Introduction

Homological techniques first arose in topology, in work of Poincaré [**174**], at the end of the 19th century. They appeared in algebra several decades later in the 1940s, when Eilenberg and Mac Lane [**59–61**] introduced homology and cohomology of groups and Hochschild [**114**] introduced homology and cohomology of algebras. Since that time, both Hochschild cohomology and group cohomology, as they came to be called, have become indispensable in algebra, algebraic topology, representation theory, and other fields. They remain active areas of research, with frequent discoveries of new applications. There are excellent books on group cohomology such as [**2**, **21**, **22**, **35**, **47**, **76**]. These are good references for those working in the field and are also important resources for those learning group cohomology in order to begin using it in their research. There are fewer such resources for Hochschild cohomology, notwithstanding some informative chapters in the books [**146**, **223**]. This book aims to begin filling the gap by providing an introduction to the basic theory of Hochschild cohomology for algebras and some of its current uses in algebra and representation theory.

Hochschild cohomology records meaningful information about rings and algebras. It is used to understand their structure and deformations, and to identify essential information about their representations. This book takes a concrete approach with many early examples that reappear later in various settings.

We begin in Chapter 1 with Hochschild's own definitions from [**114**], only slightly rephrased in modern terminology and notation, and then connect to definitions based on arbitrary resolutions under suitable conditions. We present some of the important contributions of Gerstenhaber [**82**] beginning in the 1960s that lead us now to think of a Hochschild cohomology ring as a Gerstenhaber algebra, that is, it has both an associative product and a nonassociative Lie bracket. Many properties of Hochschild cohomology rings that are essential in today's applications can be seen in these classical definitions of Hochschild and Gerstenhaber. In Chapter 2 we give detailed descriptions of many equivalent definitions of the associative product (cup product) on Hochschild cohomology. In Chapter 3 we examine several different types of examples: smooth commutative algebras, Koszul algebras, algebras defined by quivers and relations, and algebras built from others such as skew group algebras and (twisted) tensor product algebras. We present the seminal Hochschild-Kostant-Rosenberg (HKR) Theorem on Hochschild homology and cohomology of smooth finitely generated commutative algebras.

Current algebraic applications and developments in the algebraic theory of Hochschild cohomology include the following, explored in detail in the rest of the book.

Some classical geometric notions such as smoothness may be viewed as essentially homological properties of commutative function algebras, allowing interpretations of them in noncommutative settings via Hochschild cohomology. We present these and related ideas in Chapter 4, including Hochschild dimension, smoothness, noncommutative differential forms, Van den Bergh duality, Calabi-Yau algebras, the Connes differential, and Batalin-Vilkovisky structures.

Understanding how some algebras may be viewed as deformations of others calls on Hochschild cohomology, as explained in Chapter 5. There we discuss formal deformations, rigidity of algebras, the Maurer-Cartan equation, Poisson brackets, and deformation quantization. We present the fundamental Poincaré-Birkhoff-Witt (PBW) Theorem as a consequence of a more general theorem on deformations of Koszul algebras. In algebraic deformation theory, the Lie structure on Hochschild cohomology arises naturally; we spend some additional time studying this important structure in detail in Chapter 6. Further probing the associative and Lie algebra structures on Hochschild cohomology and related complexes uncovers infinity algebras. There, binary operations are layered with n-ary operations which in turn have important implications for the original algebra structure. We give a brief introduction to infinity structures and their applications to Hochschild cohomology in Chapter 7.

In representation theory, support varieties may sometimes be defined in terms of Hochschild cohomology; these are geometric spaces assigned to modules that encode representation-theoretic information. Support varieties for finite-dimensional algebras are introduced and explored in Chapter 8. This theory began in the parallel setting of finite group cohomology. There are strong connections between Hochschild cohomology and group cohomology that we analyze more generally for Hopf algebras in Chapter 9. Hopf algebras are those algebras whose categories of modules are tensor categories, and include many examples of interest such as group algebras, universal enveloping algebras of Lie algebras, and quantum groups. Relationships between Hochschild cohomology and Hopf algebra cohomology lead to better understanding of both and of all their applications. Inspecting these relationships, we connect the two first appearances of homological techniques in algebra in the form of group cohomology [**59–61**] and Hochschild cohomology [**114**].

We include an appendix with needed background material from homological algebra. The appendix is largely self-contained, however, proofs are omitted, and instead the reader is referred to standard homological algebra textbooks such as [**48**, **112**, **151**, **168**, **187**, **223**] for proofs and more details.

This introductory text is not intended to be a comprehensive treatment of the whole subject of Hochschild cohomology, which long ago expanded well beyond the reach of a single book. Necessarily many important topics are left out. For example, we do not treat Tate-Hochschild cohomology, relative Hochschild cohomology, Hochschild cohomology of presheaves and schemes, connections to cyclic homology and K-theory, Hochschild cohomology of abelian categories, topological Hochschild cohomology, Hochschild cohomology of differential graded and A_∞-algebras and categories, nor operads. Hochschild homology is an important subject in its own right, and we spend only a little time on it in this book. Also, here we will almost exclusively work with algebras over a field, both for simplicity of presentation in this introductory text and to take advantage of a great array of good properties and current applications for algebras over a field.

We provide a few references for the reader looking for details on some of the topics that are not in this book. This list is not meant to be complete, but rather a beginning, and further references may be found in each of these: more on Hochschild homology can be found in the standard references [**146**, **223**]. Tate-Hochschild cohomology, stable Hochschild cohomology, and singular Hochschild cohomology are $\mathbb{Z}$-graded theories while Hochschild cohomology itself is $\mathbb{N}$-graded; see, for example, [**29**, **74**]. Relative Hochschild cohomology and secondary Hochschild cohomology are designed for a ring and subring pair; see, for example, [**106**, **115**, **205**]. There is a version of Hochschild cohomology for coalgebras and bicomodules [**57**]. Hochschild cohomology is defined for presheaves of algebras and schemes, and used in algebraic geometry; see, for example, [**85**, **86**, **132**, **213**]. Topological Hochschild homology and cohomology are related theories in algebraic topology; see, for example, [**173**]. Hochschild cohomology is used in functional analysis, with connections to properties of Banach algebras, von Neumann algebras, and locally compact groups; see, for example, [**122**, **198**]. Many important applications of the theory of Hochschild cohomology involve its connections to cyclic homology and cohomology and algebraic K-theory; see, for example, [**146**, **223**]. Hochschild homology and cohomology can be defined for some types of categories; see, for example, [**150**, **161**]. Some operads underlie much of the structure of Hochschild cohomology, a hint of which appears in the infinity structures of Chapter 7 here; see, for example, [**152**, **153**]. Formality and Deligne's Conjecture are barely touched in Chapter 7 here, and more details may be found in the references given in Section 7.6 and in [**153**]. Hochschild cohomology may be realized as the Lie algebra of the derived Picard group of an algebra; see, for example, [**128**]. Hochschild cohomology of differential graded and A_∞-algebras and categories, for example, are in [**127**, **130**].

This book is written for graduate students and working mathematicians interested in learning about Hochschild cohomology. It can serve as a reference for many facts that are currently only found in research papers, and as a bridge to some more advanced topics that are not included here. The main prerequisite for students is a graduate course in algebra. It would also be helpful to have taken further introductory courses in homological algebra or algebraic topology and in representation theory, or else to have done some reading in these subjects. However, all of the required homological algebra background is summarized in the appendix, with references, and a motivated reader might rely solely on this as homological algebra background. Beyond the first three chapters of this book, the remaining chapters are largely independent of each other, and so there are many options for basing a one-semester graduate course on this book. A one-semester course could start with a treatment of Chapter 1 and selected sections from Chapters 2 and 3, possibly including material from the appendix depending on the background of the students. Then the course could focus on a subset of the remaining chapters: a course with a focus on noncommutative geometry could continue with Chapter 4; a course with a focus on algebraic deformation theory and related structures could instead continue with Chapter 5 and the related Chapters 6 and 7, as time allowed; a course with a focus on Hopf algebras, group algebras, or support varieties could instead continue with Chapters 8 and/or 9. A full-year course might include most of the book and time for a complete introduction to or review of homological algebra based on the appendix.

This book came into being as an aftereffect of some lecture series that I gave and through interactions with many people. I first thank Universidad de Buenos Aires, and especially Andrea Solotar and her students, postdocs, and colleagues, for hosting me for several weeks in 2010. During that time I gave a short course on Hopf algebra cohomology that led to an early version of Chapter 9 on which they gave me valuable feedback. I thank the Morningside Center in Beijing and the organizers and students of a workshop there in 2011 for the opportunity to give lectures on support varieties that expanded into the current Chapter 8. I thank the Mathematisches Forschungsinstitut Oberwolfach for its hospitality during several workshops where the idea for this book began in discussions with Karin Erdmann and Henning Krause.

Most of this book was written during the academic year 2016–17 that I spent at the University of Toronto visiting Ragnar-Olaf Buchweitz and his research group. It was with deep sadness that I learned of his death the following fall. His legacy lives on and continues to grow through the mathematical writings that are still being completed by his many collaborators, as well as others he influenced. He was a great friend and mentor to so many of us.

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