holomorphic map U, as in* (7.2.75)*, such that the general solution to* (7.2.45) *on $D_a(0) \setminus (-a, 0]$ is given by*

$$v(z) = U(z) z^{A_0} v_0, \quad v_0 \in \mathbb{C}^n. \tag{7.2.86}$$

Let us comment that, in this setting, the recursion (7.2.57)–(7.2.58) becomes

$$k U_k = [A_0, U_k] + \sum_{\ell=0}^{k-1} A_{k-\ell} U_\ell, \tag{7.2.87}$$

i.e.,

$$(kI - C_{A_0}) U_k = \sum_{\ell=0}^{k-1} A_{k-\ell} U_\ell, \tag{7.2.88}$$

for $k \geq 1$.

In cases where $A(z)$ is an even function of z, we can replace Proposition 7.2.5 by Proposition 7.2.7, and obtain the following.

Proposition 7.2.10. *In the setting of Proposition* 7.2.9*, assume $A(z)$ is even. Then the conclusion holds as long as A_0 has no two eigenvalues that differ by a nonzero even integer.*

Comparing Propositions 7.2.6 and 7.2.8 with Propositions 7.2.9 and 7.2.10, we see that the latter pair do not involve more general hypotheses on A_0 but they do have a stronger conclusion, when A_0 is not diagonalizable.

To illustrate this, we take the case $\nu = 0$ of the Bessel system (7.2.71), for which

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{7.2.89}$$

In this case, $A_0^2 = 0$, and the power series representation of $e^{\zeta A_0}$ gives

$$z^{A_0} = e^{(\log z)A_0} = I + (\log z)A_0 = \begin{pmatrix} 1 & \log z \\ 0 & 1 \end{pmatrix}. \tag{7.2.90}$$

Thus we see that (7.2.86) captures the full 2D space of solutions to this Bessel equation, and the $\log z$ factor in the behavior of $Y_0(z)$ is manifest.

On the other hand, Propositions 7.2.9–7.2.10 are not effective for the Bessel system (7.2.71) when $\nu = n$ is a nonzero integer, in which case

$$A_0 = \begin{pmatrix} 0 & 1 \\ n^2 & 0 \end{pmatrix}, \quad \operatorname{Spec} A_0 = \{-n, n\}, \quad \operatorname{Spec} C_{A_0} = \{-2n, 0, 2n\}. \tag{7.2.91}$$

We next state a result that does apply to such situations.

Proposition 7.2.11. *Let $A : D_a(0) \to M(n, \mathbb{C})$ be holomorphic, with $A(0) = A_0$. Assume*

$$A_0 \text{ is diagonalizable} \tag{7.2.92}$$

and

$$\operatorname{Spec} C_{A_0} \text{ contains exactly one positive integer } \ell. \tag{7.2.93}$$

Then there exists $B_\ell \in M(n, \mathbb{C})$ such that

$$[A_0, B_\ell] = \ell B_\ell, \tag{7.2.94}$$

and a holomorphic function $U(z)$, as in (7.2.75)*, such that the general solution to* (7.2.45) *on $D_a(0) \setminus (-a, 0]$ is given by*

$$v(z) = U(z) z^{A_0} z^{B_\ell} v_0, \quad v_0 \in \mathbb{C}^n. \tag{7.2.95}$$

Furthermore,

$$B_\ell \text{ is nilpotent.} \tag{7.2.96}$$

We refer to Proposition 11.5 in Chapter 3 of [**48**] for a proof, remarking that the crux of the matter involves modifying (7.2.77) to

$$z \frac{dw}{dz} = (A_0 + B_\ell z^\ell) w, \tag{7.2.97}$$

thus replacing (7.2.80) by

$$z \frac{dU}{dz} = A(z)U(z) - U(z)(A_0 + B_\ell z^\ell), \tag{7.2.98}$$

and consequently modifying (7.2.88) to something that works for $k = \ell$, as well as other values of $k \in \mathbb{N}$.

To say that B_ℓ is nilpotent is to say that $B_\ell^{m+1} = 0$ for some $m \in \mathbb{N}$. In such a case, we have

$$z^{B_\ell} = \sum_{k=0}^{m} \frac{1}{k!} (\log z)^k B_\ell^k. \tag{7.2.99}$$

If (7.2.45) is a 2×2 system, nilpotence implies $B_\ell^2 = 0$, and hence

$$z^{B_\ell} = I + (\log z) B_\ell, \tag{7.2.100}$$

as in (7.2.90). For larger systems, higher powers of $\log z$ can appear.

Note that if (7.2.45) is a 2×2 system and if (7.2.92) fails, then A_0 has just one eigenvalue, so Proposition 7.2.9 applies. On the other hand, if (7.2.92) holds, then

$$\operatorname{Spec} A_0 = \{\lambda_1, \lambda_2\} \Longrightarrow \operatorname{Spec} C_{A_0} = \{\lambda_1 - \lambda_2, 0, \lambda_2 - \lambda_1\}, \tag{7.2.101}$$

so either (7.2.93) holds or Proposition 7.2.9 applies.

We refer to Chapter 3 of [**48**] for further results, more general than Proposition 7.2.11, applicable to $n \times n$ systems of the form (7.2.45) when $n \geq 3$.

We have discussed differential equations with a regular singular point at 0. Similarly, a first-order system has a regular singular point at $z_0 \in \Omega$ if it is of the form

$$(z - z_0) \frac{dv}{dz} = A(z) v, \tag{7.2.102}$$

with $A : \Omega \to M(n, \mathbb{C})$ holomorphic. More generally, if $\Omega \subset \mathbb{C}$ is a connected open set, and $\mathcal{F} \subset \Omega$ is a finite subset, and if $B : \Omega \setminus \mathcal{F} \to M(n, \mathbb{C})$ is holomorphic, the system

$$\frac{dv}{dz} = B(z) v \tag{7.2.103}$$

is said to have a regular singular point at $z_0 \in \mathcal{F}$ provided $(z - z_0) B(z)$ extends to be holomorphic on a neighborhood of z_0. Related to this, if $A, B, C : \Omega \setminus \mathcal{F} \to \mathbb{C}$ are holomorphic, the second-order differential equation

$$A(z) u''(z) + B(z) u'(z) + C(z) u(z) = 0 \tag{7.2.104}$$

has a regular singular point at $z_0 \in \mathcal{F}$ provided

$$\begin{aligned} &A(z), \quad (z - z_0) B(z), \quad \text{and} \quad (z - z_0)^2 C(z) \\ &\text{extend to be holomorphic on a neighborhood of } z_0, \\ &\text{and} \quad A(z_0) \neq 0. \end{aligned} \tag{7.2.105}$$

An example is the class of Legendre equations

$$(1 - z^2) u''(z) - 2z u'(z) + \Big[\nu^2 - \frac{\mu^2}{1 - z^2}\Big] u(z) = 0, \tag{7.2.106}$$

produced in (7.4.28), which has the form (7.2.104)–(7.2.105), upon division by $1 - z^2$, with regular singular points at $z = \pm 1$, for all $\nu, \mu \in \mathbb{C}$, including $\mu = 0$, when (7.2.106) reduces to

$$(1 - z^2)u''(z) + 2zu'(z) + \nu^2 u(z) = 0. \tag{7.2.107}$$

Proposition 7.2.4 implies that if Ω is a simply connected subdomain of $\mathbb{C} \setminus \{-1, 1\}$, then (7.2.106) has a two-dimensional space of solutions that are holomorphic on Ω. Such Ω might, for example, be

$$\Omega = \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}. \tag{7.2.108}$$

Propositions 7.2.9 and 7.2.11 apply to first-order systems of the form (7.2.102), with $z_0 = \pm 1$, derivable from (7.2.106).

We next consider special functions defined by what are called hypergeometric series, and the differential equations they satisfy. The basic cases are

$$_1F_1(a; b; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!} \tag{7.2.109}$$

and

$$_2F_1(a_1, a_2; b; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b)_k} \frac{z^k}{k!}. \tag{7.2.110}$$

Here $(a)_0 = 1$ and, for $k \in \mathbb{N}$,

$$(a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, \tag{7.2.111}$$

the latter identity holding provided $a \notin \{0, -1, -2, \dots\}$. In (7.2.109)–(7.2.110), we require $b \notin \{0, -1, -2, \dots\}$. Then the series (7.2.109) converges for all $z \in \mathbb{C}$. The series (7.2.110) converges for $|z| < 1$. The function $_1F_1$ is called the confluent hypergeometric function, and $_2F_1$ is called the hypergeometric function. Differentiation of these power series implies that $_1F_1(a; b; z) = u(z)$ satisfies the differential equation

$$zu''(z) + (b - z)u'(z) - au(z) = 0, \tag{7.2.112}$$

and that $_2F_1(a_1, a_2; b; z) = u(z)$ satisfies the differential equation

$$z(1 - z)u''(z) + [b - (a_1 + a_2 + 1)z]u'(z) - a_1 a_2 u(z) = 0. \tag{7.2.113}$$

The equations (7.2.112)–(7.2.113) are called, respectively, the confluent hypergeometric equation and the hypergeometric equation. For (7.2.112), $z = 0$ is a regular singular point, and for (7.2.113), $z = 0$ and $z = 1$ are regular singular points. By Proposition 7.2.4, a solution to (7.2.113) has an analytic continuation to any simply connected domain $\Omega \subset \mathbb{C} \setminus \{0, 1\}$, such as

$$\Omega = \mathbb{C} \setminus \{(-\infty, 0] \cup [1, \infty)\}. \tag{7.2.114}$$

Such a continuation has jumps across $(-\infty, 0)$ and across $(1, \infty)$. However, we know that ${}_2F_1(a_1, a_2; b; z)$ is holomorphic in $|z| < 1$, so the jump across the segment $(-1, 0)$ in $(-\infty, 0)$ vanishes. Hence this jump vanishes on all of $(-\infty, 0)$. We deduce that, as long as $b \notin \{0, -1, -2, \dots\}$,

$$\begin{aligned} {}_2F_1(a_1, a_2; b; z) \ \text{ continues analytically to} \\ z \in \mathbb{C} \setminus [1, \infty). \end{aligned} \tag{7.2.115}$$

As we know, whenever $b \notin \{0, -1, -2, \dots\}$, both (7.2.112) and (7.2.113) have a 2D space of solutions, on their domains, respectively $\mathbb{C} \setminus (-\infty, 0]$ and (7.2.114). Concentrating on the regular singularity at $z = 0$, we see that each of these differential equations, when converted to a first-order system as in (7.2.42)–(7.2.46), yields a system of the form (7.2.45) with

$$A(0) = A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 - b \end{pmatrix}. \tag{7.2.116}$$

Clearly the eigenvalues of A_0 are 0 and $1 - b$. Given that we already require $b \notin \{0, -1, -2, \dots\}$, we see that Proposition 7.2.6 applies, with $\lambda = 1 - b$, as long as

$$b \notin \{2, 3, 4, \dots\}. \tag{7.2.117}$$

Then, as long as (7.2.117) holds, we expect to find another solution to each of (7.2.112) and (7.2.113), of the form (7.2.60), with $w(z)$ holomorphic on a neighborhood of $z = 0$ and $\lambda = 1 - b$. In fact, one can verify that, if (7.2.117) holds,

$$z^{1-b}\, {}_1F_1(1 + a - b; 2 - b; z) \tag{7.2.118}$$

is also a solution to (7.2.112). In addition,

$$z^{1-b}\, {}_2F_1(a_1 - b + 1, a_2 - b + 1; 2 - b; z) \tag{7.2.119}$$

is also a solution to (7.2.113).

One reason the hypergeometric functions are so important is that many other functions can be expressed using them. These range from elementary functions, such as

$$\begin{aligned} e^z &= {}_1F_1(1; 1; z), \\ \log(1 - z) &= -z\, {}_2F_1(1, 1; 2; z), \\ \sin^{-1} z &= z\, {}_2F_1\Bigl(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\Bigr), \end{aligned} \tag{7.2.120}$$

to special functions, such as Bessel functions,

$$J_\nu(z) = \frac{e^{-iz}}{\Gamma(\nu+1)}\Big(\frac{z}{2}\Big)^\nu {}_1F_1\Big(\nu+\frac{1}{2}; 2\nu+1; 2iz\Big), \tag{7.2.121}$$

elliptic integrals,

$$\begin{aligned}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-z^2\sin^2\theta}} &= \frac{\pi}{2}\,{}_2F_1\Big(\frac{1}{2},\frac{1}{2};1;z^2\Big),\\ \int_0^{\pi/2} \sqrt{1-z^2\sin^2\theta}\,d\theta &= \frac{\pi}{2}\,{}_2F_1\Big(-\frac{1}{2},\frac{1}{2};1;z^2\Big),\end{aligned} \tag{7.2.122}$$

and many other examples, which can be found in Chapter 9 of [**27**].

The hypergeometric functions (7.2.109)–(7.2.110) are part of a hierarchy, defined as follows. Take $p, q \in \mathbb{N},\ p \le q+1$, and

$$a = (a_1,\dots,a_p), \quad b = (b_1,\dots,b_q), \quad b_j \notin \{0,-1,-2,\dots\}. \tag{7.2.123}$$

Then we define the generalized hypergeometric functions

$${}_pF_q(a;b;z) = \sum_{k=0}^\infty \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\,\frac{z^k}{k!}, \tag{7.2.124}$$

with $(a_j)_k$ defined as in (7.2.111). Equivalently, if also $a_j \notin \{0,-1,-2,\dots\}$,

$${}_pF_q(a;b;z) = \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \sum_{k=0}^\infty \frac{\Gamma(a_1+k)\cdots\Gamma(a_p+k)}{\Gamma(b_1+k)\cdots\Gamma(b_q+k)}\,\frac{z^k}{k!}. \tag{7.2.125}$$

This power series converges for all $z \in \mathbb{C}$ if $p < q+1$, and for $|z| < 1$ if $p = q+1$. The function ${}_pF_q(a;b;z) = u(z)$ satisfies the differential equation

$$\Big[\frac{d}{dz}\prod_{j=1}^q\Big(z\frac{d}{dz}+b_j-1\Big) - \prod_{\ell=1}^p\Big(z\frac{d}{dz}+a_\ell\Big)\Big]\,u = 0, \tag{7.2.126}$$

which reduces to (7.2.112) if $p = q = 1$ and to (7.2.113) if $p = 2, q = 1$. In case $p = q+1$, we see that this differential equation has the form

$$z^q(1-z)\frac{d^pu}{dz^p} + \cdots = 0, \tag{7.2.127}$$

with singular points at $z = 0$ and $z = 1$, leading to a holomorphic continuation of ${}_pF_q(a;b;z)$, first to the domain (7.2.114) and then, since the jump across $(-1,0)$ vanishes, to $\mathbb{C}\setminus[1,\infty)$.

As an example of a special function expressible via ${}_4F_3$, we consider the Bring radical, Φ^{-1}, defined in Appendix A.6 as the holomorphic map

$$\Phi^{-1} : D_{(4/5)5^{-1/4}}(0) \longrightarrow \mathbb{C}, \quad \Phi^{-1}(0) = 0, \tag{7.2.128}$$

that inverts

$$\Phi(z) = z - z^5. \tag{7.2.129}$$

As shown in (A.6.97), we have

$$\Phi^{-1}(z) = \sum_{k=0}^{\infty} \binom{5k}{k} \frac{z^{4k+1}}{4k+1}, \tag{7.2.130}$$

for $|z| < (4/5)5^{-1/4}$. This leads to the following identity.

Proposition 7.2.12. *The Bring radical Φ^{-1} is given by*

$$\Phi^{-1}(z) = z\, {}_4F_3\Bigl(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; 5\Bigl(\frac{5z}{4}\Bigr)^4\Bigr). \tag{7.2.131}$$

Proof. To start, we write

$$\binom{5k}{k} = \frac{(5k)!}{(4k)!k!} = \frac{\Gamma(5k+1)}{\Gamma(4k+1)k!} = \frac{5}{4}\,\frac{\Gamma(5k)}{\Gamma(4k)}\,\frac{1}{k!}. \tag{7.2.132}$$

Next, the Gauss formula (4.3.46) yields

$$\frac{\Gamma(5k)}{\Gamma(4k)} = (2\pi)^{-1/2} \frac{5^{5k-1/2}}{4^{4k-1/2}}\, \frac{\Gamma(k+\frac{1}{5})\Gamma(k+\frac{2}{5})\Gamma(k+\frac{3}{5})\Gamma(k+\frac{4}{5})}{\Gamma(k+\frac{1}{4})\Gamma(k+\frac{2}{4})\Gamma(k+\frac{3}{4})}. \tag{7.2.133}$$

Hence, for $|z| < (4/5)5^{-1/4}$, $\Phi^{-1}(z)$ is equal to

$$\begin{aligned} &(2\pi)^{-1/2} \frac{5^{1/2}}{4^{1/2}} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{5}+k)\Gamma(\frac{2}{5}+k)\Gamma(\frac{3}{5}+k)\Gamma(\frac{4}{5}+k)}{\Gamma(\frac{1}{4}+k)\Gamma(\frac{2}{4}+k)\Gamma(\frac{3}{4}+k)} \cdot \frac{1}{4k+1} \\ &\qquad\qquad \times 5^k \Bigl(\frac{5}{4}\Bigr)^k z^{4k+1}. \end{aligned} \tag{7.2.134}$$

Using

$$(4k+1)\Gamma(k+\tfrac{1}{4}) = 4\Gamma(k+\tfrac{5}{4}), \tag{7.2.135}$$

we obtain

$$\begin{aligned} \Phi^{-1}(z) &= (2\pi)^{-1/2} \frac{5^{1/2}}{2}\, \frac{\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})}{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})\Gamma(\frac{3}{4})} \cdot z \\ &\qquad \times {}_4F_3\Bigl(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; 5\Bigl(\frac{5z}{4}\Bigr)^4\Bigr). \end{aligned} \tag{7.2.136}$$

It remains to evaluate the constant factor that precedes z. Using the identity

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \tag{7.2.137}$$

and $\Gamma(1/2) = \sqrt{\pi}$, we see that this factor is equal to

$$\frac{\sqrt{5}}{4} \cdot \frac{1}{\sin\frac{\pi}{5} \cdot \sin\frac{2\pi}{5}}, \tag{7.2.138}$$

since $\sin \pi/4 = 1/\sqrt{2}$. Now, by (A.6.35),

$$\cos \frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}, \quad \cos \frac{\pi}{5} = \frac{\sqrt{5}+1}{4}, \tag{7.2.139}$$

and then we can readily compute $(1-\cos^2(\pi/5))(1-\cos^2(2\pi/5))$ and show that

$$\sin \frac{\pi}{5} \cdot \sin \frac{2\pi}{5} = \frac{\sqrt{5}}{4}. \tag{7.2.140}$$

Hence (7.2.138) is equal to 1, and we have (7.2.131). □

The results on analytic continuation of ${}_pF_q$, derived above from (7.2.126)–(7.2.127), give another proof of Proposition A.6.4, on the analytic continuation of Φ^{-1}, established by a geometrical argument in Appendix A.6.

We leave special functions and return to generalities. We have seen many cases of holomorphic differential equations defined on a domain $\Omega \subset \mathbb{C}$ that is connected but not simply connected, and have seen that, given $z_0 \in \Omega$, we can analytically continue a solution from a neighborhood $D_a(z_0)$ to any other point $z \in \Omega$ along a curve $\gamma_{z_0 z}$, in a way that depends only on the homotopy class of the curve from z_0 to z. If Ω were simply connected, there would be only one such homotopy class, leading to a uniquely defined analytic continuation to a holomorphic function on Ω. If Ω is not simply connected, there can be many such homotopy classes, and one would get a "multivalued" analytic continuation. Examples of this are readily seen in the representation (7.2.60) of a solution to (7.2.45) near a regular singular point, when λ is not an integer.

One way to get a single valued analytic continuation of a solution v to a system of the form (7.2.3) is to work on the universal covering space $\widetilde{\Omega}$ of Ω,

$$\pi : \widetilde{\Omega} \longrightarrow \Omega. \tag{7.2.141}$$

Given a point $z_0 \in \Omega$, the space $\widetilde{\Omega}$ is defined so that, for $z \in \Omega$, $\pi^{-1}(z)$ consists of homotopy classes of continuous curves $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = z_0$ and $\gamma(1) = z$. The space $\widetilde{\Omega}$ has a natural structure of a Riemann surface, for which π in (7.2.141) yields local holomorphic coordinates. Then analytic continuation along a continuous curve $\gamma_{z_0 z}$, as in (7.2.36), defines a holomorphic function on $\widetilde{\Omega}$.

The obstruction to an analytic continuation of a solution to a first-order linear system on Ω being single valued on Ω, or equivalently the way in which its analytic continuation to $\widetilde{\Omega}$ differs at different points of $\pi^{-1}(z)$, for $z \in \Omega$, can be expressed in terms of monodromy, defined as follows.

Fix $z_0 \in \Omega$, and let $v : D_a(z_0) \to \mathbb{C}^n$ solve

$$\frac{dv}{dz} = A(z)v, \quad v(z_0) = v_0, \tag{7.2.142}$$

with $A : \Omega \to M(n, \mathbb{C})$ holomorphic. Consider a continuous curve

$$\gamma : [0,1] \longrightarrow \Omega, \quad \gamma(0) = \gamma(1) = z_0. \tag{7.2.143}$$

Then set

$$\varkappa(\gamma)v_0 = v_\gamma(z_0), \tag{7.2.144}$$

with v_γ defined as in (7.2.36). Clearly the dependence on v_0 is linear, so we have

$$\varkappa(\gamma) \in M(n, \mathbb{C}), \tag{7.2.145}$$

for each continuous path γ of the form (7.2.143). By Proposition 7.2.3, $\varkappa(\gamma_0) = \varkappa(\gamma_1)$ if γ_0 and γ_1 are homotopic curves satisfying (7.2.143). Hence $\varkappa$ induces a well defined map

$$\varkappa : \pi_1(\Omega, z_0) \longrightarrow M(n, \mathbb{C}), \tag{7.2.146}$$

where $\pi_1(\Omega, z_0)$ consists of homotopy classes of continuous curves satisfying (7.2.143). This is the monodromy map associated with the holomorphic system (7.2.142).

We can define a product on paths of the form (7.2.143), as follows. If γ_0 and γ_1 satisfy (7.2.143), we set

$$\begin{aligned} \gamma_1 \circ \gamma_0(t) = \gamma_0(2t), &\qquad 0 \le t \le \frac{1}{2}, \\ \gamma_1(2t-1), &\quad \frac{1}{2} \le t \le 1. \end{aligned} \tag{7.2.147}$$

We have

$$\varkappa(\gamma_1 \circ \gamma_0) = \varkappa(\gamma_1)\varkappa(\gamma_0). \tag{7.2.148}$$

If γ_j are homotopic to σ_j, satisfying (7.2.143), then $\gamma_1 \circ \gamma_0$ is homotopic to $\sigma_1 \circ \sigma_0$, so we have a product on $\pi_1(\Omega, z_0)$, and $\varkappa$ in (7.2.146) preserves products.

Note that the constant map

$$\varepsilon : [0,1] \longrightarrow \Omega, \quad \varepsilon(t) \equiv z_0, \tag{7.2.149}$$

acts as a multiplicative identity on $\pi_1(\Omega, z_0)$, in that for each γ satisfying (7.2.143), γ, $\varepsilon \circ \gamma$, and $\gamma \circ \varepsilon$ are homotopic. Clearly

$$\varkappa(\varepsilon) = I. \tag{7.2.150}$$

Furthermore, given γ as in (7.2.143), if we set

$$\gamma^{-1} : [0,1] \longrightarrow \Omega, \quad \gamma^{-1}(t) = \gamma(1-t), \tag{7.2.151}$$

then $\gamma^{-1}\circ\gamma$ and $\gamma\circ\gamma^{-1}$ are homotopic to ε. Hence

$$\varkappa(\gamma^{-1}\circ\gamma)=\varkappa(\gamma^{-1})\varkappa(\gamma)=I, \tag{7.2.152}$$

so

$$\varkappa(\gamma^{-1})=\varkappa(\gamma)^{-1}, \tag{7.2.153}$$

and in particular $\varkappa(\gamma)\in M(n,\mathbb{C})$ is invertible. We have

$$\varkappa:\pi_1(\Omega,z_0)\longrightarrow Gl(n,\mathbb{C}), \tag{7.2.154}$$

where $Gl(n,\mathbb{C})$ denotes the set of invertible matrices in $M(n,\mathbb{C})$.

One additional piece of structure on $\pi_1(\Omega,z_0)$ worth observing is that, if γ_1,γ_2, and γ_3 satisfy (7.2.143), then

$$\gamma_3\circ(\gamma_2\circ\gamma_1)\text{ and }(\gamma_3\circ\gamma_2)\circ\gamma_1\text{ are homotopic}, \tag{7.2.155}$$

and hence define the same element of $\pi_1(\Omega,z_0)$. We say that the product in $\pi_1(\Omega,z_0)$ is associative. A set with an associative product, and a multiplicative identity element, with the property that each element has a multiplicative inverse is called a *group*. See Appendix 5.13 for a general discussion of groups. The group $\pi_1(\Omega,z_0)$ is called the fundamental group of Ω. Matrix multiplication also makes $Gl(n,\mathbb{C})$ a group. The fact that $\varkappa$ in (7.2.154) satisfies (7.2.148) and (7.2.150) makes it a *group homomorphism.*

To illustrate the monodromy map, let us take

$$\Omega=D_a(0)\setminus 0, \tag{7.2.156}$$

and consider the system (7.2.45), with A holomorphic on $D_a(0)$. Take

$$z_0=r\in(0,a), \tag{7.2.157}$$

and set

$$\beta(t)=re^{2\pi it},\quad 0\le t\le 1, \tag{7.2.158}$$

so $\beta(0)=\beta(1)=z_0$. Note that

$$\begin{aligned}A(z)\equiv A_0&\Longrightarrow v(z)=z^{A_0}(r^{-A_0}v_0)\\&\Longrightarrow v(re^{2\pi it})=e^{2\pi itA_0}v_0\\&\Longrightarrow \varkappa(\beta)=e^{2\pi iA_0}.\end{aligned} \tag{7.2.159}$$

When $A(z)$ is not constant, matters are more complicated. Here is a basic case.

Proposition 7.2.13. *Assume that Proposition* 7.2.9 *applies to the system* (7.2.45)*, so* (7.2.86) *holds. Then*

$$\varkappa(\beta)=U(r)e^{2\pi iA_0}U(r)^{-1}. \tag{7.2.160}$$

Proof. From (7.2.86) we have

$$v(r) = U(r)r^{A_0}v_1, \quad v_1 \in \mathbb{C}^n, \tag{7.2.161}$$

and

$$\begin{aligned} v(e^{2\pi it}r) &= U(e^{2\pi it}r)r^{A_0}e^{2\pi itA_0}v_1 \\ &= U(e^{2\pi it}r)r^{A_0}e^{2\pi itA_0}r^{-A_0}U(r)^{-1}v(r) \\ &= U(e^{2\pi it}r)e^{2\pi itA_0}U(r)^{-1}v(r), \end{aligned} \tag{7.2.162}$$

and taking $t \to 1$ gives (7.2.160). □

Suppose you have a second-order differential equation for which $z = 0$ is a regular singular point, as in (7.2.42). Then the conversion to the system (7.2.45) is accomplished by the substitution (7.2.44), and Proposition 7.2.13 bears on this system, in most cases. However, it might be preferable to instead use the substitution

$$v(z) = \begin{pmatrix} u(z) \\ u'(z) \end{pmatrix}, \tag{7.2.163}$$

producing a different first-order system, with a different monodromy map, though the two monodromy maps are related in an elementary fashion.

We briefly discuss the monodromy map associated to the hypergeometric equation (7.2.113), which is holomorphic on

$$\Omega = \mathbb{C} \setminus \{0, 1\}, \tag{7.2.164}$$

with regular singular points at $z = 0$ and $z = 1$. In this case, we take

$$z_0 = \frac{1}{2}, \quad \beta_0(t) = \frac{1}{2}e^{2\pi it}, \quad \beta_1(t) = 1 - \frac{1}{2}e^{2\pi it}. \tag{7.2.165}$$

We convert (7.2.113) to a first-order 2×2 system, using (7.2.163). In this case, every continuous path γ satisfying (7.2.143) is homotopic to a product

$$\beta_{j_1} \circ \beta_{j_2} \circ \cdots \circ \beta_{j_M}, \quad j_\nu \in \{0, 1\}. \tag{7.2.166}$$

(This product is not commutative!) Thus the monodromy map (7.2.154) is determined by

$$\varkappa(\beta_0),\ \varkappa(\beta_1) \in Gl(2, \mathbb{C}). \tag{7.2.167}$$

Exercises

1. The following equation is called the Airy equation:

$$u''(z) - zu(z) = 0, \quad u(0) = u_0,\ u'(0) = u_1.$$

Show that taking $v = (u, u')^t$ yields the first-order system

$$\frac{dv}{dz} = (A_0 + A_1 z)v, \quad v(0) = v_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

with

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Using the recursion (7.2.14), show that the coefficients v_k in the power series

$$v(z) = \sum_{k=0}^{\infty} v_k z^k$$

satisfy

$$v_{k+3} = \frac{1}{k+3} \begin{pmatrix} \frac{1}{k+2} & 0 \\ 0 & \frac{1}{k+1} \end{pmatrix} v_k.$$

Use this to establish directly that this series converges for all $z \in \mathbb{C}$.

2. Let $v(z)$ solve the first-order system

$$z\frac{dv}{dz} = A(z)v(z), \tag{7.2.168}$$

on $|z| > R$, with $A(z)$ holomorphic in z in this region, and set

$$\tilde{v}(z) = v(z^{-1}), \quad 0 < |z| < R^{-1}. \tag{7.2.169}$$

Show that $\tilde{v}$ satisfies

$$z\frac{d\tilde{v}}{dz} = -A(z^{-1})\tilde{v}(z), \tag{7.2.170}$$

for $0 < |z| < R^{-1}$. We say (7.2.168) has a regular singular point at ∞ provided (7.2.170) has a regular singular point at 0. Note that this holds if and only if $\widetilde{A}(z) = A(z^{-1})$ extends to be holomorphic on a neighborhood of $z = 0$, hence if and only if $A(z)$ is bounded as $z \to \infty$.

3. Let $u(z)$ solve a second-order linear differential equation, e.g., (7.2.32), with coefficients that are holomorphic for $|z| > R$. We say this equation has a regular singular point at ∞ provided the first-order system, of the form (7.2.168), produced via (7.2.44)–(7.2.46), has a regular singular point at ∞. Show that the hypergeometric equation (7.2.113) has a regular singular point at ∞. Show that the confluent hypergeometric equation (7.2.112) does not have a regular singular point at ∞. Neither does the Bessel equation, (7.2.2).

4. Show that the function (7.2.119) solves the hypergeometric equation (7.2.113).

5. Demonstrate the identity (7.2.121) relating J_ν and ${}_1F_1$.

6. Demonstrate the identities (7.2.122) relating elliptic integrals and ${}_2F_1$.

7. Show that if $b \notin \{0, -1, -2, \dots\}$,

$${}_1F_1(a; b; z) = \lim_{c \nearrow \infty} {}_2F_1(a, c; b; c^{-1}z).$$